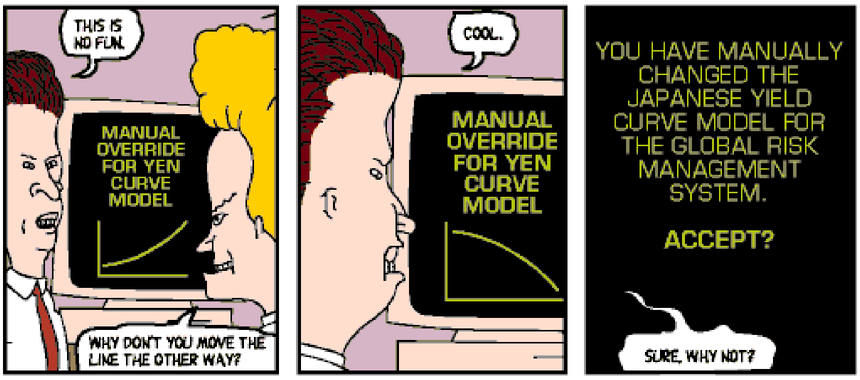
4

Interest rates



Time is what stops everything happening at once.

—John Wheeler

... interest, which means the birth of money from money, is applied to the breeding of money because the offspring resembles the parent. Wherefore, of all modes of getting wealth, this is the most unnatural

—Aristotle

Overview

- Types of rates
- Measuring interest rates
- Zero rates
- Bond pricing
- Determining zero rates
- Forward rates
- Forward rate agreements
- Duration
- Convexity
- Theories of the term structure

Moved to Chapter 6 in Hull 6th ed

- Day count conventions
- Quotations for Treasury bonds
- Treasury bond futures
- Eurodollar futures
- Duration-based hedging strategies
- Hedging portfolios of assets and liabilities

Introduction

- Understanding IRs essential
- How to measure and analyse
- Rates:
  - compounding frequency
  - continuously compounded
  - zero rates, par yields, yield curves
- Bonds
  - bond pricing
  - bootstrapping the (zero coupon treasury) yield curve
- Forward
  - forward rates, FRAs
- Duration and convexity
- Ignore day count conventions

Types of rates

Introduction

- IR is the price of money
- Money borrower promises to pay lender
- Different
  - for each FX
  - within each FX due to credit risk

Treasury rates

- Government in its own country
- e.g. $\left\{\begin{matrix}\text{US}\\ \text{Japanese}\end{matrix}\right\}$ Treasury rates for $\left\{\begin{matrix}\text{US}\\ \text{Japanese}\end{matrix}\right\}$ government in $\left\{\begin{matrix}\text{USD \$}\\ \text{JPY ¥}\end{matrix}\right\}$
- Usually assumed that government will not default on an obligation denominated in own currency (why?). Hence, (credit) "risk-free" rate (see below)
- T rates useful to price T bonds
- Derivative pricing, used

to define payoff	✓
discounting	✗*

  *Use LIBOR instead

LIBOR

- *London interbank offer rate*
- International banks trade on 1, 3, 6, 12-month deposits denominated in all of the world's major currencies
- For large, wholesale deposits
- Citibank, AUD quote bid 6.25%, offer 6.375
- You need a good credit rating to make □ or accept □ LIBOR quotes (Which?)
- To AA rated instution, LIBOR is short-term *opportunity cost of capital*
- Also LIBID = "*London interbank bid rate*"
- From point of view of bank:

Buy		Sell
Borrow		Lend / advance
Bid		Offer
Pay on deposits		Receive
LIBID	<	LIBOR

- LIBOR > Treasury (why?)
- Set by supply and demand in interbank market

- Usually taken as the risk-free rate rather than T-rates (why?)
- LIBID and LIBOR trade in *Eurocurrency market* (no government)
- For €, *Euribor* (Euro Interbank Offer Rate)

Repo rates

- Investment dealer funds trading with *repurchase agreement.*
- Sell and buy back at higher price
- Difference in prices between sell and rebuy is interest
- Low credit risk
- Commonest is *overnight repo*, but *term repos* too

Measuring interest rates

- The compounding frequency used for an interest rate is the unit of measurement
- Quarterly vs annual compounding cf. miles vs. kilometers

Quoting convention for discrete compounding

A rate R_m with discrete compounding rate m, indicates that over a time period $\frac{1}{m}$, a unit of money grows to $\left(1+\frac{R_m}{m}\right)$.

- E.g. when the rate is 10% , compounded
  - **annually**, over 12 months, £100 grows to $£100 + £\,\frac{10}{1} = £110.$
  - **semi annually**, over 6 months, £100 grows to $£100 + £\,\frac{10}{2} = £105.$
  - **quarterly**, over 3 months, £100 grows to $£100 + £\,\frac{10}{4} = £102.50.$
  - **monthly**, over 1 months, £100 grows to $£100 + £\,\frac{10}{12} = £100.83.$
- Over a full year is
  - **annually**, £100 grows to $£100 \times 1.1 = £110.$
  - **semi annually**, £100 grows to $£100 \times 1.05 \times 1.05 = £110.25.$
  - **quarterly**, £100 grows to $£100 \times 1.025 \times 1.025 \times 1.025 \times 1.025 = £110.381.$

Effect of compounding

Code

Output

Table 4.1. Effect of compounding frequency on the value of $100 at the end of one year, when the interest rate is 10%.

	Compounding frequency	*Value \$100 at end of year*
Annually	1	110.
Semi−annually	2	110.25
Quarterly	4	110.381
Monthly	12	110.471
Weekly	52	110.506
Daily	365	110.516

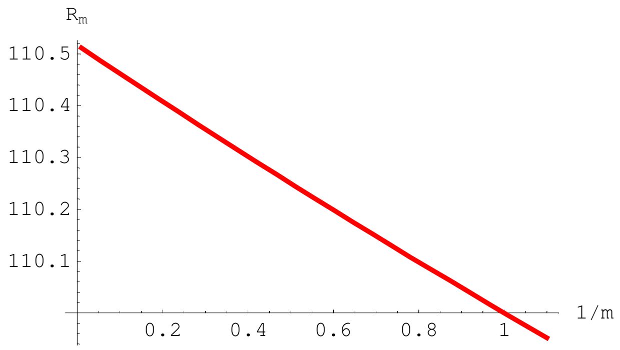


Figure 4.1: Effect of compounding frequency on the value of £100 after one year for an annualised interest rate of 10%.

Continuous Compounding

- Hull page 79
- In limit $m \to \infty$, obtain continuously compounded interest rates
- \$100 grows to $\$100 e^{RT}$ when invested at a cc rate R for time T
- \$100 received at time T discounts to $\$100 e^{-RT}$ at time zero when the cc discount rate is R

Conversion Formulas

Notation

R_c	continuously compounded rate
R_m	same rate with compounding m times per year

Equations relating discrete and continuous compounding rates

$$\begin{aligned} R_c &= m \ln(1 + \tfrac{R_m}{m}) \\ R_m &= m\,(e^{R_c/m} - 1) \end{aligned} \tag{4.1}$$

Example 4.1. Find the equivalent continuously compounded interest rate corresponding to 10% with semiannual compounding.

$$R_C = m \ln\left(1 + \frac{R_m}{m}\right) = 2 \ln\left(1 + \frac{0.1}{2}\right) = 0.09758$$

Example 4.2. Find the equivalent quarterly compounded interest rate corresponding to 8% with continuous compounding.

$$R_m = m\left(e^{R_c/m} - 1\right) = 4\left(e^{0.08/4} - 1\right) = 0.0808$$

Zero Rates

Diagram of cashflows of zero-coupon and coupon-paying bonds

Code

Output

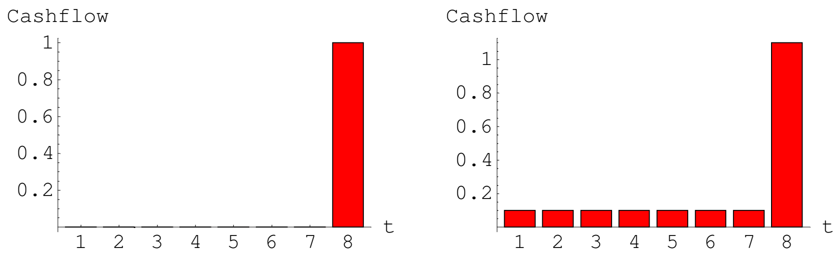


Figure 4.2: Cash flow diagrams for zero coupon (left) and coupon paying (right) bonds.

- *n-year zero rate* (*zero-coupon* rate, *spot* rate), starts today, lasts *n*-years
- interest and *principal* (or *face value* or *nominal value* or *par values*) realized at the end of *n* years — no intermediate payments
- interest earned on a *zero coupon bond* (zcb) or *pure discount bond* (pdb), whose value we denote by $B(t, T)$
- Note that zcbs are a useful abstraction, but that most traded bonds are coupon bearing

Zero rate

Definition 4.1. A *zero rate* (or *spot rate*), for maturity T is the rate of interest earned on an investment that provides a payoff only at time T

Table 4.2. Treasury zero rates

Maturity (years)	*Zero Rate(% cont comp)*
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

- Hull Table 4.2, page 81

Sneak preview

- Why these ideas are so crucial to IR modelling
- Hull Chapters 28 & 29
- Three alternative representations for the term structure

$P(t,T)$	price of zero-coupon bond maturing at time T, at any instant $t \leq T$
$Y(t,T)$	yield to maturity of a zcb — is the slope of the chord to $\ln P(t,T)$
$f(t,T)$	instantaneous forward rate — is the slope of the tangent to $\ln P(t,T)$

are related

$$\begin{aligned} P(t, T) &= e^{-Y(t,T)(T-t)} = e^{-\int_t^T f(t,s)\,ds} \\ Y(t, T) &= \frac{-1}{T-t} \ln P(t, T) = \frac{1}{T-t} \int_t^T f(t, s)\, ds \\ f(t, T) &= -\frac{1}{P(t, T)} \frac{\partial P(t, T)}{\partial T} = Y(t, T) + (T-t) \frac{\partial Y(t, T)}{\partial T} \end{aligned} \tag{4.2}$$

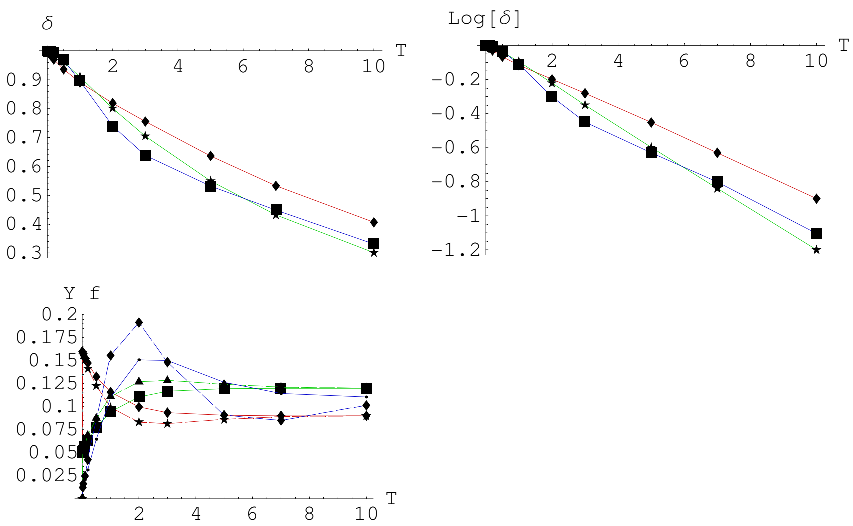


Figure 4.3: ZCB bond price, log of bond price and yield & forward rates

- To be continued in FM07...

Bond pricing

Overview

- Price of coupon paying bond
- Bond yield
- Par yield

Price of coupon paying bond

- To calculate the cash price of a bond we discount each cash flow at the appropriate zero rate

Example 4.3. Given the zero rates in the table above, price a 2-year Treasury bond, with principal $100 that pays a 6% per annum semi-annual coupon.

To calculate the PV of the $\begin{pmatrix}\text{first}\\ \text{second}\\ \dots\end{pmatrix}$ cashflow, we discount at $\begin{pmatrix}5.0\\ 5.8\\ \dots\end{pmatrix}$% for $\begin{pmatrix}\text{6 months}\\ \text{1 yr}\\ \dots\end{pmatrix}$.

$$\$\ \overbrace{3\,e^{-0.05\times 0.5} + 3\,e^{-0.058\times 1.0} + 3\,e^{-0.064\times 1.5}}^{\text{Coupons prior to maturity}} + \overbrace{103}^{\text{Coupon + principal}}\ e^{-0.068\times 2.0} = \$98.39$$

Mathematica demonstration

- We can program *Mathematica* to price a bond given
  - yield curve in the form of a matrix of times and yields
  - coupon, *cm* per annum
  - compounding frequency, *m*

```
CouponBond(yieldCurve_?MatrixQ, cm_, m_) :=
  Module[{T = yieldCurve[[1]], Y = yieldCurve[[2]], n = Dimensions[yieldCurve][[2]]},
```

$$100\left(e^{-Y[\![n]\!]\,T[\![n]\!]}\left(\frac{cm}{m}+1\right)+\sum_{i=1}^{n-1}\frac{cm}{m}\,e^{-Y[\![i]\!]\,T[\![i]\!]}\right)]$$

- E.g. let us check the example above

```
With[{yieldCurve = ( 0.5`    1       1.5`    2.`
                     0.05`   0.058`  0.064`  0.068` ), cm = 0.06, m = 2},
 CouponBond[yieldCurve, cm, m]]
```

```
98.3851
```

- Programming details are not important here and are not examinable. Code is for demonstration purposes only.

Bond yield

Definition 4.2. The *bond yield* for a bond is the discount rate that makes the present value of the cash flows equal to the market price.

- Also *yield to maturity* or *bond equivalent yield*

Equation for bond yield

- The bond yield (continuously compounded) is given by solving

$$\overbrace{\sum_{i=1}^{n} c\, e^{-Y\,T_i}}^{\text{coupons}} + \overbrace{1\, e^{-Y\,T_n}}^{\text{principal}} = P \tag{4.3}$$

where

n	number of cashflows
c	coupon payment cashflow
T_i	time of ith payment (coupon or principal) payment, $i=1,\ldots,n$
P	given (market) price of bond
Y	bond yield

- Note, $\sum_{i=1}^{n} c\, e^{-Y\,T_i} + e^{-Y\,T_n} = \sum_{i=1}^{n-1} c\, e^{-Y\,T_i} + (1+c)\, e^{-Y\,T_n}$

Example

Example 4.4. Find the bond yield for a 2-year Treasury bond, with principal \$100 with market price equal to \$98.39.

Numerical root search. Could use Solver in Excel. Here we will use *Mathematica*'s FindRoot function....

Mathematica experiment

- We can program *Mathematica* to price a bond given
  - times for coupon payments (vector)
  - yield (scalar, to apply to all maturities, corresponding to a flat yield curve)
  - coupon, c_m, annualised
  - compounding frequency, m

CouponBondFlatYield(T_List, Y_, cm_, m_) :=

$$\text{Module}\left[\{n = \text{Length}[T]\}, 100\left(e^{-Y\,T[[n]]}\left(\frac{cm}{m}+1\right)+\sum_{i=1}^{n-1}\frac{cm\, e^{-Y\,T[[i]]}}{m}\right)\right]$$

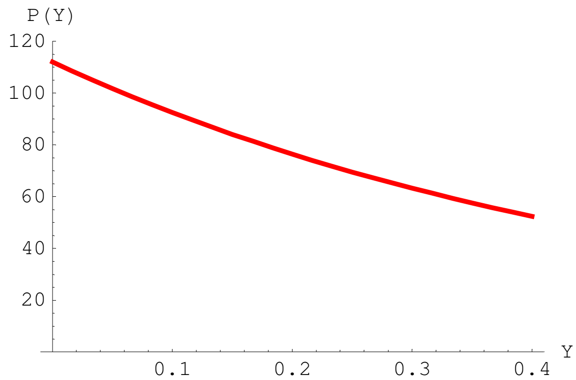
- Test the definition for general Y

```
CouponBondFlatYield[{0.5`, 1, 1.5`, 2.`}, Y, 0.06, 2]
```

$100\,(1.03\,e^{-2.\,Y} + 0.03\,e^{-1.5\,Y} + 0.03\,e^{-Y} + 0.03\,e^{-0.5\,Y})$

- Plot this

```
Plot[CouponBondFlatYield[{0.5`, 1, 1.5`, 2.`}, Y, 0.06, 2],
  {Y, 0, 0.4}, PlotStyle -> {Thickness[0.01], Hue[0]},
  AxesLabel -> {"Y", "P(Y)"}, PlotRange -> {0, 120}];
```



- Numerically solve for the bond yield, Y, such that our calculated bond price agrees with the market price

```
FindRoot[CouponBondFlatYield[{0.5`, 1, 1.5`, 2.`}, Y, 0.06, 2] == 98.39,
 {Y, 0.1}]
```

{Y → 0.0675982}

- Check: when we take the bond yield to be 6.76%, is the calculated bond price \$98.39 as we require?

```
CouponBondFlatYield[{0.5`, 1, 1.5`, 2.`}, 0.067598, 0.06, 2]
```

98.39

Answer

...

Bond yield is $y = 0.0676$ or 6.76 %.

Discrete case

- Hull develops ideas for price of zcb using continuous compounding "$B(0,\,T) = e^{-Y_C(0,T)\,T}$".
- Analogous expressions exist for discrete case "$B(0,\,i) = \frac{1}{(1+Y_D(0,i))^i}$"

Par yield

Definition

Definition 4.3. The *par yield* for a certain maturity is the coupon rate that causes the bond price to equal its face value.

Equation

$$100\left(\sum_{i=1}^{n-1} c\, e^{-Y_i T_i} + (1 + c)\, e^{-Y_i T_i}\right) = 100 \tag{4.4}$$

where

n	number of cashflows, $(n-1)$ of which are coupons, and the nth is coupon + principal
c	coupon payment
T_i	time of ith payment (coupon or principal) payment, $i=1,\ldots,n$
Y_i	bond yield (zero rate) for time T_i ←Typo

- Solving for c

$$c = \frac{1 - e^{-Y_n T_n}}{\sum_{i=1}^{n} e^{-Y_i T_i}} \tag{4.5}$$

- Here c is the actual coupon payment
- Often coupon rates are quoted assuming a frequency m, in which case the annualised coupon rate c_m satisfies $c = c_m / m$.
- If we name $d = e^{-Y_n T_n}$ and $A = \sum_{i=1}^{n} e^{-Y_i T_i}$, we obtain

$$c_m = \frac{(1 - d)\, m}{A} \tag{4.6}$$

Example 4.5. Find the annualised par yield for a 2-year Treasury bond, making semi-annual coupon payments, with principal \$100 with market price equal to \$98.39.

$\frac{c_2}{2}\, e^{-0.05\times0.5} + \frac{c_2}{2}\, e^{-0.058\times1.0} + \frac{c_2}{2}\, e^{-0.064\times1.5} + \left(1 + \frac{c_2}{2}\right) e^{-0.068\times2.0} = 1$

$c_2 = \frac{2\left(1 - e^{-0.068\times2.0}\right)}{e^{-0.05\times0.5} + e^{-0.058\times1.0} + e^{-0.064\times1.5} + e^{-0.068\times2.0}} = 6.87\,\%$ (annualized)

or taking $A = e^{-0.05\times0.5} + e^{-0.058\times1.0} + e^{-0.064\times1.5} + e^{-0.068\times2.0} = 3.70027$ and $d = e^{-0.068\times2.0} = 0.87284$

$c_2 = \frac{2\,(1-d)}{A} = 6.87\,\%$

- Check algebra

```
With[{A = e^(-0.05 0.5) + e^(-0.058 1.0) + e^(-0.064 1.5) + e^(-0.068 2.0), d = e^(-0.068 2.0)},
 {A, d, 2 (1 - d)/A}]
```

$$\text{With}\left[\left\{A = e^{-0.05\,0.5} + e^{-0.058\,1.0} + e^{-0.064\,1.5} + e^{-0.068\,2.0},\ d = e^{-0.068\,2.0}\right\},\ \left\{A,\ d,\ 2\frac{(1-d)}{A}\right\}\right]$$

```
{3.70027, 0.872843, 0.0687288}
```

Mathematica **experiment**

- Use the function for the coupon paying bond in terms of the yield curve

$$\text{With}\left[\left\{\text{yieldCurve} = \begin{pmatrix} 0.5\text{`} & 1 & 1.5\text{`} & 2.\text{`} \\ 0.05\text{`} & 0.058\text{`} & 0.064\text{`} & 0.068\text{`} \end{pmatrix},\ m = 2\right\},\right.$$

```
 Solve[CouponBond[yieldCurve, cm, m] == 100, cm]]
```

```
{{cm → 0.0687288}}
```

Determining zero rates

- How to calculate Treasury zero rates from the prices of Treasury bonds

Sample data

- Hull Table 4.3, page 82
- Data for 5 bonds, 2 paying coupons semi-annually
  - Principal
  - Time to maturity
  - Coupon
  - Market price

Make table

Code

Output

Table 4.3. Data for 5 bonds, 2 paying a semi-annual coupon

Principal	Time to maturity	Coupon	Cash price
100	0.25	0	97.5
100	0.5	0	94.9
100	1.	0	90.
100	1.5	8.	96.
100	2.	12.	101.6

The bootstrap method

Example 4.6. From the prices of the 5 bonds above, infer the zero rate (yield) curve.

- An amount 2.5 can be earned on 97.5 during 3 months. The annualised 3-month rate is $4\,\frac{100-97.5}{97.5} = 4\,\frac{2.5}{97.5} = 0.10256$ or 10.256% with quarterly compounding. This is 10.127% with continuous compounding

$$R_c \;=\; m\ln\left(1+\frac{R_m}{m}\right) = 4\,\text{Log}\left[1+\frac{0.10256}{4}\right] = 0.101267$$

- Similarly the 6 month and 1 year rates are 10.469% and 10.536% with continuous compounding

$$2\,\text{Log}\left[1+\frac{2\,(100-94.9)/94.9}{2}\right] = 0.104693$$

$$1\,\text{Log}\left[1+\frac{1\,(100-90)/90}{1}\right] = 0.105361$$

- To calculate the 1.5 year rate we solve

$$4\,e^{-0.10469\times 0.5} + 4\,e^{-0.10536\times 1.0} + 104\,e^{-R\times 1.5} = 96$$

$$R = -\frac{1}{1.5}\,\text{Log}\left[\frac{96-(4\,e^{-0.10469\times 0.5}+4\,e^{-0.10536\times 1.0})}{104}\right]$$

to get $R = 0.10681$ or 10.681 %

- Similarly the two-year rate is 10.808%

$$6\,e^{-0.10469\times 0.5} + 6\,e^{-0.10536\times 1.0} + 6\,e^{-0.10681\times 1.5} + 106\,e^{-R\times 2.0} = 101.6$$

$$R = -\frac{1}{2.0}\,\text{Log}\left[\frac{101.6-(6\,e^{-0.10469\times 0.5}+6\,e^{-0.10536\times 1.0}+6\,e^{-0.10681\times 1.5})}{106}\right]$$

Some checks

Figure of zero curve

Code

Output

Table 4.4. Yield curve obtained by bootstrapping

Time to maturity	Zero rate
0.25	0.10127
0.5	0.10469
1	0.10536
1.5	0.10681
2.	0.10808

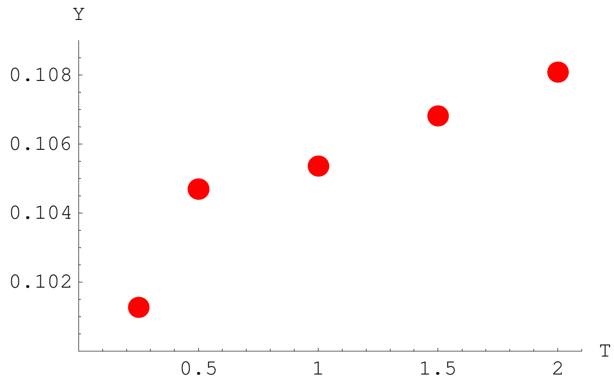


Figure 4.4: Yield curve obtained by bootstrapping

Interpolation

- Intermediate points on curve obtained by interpolation

- Simple approach: linear (straight lines between points)
- Splines etc.

Forward rates

Definition

Definition 4.4. The *forward rate* is the future zero rate implied by today's term structure of interest rates

- Basic idea:
  - given time 0 cost of \$1 at time T_1
  - given time 0 cost of \$1 at time T_2
  - at time 0 lock in the time T_1 cost of \$1 at T_2:
    "$B(0, T_2) = B(0, T_1)\, B(0, T_1, T_2)$"
- Each bond (spot or forward) has a rate associated with it (spot or forward)
  "$e^{-T_2\, Y(0,T_2)} = e^{-T_1\, Y(0,T_1)}\, e^{-(T_2-T_1)\, Y(0,T_1,T_2)}$"

Calculation of forward rates

- Hull Table 4.5, page 85

Example 4.7. The continuously compounded zero rates for n-year investment are given by the following matrix (upper row is n, lower row is zero rate):
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3.0 & 4.0 & 4.6 & 5.0 & 5.3 \end{pmatrix}.$$
• What does it mean to say that the 1-year rate is 3.0% and the 2-year rate is 4.%?
• What rate of interest (the "forward" interest) holds over year 2 such that if combined with the rate of interest over year 1 equals the overall rate of interest over the two years?
• Find the forward rates for borrowing over the nth year for $n = 2, \ldots 5$.

• The 1-year continuously compounded rate is 3.0% means that £100 invested now will grow to £$100e^{0.03} =$ £103.05 over 1 yr

• The 2-year continuously compounded rate is 4.0% means that £100 invested now will grow to £$100e^{0.04\times 2} =$ £108.33 over 2 yrs

• $100\, \overset{\text{spot}}{e^{-0.03\times 1}}\, \overset{\text{forward}}{e^{-R_F\times 1}} = 100\, \overset{\text{spot}}{e^{-0.04\times 2}}$

$$R_F = \frac{0.04\times 2 - 0.03\times 1}{1} = 0.05$$

Forward rate over year 1 is 5%

• Forward rates over interval $(T, T+1)$

T	$R_F(T, T+1)$
2	$\frac{0.04\times2-0.03\times1}{1} = 0.05$
3	$\frac{0.046\times3-0.04\times2}{1} = 0.058$
4	$\frac{0.05\times4-0.046\times3}{1} = 0.062$
5	$\frac{0.053\times5-0.05\times4}{1} = 0.065$

- Check calculation with *Mathematica*

$$\left\{\frac{0.04\,2-0.03\,1}{1}, \frac{0.046\,3-0.04\,2}{1}, \frac{0.05\,4-0.046\,3}{1}, \frac{0.053\,5-0.05\,4}{1}\right\}$$

```
{0.05, 0.058, 0.062, 0.065}
```

Check

Code

Output

```
table: fwd rates
```

Table 4.5. Zero rates for n-year investment and forward rates for borrowing for one year inferred from these.

T	Zero rate	Fwd rate
1	3.	0
2	4.	5.
3	4.6	5.8
4	5.	6.2
5	5.3	6.5

Formula for forward rates

- Suppose that the zero rates for time periods T_1 and T_2 are R_1 and R_2 with both rates continuously compounded.
- The forward rate for the period between times T_1 and T_2 is

$$R_F = \frac{R_2\,T_2 - R_1\,T_1}{T_2 - T_1} \tag{4.7}$$

Derivation

- Not in Hull
- Result is more intuitive using a different notation:

T_i	maturity times of bonds, i=1,2, $0<T_1<T_2$
$B(0,T_i)$	cost at time 0 for a zcb maturing at time T_i, i=1,2
$B(0,T_1,T_2)$	time 0 forward price for a bond at time T_1 maturing at time T_2
$Y(0,T_i)$	spot yield over interval $(0,T_i)$ — i.e. R_i
$Y(0,T_1,T_2)$	time 0 forward yield for interval (T_1,T_2) — i.e. R_F

- Relationship between spot and forward bond prices

$$B(0, T_2) = B(0, T_1)\, B(0, T_1, T_2)$$

- Implies relationship between spot and forward yields

$$e^{-T_2\, Y(0,T_2)} = e^{-T_1\, Y(0,T_1)}\, e^{-(T_2 - T_1)\, Y(0,T_1,T_2)}$$

- Solve for $Y(0, T_1, T_2)$

Instantaneous forward rate

Definition 4.5. The *instantaneous forward rate* for a maturity T is the forward rate that applies for a very short time period starting at T. It is $R_F = R + T\,\frac{\partial R}{\partial T}$ where R is the T-year rate

$$R_F = R + T\,\frac{\partial R}{\partial T} \qquad (4.8)$$

Upward vs downward sloping yield curve

- Yield curve slopes:
  - *upward*

    Fwd Rate > Zero Rate > Par Yield

  - *downward*

    Par Yield > Zero Rate > Fwd Rate

Forward rate agreement

- Might be used by company that wishes to borrow cash at a future date T_1 for the period $[T_1, T_2]$
- Locks in rate of interest
- Cash flow determined by
  - length of time-period
  - interest rates:
    - prespecified
    - prevailing at the future time
- Forward contract on uncertain future cash flow
- Can synthesise with *forward-forward loan* – LT loan + ST deposit
- OTC

Definition

Definition 4.6. A *forward rate agreement* (FRA) is an agreement that a certain rate will apply to a certain principal during a certain future time period.

- Interest payments exchanged – predetermined R_K $\leftrightarrow$ market R_F

FRA valued by assuming that the forward interest rate is certain to be realized

Payoffs

- Hull Equations 4.9 and 4.10 page 88
- Notation

R_K	IR agreed to in FRA
R_F	forward LIBOR IR for period $[T_1, T_2]$ calculated today
R_M	actual LIBOR IR observed in the market at T_1 for interval $[T_1, T_2]$
L	principal

- Compounding frequency reflects maturity
- X lends to Y
- Normally at LIBOR R_M
- FRA $\Rightarrow$ X earns R_K instead
- Extra interest due to FRA

$$L(R_K - R_M)(T_2 - T_1) \tag{4.9}$$

- Y gets minus this
- X receives fixed, pays market; Y pays fixed, receives market,
- FRAs settled at T_2

Settlement in advance

- Usually, settled at time T_1
- Payment at T_2, discounted back to T_1
- Company X's payoff at T_1:

$$\frac{L(R_K - R_M)(T_1 - T_2)}{1 + R_M(T_2 - T_1)} \tag{4.10}$$

- Y gets minus this.

Example

Example 4.8. A company enters into an FRA so that it will receive 4% fixed on a principal of $1 million for a 3-month period starting in 3 years. If LIBOR turns out to be 4.5%, find the cashflow to the lender at the end of the borrowing period and the equivalent cashflow at the start of the borrowing period, expressing all interest rates using quarterly compounding.

Cashflows

At end of period: T_2

$$\$10^6 \times (0.04 - 0.045) \times 0.25 = -\$1250$$

At start of period: T_1

$$-\$\frac{1250}{1+0.045\times 0.25} = \$1236.09$$

Valuation

- Key observation:

FRA worth zero when $R_K = R_F$

- Why buy an FRA for $$ if you can lock in price of forward borrowing
- E.g. borrow $[0, T_1]$, lend $[0, T_2]$; borrowing costs $[T_1, T_2]$ locked at time 0.
- Go long FRA with rate R_K, short FRA with rate R_F
- Costs are V_{FRA} and 0 respectively
- Net (deterministic) payoff at T_2 is $L((R_K - R_M) - (R_F - R_M))(T_1 - T_2)$
- Value of FRA where a fixed rate R_K will be *received* on a principal L between times T_1 and T_2 is

$$V_{\mathrm{FRA}} = \overbrace{L(R_K - R_F)(T_1 - T_2)}^{\text{Cash flow at time } T_2} e^{-R_2 T_2} \tag{4.11}$$

- Value of FRA where a fixed rate is *paid* is minus this
- R_F is the forward rate for the period and R_2 is the zero rate for maturity T_2

Example

Example 4.9. Using the LIBOR zero and forward rates from before, value an FRA where we will receive 6% (annual compounding) on a principal of $1 million between the end of year 1 and the end of year 2.

$$R_m = m\left(e^{R_c/m}-1\right) = 1\left(e^{\overset{\text{Fwd}}{0.05}/1}-1\right) = 0.05127 = R_F$$

$$V_{FRA} = L(R_K - R_F)(T_2 - T_1)\, e^{-R_2 T_2} = \$10^6 \times (0.06 - 0.05127)\, e^{-\overset{\text{2 yr spot}}{0.04} \times 2} = \$8058$$

Duration

- Hull page 89
- Name ⇒ average time to receive payments

Definition

Definition 4.7. The *duration* of a bond that provides cash flow c_i at time t_i is

$$D = \sum_{i=1}^{n} t_i \frac{c_i\, e^{-y t_i}}{B}$$

where B is its price and y is its yield (continuously compounded)

- Cf. "centre of mass"

Sensitivity

- This leads to

$$\frac{\Delta B}{B} = -D\,\Delta y \tag{4.12}$$

Example

Example 4.10. Express the price of a bond, B, in terms of a sum of n discounted cash flows, c_i, paid at time t_i, $t = 1, \ldots, n$, assuming a constant, continuously compounded yield y. Differentiate this w.r.t. y to obtain an expression for the duration $D = -\frac{1}{B}\frac{dB}{dy}$. In what sense is this the average maturity of the instrument?

$$B = \sum_{i=1}^{n} c_i\, e^{-y t_i}$$

$$\frac{dB}{dy} = -\sum_{i=1}^{n} t_i\, c_i\, e^{-y t_i}$$

$$D = -\frac{1}{B}\frac{dB}{dy} = \sum_{i=1}^{n} t_i \overbrace{\frac{c_i\, e^{-y t_i}}{B}}^{\text{Weights sum to one}}$$

Interpretation

Duration is the proportional change in the bond price per unit (parallel) shift in the yield curve

Example

Example 4.11. Calculate the duration of a 3-year bond paying semi-annuallly a coupon of 10% (annualized), with a face value of \$100 and a yield of 12% (continuously compounded).

Time	Cash flow	PV	Wt	Wt×time
0.5	5	$5 \times e^{-0.12\times0.5} = 4.709$	$\frac{4.709}{94.213} = 0.050$	0.025
1.0	5	$5 \times e^{-0.12\times1.0} = 4.435$	$\frac{4.435}{94.213} = 0.047$	0.047
1.5	5	$5 \times e^{-0.12\times1.5} = 4.176$	$\frac{4.176}{94.213} = 0.044$	0.066
2.0	5	$5 \times e^{-0.12\times2.0} = 3.933$	$\frac{3.933}{94.213} = 0.042$	0.083
2.5	5	$5 \times e^{-0.12\times2.5} = 3.704$	$\frac{3.704}{94.213} = 0.039$	0.098
3.0	105	$105 \times e^{-0.12\times3.0} = 73.256$	$\frac{73.256}{94.213} = 0.778$	2.333
Total	130	$B = 94.213$	1.000	$D = 2.653$

Example

Example 4.12. Using the bond from the previous example, find the change in the bond price for a 10 basis point increase in the yield
i) approximately using your knowledge of the duration
ii) exactly

i) $B + \Delta B = B - B\,D\,\Delta y = 94.213 - 94.213 \times 2.653 \times 0.001 = \underbrace{94.213}_{\text{bond price 12 \% yield}} \underbrace{-\,0.250}_{\text{change}} = \underbrace{93.963}_{\text{bond price 12.1 \% yield}}$

ii) $\sum_{i=1}^{n} c_i\, e^{-y\, t_i}\, |_{y=0.121} = 5 \times e^{-0.121\times0.5} + \ldots + 105 \times e^{-0.121\times3.0} = 93.963$

Same to 3dp!

Modified duration

- So far, continuous compounding
- When the yield y is expressed with compounding m times per year

$$\Delta B = -\frac{B\,D\,\Delta y}{1 + \frac{y}{m}} \tag{4.13}$$

Definition 4.8. The *modified duration* is given by the expression
$\tilde{D} = \frac{D}{1+\frac{y}{m}}$

Example

Example 4.13. For the bond in the previous examples, find the yield for semi-annual compounding, the modified duration, the approximate decrease in the bond price if the yield increases by 10 basis points, and the new bond price.

$R_m = m\left(e^{R_c/m} - 1\right) = 2\left(e^{0.12/2} - 1\right) = 0.123673$

$\tilde{D} = \frac{D}{1+\frac{y}{m}} = \frac{2.653}{1+\frac{0.123673}{2}} = 2.4985$

$\Delta B = -B\,\tilde{D}\,\Delta y = -94.213 \times 2.4985 \times 0.001 = -0.234$

$B' = 94.213 - 0.235 = 93.978$

Convexity

- The convexity of a bond is defined as

$$C = \frac{1}{B}\frac{\partial^2 B}{\partial y^2} = \sum_{i=1}^{n} t_i^2 \frac{c_i e^{-y t_i}}{B} \tag{4.14}$$

so that

$$\frac{\Delta B}{B} = -D\,\Delta y + \frac{1}{2} C (\Delta y)^2 \tag{4.15}$$

Theories of the term structure

- *Expectations Theory*: forward rates equal expected future zero rates [long IRs] = $\mathbb{E}$ [short IRs]
  - No reason to believe a downward sloping yield curve implies falling rates.
  - Is true under RNM $\mathbb{Q}$, in arb-free market
- *Market Segmentation*: short, medium and long rates determined independently of each other
  - Supply and demand for short / medium / long term debt
- *Liquidity Preference Theory*: forward rates higher than expected future zero rates
  - ET $\Rightarrow$ long IRs = $\mathbb{E}$ short IRs; however:
  - investors – preserve liquidity so prefer short end
  - borrowers – fixed rates for long periods
  - to clear market, short ↓, long ↑
  - $\Rightarrow$ forward rates $>$ $\mathbb{E}$ future zero-rates
  - usually observed
- To prevent arbitrage opportunities, different parts of curve are related; empirically are correlated

- Hull Page 93, Musiela and Rutkowski P332

Summary

- Important rates for derivatives
  - Treasury
  - LIBOR
- Rate depends on compounding rates; as frequency → ∞, continuous compounding
- Rates
  - *n*-year zero – investment lasts *n* years, all return realized at end
  - bond yield – for bond with particular maturity, is flat yield s.t. theoretical and market prices match
  - par yield – for bond with particular maturity, is coupon s.t. theoretical and par prices match
  - forward – rates for intervals of time in future implied by current zero rates
- Bootstrapping zero curve from prices of coupon bonds
- Forward rate agreement (FRA)
  - OTC agreement; certain IR for principal in return for LIBOR over future interval
  - Value – assume fwd rates realized and discount payoff
- Duration – sensitivity of value of bond (portfolio) to parallel shift in yield curve
- Convexity – 2nd order sensitivity of value of bond (portfolio) to parallel shift in yield curve; “curvature"
  $$\frac{\Delta B}{B} = -D\,\Delta y + \frac{1}{2}\,C\,(\Delta y)^2$$
- Theories of the TS: expectations, market segmentation, liquidity preference