FX Options and Structured Products

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To Ansua

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Preface

0.1 Scope of this Book

Treasury management of international corporates involves dealing with cash flows in different currencies. Therefore the natural service of an investment bank consists of a variety of money market and foreign exchange products. This book explains the most popular products and strategies with a focus on everything beyond vanilla options.

It explains all the FX options, common structures and tailor-made solutions in examples with a special focus on the application with views from traders and sales as well as from a corporate client perspective.

It contains actually traded deals with corresponding motivations explaining why the structures have been traded. This way the reader gets a feeling how to build new structures to suit clients' needs.

The exercises are meant to practice the material. Several of them are actually difficult to solve and can serve as incentives to further research and testing. Solutions to the exercises are not part of this book, however they will be published on the web page of the book, www.mathfinance.com/FXOptions/.

0.2 The Readership

Prerequisite is some basic knowledge of FX markets as for example taken from the Book *Foreign Exchange Primer* by Shami Shamah, Wiley 2003, see [90]. The target readers are

- Graduate students and Faculty of Financial Engineering Programs, who can use this book as a textbook for a course named *structured products* or *exotic currency options*.

- Traders, Trainee Structurers, Product Developers, Sales and Quants with interest in the FX product line. For them it can serve as a source of ideas and as well as a reference guide.
- Treasurers of corporates interested in managing their books. With this book at hand they can structure their solutions themselves.

The readers more interested in the quantitative and modeling aspects are recommended to read *Foreign Exchange Risk* by J. Hakala and U. Wystup, Risk Publications, London, 2002, see [50]. This book explains several exotic FX options with a special focus on the underlying models and mathematics, but does not contain any structures or corporate clients' or investors' view.

0.3 About the Author

Figure 1: Uwe Wystup, professor of Quantitative Finance at HfB Business School of Finance and Management in Frankfurt, Germany.

Uwe Wystup is also CEO of MathFinance AG, a global network of quants specializing in Quantitative Finance, Exotic Options advisory and Front Office Software Production. Previously he was a Financial Engineer and Structurer in the FX Options Trading Team at Commerzbank. Before that he worked for Deutsche Bank, Citibank, UBS and Sal. Oppenheim jr. & Cie. He is founder and manager of the web site MathFinance.de and the MathFinance Newsletter. Uwe holds a PhD in mathematical finance from Carnegie Mellon University. He also lectures on mathematical finance for Goethe University Frankfurt, organizes the *Frankfurt MathFinance Colloquium* and is founding director of the *Frankfurt MathFinance Institute*. He has given several seminars on exotic options, computational finance and volatility modeling. His area of specialization are the quantitative aspects and the design of structured products of foreign exchange markets. He published a book on *Foreign Exchange Risk* and articles in *Finance and Stochastics* and the *Journal of Derivatives*. Uwe has given many presentations at both universities and banks around the world. Further information on his curriculum vitae and a detailed publication list is available at www.mathfinance.com/wystup/.

0.4 Acknowledgments

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Chapter 1

Foreign Exchange Options

FX Structured Products are tailor-made linear combinations of FX Options including both vanilla and exotic options. We recommend the book by Shamah [90] as a source to learn about FX Markets with a focus on market conventions, spot, forward and swap contracts, vanilla options. For pricing and modeling of exotic FX options we suggest Hakala and Wystup [50] or Lipton [71] as useful companions to this book.

The market for structured products is restricted to the market of the necessary ingredients. Hence, typically there are mostly structured products traded the currency pairs that can be formed between USD, JPY, EUR, CHF, GBP, CAD and AUD. In this chapter we start with a brief history of options, followed by a technical section on vanilla options and volatility, and deal with commonly used linear combinations of vanilla options. Then we will illustrated the most important ingredients for FX structured products: the first and second generation exotics.

1.1 A Journey through the History Of Options

The very first options and futures were traded in ancient Greece, when olives were sold before they had reached ripeness. Thereafter the market evolved in the following way.

16th century Ever since the 15th century tulips, which were liked for their exotic appearance, were grown in Turkey. The head of the royal medical gardens in Vienna, Austria, was the first to cultivate those Turkish tulips successfully in Europe. When he fled to Holland because of religious persecution, he took the bulbs along. As the new head of the botanical gardens of Leiden, Netherlands, he cultivated several new strains. It was from these gardens that avaricious traders stole the bulbs to commercialize them, because tulips were a great status symbol.

17th century The first futures on tulips were traded in 1630. As of 1634, people could buy special tulip strains by the weight of their bulbs, for the bulbs the same value was chosen as for gold. Along with the regular trading, speculators entered the market and the prices skyrocketed. A bulb of the strain "Semper Octavian" was worth two wagonloads of wheat, four loads of rye, four fat oxen, eight fat swine, twelve fat sheep, two hogsheads of wine, four barrels of beer, two barrels of butter, 1,000 pounds of cheese, one marriage bed with linen and one sizable wagon. People left their families, sold all their belongings, and even borrowed money to become tulip traders. When in 1637, this supposedly risk-free market crashed, traders as well as private individuals went bankrupt. The government prohibited speculative trading; the period became famous as Tulipmania.

18th century In 1728, the Royal West-Indian and Guinea Company, the monopolist in trading with the Caribbean Islands and the African coast issued the first stock options. Those were options on the purchase of the French Island of Ste. Croix, on which sugar plantings were planned. The project was realized in 1733 and paper stocks were issued in 1734. Along with the stock, people purchased a relative share of the island and the valuables, as well as the privileges and the rights of the company.

19th century In 1848, 82 businessmen founded the Chicago Board of Trade (CBOT). Today it is the biggest and oldest futures market in the entire world. Most written documents were lost in the great fire of 1871, however, it is commonly believed that the first standardized futures were traded as of 1860. CBOT now trades several futures and forwards, not only T-bonds and treasury bonds, but also options and gold.

In 1870, the New York Cotton Exchange was founded. In 1880, the gold standard was introduced.

20th century

- In 1914, the gold standard was abandoned because of the war.
- In 1919, the Chicago Produce Exchange, in charge of trading agricultural products was renamed to Chicago Mercantile Exchange. Today it is the most important futures market for Eurodollar, foreign exchange, and livestock.
- In 1944, the Bretton Woods System was implemented in an attempt to stabilize the currency system.
- In 1970, the Bretton Woods System was abandoned for several reasons.
- In 1971, the Smithsonian Agreement on fixed exchange rates was introduced.
- In 1972, the International Monetary Market (IMM) traded futures on coins, currencies and precious metal.

- In 1973, the CBOE (Chicago Board of Exchange) firstly traded call options; four years later also put options. The Smithsonian Agreement was abandoned; the currencies followed managed floating.
- In 1975, the CBOT sold the first interest rate future, the first future with no "real" underlying asset.
- In 1978, the Dutch stock market traded the first standardized financial derivatives.
- In 1979, the European Currency System was implemented, and the European Currency Unit (ECU) was introduced.
- In 1991, the Maastricht Treaty on a common currency and economic policy in Europe was signed.
- In 1999, the Euro was introduced, but the countries still used cash of their old currencies, while the exchange rates were kept fixed.

21th century

In 2002, the Euro was introduced as new money in the form of cash.

1.2 Technical Issues for Vanilla Options

We consider the model *geometric Brownian motion*

$$dS_t = (r_d - r_f)S_t\,dt + \sigma S_t\,dW_t \tag{1.1}$$

for the underlying exchange rate quoted in FOR-DOM (foreign-domestic), which means that one unit of the foreign currency costs FOR-DOM units of the domestic currency. In case of EUR-USD with a spot of 1.2000, this means that the price of one EUR is 1.2000 USD. The notion of *foreign* and *domestic* do not refer the location of the trading entity, but only to this quotation convention. We denote the (continuous) foreign interest rate by r_f and the (continuous) domestic interest rate by r_d. In an equity scenario, r_f would represent a continuous dividend rate. The volatility is denoted by σ, and W_t is a standard Brownian motion. The sample paths are displayed in Figure 1.1. We consider this standard model, not because it reflects the statistical properties of the exchange rate (in fact, it doesn't), but because it is widely used in practice and front office systems and mainly serves as a tool to communicate prices in FX options. These prices are generally quoted in terms of volatility in the sense of this model.

Applying Itô's rule to $\ln S_t$ yields the following solution for the process S_t

$$S_t = S_0 \exp\left\{(r_d - r_f - \frac{1}{2}\sigma^2)t + \sigma W_t\right\}, \tag{1.2}$$

which shows that S_t is log-normally distributed, more precisely, $\ln S_t$ is normal with mean $\ln S_0 + (r_d - r_f - \frac{1}{2}\sigma^2)t$ and variance $\sigma^2 t$. Further model assumptions are

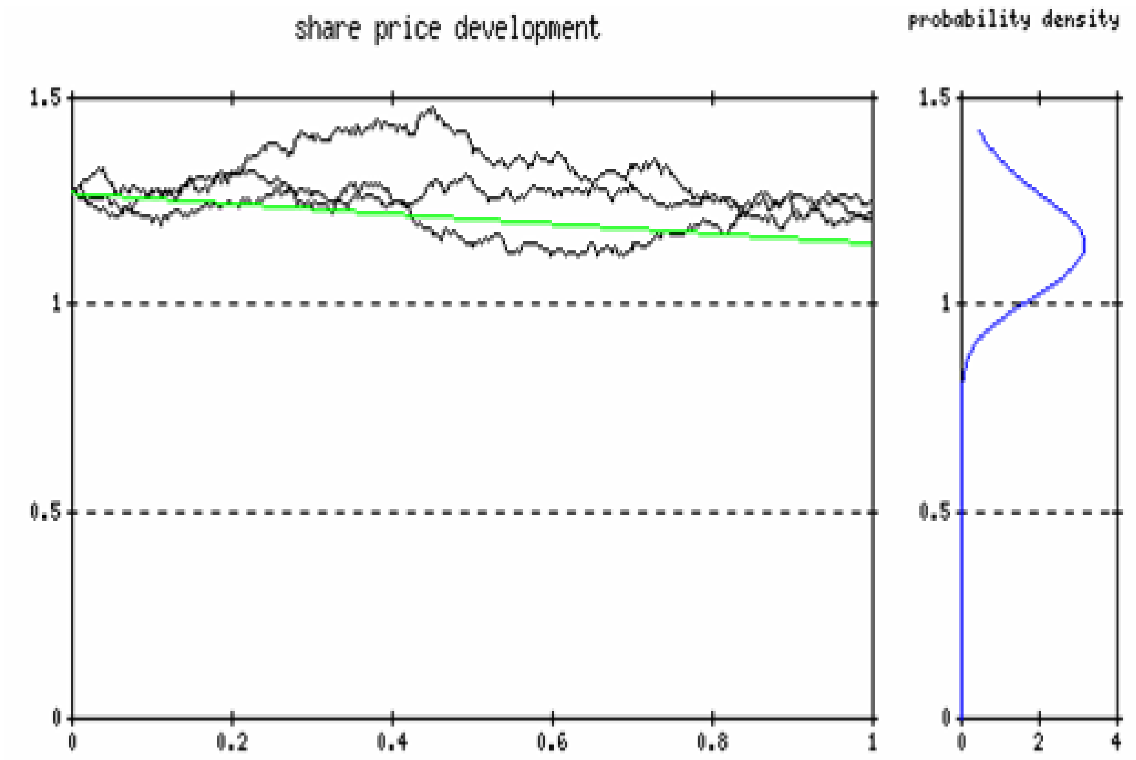


Figure 1.1: Simulated paths of a geometric Brownian motion. The distribution of the spot S_T at time T is log-normal.

1. There is no arbitrage
2. Trading is frictionless, no transaction costs
3. Any position can be taken at any time, short, long, arbitrary fraction, no liquidity constraints

The payoff for a vanilla option (European put or call) is given by

$$F = [\phi(S_T - K)]^+, \tag{1.3}$$

where the contractual parameters are the strike K, the expiration time T and the type ϕ, a binary variable which takes the value $+1$ in the case of a call and -1 in the case of a put. The symbol x^+ denotes the positive part of x, i.e., $x^+ \stackrel{\Delta}{=} \max(0, x) \stackrel{\Delta}{=} 0 \vee x$.

1.2.1 Value

In the Black-Scholes model the value of the payoff F at time t if the spot is at x is denoted by $v(t, x)$ and can be computed either as the solution of the *Black-Scholes partial differential* *equation*

$$v_t - r_d v + (r_d - r_f) x v_x + \frac{1}{2}\sigma^2 x^2 v_{xx} = 0, \quad (1.4)$$

$$v(T, x) = F. \quad (1.5)$$

or equivalently (*Feynman-Kac-Theorem*) as the discounted expected value of the payoff-function,

$$v(x, K, T, t, \sigma, r_d, r_f, \phi) = e^{-r_d \tau} I\!E[F]. \quad (1.6)$$

This is the reason why basic financial engineering is mostly concerned with solving partial differential equations or computing expectations (numerical integration). The result is the *Black-Scholes formula*

$$v(x, K, T, t, \sigma, r_d, r_f, \phi) = \phi e^{-r_d \tau} [f \mathcal{N}(\phi d_+) - K \mathcal{N}(\phi d_-)]. \quad (1.7)$$

We abbreviate

- x: current price of the underlying
- $\tau \stackrel{\Delta}{=} T - t$: time to maturity
- $f \stackrel{\Delta}{=} I\!E[S_T | S_t = x] = x e^{(r_d - r_f)\tau}$: forward price of the underlying
- $\theta_\pm \stackrel{\Delta}{=} \frac{r_d - r_f}{\sigma} \pm \frac{\sigma}{2}$
- $d_\pm \stackrel{\Delta}{=} \frac{\ln \frac{x}{K} + \sigma \theta_\pm \tau}{\sigma \sqrt{\tau}} = \frac{\ln \frac{f}{K} \pm \frac{\sigma^2}{2}\tau}{\sigma\sqrt{\tau}}$
- $n(t) \stackrel{\Delta}{=} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2} = n(-t)$
- $\mathcal{N}(x) \stackrel{\Delta}{=} \int_{-\infty}^{x} n(t)\, dt = 1 - \mathcal{N}(-x)$

The Black-Scholes formula can be derived using the integral representation of Equation (1.6)

$$\begin{aligned} v &= e^{-r_d \tau} I\!E[F] \\ &= e^{-r_d \tau} I\!E[[\phi(S_T - K)]^+] \\ &= e^{-r_d \tau} \int_{-\infty}^{+\infty} \left[\phi\left(x e^{(r_d - r_f - \frac{1}{2}\sigma^2)\tau + \sigma\sqrt{\tau} y} - K\right)\right]^+ n(y)\, dy. \end{aligned} \quad (1.8)$$

Next one has to deal with the positive part and then complete the square to get the Black-Scholes formula. A derivation based on the partial differential equation can be done using results about the well-studied *heat-equation*.

1.2.2 A Note on the Forward

The *forward price* f is the strike which makes the time zero value of the *forward contract*

$$F = S_T - f \tag{1.9}$$

equal to zero. It follows that $f = \mathbb{E}[S_T] = xe^{(r_d - r_f)T}$, i.e. the forward price is the expected price of the underlying at time T in a risk-neutral setup (drift of the geometric Brownian motion is equal to cost of carry $r_d - r_f$). The situation $r_d > r_f$ is called *contango*, and the situation $r_d < r_f$ is called *backwardation*. Note that in the Black-Scholes model the class of forward price curves is quite restricted. For example, no seasonal effects can be included. Note that the value of the forward contract after time zero is usually different from zero, and since one of the counterparties is always short, there may be risk of default of the short party. A *futures contract* prevents this dangerous affair: it is basically a forward contract, but the counterparties have to a *margin account* to ensure the amount of cash or commodity owed does not exceed a specified limit.

1.2.3 Greeks

Greeks are derivatives of the value function with respect to model and contract parameters. They are an important information for traders and have become standard information provided by front-office systems. More details on Greeks and the relations among Greeks are presented in Hakala and Wystup [50] or Reiss and Wystup [84]. For vanilla options we list some of them now.

(Spot) Delta.

$$\frac{\partial v}{\partial x} = \phi e^{-r_f \tau} \mathcal{N}(\phi d_+) \tag{1.10}$$

Forward Delta.

$$\frac{\partial v}{\partial f} = \phi e^{-r_d \tau} \mathcal{N}(\phi d_+) \tag{1.11}$$

Driftless Delta.

$$\phi \mathcal{N}(\phi d_+) \tag{1.12}$$

Gamma.

$$\frac{\partial^2 v}{\partial x^2} = e^{-r_f \tau} \frac{n(d_+)}{x \sigma \sqrt{\tau}} \tag{1.13}$$

Speed.

$$\frac{\partial^3 v}{\partial x^3} = -e^{-r_f\tau}\frac{n(d_+)}{x^2\sigma\sqrt{\tau}}\left(\frac{d_+}{\sigma\sqrt{\tau}}+1\right) \tag{1.14}$$

Theta.

$$\begin{aligned}\frac{\partial v}{\partial t} &= -e^{-r_f\tau}\frac{n(d_+)x\sigma}{2\sqrt{\tau}} \\ &+ \phi[r_f x e^{-r_f\tau}\mathcal{N}(\phi d_+) - r_d K e^{-r_d\tau}\mathcal{N}(\phi d_-)]\end{aligned} \tag{1.15}$$

Charm.

$$\frac{\partial^2 v}{\partial x\partial\tau} = -\phi r_f e^{-r_f\tau}\mathcal{N}(\phi d_+) + \phi e^{-r_f\tau}n(d_+)\frac{2(r_d-r_f)\tau - d_-\sigma\sqrt{\tau}}{2\tau\sigma\sqrt{\tau}} \tag{1.16}$$

Color.

$$\frac{\partial^3 v}{\partial x^2\partial\tau} = -e^{-r_f\tau}\frac{n(d_+)}{2x\tau\sigma\sqrt{\tau}}\left[2r_f\tau + 1 + \frac{2(r_d-r_f)\tau - d_-\sigma\sqrt{\tau}}{2\tau\sigma\sqrt{\tau}}d_+\right] \tag{1.17}$$

Vega.

$$\frac{\partial v}{\partial\sigma} = xe^{-r_f\tau}\sqrt{\tau}n(d_+) \tag{1.18}$$

Volga.

$$\frac{\partial^2 v}{\partial\sigma^2} = xe^{-r_f\tau}\sqrt{\tau}n(d_+)\frac{d_+d_-}{\sigma} \tag{1.19}$$

Volga is also sometimes called *vomma* or *volgamma*.

Vanna.

$$\frac{\partial^2 v}{\partial\sigma\partial x} = -e^{-r_f\tau}n(d_+)\frac{d_-}{\sigma} \tag{1.20}$$

Rho.

$$\frac{\partial v}{\partial r_d} = \phi K\tau e^{-r_d\tau}\mathcal{N}(\phi d_-) \tag{1.21}$$

$$\frac{\partial v}{\partial r_f} = -\phi x\tau e^{-r_f\tau}\mathcal{N}(\phi d_+) \tag{1.22}$$

Dual Delta.

$$\frac{\partial v}{\partial K} = -\phi e^{-r_d \tau} \mathcal{N}(\phi d_-) \tag{1.23}$$

Dual Gamma.

$$\frac{\partial^2 v}{\partial K^2} = e^{-r_d \tau} \frac{n(d_-)}{K\sigma\sqrt{\tau}} \tag{1.24}$$

Dual Theta.

$$\frac{\partial v}{\partial T} = -v_t \tag{1.25}$$

1.2.4 Identities

$$\frac{\partial d_\pm}{\partial \sigma} = -\frac{d_\mp}{\sigma} \tag{1.26}$$

$$\frac{\partial d_\pm}{\partial r_d} = \frac{\sqrt{\tau}}{\sigma} \tag{1.27}$$

$$\frac{\partial d_\pm}{\partial r_f} = -\frac{\sqrt{\tau}}{\sigma} \tag{1.28}$$

$$x e^{-r_f \tau} n(d_+) = K e^{-r_d \tau} n(d_-). \tag{1.29}$$

$$\mathcal{N}(\phi d_-) = I\!P[\phi S_T \geq \phi K] \tag{1.30}$$

$$\mathcal{N}(\phi d_+) = I\!P\left[\phi S_T \leq \phi \frac{f^2}{K}\right] \tag{1.31}$$

The *put-call-parity* is the relationship

$$v(x, K, T, t, \sigma, r_d, r_f, +1) - v(x, K, T, t, \sigma, r_d, r_f, -1) = x e^{-r_f \tau} - K e^{-r_d \tau}, \tag{1.32}$$

which is just a more complicated way to write the trivial equation $x = x^+ - x^-$.
The *put-call delta parity* is

$$\frac{\partial v(x, K, T, t, \sigma, r_d, r_f, +1)}{\partial x} - \frac{\partial v(x, K, T, t, \sigma, r_d, r_f, -1)}{\partial x} = e^{-r_f \tau}. \tag{1.33}$$

In particular, we learn that the absolute value of a put delta and a call delta are not exactly adding up to one, but only to a positive number $e^{-r_f \tau}$. They add up to one approximately if either the time to expiration τ is short or if the foreign interest rate r_f is close to zero.

Whereas the choice $K = f$ produces identical values for call and put, we seek the *delta-symmetric strike* $\check{K}$ which produces absolutely identical deltas (spot, forward or driftless). This condition implies $d_+ = 0$ and thus

$$\check{K} = fe^{\frac{\sigma^2}{2}T}, \tag{1.34}$$

in which case the absolute delta is $e^{-r_f\tau}/2$. In particular, we learn, that always $\check{K} > f$, i.e., there can't be a put and a call with identical values *and* deltas. Note that the strike $\check{K}$ is usually chosen as the middle strike when trading a straddle or a butterfly. Similarly the dual-delta-symmetric strike $\hat{K} = fe^{-\frac{\sigma^2}{2}T}$ can be derived from the condition $d_- = 0$.

1.2.5 Homogeneity based Relationships

We may wish to measure the value of the underlying in a different unit. This will obviously effect the option pricing formula as follows.

$$av(x, K, T, t, \sigma, r_d, r_f, \phi) = v(ax, aK, T, t, \sigma, r_d, r_f, \phi) \text{ for all } a > 0. \tag{1.35}$$

Differentiating both sides with respect to a and then setting $a = 1$ yields

$$v = xv_x + Kv_K. \tag{1.36}$$

Comparing the coefficients of x and K in Equations (1.7) and (1.36) leads to suggestive results for the delta v_x and dual delta v_K. This *space-homogeneity* is the reason behind the simplicity of the delta formulas, whose tedious computation can be saved this way.
We can perform a similar computation for the time-affected parameters and obtain the obvious equation

$$v(x, K, T, t, \sigma, r_d, r_f, \phi) = v(x, K, \frac{T}{a}, \frac{t}{a}, \sqrt{a}\sigma, ar_d, ar_f, \phi) \text{ for all } a > 0. \tag{1.37}$$

Differentiating both sides with respect to a and then setting $a = 1$ yields

$$0 = \tau v_t + \frac{1}{2}\sigma v_\sigma + r_d v_{r_d} + r_f v_{r_f}. \tag{1.38}$$

Of course, this can also be verified by direct computation. The overall use of such equations is to generate double checking benchmarks when computing Greeks. These homogeneity methods can easily be extended to other more complex options.
By *put-call symmetry* we understand the relationship (see [6], [7],[16] and [19])

$$v(x, K, T, t, \sigma, r_d, r_f, +1) = \frac{K}{f}v(x, \frac{f^2}{K}, T, t, \sigma, r_d, r_f, -1). \tag{1.39}$$

The strike of the put and the strike of the call result in a geometric mean equal to the forward f. The forward can be interpreted as a *geometric mirror* reflecting a call into a certain number of puts. Note that for at-the-money options ($K = f$) the put-call symmetry coincides with the special case of the put-call parity where the call and the put have the same value.
Direct computation shows that the *rates symmetry*

$$\frac{\partial v}{\partial r_d} + \frac{\partial v}{\partial r_f} = -\tau v \tag{1.40}$$

holds for vanilla options. This relationship, in fact, holds for all European options and a wide class of path-dependent options as shown in [84].
One can directly verify the relationship the *foreign-domestic symmetry*

$$\frac{1}{x} v(x, K, T, t, \sigma, r_d, r_f, \phi) = K v\left(\frac{1}{x}, \frac{1}{K}, T, t, \sigma, r_f, r_d, -\phi\right). \tag{1.41}$$

This equality can be viewed as one of the faces of put-call symmetry. The reason is that the value of an option can be computed both in a domestic as well as in a foreign scenario. We consider the example of S_t modeling the exchange rate of EUR/USD. In New York, the call option $(S_T - K)^+$ costs $v(x, K, T, t, \sigma, r_{usd}, r_{eur}, 1)$ USD and hence $v(x, K, T, t, \sigma, r_{usd}, r_{eur}, 1)/x$ EUR. This EUR-call option can also be viewed as a USD-put option with payoff $K\left(\frac{1}{K} - \frac{1}{S_T}\right)^+$. This option costs $Kv(\frac{1}{x}, \frac{1}{K}, T, t, \sigma, r_{eur}, r_{usd}, -1)$ EUR in Frankfurt, because S_t and $\frac{1}{S_t}$ have the same volatility. Of course, the New York value and the Frankfurt value must agree, which leads to (1.41). We will also learn later, that this symmetry is just one possible result based on *change of numeraire*.

1.2.6 Quotation

Quotation of the Underlying Exchange Rate

Equation (1.1) is a model for the exchange rate. The quotation is a permanently confusing issue, so let us clarify this here. The exchange rate means how much of the *domestic* currency are needed to buy one unit of *foreign* currency. For example, if we take EUR/USD as an exchange rate, then the default quotation is EUR-USD, where USD is the domestic currency and EUR is the foreign currency. The term *domestic* is in no way related to the location of the trader or any country. It merely means the *numeraire* currency. The terms *domestic*, *numeraire* or *base currency* are synonyms as are *foreign* and *underlying* . Throughout this book we denote with the slash (/) the currency pair and with a dash (-) the quotation. The slash (/) does *not* mean a division. For instance, EUR/USD can also be quoted in either EUR-USD, which then means how many USD are needed to buy one EUR, or in USD-EUR, which then means how many EUR are needed to buy one USD. There are certain market standard quotations listed in Table 1.1.

currency pair	default quotation	sample quote
GBP/USD	GPB-USD	1.8000
GBP/CHF	GBP-CHF	2.2500
EUR/USD	EUR-USD	1.2000
EUR/GBP	EUR-GBP	0.6900
EUR/JPY	EUR-JPY	135.00
EUR/CHF	EUR-CHF	1.5500
USD/JPY	USD-JPY	108.00
USD/CHF	USD-CHF	1.2800

Table 1.1: Standard market quotation of major currency pairs with sample spot prices

Trading Floor Language

We call one million a *buck*, one billion a *yard*. This is because a billion is called 'milliarde' in French, German and other languages. For the British Pound one million is also often called a *quid*.

Certain currency pairs have names. For instance, GBP/USD is called *cable*, because the exchange rate information used to be sent through a cable in the Atlantic ocean between America and England. EUR/JPY is called the *cross* , because it is the cross rate of the more liquidly traded USD/JPY and EUR/USD.

Certain currencies also have names, e.g. the New Zealand Dollar NZD is called a *kiwi* , the Australian Dollar AUD is called *Aussie*, the Scandinavian currencies DKR, NOK and SEK are called *Scandies*.

Exchange rates are generally quoted up to five relevant figures, e.g. in EUR-USD we could observe a quote of 1.2375. The last digit '5' is called the *pip*, the middle digit '3' is called the *big figure*, as exchange rates are often displayed in trading floors and the big figure, which is displayed in bigger size, is the most relevant information. The digits left to the big figure are known anyway, the pips right of the big figure are often negligible. To make it clear, a rise of USD-JPY 108.25 by 20 pips will be 108.45 and a rise by 2 big figures will be 110.25.

Quotation of Option Prices

Values and prices of vanilla options may be quoted in the six ways explained in Table 1.2.

name	symbol	value in units of	example
domestic cash	**d**	DOM	29,148 USD
foreign cash	**f**	FOR	24,290 EUR
% domestic	**% d**	DOM per unit of DOM	2.3318% USD
% foreign	**% f**	FOR per unit of FOR	2.4290% EUR
domestic pips	**d pips**	DOM per unit of FOR	291.48 USD pips per EUR
foreign pips	**f pips**	FOR per unit of DOM	194.32 EUR pips per USD

Table 1.2: Standard market quotation types for option values. In the example we take FOR=EUR, DOM=USD, $S_0 = 1.2000$, $r_d = 3.0\%$, $r_f = 2.5\%$, $\sigma = 10\%$, $K = 1.2500$, $T = 1$ year, $\phi = +1$ (call), notional $= 1,000,000$ EUR $= 1,250,000$ USD. For the pips, the quotation 291.48 USD pips per EUR is also sometimes stated as 2.9148% USD per 1 EUR. Similarly, the 194.32 EUR pips per USD can also be quoted as 1.9432% EUR per 1 USD.

The Black-Scholes formula quotes **d pips**. The others can be computed using the following instruction.

$$\mathbf{d\ pips} \stackrel{\times\frac{1}{S_0}}{\longrightarrow} \%\mathbf{f} \stackrel{\times\frac{S_0}{K}}{\longrightarrow} \%\mathbf{d} \stackrel{\times\frac{1}{S_0}}{\longrightarrow} \mathbf{f\ pips} \stackrel{\times S_0 K}{\longrightarrow} \mathbf{d\ pips} \tag{1.42}$$

Delta and Premium Convention

The spot delta of a European option without premium is well known. It will be called *raw spot delta* δ_{raw} now. It can be quoted in either of the two currencies involved. The relationship is

$$\delta_{raw}^{reverse} = -\delta_{raw}\frac{S}{K}. \tag{1.43}$$

The delta is used to buy or sell spot in the corresponding amount in order to hedge the option up to first order.

For consistency the premium needs to be incorporated into the delta hedge, since a premium in foreign currency will already hedge part of the option's delta risk. To make this clear, let us consider EUR-USD. In the standard arbitrage theory, $v(x)$ denotes the value or premium in USD of an option with 1 EUR notional, if the spot is at x, and the raw delta v_x denotes the number of EUR to buy for the delta hedge. Therefore, xv_x is the number of USD to sell. If now the premium is paid in EUR rather than in USD, then we already have $\frac{v}{x}$ EUR, and the number of EUR to buy has to be reduced by this amount, i.e. if EUR is the premium currency, we need to buy $v_x - \frac{v}{x}$ EUR for the delta hedge or equivalently sell $xv_x - v$ USD.

The entire FX quotation story becomes generally a mess, because we need to first sort out which currency is domestic, which is foreign, what is the notional currency of the option, and what is the premium currency. Unfortunately this is not symmetric, since the counterpart might have another notion of domestic currency for a given currency pair. Hence in the professional inter bank market there is one notion of delta per currency pair. Normally it is the left hand side delta of the *Fenics* screen if the option is traded in left hand side premium, which is normally the standard and right hand side delta if it is traded with right hand side premium, e.g. EUR/USD lhs, USD/JPY lhs, EUR/JPY lhs, AUD/USD rhs, etc... Since OTM options are traded most of time the difference is not huge and hence does not create a huge spot risk.

Additionally the standard delta per currency pair [left hand side delta in *Fenics* for most cases] is used to quote options in volatility. This has to be specified by currency.

This standard inter bank notion must be adapted to the real delta-risk of the bank for an automated trading system. For currencies where the risk–free currency of the bank is the base currency of the currency it is clear that the delta is the raw delta of the option and for risky premium this premium must be included. In the opposite case the risky premium and the market value must be taken into account for the base currency premium, such that these offset each other. And for premium in underlying currency of the contract the market-value needs to be taken into account. In that way the delta hedge is invariant with respect to the risky currency notion of the bank, e.g. the delta is the same for a USD-based bank and a EUR-based bank.

Example

We consider two examples in Table 1.3 and 1.4 to compare the various versions of deltas that are used in practice.

delta ccy	prem ccy	Fenics	formula	delta
% EUR	EUR	lhs	$\delta_{raw} - P$	44.72
% EUR	USD	rhs	δ_{raw}	49.15
% USD	EUR	rhs [flip F4]	$-(\delta_{raw} - P)S/K$	-44.72
% USD	USD	lhs [flip F4]	$-(\delta_{raw})S/K$	-49.15

Table 1.3: 1y EUR call USD put strike $K = 0.9090$ for a EUR–based bank. Market data: spot $S = 0.9090$, volatility $\sigma = 12\%$, EUR rate $r_f = 3.96\%$, USD rate $r_d = 3.57\%$. The raw delta is 49.15%EUR and the value is 4.427%EUR.

delta ccy	prem ccy	Fenics	formula	delta
% EUR	EUR	lhs	$\delta_{raw} - P$	72.94
% EUR	USD	rhs	δ_{raw}	94.82
% USD	EUR	rhs [flip F4]	$-(\delta_{raw} - P)S/K$	-94.72
% USD	USD	lhs [flip F4]	$-\delta_{raw}S/K$	-123.13

Table 1.4: 1y call EUR call USD put strike $K = 0.7000$ for a EUR–based bank. Market data: spot $S = 0.9090$, volatility $\sigma = 12\%$, EUR rate $r_f = 3.96\%$, USD rate $r_d = 3.57\%$. The raw delta is 94.82%EUR and the value is 21.88%EUR.

1.2.7 Strike in Terms of Delta

Since $v_x = \Delta = \phi e^{-r_f\tau}\mathcal{N}(\phi d_+)$ we can retrieve the strike as

$$K = x \exp\left\{-\phi\mathcal{N}^{-1}(\phi\Delta e^{r_f\tau})\sigma\sqrt{\tau} + \sigma\theta_+\tau\right\}. \tag{1.44}$$

1.2.8 Volatility in Terms of Delta

The mapping $\sigma \mapsto \Delta = \phi e^{-r_f\tau}\mathcal{N}(\phi d_+)$ is not one-to-one. The two solutions are given by

$$\sigma_\pm = \frac{1}{\sqrt{\tau}}\left\{\phi\mathcal{N}^{-1}(\phi\Delta e^{r_f\tau}) \pm \sqrt{(\mathcal{N}^{-1}(\phi\Delta e^{r_f\tau}))^2 - \sigma\sqrt{\tau}(d_+ + d_-)}\right\}. \tag{1.45}$$

Thus using just the delta to retrieve the volatility of an option is not advisable.

1.2.9 Volatility and Delta for a Given Strike

The determination of the volatility and the delta for a given strike is an iterative process involving the determination of the delta for the option using at-the-money volatilities in a first step and then using the determined volatility to re–determine the delta and to continuously iterate the delta and volatility until the volatility does not change more than $\epsilon = 0.001\%$ between iterations. More precisely, one can perform the following algorithm. Let the given strike be K.

1. Choose σ_0 = at-the-money volatility from the volatility matrix.
2. Calculate $\Delta_{n+1} = \Delta(\text{Call}(K, \sigma_n))$.
3. Take $\sigma_{n+1} = \sigma(\Delta_{n+1})$ from the volatility matrix, possibly via a suitable interpolation.
4. If $|\sigma_{n+1} - \sigma_n| < \epsilon$, then quit, otherwise continue with step 2.

In order to prove the convergence of this algorithm we need to establish convergence of the recursion

$$\begin{aligned} \Delta_{n+1} &= e^{-r_f\tau}\mathcal{N}(d_+(\Delta_n)) \\ &= e^{-r_f\tau}\mathcal{N}\left(\frac{\ln(S/K)+(r_d-r_f+\frac{1}{2}\sigma^2(\Delta_n))\tau}{\sigma(\Delta_n)\sqrt{\tau}}\right) \end{aligned} \tag{1.46}$$

for sufficiently large $\sigma(\Delta_n)$ and a sufficiently smooth volatility smile surface. We must show that the sequence of these Δ_n converges to a fixed point $\Delta^* \in [0,1]$ with a fixed volatility $\sigma^* = \sigma(\Delta^*)$.

This proof has been carried out in [15] and works like this. We consider the derivative

$$\frac{\partial \Delta_{n+1}}{\partial \Delta_n} = -e^{-r_f\tau}n(d_+(\Delta_n))\frac{d_-(\Delta_n)}{\sigma(\Delta_n)} \cdot \frac{\partial}{\partial \Delta_n}\sigma(\Delta_n). \tag{1.47}$$

The term

$$-e^{-r_f\tau}n(d_+(\Delta_n))\frac{d_-(\Delta_n)}{\sigma(\Delta_n)}$$

converges rapidly to zero for very small and very large spots, being an argument of the standard normal density n. For sufficiently large $\sigma(\Delta_n)$ and a sufficiently smooth volatility surface in the sense that $\frac{\partial}{\partial \Delta_n}\sigma(\Delta_n)$ is sufficiently small, we obtain

$$\left|\frac{\partial}{\partial \Delta_n}\sigma(\Delta_n)\right| \stackrel{\Delta}{=} q < 1. \tag{1.48}$$

Thus for any two values $\Delta_{n+1}^{(1)}$, $\Delta_{n+1}^{(2)}$, a continuously differentiable smile surface we obtain

$$|\Delta_{n+1}^{(1)} - \Delta_{n+1}^{(2)}| < q|\Delta_n^{(1)} - \Delta_n^{(2)}|, \tag{1.49}$$

due to the mean value theorem. Hence the sequence Δ_n is a contraction in the sense of the fixed point theorem of Banach. This implies that the sequence converges to a unique fixed point in $[0,1]$, which is given by $\sigma^* = \sigma(\Delta^*)$.

1.2.10 Greeks in Terms of Deltas

In Foreign Exchange markets the moneyness of vanilla options is always expressed in terms of deltas and prices are quoted in terms of volatility. This makes a ten-delta call a financial object as such independent of spot and strike. This method and the quotation in volatility makes objects and prices transparent in a very intelligent and user-friendly way. At this point we list the Greeks in terms of deltas instead of spot and strike. Let us introduce the quantities

$$\Delta_+ \stackrel{\Delta}{=} \phi e^{-r_f\tau}\mathcal{N}(\phi d_+) \text{ spot delta,} \tag{1.50}$$

$$\Delta_- \stackrel{\Delta}{=} -\phi e^{-r_d\tau}\mathcal{N}(\phi d_-) \text{ dual delta,} \tag{1.51}$$

which we assume to be given. From these we can retrieve

$$d_+ = \phi\mathcal{N}^{-1}(\phi e^{r_f\tau}\Delta_+), \quad (1.52)$$
$$d_- = \phi\mathcal{N}^{-1}(-\phi e^{r_d\tau}\Delta_-). \quad (1.53)$$

Interpretation of Dual Delta

The dual delta introduced in (1.23) as the sensitivity with respect to strike has another - more practical - interpretation in a foreign exchange setup. We have seen in Section 1.2.5 that the domestic value

$$v(x, K, \tau, \sigma, r_d, r_f, \phi) \quad (1.54)$$

corresponds to a foreign value

$$v\left(\frac{1}{x}, \frac{1}{K}, \tau, \sigma, r_f, r_d, -\phi\right) \quad (1.55)$$

up to an adjustment of the nominal amount by the factor xK. From a foreign viewpoint the delta is thus given by

$$\begin{aligned} & -\phi e^{-r_d\tau}\mathcal{N}\left(-\phi\frac{\ln(\frac{K}{x}) + (r_f - r_d + \frac{1}{2}\sigma^2\tau)}{\sigma\sqrt{\tau}}\right) \\ = & -\phi e^{-r_d\tau}\mathcal{N}\left(\phi\frac{\ln(\frac{x}{K}) + (r_d - r_f - \frac{1}{2}\sigma^2\tau)}{\sigma\sqrt{\tau}}\right) \\ = & \ \Delta_-, \end{aligned} \quad (1.56)$$

which means the dual delta is the delta from the foreign viewpoint. We will see below that foreign rho, vega and gamma do not require to know the dual delta. We will now state the Greeks in terms of $x, \Delta_+, \Delta_-, r_d, r_f, \tau, \phi$.

Value.

$$v(x, \Delta_+, \Delta_-, r_d, r_f, \tau, \phi) = x\Delta_+ + x\Delta_-\frac{e^{-r_f\tau}n(d_+)}{e^{-r_d\tau}n(d_-)} \quad (1.57)$$

(Spot) Delta.

$$\frac{\partial v}{\partial x} = \Delta_+ \quad (1.58)$$

Forward Delta.

$$\frac{\partial v}{\partial f} = e^{(r_f - r_d)\tau}\Delta_+ \quad (1.59)$$

Gamma.

$$\frac{\partial^2 v}{\partial x^2} = e^{-r_f\tau}\frac{n(d_+)}{x(d_+ - d_-)} \tag{1.60}$$

Taking a trader's gamma (change of delta if spot moves by 1%) additionally removes the spot dependence, because

$$\Gamma_{trader} = \frac{x}{100}\frac{\partial^2 v}{\partial x^2} = e^{-r_f\tau}\frac{n(d_+)}{100(d_+ - d_-)} \tag{1.61}$$

Speed.

$$\frac{\partial^3 v}{\partial x^3} = -e^{-r_f\tau}\frac{n(d_+)}{x^2(d_+ - d_-)^2}(2d_+ - d_-) \tag{1.62}$$

Theta.

$$\begin{aligned}\frac{1}{x}\frac{\partial v}{\partial t} &= -e^{-r_f\tau}\frac{n(d_+)(d_+ - d_-)}{2\tau} \\ &+ \left[r_f\Delta_+ + r_d\Delta_-\frac{e^{-r_f\tau}n(d_+)}{e^{-r_d\tau}n(d_-)}\right]\end{aligned} \tag{1.63}$$

Charm.

$$\frac{\partial^2 v}{\partial x \partial \tau} = -\phi r_f e^{-r_f\tau}\mathcal{N}(\phi d_+) + \phi e^{-r_f\tau}n(d_+)\frac{2(r_d - r_f)\tau - d_-(d_+ - d_-)}{2\tau(d_+ - d_-)} \tag{1.64}$$

Color.

$$\frac{\partial^3 v}{\partial x^2 \partial \tau} = -\frac{e^{-r_f\tau}n(d_+)}{2x\tau(d_+ - d_-)}\left[2r_f\tau + 1 + \frac{2(r_d - r_f)\tau - d_-(d_+ - d_-)}{2\tau(d_+ - d_-)}d_+\right] \tag{1.65}$$

Vega.

$$\frac{\partial v}{\partial \sigma} = xe^{-r_f\tau}\sqrt{\tau}n(d_+) \tag{1.66}$$

Volga.

$$\frac{\partial^2 v}{\partial \sigma^2} = xe^{-r_f\tau}\tau n(d_+)\frac{d_+ d_-}{d_+ - d_-} \tag{1.67}$$

Vanna.

$$\frac{\partial^2 v}{\partial \sigma \partial x} = -e^{-r_f \tau} n(d_+) \frac{\sqrt{\tau} d_-}{d_+ - d_-} \tag{1.68}$$

Rho.

$$\frac{\partial v}{\partial r_d} = -x\tau\Delta_- \frac{e^{-r_f\tau} n(d_+)}{e^{-r_d\tau} n(d_-)} \tag{1.69}$$

$$\frac{\partial v}{\partial r_f} = -x\tau\Delta_+ \tag{1.70}$$

Dual Delta.

$$\frac{\partial v}{\partial K} = \Delta_- \tag{1.71}$$

Dual Gamma.

$$K^2 \frac{\partial^2 v}{\partial K^2} = x^2 \frac{\partial^2 v}{\partial x^2} \tag{1.72}$$

Dual Theta.

$$\frac{\partial v}{\partial T} = -v_t \tag{1.73}$$

As an important example we consider vega.

Vega in Terms of Delta

The mapping $\Delta \mapsto v_\sigma = xe^{-r_f\tau}\sqrt{\tau} n(\mathcal{N}^{-1}(e^{r_f\tau}\Delta))$ is important for trading vanilla options. Observe that this function does not depend on r_d or σ, just on r_f. Quoting vega in % foreign will additionally remove the spot dependence. This means that for a moderately stable foreign term structure curve, traders will be able to use a moderately stable vega matrix. For $r_f = 3\%$ the vega matrix is presented in Table 1.5.

1.3 Volatility

Volatility is the *annualized standard deviation of the log-returns*. It is *the* crucial input parameter to determine the value of an option. Hence, the crucial question is where to derive the volatility from. If no active option market is present, the only source of information is estimating the historic volatility. This would give some clue about the *past*. In liquid currency

Mat/Δ	50%	45%	40%	35%	30%	25%	20%	15%	10%	5%
1D	2	2	2	2	2	2	1	1	1	1
1W	6	5	5	5	5	4	4	3	2	1
1W	8	8	8	7	7	6	5	5	3	2
1M	11	11	11	11	10	9	8	7	5	3
2M	16	16	16	15	14	13	11	9	7	4
3M	20	20	19	18	17	16	14	12	9	5
6M	28	28	27	26	24	22	20	16	12	7
9M	34	34	33	32	30	27	24	20	15	9
1Y	39	39	38	36	34	31	28	23	17	10
2Y	53	53	52	50	48	44	39	32	24	14
3Y	63	63	62	60	57	53	47	39	30	18

Table 1.5: Vega in terms of Delta for the standard maturity labels and various deltas. It shows that one can vega hedge a long 9M 35 delta call with 4 short 1M 20 delta puts.

pairs volatility is often a traded quantity on its own, which is quoted by traders, brokers and real-time data pages. These quotes reflect views of market participants about the *future*.

Since volatility normally does not stay constant, option traders are highly concerned with hedging their volatility exposure. Hedging vanilla options' vega is comparatively easy, because vanilla options have convex payoffs, whence the vega is always positive, i.e. the higher the volatility, the higher the price. Let us take for example a EUR-USD market with spot 1.2000, USD- and EUR rate at 2.5%. A 3-month at-the-money call with 1 million EUR notional would cost 29,000 USD at at volatility of 12%. If the volatility now drops to a value of 8%, then the value of the call would be only 19,000 USD. This monotone dependence is not guaranteed for non-convex payoffs as we illustrate in Figure 1.2.

1.3.1 Historic Volatility

We briefly describe how to compute the historic volatility of a time series

$$S_0, S_1, \ldots, S_N \tag{1.74}$$

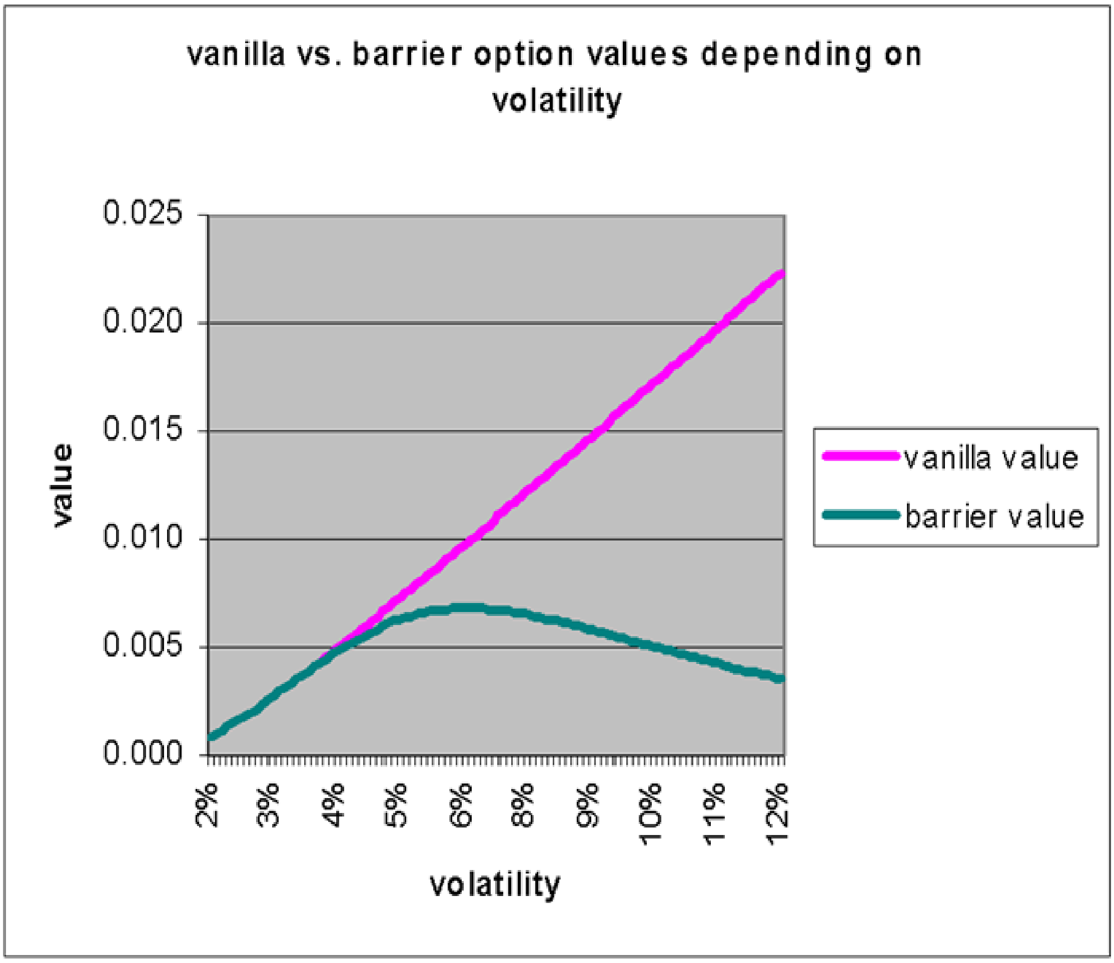


Figure 1.2: Dependence of a vanilla call and a reverse knock-out call on volatility. The vanilla value is monotone in the volatility, whereas the barrier value is not. The reason is that as the spot gets closer to the upper knock-out barrier, an increasing volatility would increase the chance of knock-out and hence decrease the value.

of daily data. First, we create the sequence of log-returns

$$r_i = \ln \frac{S_i}{S_{i-1}}, \quad i = 1, \ldots, N. \tag{1.75}$$

Then, we compute the average log-return

$$\bar{r} = \frac{1}{N} \sum_{i=1}^{N} r_i, \tag{1.76}$$

their variance

$$\hat{\sigma}^2 = \frac{1}{N-1}\sum_{i=1}^{N}(r_i - \bar{r})^2, \tag{1.77}$$

and their standard deviation

$$\hat{\sigma} = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(r_i - \bar{r})^2}. \tag{1.78}$$

The annualized standard deviation, which is the volatility, is then given by

$$\hat{\sigma}_a = \sqrt{\frac{B}{N-1}\sum_{i=1}^{N}(r_i - \bar{r})^2}, \tag{1.79}$$

where the *annualization factor* B is given by

$$B = \frac{N}{k}d, \tag{1.80}$$

and k denotes the number of calendar days within the time series and d denotes the number of calendar days per year. The is done to press the trading days into the calendar days.

Assuming normally distributed log-returns, we know that $\hat{\sigma}^2$ is χ^2-distributed. Therefore, given a confidence level of p and a corresponding error probability $\alpha = 1 - p$, the p-confidence interval is given by

$$\left[\hat{\sigma}_a\sqrt{\frac{N-1}{\chi^2_{N-1;1-\frac{\alpha}{2}}}}, \hat{\sigma}_a\sqrt{\frac{N-1}{\chi^2_{N-1;\frac{\alpha}{2}}}}\right], \tag{1.81}$$

where $\chi^2_{n;p}$ denotes the p-quantile of a χ^2-distribution[1] with n degrees of freedom.

As an example let us take the 256 ECB-fixings of EUR-USD from 4 March 2003 to 3 March 2004 displayed in Figure 1.3. We get $N = 255$ log-returns. Taking $k = d = 365$, we obtain

$$\bar{r} = \frac{1}{N}\sum_{i=1}^{N} r_i = 0.0004166,$$

$$\hat{\sigma}_a = \sqrt{\frac{B}{N-1}\sum_{i=1}^{N}(r_i - \bar{r})^2} = 10.85\%,$$

and a 95% confidence interval of $[9.99\%, 11.89\%]$.

[1]values and quantiles of the χ^2-distribution and other distributions can be computed on the internet, e.g. at http://www.wiso.uni-koeln.de/ASPSamp/eswf/html/allg/surfstat/tables.htm.

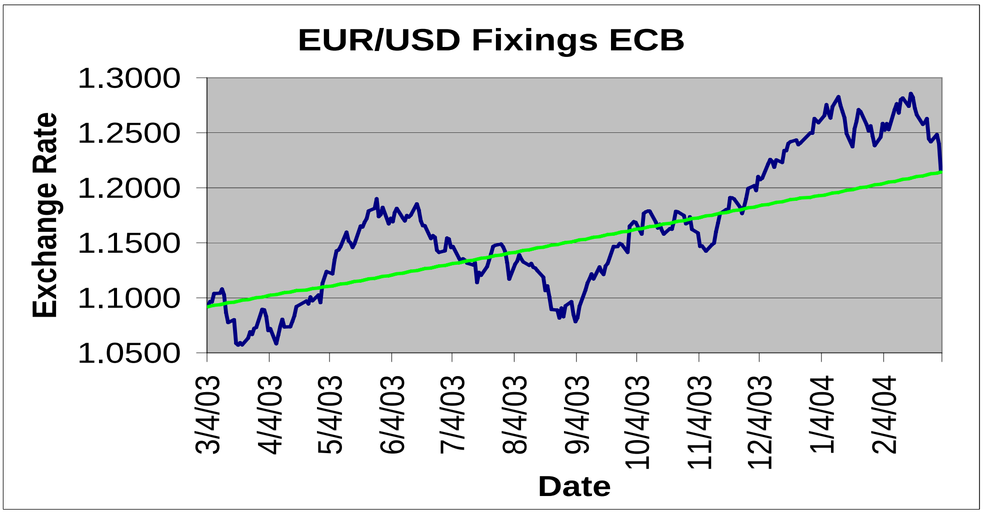


Figure 1.3: ECB-fixings of EUR-USD from 4 March 2003 to 3 March 2004 and the line of average growth

1.3.2 Historic Correlation

As in the preceding section we briefly describe how to compute the historic correlation of two time series

$$x_0, x_1, \ldots, x_N,$$
$$y_0, y_1, \ldots, y_N,$$

of daily data. First, we create the sequences of log-returns

$$\begin{aligned} X_i &= \ln \frac{x_i}{x_{i-1}}, \quad i = 1, \ldots, N, \\ Y_i &= \ln \frac{y_i}{y_{i-1}}, \quad i = 1, \ldots, N. \end{aligned} \tag{1.82}$$

Then, we compute the average log-returns

$$\begin{aligned} \bar{X} &= \frac{1}{N} \sum_{i=1}^{N} X_i, \\ \bar{Y} &= \frac{1}{N} \sum_{i=1}^{N} Y_i, \end{aligned} \tag{1.83}$$

their variances and covariance

$$\hat{\sigma}_X^2 = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2, \quad (1.84)$$

$$\hat{\sigma}_Y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(Y_i - \bar{Y})^2, \quad (1.85)$$

$$\hat{\sigma}_{XY} = \frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})(Y_i - \bar{Y}), \quad (1.86)$$

and their standard deviations

$$\hat{\sigma}_X = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(X_i - \bar{X})^2}, \quad (1.87)$$

$$\hat{\sigma}_Y = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(Y_i - \bar{Y})^2}. \quad (1.88)$$

The estimate for the correlation of the log-returns is given by

$$\hat{\rho} = \frac{\hat{\sigma}_{XY}}{\hat{\sigma}_X \hat{\sigma}_Y}. \quad (1.89)$$

This correlation estimate is often not very stable, but on the other hand, often the only available information. More recent work by Jäkel [37] treats robust estimation of correlation. We will revisit FX correlation risk in Section 1.6.7.

1.3.3 Volatility Smile

The Black-Scholes model assumes a constant volatility throughout. However, market prices of traded options imply different volatilities for different maturities and different deltas. We start with some technical issues how to imply the volatility from vanilla options.

Retrieving the Volatility from Vanilla Options

Given the value of an option. Recall the Black-Scholes formula in Equation (1.7). We now look at the function $v(\sigma)$, whose derivative (vega) is

$$v'(\sigma) = xe^{-r_f T}\sqrt{T}n(d_+). \quad (1.90)$$

The function $\sigma \mapsto v(\sigma)$ is

1. strictly increasing,
2. concave up for $\sigma \in [0, \sqrt{2|\ln F - \ln K|/T})$,
3. concave down for $\sigma \in (\sqrt{2|\ln F - \ln K|/T}, \infty)$

and also satisfies

$$
\begin{aligned}
v(0) &= [\phi(xe^{-r_fT} - Ke^{-r_dT})]^+, & (1.91)\\
v(\infty, \phi = 1) &= xe^{-r_fT}, & (1.92)\\
v(\sigma = \infty, \phi = -1) &= Ke^{-r_dT}, & (1.93)\\
v'(0) &= xe^{-r_fT}\sqrt{T}/\sqrt{2\pi}I\!I_{\{F=K\}}, & (1.94)
\end{aligned}
$$

In particular the mapping $\sigma \mapsto v(\sigma)$ is invertible. However, the starting guess for employing Newton's method should be chosen with care, because the mapping $\sigma \mapsto v(\sigma)$ has a saddle point at

$$
\left(\sqrt{\frac{2}{T}|\ln\frac{F}{K}|}, \phi e^{-r_dT}\left\{F\mathcal{N}\left(\phi\sqrt{2T[\ln\frac{F}{K}]^+}\right) - K\mathcal{N}\left(\phi\sqrt{2T[\ln\frac{K}{F}]^+}\right)\right\}\right), \qquad (1.95)
$$

as illustrated in Figure 1.4.
To ensure convergence of Newton's method, we are advised to use initial guesses for σ on the same side of the saddle point as the desired implied volatility. The danger is that a large initial guess could lead to a negative successive guess for σ. Therefore one should start with small initial guesses at or below the saddle point. For at-the-money options, the saddle point is degenerate for a zero volatility and small volatilities serve as good initial guesses.

Visual Basic Source Code

```
Function VanillaVolRetriever(spot As Double, rd As Double, _
rf As Double, strike As Double, T As Double, _
type As Integer, GivenValue As Double) As Double
Dim func As Double
Dim dfunc As Double
Dim maxit As Integer 'maximum number of iterations
Dim j As Integer
Dim s As Double
'first check if a volatility exists, otherwise set result to zero
If GivenValue < Application.Max _
   (0, type * (spot * Exp(-rf * T) - strike * Exp(-rd * T))) Or _
   (type = 1 And GivenValue > spot * Exp(-rf * T)) Or _
   (type = -1 And GivenValue > strike * Exp(-rd * T)) Then
```

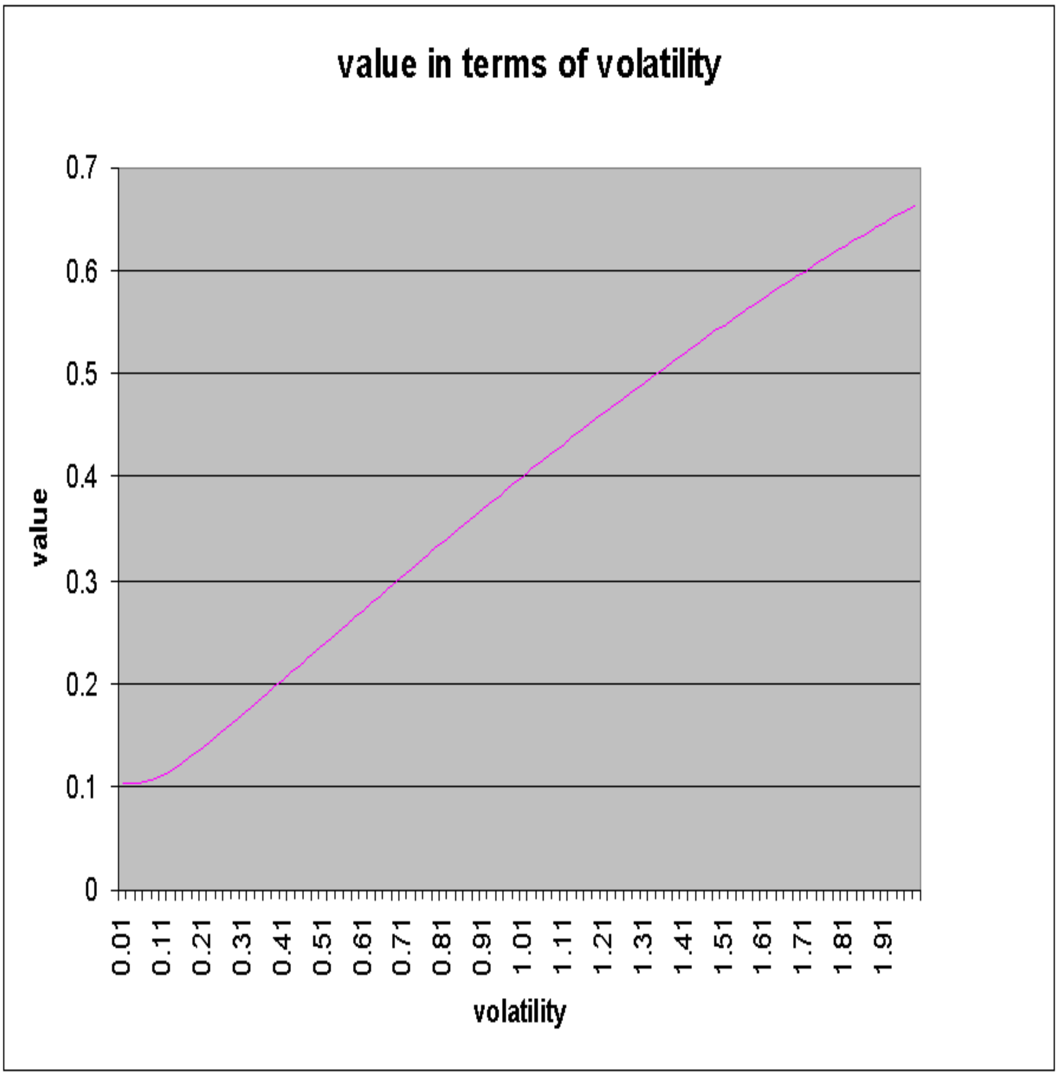


Figure 1.4: Value of a European call in terms of volatility with parameters $x = 1$, $K = 0.9$, $T = 1$, $r_d = 6\%$, $r_f = 5\%$. The saddle point is at $\sigma = 48\%$.

```
    VanillaVolRetriever = 0
Else
    ' there exists a volatility yielding the given value,
    ' now use Newton's method:
    ' the mapping vol to value has a saddle point.
    ' First compute this saddle point:
    saddle = Sqr(2 / T * Abs(Log(spot / strike) + (rd - rf) * T))
```

```
   If saddle > 0 Then
      VanillaVolRetriever = saddle * 0.9
   Else
      VanillaVolRetriever = 0.1
   End If
   maxit = 100
   For j = 1 To maxit Step 1
      func = Vanilla(spot, strike, VanillaVolRetriever, _
      rd, rf, T, type, value) - GivenValue
      dfunc = Vanilla(spot, strike, VanillaVolRetriever, _
      rd, rf, T, type, vega)
      VanillaVolRetriever = VanillaVolRetriever - func / dfunc
      If VanillaVolRetriever <= 0 Then VanillaVolRetriever = 0.01
      If Abs(func / dfunc) <= 0.0000001 Then j = maxit
   Next j
End If
End Function
```

Market Data

Now that we know how to imply the volatility from a given value, we can take a look at the market. We take EUR/GBP at the beginning of April 2005. The at-the-money volatilities for various maturities are listed in Table 1.6. We observe that implied volatilities are not constant, but depend on the time to maturity of the option as well as on the current time. This shows that the Black-Scholes assumption of a constant volatility is not fully justified looking at market data. We have a *term structure of volatility* as well as a stochastic nature of the term structure curve as time passes.
Besides the dependence on the time to maturity (term structure) we also observe different implied volatilities for different degrees of moneyness. This effect is called the *volatility smile*. The term structure and smile together are called a *volatility matrix* or *volatility surface*, if it is graphically displayed. Various possible reasons for this empirical phenomenon are discussed among others by Bates, e.g. in [7].

In Foreign Exchange Options markets implied volatilities are generally quoted and plotted against the deltas of out-of-the-money call and put options. This allows market participants to ask various partners for quotes on a 25-Delta call, which is spot independent. The actual strike will be set depending on the spot if the trade is close to being finalized. The at-the-money option is taken to be the one that has a strike equal to the forward, which is equivalent to the value of the call and the put being equal. Other types of *at-the-money* are discussed

Date	Spot	1 Week	1 Month	3 Month	6 Month	1 Year	2 Years
1-Apr-05	0.6864	4.69	4.83	5.42	5.79	6.02	6.09
4-Apr-05	0.6851	4.51	4.88	5.34	5.72	5.99	6.07
5-Apr-05	0.6840	4.66	4.95	5.34	5.70	5.97	6.03
6-Apr-05	0.6847	4.65	4.91	5.39	5.79	6.05	6.12
7-Apr-05	0.6875	4.78	4.97	5.39	5.79	6.01	6.10
8-Apr-05	0.6858	4.76	5.00	5.41	5.78	6.00	6.09

Table 1.6: EUR/GBP implied volatilities in % for at-the-money vanilla options. Source: BBA (British Bankers Association), http://www.bba.org.uk.

in Section 1.3.6. Their delta is

$$\frac{\partial v}{\partial x} = \phi e^{-r_f \tau} \mathcal{N}(\phi \frac{1}{2} \sigma \sqrt{\tau}), \tag{1.96}$$

for a small volatility σ and short time to maturity τ, a number near $\phi 50\%$. This is no more true for long-term vanilla options. Further market information are the implied volatilities for puts and calls with a delta of $\phi 25\%$. Other or additional implied volatilities for other deltas such as $\phi 10\%$ and $\phi 35\%$ are also quoted. Volatility matrices for more delta pillars are usually interpolated.

Symmetric Decomposition

Generally in Foreign Exchange, volatilities are decomposed into a *symmetric* part of the smile reflecting the *convexity* and a *skew-symmetric* part of the smile reflecting the *skew*. The way this works is that the market quotes *risk reversals (RR)* and *butterflies (BF)* or strangles, see Sections 1.4.2 and 1.4.5 for the description of the *products* and Figure 1.5 for the payoffs. Here we are talking about the respective *volatilities* to use to price the products. Sample quotes are listed in Tables 1.7 and 1.8. The relationship between risk reversal and strangle/butterfly quotes and the volatility smile are graphically explained in Figure 1.6.

The relationship between risk reversal quoted in terms of volatility (RR) and butterfly/strangle

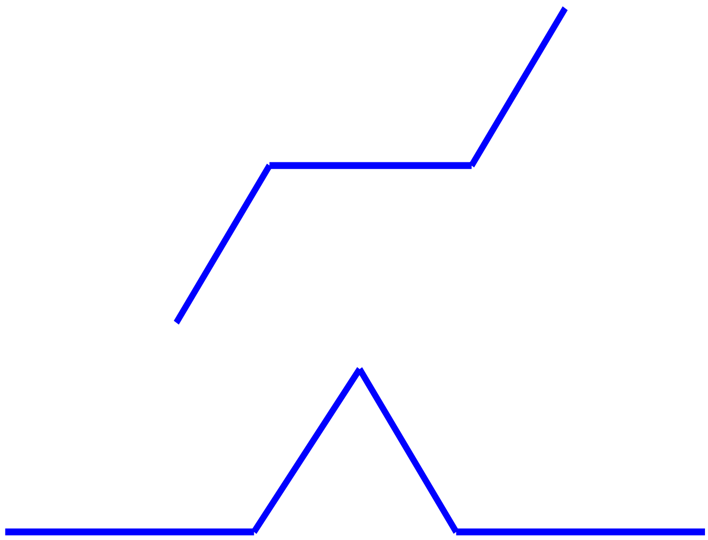


Figure 1.5: The risk reversal (upper payoff) is a skew symmetric product, the butterfly (lower payoff) a symmetric product.

(BF) quoted in terms of volatility and the volatilities of 25-delta calls and puts are given by

$$\sigma_+ = \text{ATM} + BF + \frac{1}{2}RR, \quad (1.97)$$

$$\sigma_- = \text{ATM} + BF - \frac{1}{2}RR, \quad (1.98)$$

$$RR = \sigma_+ - \sigma_-, \quad (1.99)$$

$$BF = \frac{\sigma_+ + \sigma_-}{2} - \sigma_0, \quad (1.100)$$

where σ_0 denotes the at-the-money volatility of both put and call, σ_+ the volatility of an out-of-the-money call (usually $25 - \Delta$) and σ_- the volatility of an out-of-the-money put (usually $25 - \Delta$). Our sample market data is given in terms of RR and BF. Translated into implied volatilities of vanillas we obtain the data listed in Table 1.9 and Figure 1.7.

Date	Spot	1 Month	3 Month	1 Year
1-Apr-05	0.6864	0.18	0.23	0.30
4-Apr-05	0.6851	0.15	0.20	0.29
5-Apr-05	0.6840	0.11	0.19	0.28
6-Apr-05	0.6847	0.08	0.19	0.28
7-Apr-05	0.6875	0.13	0.19	0.28
8-Apr-05	0.6858	0.13	0.19	0.28

Table 1.7: EUR/GBP 25 Delta Risk Reversal in %. Source: BBA (British Bankers Association). This means that for example on 4 April 2005, the 1-month 25-delta EUR call was priced with a volatility of 0.15% higher than the EUR put. At that moment the market apparently favored calls indicating a belief in an upward movement.

Date	Spot	1 Month	3 Month	1 Year
1-Apr-05	0.6864	0.15	0.16	0.16
4-Apr-05	0.6851	0.15	0.16	0.16
5-Apr-05	0.6840	0.15	0.16	0.16
6-Apr-05	0.6847	0.15	0.16	0.16
7-Apr-05	0.6875	0.15	0.16	0.16
8-Apr-05	0.6858	0.15	0.16	0.16

Table 1.8: EUR/GBP 25 Delta Strangle in %. Source: BBA (British Bankers Association). This means that for example on 4 April 2005, the 1-month 25-delta EUR call and the 1-month 25-delta EUR put are on average quoted with a volatility of 0.15% higher than the 1-month at-the-money calls and puts. The result is that the 1-month EUR call is quoted with a volatility of $4.88\% + 0.075\%$ and the 1-month EUR put is quoted with a volatility of $4.88\% - 0.075\%$.

1.3.4 At-The-Money Volatility Interpolation

The interpolation takes into account the effect of reduced volatility on weekends and on days closed in the global main trading centers London or New York and the local market, e.g. Tokyo for JPY–trades. The change is done for the one–day forward volatility. There is a reduction in the one–day forward variance of 25% for each London and New York closed day. For

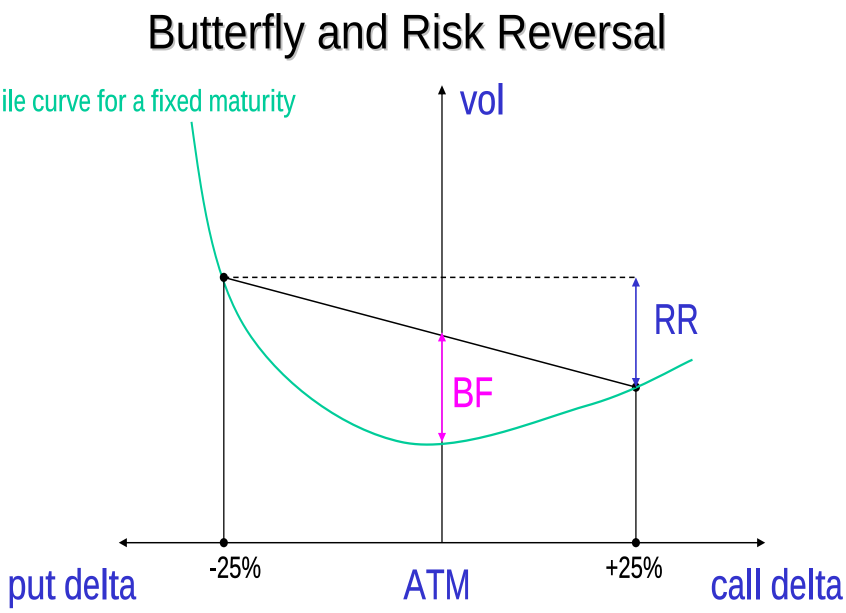


Figure 1.6: Risk reversal and butterfly in terms of volatility for a given FX vanilla option smile

Maturity	25 delta put	at-the-money	25 delta call
1M	4.890	4.830	5.070
3M	5.465	5.420	5.695
1Y	6.030	6.020	6.330

Table 1.9: EUR/GBP implied volatilities in % of 1 April 2005. Source: BBA (British Bankers Association). They are computed based on the market data displayed in Tables 1.6, 1.7 and 1.8 using Equations (1.97) and (1.98).

local market holidays there is a reduction of 25%, where local holidays for EUR are ignored. Weekends are accounted by a reduction to 15% variance. The variance on trading days is adjusted to match the volatility on the pillars of the ATM–volatility curve exactly.

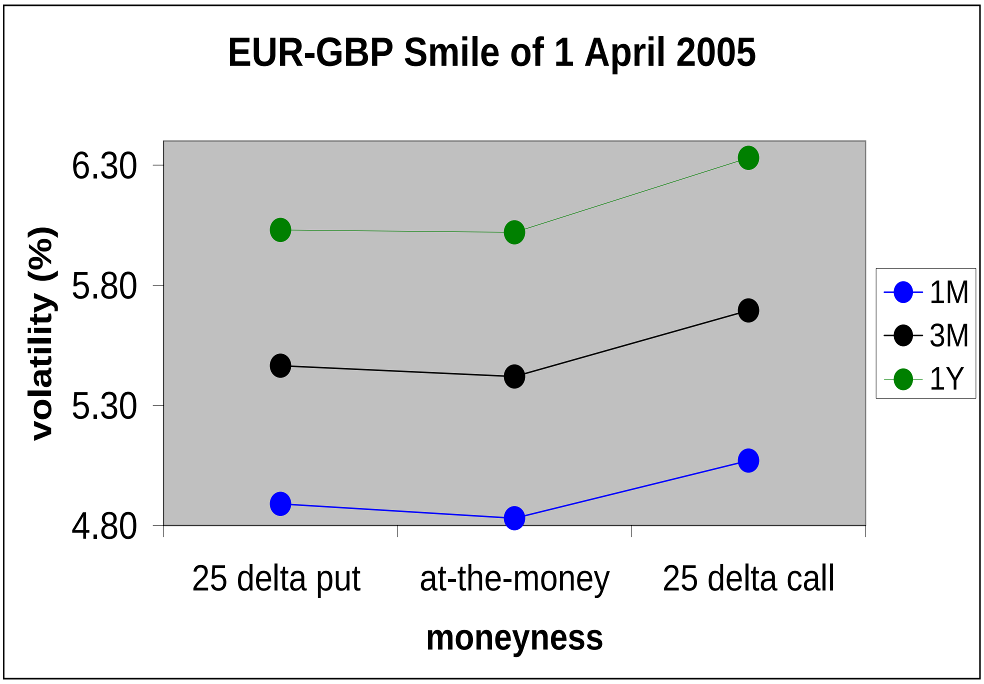


Figure 1.7: Implied volatilities for EUR-GBP vanilla options as of 1 April 2005. Source: BBA(British Bankers Association).

The procedure starts from the two pillars t_1, t_2 surrounding the date t_r in question. The ATM forward volatility for the period is calculated based on the consistency condition

$$\sigma^2(t_1)(t_1 - t_0) + \sigma_f^2(t_1, t_2)(t_2 - t_1) = \sigma^2(t_2)(t_2 - t_0), \tag{1.101}$$

whence

$$\sigma_f(t_1, t_2) = \sqrt{\frac{\sigma^2(t_2)(t_2 - t_0) - \sigma^2(t_1)(t_1 - t_0))}{t_2 - t_1}}. \tag{1.102}$$

For each day the factor is determined and from the constraint that the sum of one-day forward variances matches exactly the total variance the factor for the enlarged one day business variances $\alpha(t)$ with t business day is determined.

$$\sigma^2(t_1,t_2)(t_2-t_1)=\sum_{t=t_1}^{t_r}\alpha(t)\sigma_f^2(t,t+1) \tag{1.103}$$

The variance for the period is the sum of variances to the start and sum of variances to the required date.

$$\sigma^2(t_r)=\sqrt{\frac{\sigma^2(t_1)(t_1-t_0)+\sum_{t=t_1}^{t_r}\alpha(t)\sigma_f^2(t,t+1)}{t_r-t_0}} \tag{1.104}$$

1.3.5 Volatility Smile Conventions

The volatility smile is quoted in terms of delta and one at-the-money pillar. We recall that there are several notions of delta

- spot delta $e^{-r_f\tau}N(d_+)$,
- forward delta $e^{-r_d\tau}N(d_+)$,
- driftless delta $N(d_+)$,

and there is the premium which might be included in the delta. It is important to specify the notion that is used to quote the smile. There are three different deltas concerning plain vanilla options.

1.3.6 At-The-Money Definition

There is one specific at-the-money pillar in the middle. There are at least three notions for the meaning of *at-the-money (ATM)*.

Delta parity: delta call = - delta put

Value parity: call value = put value

Fifty delta: call delta = 50% and put delta = 50%

Moreover, these notions use different versions of delta, namely either spot, forward, or driftless and premium included or excluded.
The standard for all currencies one can stick to is spot delta parity with premium included [left hand side *Fenics* delta for call and put is the same] or excluded [right hand side *Fenics* delta] is used.

1.3.7 Interpolation of the Volatility on Maturity Pillars

To determine the spread to at-the-money we can take a kernel interpolation in one dimension to compute the volatility on the delta pillars. Given N points $(X_n, y_n), n = 1, \ldots, N$, where $X = (x^1, x^2) \in \mathcal{R}^2$ and $y \in \mathcal{R}$, a "smooth" interpolation of these points is given by a "smooth" function
$g : \mathcal{R}^2 \to \mathcal{R}$ which suffices

$$g(X_n) = y_n \quad (n = 1, \ \ldots, N)\,. \tag{1.105}$$

The kernel approach is

$$g(X) = g_{[\lambda,\alpha_1,\ldots,\alpha_N]}(X) \triangleq \frac{1}{\Gamma_\lambda(X)} \sum_{n=1}^{N} \alpha_n K_\lambda(\|X - X_n\|), \tag{1.106}$$

where

$$\Gamma_\lambda(x) \triangleq \sum_{n=1}^{N} K_\lambda(\|X - X_n\|) \tag{1.107}$$

and $\|.\|$ denotes the Euclidean norm. The required smoothness may be achieved by using analytic kernels K_λ, for instance $K_\lambda(u) \triangleq e^{-\frac{u^2}{2\lambda^2}}$.
The idea behind this approach is as follows. The parameters which solve the interpolation conditions (1.105) are $\alpha_1, \ldots, \alpha_n$. The parameter λ determines the "smoothness" of the resulting interpolation g and should be fixed according to the nature of the points (X_n, y_n). If these points yield a smooth surface, a "large" λ might yield a good fit, whereas in the opposite case when for neighboring points X_k, X_n the appropriate values y_k, y_n vary significantly, only a small λ, that means $\lambda << \min_{n,k} \|X_k - X_n\|$, can provide the needed flexibility.
For the set of delta pillars of $10\%, 25\%, ATM, -25\%, -10\%$ one can use $\lambda = 25\%$ for a smooth interpolation.

1.3.8 Interpolation of the Volatility Spread between Maturity Pillars

The interpolation of the volatility spread to ATM uses the interpolation of the spread on the two surrounding maturity pillars for the initial Black–Scholes delta of the option. The spread is interpolated using square root of time where $\tilde{\sigma}$ is the volatility spread,

$$\tilde{\sigma}(t) = \tilde{\sigma}_1 + \frac{\sqrt{t} - \sqrt{t_1}}{\sqrt{t_2} - \sqrt{t_1}}(\tilde{\sigma}_2 - \tilde{\sigma}_1). \tag{1.108}$$

The spread is added to the interpolated ATM volatility as calculated above.

1.3.9 Volatility Sources

1. BBA, the *British Bankers Association*, provides historic smile data for all major currency pairs in spread sheet format at http://www.bba.org.uk.

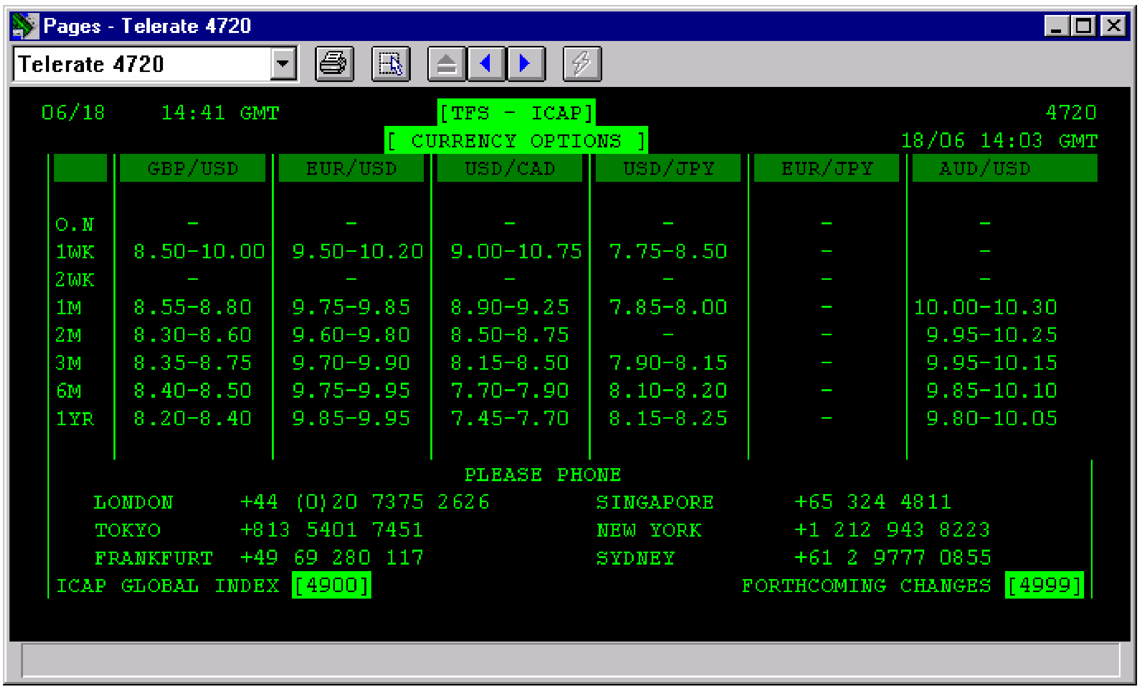
2. Olsen Associates (http://www.olson.ch) can provide tic data of historic spot rates, from which the historic volatilities can be computed.

3. Bloomberg, not really the traditional FX data source, contains both implied volatilities and historic volatilities.

4. Reuters pages such as FXMOX, SGFXVOL01, and others are commonly used and contain mostly implied volatilities. JYSKEOPT is a common reference for volatilities of Scandinavia (scandie-vols). NMRC has some implied volatilities for precious metals.

5. Telerate pages such as 4720, see Figure 1.8, delivers implied volatilies.

6. Cantorspeed 90 also provides implied volatilities.



Pages - Telerate 4720

Telerate 4720

06/18 14:41 GMT [TFS - ICAP] 4720

[CURRENCY OPTIONS] 18/06 14:03 GMT

	GBP/USD	EUR/USD	USD/CAD	USD/JPY	EUR/JPY	AUD/USD
O.N	-	-	-	-	-	-
1WK	8.50-10.00	9.50-10.20	9.00-10.75	7.75-8.50	-	-
2WK	-	-	-	-	-	-
1M	8.55-8.80	9.75-9.85	8.90-9.25	7.85-8.00	-	10.00-10.30
2M	8.30-8.60	9.60-9.80	8.50-8.75	-	-	9.95-10.25
3M	8.35-8.75	9.70-9.90	8.15-8.50	7.90-8.15	-	9.95-10.15
6M	8.40-8.50	9.75-9.95	7.70-7.90	8.10-8.20	-	9.85-10.10
1YR	8.20-8.40	9.85-9.95	7.45-7.70	8.15-8.25	-	9.80-10.05

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ICAP GLOBAL INDEX [4900] FORTHCOMING CHANGES [4999]

Figure 1.8: Telerate page 4720 quoting currency option volatilities

1.3.10 Volatility Cones

Volatility cones visualize whether current at-the-money volatility levels for various maturities are high or low compared to a recent history of these implied volatilities. This indicates to a trader or risk taker whether it is currently advisable to buy volatility or sell volatility, i.e. to buy vanilla options or to sell vanilla options. We fix a time horizon of historic observations of mid market at-the-money implied volatility and look at the maximum, the minimum that traded over this time horizon and compare this with the current volatility level. Since long term volatilities tend to fluctuate less than short term volatility levels, the chart of the minimum and the maximum typically looks like a part of a cone. We illustrate this in Figure 1.9 based on the data provided in Table 1.10.

maturity	low	high	current
1M	7.60	13.85	11.60
2M	7.80	12.40	11.10
3M	7.90	11.40	10.85
6M	8.00	11.00	10.40
12M	8.20	10.75	10.00

Table 1.10: Sample data of a volatility cone in USD-JPY for a 9 months time horizon from 6 Sept 2003 to 24 Feb 2005

1.3.11 Stochastic Volatility

Stochastic volatility models are very popular in FX Options, whereas *jump diffusion models* can be considered as the cherry on the cake. The most prominent reason for the popularity is very simple: FX volatility *is* stochastic as is shown for instance in Figure 1.10. Treating stochastic volatility in detail here is way beyond the scope of this book. A more recent overview can be found in the article *The Heston Model and the Smile* by Weron and Wystup [23].

1.3.12 Exercises

1. For the market data in Tables 1.6, 1.7 and 1.8 determine a smile matrix for at-the-money and the 25-deltas. Also compute the corresponding strikes for the three pillars or moneyness.

2. Taking the smile of the previous exercise, implement the functions for interpolation to generate a suitable implied volatility for any given time to maturity and any strike or delta.

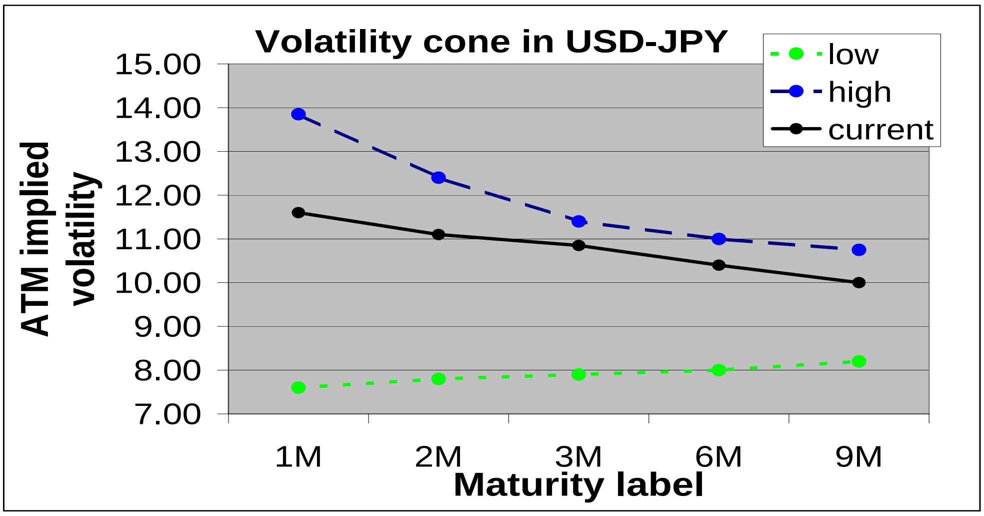

Figure 1.9: Example of a volatility cone in USD-JPY for a 9 months time horizon from 6 Sept 2003 to 24 Feb 2005

3. Using the historic data, generate a volatility cone for USD-JPY.

4. It is often believed that an at-the-money (in the sense that the strike is set equal to the forward) vanilla call has a delta near 50%. What can you say about the delta of a 15 year at-the-money USD-JPY call if USD rates are at 5%, JPY rates are at 1% and the volatility is at 11%?

1.4 Basic Strategies containing Vanilla Options

Linear Combinations of vanillas are quite well known and have been explained in several text books including the one by Spies [93]. Therefore, we will restrict our attention in this section to the most basic strategies.

1.4.1 Call and Put Spread

A Call Spread is a combination of a long and a short Call option. It is also called *capped call*. The motivation to do this is the fact that buying a simple call may be too expensive and the buyer wishes to lower the premium. At the same time he does not expect the underlying exchange rate to appreciate above the strike of the short Call option.

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Index

digital(ϕ, K)	$\lim_{n\to\infty} n$[vanilla(K) $-$ vanilla($K + \phi/n$)]
knock-in	vanilla $-$ knock-out
EKO(ϕ, K, B)	vanilla(ϕ, K) $-$ vanilla(ϕ, B) $-$ digital(B)$\phi(B - K)$
EDKOCall(K, L, H) ($K < L < H$)	call(L) $-$ call(H) $+ (L - K)$[digital(L) $-$ digital(H)]
vanilla(K)	digital$_{for}$ $- K \cdot$ digital$_{dom}$
RKO(ϕ, K, B)	KO($-\phi, K, B$) $-$ KO($-\phi, B, B$) $- \phi(B - K)$NT(B)
(D)OT	e^{-rT}- (D)NT
DOT$_{for}$($L, H, S_0, r_d, r_f, \sigma$)	S_0DOT$_{dom}$($\frac{1}{H}, \frac{1}{L}, \frac{1}{S_0}, r_f, r_d, \sigma$)
DOT$_{dom}$(L, H)	[DKOCall($K = L, L, H$) + DKOPut($K = H, L, H$)]/[$H - L$]
DOT$_{for}$(L, H)	[$H \cdot$ DKOCall($K = L, L, H$) + $L \cdot$ DKOPut($K = H, L, H$)]/[$(H - L)S_0$]
EDNT(L, H)	digital(L) $-$ digital(H)
two-touch(L, H)	OT(L) + OT(H) $-$ DOT(L, H)
second DNT($A < B < C < D$)	DNT(A, C) + DNT(B, D) $-$ DNT(B, C)
KIKO($L = ko, H = ki$)	KO(L) $-$ DKO(L, H)
forward(K)	call(K) $-$ put(K)
paylater premium	vanilla/digital
spread(ϕ)	vanilla(K, ϕ) $-$ vanilla($K + \phi \cdot$ spread$, \phi$)
risk reversal	call(K_+) $-$ put(K_-)
straddle(K)	call(K) + put(K)
strangle	call(K_+) + put(K_-)
butterfly	call(K_+) + put(K_-) $-$ call$_{ATM}$ $-$ put$_{ATM}$
shark forward	forward + RKO
bonus forward	forward + DNT
butterfly forward	forward + DKO straddle
accrued forward	forward + corridor
participating forward	call $- P$%put
fade-in forward	forward + fade-in vanilla
dcd($r >$ market)	deposit($r =$ market) $-$ vanilla
range deposit($r >$ market)	deposit($r <$ market) + DNT
performance note($r_{max} >$ market)	deposit($r <$ market) + call

Table 5.1: Common Replication Strategies and Structures