FX Volatility Smile Construction

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Abstract

The foreign exchange options market is one of the largest and most liquid OTC derivative markets in the world. Surprisingly, very little is known in the academic literature about the construction of the most important object in this market: The implied volatility smile. The smile construction procedure and the volatility quoting mechanisms are FX specific and differ significantly from other markets. We give a brief overview of these quoting mechanisms and provide a comprehensive introduction to the resulting smile construction problem. Furthermore, we provide a new formula which can be used for an efficient and robust FX smile construction.

Keywords

FX Quotations, FX Smile Construction, Risk Reversal, Butterfly, Strangle, Delta Conventions, Malz Formula

1 FX Market Conventions

Introduction

It is common market practice to summarize the information of the vanilla options market in a volatility smile table which includes Black-Scholes implied volatilities for different maturities and moneyness levels. The degree of moneyness of an option can be represented by the strike or any linear or non-linear transformation of the strike (forward-moneyness, log-moneyness, delta). The implied volatility as a function of moneyness for a single time to maturity is generally referred to as the smile. To be more precise, the volatility smile is a mapping

$$X \mapsto \sigma(X) \in [0, \infty)$$

with X being the moneyness variable. The function value $\sigma(\overline{X})$ for a given moneyness $\overline{X}$ and time to maturity $\overline{T}$ represents the implied volatility which is the crucial input variable for the well known Black-Scholes formula (Black and Scholes (1973)). The volatility smile is the crucial input into pricing and risk management procedures since it is used to price vanilla, as well as exotic option books. In the FX OTC derivative market the volatility smile is not directly observable, as opposed to the equity markets, where strike-price or strike-volatility pairs can be observed. In foreign exchange OTC derivative markets, the only volatility inputs are currency pair specific risk reversal σ_{RR}, quoted strangle σ_{S-Q} and at-the-money volatility σ_{ATM} quotes (see market sample in Table 1). The risk reversal and strangle quotes are assigned to a delta such as 0.25 which is incorporated in the notation in Table 1. These quotes can be used to construct a complete volatility smile, from which one can extract the volatility for any strike. In this section, a brief overview of basic FX terminology will be introduced which will be used in the remaining part of the paper. In the next section, the market implied information for quotes such as those given in Table 1 will be explained in detail. Finally, a new implied volatility function will be introduced which accounts for this information in an efficient manner.

FX Terminology

Before explaining the market quotes, we will briefly introduce the common FX terminology. For a more detailed introduction, we refer the reader to Beier and Renner (2010), Castagna (2010), Clark (2010), Reiswich and Wystup (2010), Reiswich (2010).

The FX spot rate S_t = FOR-DOM is the exchange rate at time t representing the number of units of domestic currency which are needed to buy one unit of foreign currency. We will refer to the "domestic" currency in the sense of a base (numeraire) currency in relation to which "foreign" amounts of money are measured, see Wystup (2006).

The holder of an FX option obtains the right to exchange a specified amount of money in domestic currency for a specified amount of money in foreign currency at an agreed exchange rate K at maturity time T. This is equivalent to receiving the asset and paying a predefined amount of money in the equity market. Under standard Black-Scholes assumptions (see Black

Table 1: FX Market data for a maturity of 1 month. Retrieved for January, 20th 2009 from Bloomberg database.

	EUR-USD	USD-JPY
σ_{ATM}	21.6215%	21.00%
σ_{25-RR}	–0.5%	–5.3%
$\sigma_{25-S-RR}$	0.7375%	0.184%

and Scholes (1973)) , the value of a vanilla option with strike K and expiry time T is given by

$$\begin{aligned} v(K,\sigma,\phi) &= \phi e^{-r_d\tau}\left[f(t,T)\,N(\phi d_+) - K\,N(\phi d_-)\right] \text{ where} \\ d_\pm &= \frac{\ln\left(\frac{f(t,T)}{K}\right) \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \\ K &: \text{ strike of the FX option,} \\ f(t,T) &: \text{ FX forward rate,} \\ \sigma &: \text{ Black-Scholes volatility,} \\ \phi &= +1 \text{ for a call }, \phi = -1 \text{ for a put,} \\ r_d, r_f &: \text{continuously compounded domestic or foreign rate,} \\ \tau &= T - t : \text{time to maturity.} \end{aligned} \tag{1}$$

In the formula above v depends on the variables K, σ, ϕ only since these will be the variables of interest in the rest of this work.

Deltas

This section gives a brief introduction to FX delta conventions. We follow the same brief summary as provided in Reiswich (2011). For a more detailed introduction on delta and at-the-money conventions we refer the reader to Beier and Renner (2010), Reiswich and Wystup (2010).

The standard Black-Scholes delta has an intuitive interpretation from a hedging point of view. It denotes the number of stocks an option seller has to buy to be hedged with respect to spot movements. Foreign exchange options have a nominal amount, such as 1,000,000 EUR. In this particular case, an option buyer of a EUR-USD option with a strike of $K = 1{,}3500$ will receive 1,000,000 EUR and pay 1,350,000 USD at maturity. The option value is by default measured in domestic currency units. In the example it is a dollar value such as 102,400 USD. A delta of 0.6 would imply that buying 60% = 600,000 EUR of the foreign notional is appropriate to hedge a sold option position. This is the standard Black-Scholes delta and is called *spot delta*.

Alternatively, we could consider a hedge in the forward market. This would require to take FX forward positions, which are influenced by spot and interest rate movements. A forward delta of 50% would imply to enter a forward contract with a nominal of 500,000 EUR to hedge the sold option with a nominal of 1,000,000 EUR. This delta type is called *forward delta*.

There are two more delta types, which can be explained by considering again the previously described spot delta hedge. The option was sold for 102,400 USD and 600,000 EUR were bought for the hedge. One can reduce the quantity of the hedge by converting the 102,400 USD received as the premium to a EUR amount at a given rate of $S_0 = 1.3900$ EUR-USD. This USD premium is equivalent to 73,669 EUR. Consequently, the final hedge quantity will be 526,331 EUR which is the original delta quantity of 600,000 reduced by the received premium measured in EUR. This procedure implies adjusting the delta by the premium such that approximately 52.6% of the notional needs to be bought for the hedge (instead of 60%). The quantity 52.6% can also be viewed as a delta and is called the *premium-adjusted spot delta*. The same argument holds for a hedge in the forward market, the resulting delta is called *premium-adjusted forward delta*.

Table 2: Delta Formulas. *S* denotes spot, *f* forward, *pa* premium-adjusted, ϕ = 1 for a call and ϕ = −1 for a put.

Δ_S	Δ_f	$\Delta_{S,pa}$	$\Delta_{f,pa}$
$\phi e^{-r_f\tau}N(\phi d_+)$	$\phi N(\phi d_+)$	$\phi e^{-r_f\tau}\,\frac{K}{f}N(\phi d_-)$	$\phi\frac{K}{f}N(\phi d_-)$

The introduced deltas can be stated as Black-Scholes type of formulas for puts and calls. For example, the premium-adjusted spot delta can be deduced from

$$\Delta_{S,pa}(K,\sigma,\phi) = \Delta_S(K,\sigma,\phi) - \frac{v(K,\sigma,\phi)}{S_t}$$

where Δ_S is the standard Black-Scholes delta. The resulting formulas are summarized in Table 2.

At-the-Money

The at-the-money definition is not as obvious as one might think. If a volatility σ_{ATM} is quoted, and no corresponding strike, one has to identify which at-the-money quotation is used. Some common at-the-money definitions are

ATM-spot $K = S_0$
ATM-forward $K = f$

ATM-delta-neutral K such that call delta = −put delta.

In addition to that, all notions of ATM involving delta will have sub-categories depending on which delta convention is used. The at-the-money spot quotation is well known. **ATM-forward** is very common for currency pairs with a large interest rate differential (emerging markets) or a large time to maturity. Choosing the strike in the **ATM-delta-neutral** sense ensures that a straddle with this strike has a zero delta (where delta has to be specified). This convention is considered as the default ATM notion for short-dated FX options. The formulas for different at-the-money strikes can be found in Beier and Renner (2010), Beneder and Elkenbracht-Huizing (2003), Reiswich and Wystup (2010).

2 Construction of Implied Volatility Smiles

The previous section introduced FX specific delta and ATM conventions. This knowledge is crucial to understand the volatility construction procedure in FX markets. In FX option markets it is common to use the delta to measure the degree of moneyness. Consequently, volatilities are assigned to deltas (for any delta type), rather than strikes. For example, it is common to quote the volatility for an option which has a premium-adjusted delta of 0.25. These quotes are often provided by market data vendors to their customers. However, the volatility-delta version of the smile is translated by the vendors after using the smile construction procedure discussed below. Other vendors do not provide delta-volatility quotes. In this case, the customers have to employ the smile construction procedure. Related sources covering this subject can be found in Bossens et al. (2009), Castagna (2010), Clark (2010).

Unlike in other markets, the FX smile is given implicitly as a set of restrictions implied by market instruments. This is FX-specific, as other markets quote volatility versus strike directly. A consequence is that one has to

Table 3: Market data for a maturity of 1 month, as of January, 20th 2009.		
	EUR-USD	**USD-JPY**
S_0	1.3088	90.68
r_d	0.3525%	0.42875%
r_f	2.0113%	0.3525%
σ_{ATM}	21.6215%	21.00%
σ_{25-RR}	–0.5%	–5.3%
σ_{25-S-Q}	0.7375%	0.184%

employ a calibration procedure to construct a volatility vs. delta or volatility vs. strike smile. This section introduces the set of restrictions implied by market instruments. The next section proposes a new method with an efficient and robust calibration.

Suppose the mapping of a strike to the corresponding implied volatility

$$K \mapsto \sigma(K)$$

is given. We will not specify $\sigma(K)$ here but assume that it represents a general, well behaving function that characterizes key aspects of volatility smiles (positivity, continuity). Crucial in the construction of the FX volatility smile is to build $\sigma(K)$ such that it matches the volatilities/prices implied by market quotes. The general FX market convention is to use three volatility quotes for a given delta such as $\Delta = \pm 0.25$[1]:

an at-the-money volatility σ_{ATM},
a risk reversal volatility σ_{25-RR},
a quoted strangle volatility σ_{25-S-Q} (also called broker strangle).

A sample of market quotes for the EUR-USD and USD-JPY currency pairs is given in Table 3. Before starting the smile construction it is important to analyze the exact characteristics of the quotes in Table 3. In particular, one has to identify first

which at-the-money convention is used,
which delta type is used.

For example, Figure 1 shows market consistent smiles based on the EUR-USD and USD-JPY market data from Table 3, assuming that this data refers to different deltas, a simple or premium-adjusted one. The smile functions have significantly different shapes, in particular for out-of-the-money and in-the-money volatilities. Therefore, it is crucial to know how the deltas are defined when such quotes are obtained. Otherwise, option prices will be incorrect.

The quotes in the given market sample refer to a spot delta for the currency pair EUR-USD and a premium-adjusted spot delta for the currency pair USD-JPY. Both currency pairs use the forward delta neutral at-the-money quotation.

At-the-Money Volatility

After identifying the at-the-money type, we can extract the at-the-money strike K_{ATM} using the formulas provided in Beier and Renner (2010), Beneder and Elkenbracht-Huizing (2003), Reiswich and Wystup (2010). For the market sample data in Table 3 the corresponding strikes are summarized in Table 4. To clarify how the strikes are calculated, consider the EUR-USD data in Table 3. As stated earlier, the at-the-money data refers to the forward delta-neutral definition. The corresponding at-the-money strike formula is:

$$K_{ATM} = fe^{\frac{1}{2}\sigma^2\tau}.$$

The variables corresponding to the market sample are $\tau = 0.0849$, $f = 1.3070$, $\sigma = 0.216215$. Consequently, one can calculate

$$K_{ATM} = 1.3070 e^{\frac{1}{2} 0.216215^2 0.0849} = 1.3096,$$

which is the EUR-USD at-the-money strike in Table 4. Independent of the choice of $\sigma(K)$, it has to be ensured that the volatility for the at-the-money strike is σ_{ATM}. Consequently, the construction procedure for $\sigma(K)$ has to guarantee that the following Equation

$$\sigma(K_{ATM}) = \sigma_{ATM} \tag{2}$$

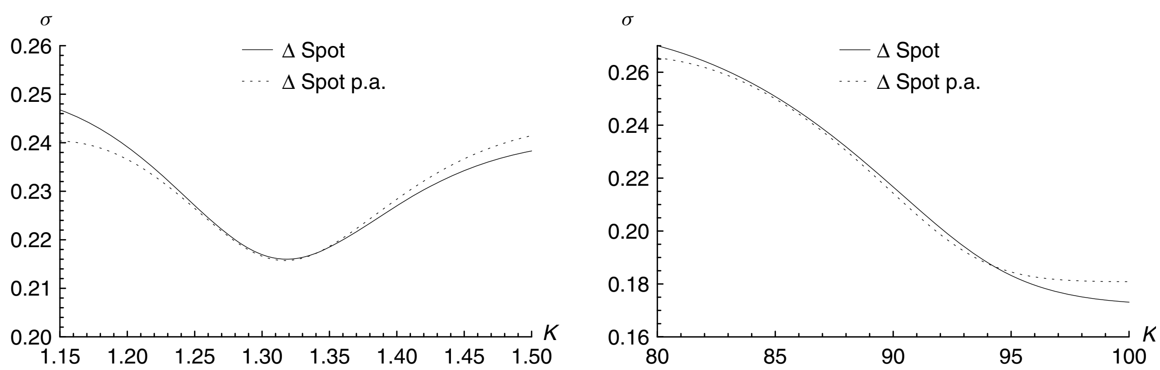
holds. A market consistent smile function $\bar{\sigma}(K)$ for the EUR-USD currency pair thus has to yield

$$\bar{\sigma}(1.3096) = 21.6215\%$$

for the market data in Table 3. We will show later how to calibrate $\sigma(K)$ to retrieve $\bar{\sigma}(K)$, so assume for the moment that the calibrated, market consistent smile function $\bar{\sigma}(K)$ is given.

Table 4: At-the-money strikes for market sample.		
	EUR-USD	**USD-JPY**
K_{ATM}	1.3096	90.86

Figure 1: Smile construction with EUR-USD (left graph) and USD-JPY (right graph) market data from Table 3, assuming different delta types.



Risk Reversal

The risk reversal quotation σ_{25-RR} is the difference between two volatilities:

- the implied volatility of a call with a delta of 0.25 and
- the implied volatility of a put with a delta of –0.25.

It measures the skewness of the smile, the extra volatility which is added to the 0.25 Δ put volatility compared to a call volatility which has the same absolute delta. Clearly, the delta type has to be specified in advance. For example, the implied volatility of a USD call JPY put with a premium-adjusted spot delta of 0.25 could be considered. Given $\sigma(K)$, it is possible to extract strike-volatility pairs[2] for a call and a put

$$\left(K_{25C}, \sigma(K_{25C})\right) \qquad \left(K_{25P}, \sigma(K_{25P})\right)$$

which yield a delta of 0.25 and –0.25 respectively:

$$\Delta(K_{25C}, \sigma(K_{25C}), 1) = 0.25$$

$$\Delta(K_{25P}, \sigma(K_{25P}), -1) = -0.25$$

In the equation system above, Δ denotes a general delta which has to be specified to Δ_S, $\Delta_{S,pa}$ or Δ_f, $\Delta_{f,pa}$. The market consistent smile function $\overline{\sigma}(K)$ has to match the information implied in the risk reversal. Consequently, it has to fulfill

$$\overline{\sigma}(K_{25C}) - \overline{\sigma}(K_{25P}) = \sigma_{25-RR}. \tag{3}$$

Examples of such 0.25Δ strike-volatility pairs for the market data in Table 3 and a calibrated smile function $\overline{\sigma}(K)$ are given in Table 5. For the currency pair EUR-USD we can calculate the difference of the 0.25Δ call and put volatilities as

$$\overline{\sigma}(1.3677) - \overline{\sigma}(1.2530) = 22.1092\% - 22.6092\% = -0.5\%$$

which is consistent with the risk reversal quotation in Table 3. It can also be verified that

$$\Delta_S\,(1.3677, 22.1092\%, 1) = 0.25 \text{ and } \Delta_S(1.2530, 22.6092\%, -1) = -0.25.$$

The value of the actually traded instrument is

$$v_{RR} = v(K_{25C}, \sigma(K_{25C}), 1) - v(K_{25P}, \sigma(K_{25P}), -1),$$

see also Castagna (2010). This price can be obtained from the smile after calibration. The market quotes a difference of implied volatilities rather than a concrete price for the risk reversal. This is different in the case of the market strangle discussed below.

Market Strangle

The strangle is the third restriction on the function $\sigma(K)$. Its quotation procedure leads to a lot of confusion among academics and practitioners so it is worth spending some time explaining it. Define the market strangle volatility σ_{25-S-M} as

$$\sigma_{25-S-M} = \sigma_{ATM} + \sigma_{25-S-Q}. \tag{4}$$

As before, the variable σ_{25-S-Q} denotes the quoted strangle volatility[3]. For the market sample from Table 3 and the USDJPY case this would correspond to

$$\sigma_{25-S-M} = 21.00\% + 0.184\% = 21.184\%.$$

Given this single volatility, we can extract a call strike $K_{25C-S-M}$ and a put strike $K_{25P-S-M}$ which - using σ_{25-S-M} as the volatility - yield a delta of 0.25 and –0.25 respectively. The procedure to extract a strike given a delta and volatility can be found in Reiswich (2010). The resulting strikes will then fulfill

$$\Delta(K_{25C-S-M}, \sigma_{25-S-M}, 1) = 0.25$$

$$\Delta(K_{25P-S-M}, \sigma_{25-S-M}, -1) = -0.25. \tag{5}$$

The strikes corresponding to the market sample are summarized in Table 6. For the USDJPY case the strike volatility combinations given in Table 6 fulfill

$$\Delta_{S,pa}(94.55, 21.184\%, 1) = 0.25 \tag{6}$$

$$\Delta_{S,pa}(87.00, 21.184\%, -1) = -0.25 \tag{7}$$

where $\Delta_{S,pa}(K, \sigma, \phi)$ is the premium-adjusted spot delta. Given the strikes $K_{25C-S-M}$, $K_{25P-S-M}$ and the volatility σ_{25-S-M}, one can calculate the price of an option position of a long call with a strike of $K_{25C-S-M}$ and a volatility of σ_{25-S-M} and a long put with a strike of $K_{25P-S-M}$ and the same volatility. The resulting price v_{25-S-M} is

$$v_{25-S-M} = v(K_{25C-S-M}, \sigma_{25-S-M}, 1) + v(K_{25P-S-M}, \sigma_{25-S-M}, -1) \tag{8}$$

and is the final variable one is interested in. This is the third information implied by the market: The sum of the call option with a strike of $K_{25C-S-M}$ and the put option with a strike of $K_{25P-S-M}$ has to be v_{25-S-M}. This information has to be incorporated by a market consistent volatility function $\overline{\sigma}(K)$ which can have different volatilities at the strikes $K_{25C-S-M}$, $K_{25P-S-M}$ but should guarantee that the corresponding option prices at these strikes add up to v_{25-S-M}. The delta of these options with the smile volatilities is not restricted to yield 0.25 or – 0.25. To summarize,

$$v_{25-S-M} = v(K_{25C-S-M}, \sigma(K_{25C-S-M}), 1) + v(K_{25P-S-M}, \sigma(K_{25P-S-M}), -1) \tag{9}$$

is the last restriction on the volatility smile. Taking again the USDJPY as an example yields that the strangle price to be matched is

$$v_{25-S-M} = v(94.55, 21.184\%, 1) + v(87.00, 21.184\%, -1) = 1.67072. \tag{10}$$

The resulting price v_{25-S-M} is in the domestic currency, JPY in this case. One can then extract the volatilities from a calibrated smile $\overline{\sigma}(K)$ — as in Table 6 — and calculate the strangle price with volatilities given by the calibrated smile function $\overline{\sigma}(K)$

$$v(94.55, 18.5435\%, 1) + v(87.00, 23.7778\%, -1) = 1.67072. \tag{11}$$

This is the same price as the one implied by the market in Equation (10).

Table 5: 0.25 Δ strikes.

	EUR-USD	USD-JPY
K_{25C}	1.3677	94.10
K_{25P}	1.2530	86.51
$\overline{\sigma}(K_{25C})$	22.1092%	18.7693%
$\overline{\sigma}(K_{25P})$	22.6092%	24.0693%

Table 6: Market Strangle data.

	EUR-USD	USD-JPY
$K_{25C\text{-}S\text{-}M}$	1.3685	94.55
$K_{25P\text{-}S\text{-}M}$	1.2535	87.00
$\overline{\sigma}(K_{25C\text{-}S\text{-}M})$	22.1216%	18.5435%
$\overline{\sigma}(K_{25P\text{-}S\text{-}M})$	22.5953%	23.7778%
$v_{25\text{-}S\text{-}M}$	0.0254782	1.67072

To summarize: The market uses a single volatility (quoted strangle plus the at-the-money volatility) to convey information about a price of a strangle with certain strikes. When the smile is constructed, one must choose the volatilities at these strikes such that the price of the corresponding strangle matches the one calculated with a single volatility.

Remark 1 (A Negative Broker Strangle?). We would like to note that it is totally valid for the broker strangle σ_{25-S-Q} to be **negative**. A negative strangle can be observed quite regularly in the market for specific currency pairs. The only limitation is that the absolute value of a negative broker strangle can not be larger than the at-the-money volatility as this would then lead to a negative market strangle volatility in Equation (4).

Remark 2 (Market Strangle Intuition). The market strangle quotation often confuses people. A standard question is, why this type of quotation is used. As already stated, the smile consistent deltas are not necessarily ±0.25 leading to a potentially non delta-neutral strangle. The major driver behind this quotation seems to have historical reasons. In the early times of the FX derivative market, the smiles were more symmetric and there was no significant skew. Consequently, it was convenient to quote a single volatility for a two leg option position. In todays markets smiles are often asymmetric as one can observe significant risk reversals. However, the quoting mechanism for the strangle did not change. The quoted single volatility can now be considered as an average of the implied volatility of both, the call and the put (as will be shown below). The single volatility allows an easier comparison of prices if the strikes are fixed: It is a common procedure in the market to ask a broker for a single strangle volatility and specify two strikes for a call and put (the strikes not necessarily being 25 delta strikes). This is slightly different from the standard 25 delta strangle quote described above. The broker would then quote a single volatility which one could use to retrieve the total price. The higher the single volatility, the higher the price of the strangle which allows an easy price comparison. Comparing two prices would be less easy if two volatilities would be quoted: One broker may quote a higher call volatility, but a lower put volatility than some other broker and it would not be clear who charges the higher premium without actually calculating it.

Market Strangle as an Average of Smile Volatilities

In Bossens et al. (2009), the authors use a first order Taylor expansion around the at-the-money volatility to show that the market strangle can be represented as a vega weighted sum of smile consistent volatilities. In opposite to the derivation in Bossens et al. (2009), we will expand around the market strangle volatility σ_{25-S-M} which yields[4]:

$$\begin{aligned} &v(K_{25C-S-M}, \sigma_{25-S-M}, 1) + v(K_{25P-S-M}, \sigma_{25-S-M}, -1) \\ &= v(K_{25C-S-M}, \sigma(K_{25C-S-M}), 1) + v(K_{25P-S-M}, \sigma(K_{25P-S-M}), -1) \qquad (12) \\ &\approx v(K_{25C-S-M}, \sigma_{25-S-M}, 1) + v(K_{25P-S-M}, \sigma_{25-S-M}, -1) \\ &+ \frac{\partial v(K_{25C-S-M}, \sigma_{25-S-M})}{\partial \sigma}(\sigma(K_{25C-S-M}) - \sigma_{25-S-M}) \\ &+ \frac{\partial v(K_{25P-S-M}, \sigma_{25-S-M})}{\partial \sigma}(\sigma(K_{25P-S-M}) - \sigma_{25-S-M})) \qquad (13) \end{aligned}$$

Equaling Equation (12) and (13) yields

$$\sigma_{25-S-M} \approx \frac{\frac{\partial v(K_{25C-S-M},\sigma_{25-S-M})}{\partial \sigma}\sigma(K_{25C-S-M}) + \frac{\partial v(K_{25P-S-M},\sigma_{25-S-M})}{\partial \sigma}\sigma(K_{25P-S-M})}{\frac{\partial v(K_{25C-S-M},\sigma_{25-S-M})}{\partial \sigma} + \frac{\partial v(K_{25P-S-M},\sigma_{25-S-M})}{\partial \sigma}} \qquad (14)$$

This shows, that the market strangle is a vega weighted sum of smile consistent call and put volatilities (at market strangle strikes). However, we can proceed a step further and use the fact that, if the absolute Black Scholes spot or forward delta of a call and a put is equal, so are the vegas. The following Lemma can be found in Reiswich (2010) (which is based on Castagna (2010)):

Lemma 1. *Let a volatility-strike function $\sigma(K)$ be given and*

$$\frac{\partial v}{\partial \sigma}(K, \sigma(K))$$

denote the Black-Scholes Vega at strike K. Given a call strike K_C and a put strike K_P such that

$$\Delta(K_C, \sigma(K_C), 1) = |\Delta(K_P, \sigma(K_P), -1)|,$$

where Δ is either a spot or forward delta, implies

$$\frac{\partial v}{\partial \sigma}(K_C, \sigma(K_C)) = \frac{\partial v}{\partial \sigma}(K_P, \sigma(K_P)).$$

Remembering that the market strangle strikes are chosen such that the absolute delta, using a single volatility, is 0.25 (see Equation (5)) implies that

$$\frac{\partial v(K_{25C-S-M}, \sigma_{25-S-M})}{\partial \sigma} = \frac{\partial v(K_{25P-S-M}, \sigma_{25-S-M})}{\partial \sigma} \qquad (15)$$

which we plug into Equation (14) to get

$$\sigma_{25-S-M} \approx \frac{1}{2}\left[\sigma(K_{25C-S-M}) + \sigma(K_{25P-S-M})\right]. \qquad (16)$$

Note that this is only proved for spot and forward delta, and not for the premium adjusted equivalents. The approximation breaks down with significant second order derivative (i.e. the Volga[5]) and for large maturities.

The Simplified Formula

As opposed to the approach we have shown, previous research has tended to use a more ad hoc procedure to determine 0.25Δ volatilities. This simplified procedure is used by Malz (1997), Wang (2009), Chalamandaris and Tsekrekos (2010), Bakshi et al. (2008), Galati et al. (2005). We will discuss the potential problems with this approach if it is used as a "quick and dirty" way to construct the implied volatility smile. Let σ_{25C} be the market consistent call volatility corresponding to a delta of 0.25 and σ_{25P} the market consistent –0.25 delta put volatility. Let K_{25C} and K_{25P} denote the corresponding strikes. The simplified formula states that

$$\sigma_{25C} = \sigma_{ATM} + \frac{1}{2}\sigma_{25-RR} + \sigma_{25-S-Q}$$
$$\sigma_{25P} = \sigma_{ATM} - \frac{1}{2}\sigma_{25-RR} + \sigma_{25-S-Q}. \tag{17}$$

This would allow a simple calculation of the 0.25Δ volatilities σ_{25C}, σ_{25P} with market quotes as given in Table 3. Including the at-the-money volatility would result in a smile with three anchor points which can then be interpolated in the usual way. In this case, no calibration procedure is needed. Note, that

$$\sigma_{25C} - \sigma_{25P} = \sigma_{25-RR} \tag{18}$$

such that the 0.25Δ volatility difference automatically matches the quoted risk reversal volatility. The simplified formula can be reformulated to calculate σ_{25-S-Q}, given σ_{25C}, σ_{25P} and σ_{ATM} quotes. This yields

$$\sigma_{25-S-Q} = \frac{\sigma_{25C} + \sigma_{25P}}{2} - \sigma_{ATM}, \tag{19}$$

which presents the strangle as a convexity parameter. Unfortunately, this approach can fail to match the market strangle given in Equation (8), which is repeated here for convenience

$$v_{25-S-M} = v(K_{25C-S-M}, \sigma_{25-S-M}, 1) + v(K_{25P-S-M}, \sigma_{25-S-M}, -1).$$

Interpolating the smile from the three anchor points given by the simplified formula and calculating the market strangle with the corresponding volatilities at $K_{25P-S-M}$ and $K_{25C-S-M}$ does not necessarily lead to the matching of v_{25-S-M}. For example, consider the USD-JPY data from Table 3. In the previous section we have shown that the USD-JPY market strangle price v_{25-S-M} implied by the market is 1.67072. However, when the simplified approach in Malz (1997) is used, the resulting market strangle price is 1.62801. The prices differ significantly. The reason why the formula is used often is that the corresponding procedure significantly simplifies the smile construction. The market strangle matching in the simplified approach works for small risk reversal volatilities σ_{25-RR}. To be more precise, assume that σ_{25-RR} is zero. The simplified formula (17) then reduces to

$$\sigma_{25C} = \sigma_{ATM} + \sigma_{25-S-Q},$$
$$\sigma_{25P} = \sigma_{ATM} + \sigma_{25-S-Q}.$$

This implies, that the volatility corresponding to a delta of 0.25 is the same as the volatility corresponding to a delta of –0.25, which is the same as the market strangle volatility σ_{25-S-M} introduced in Equation (4). Assume that in case of a zero risk reversal the smile is built using three anchor points given by the simplified formula and a strangle is priced with strikes $K_{25C-S-M}$ and $K_{25P-S-M}$. Given the volatility $\sigma_{25C} = \sigma_{ATM} + \sigma_{25-S-Q}$ and a delta of 0.25 would result in $K_{25C-S-M}$ as the corresponding strike. Consequently, we would assign $\sigma_{ATM} + \sigma_{25-S-Q}$ to the strike $K_{25C-S-M}$ if we move from delta to the strike space. Similarly, a volatility of $\sigma_{ATM} + \sigma_{25-S-Q}$ would be assigned to $K_{25P-S-M}$. The resulting strangle from the three anchor smile would be

$$v(K_{25C-S-M}, \sigma_{ATM} + \sigma_{25-S-Q}, 1) + v(K_{25P-S-M}, \sigma_{ATM} + \sigma_{25-S-Q}, -1)$$

which is exactly the market strangle price v_{25-S-M}. In this particular case, we have

$$K_{25C-S-M} = K_{25C},$$
$$K_{25P-S-M} = K_{25P}.$$

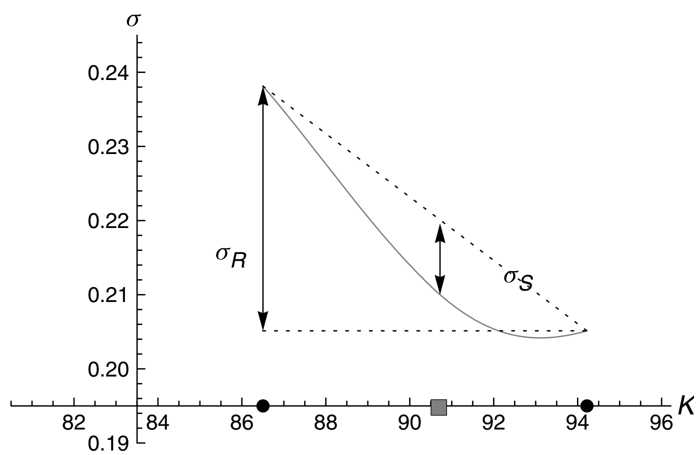
Using the simplified smile construction procedure yields a market strangle consistent smile setup in case of a zero risk reversal (or a small absolute risk reversal <1%, see Bossens et al. (2009)). The other market matching requirements are met by default. For example, consider the EUR-USD market strangle price in Table 6, which is 0.0254782. Table 3 shows that the EUR-USD risk reversal is –0.5%, which is less than 1%. The market strangle price with the approach in Malz (1997) can be calculated as 0.0254778. In this case, the difference between the prices is small. In any other case, the strangle price might not be matched which leads to a non market consistent setup of the volatility smile. Note that in the special case where the formula can be used there is still an issue which has to be taken care of: One has to carefully consider the delta type when switching to the strike-volatility space. This is a common misconception in the literature when the FX smile is constructed.

The simplified formula can still yield accurate results, even for large risk reversals, if σ_{25-S-Q} is replaced by an alternative strangle definition. This parameter can be extracted after finishing the market consistent smile construction and is calculated in a similar manner to Equation (19). Assume that the 0.25 delta volatilities $\sigma_{25C} = \overline{\sigma}(K_{25C})$ and $\sigma_{25P} = \overline{\sigma}(K_{25P})$ are given by the **calibrated** smile function $\overline{\sigma}(K)$. We can then calculate another strangle, called the smile strangle via

$$\sigma_{25-S-S} = \frac{\overline{\sigma}(K_{25C}) + \overline{\sigma}(K_{25P})}{2} - \sigma_{ATM}. \tag{20}$$

The smile strangle measures the convexity of the calibrated smile function and is plotted in Figure 2. It is approximately the difference between a straight line between the ±25 Δ put and call volatilities and the at-the-money volatility, evaluated at Δ_{ATM}[6]. This is equivalent to Equation (19), but in this case we are using out-of-the-money volatilities obtained from the calibrated smile and not from the simplified formula. Given σ_{25-S-S}, the simplified Equation (17) can still be used if the quoted strangle volatility σ_{25-S-Q} is replaced by the smile strangle volatility σ_{25-S-S}. Clearly, σ_{25-S-S} is not known a priori but is obtained after calibration[7]. Thus, one obtains a correct simplified formula as

Figure 2: Smile strangle for random market data. Filled circles indicate K_{25P}, K_{25C} strikes. Rectangle indicates K_{ATM}. σ_R denotes the risk reversal, σ_S the smile strangle.



$$\sigma_{25C} = \sigma_{ATM} + \frac{1}{2}\sigma_{25-RR} + \sigma_{25-S-S},$$
$$\sigma_{25P} = \sigma_{ATM} - \frac{1}{2}\sigma_{25-RR} + \sigma_{25-S-S}. \tag{21}$$

In general, the smile strangle is not observable in the market. However, some FX option market participants are moving towards a quotation of the smile strangle in their systems. In this case, Equation (21) is appropriate and the simple smile construction procedure can be employed. Confusion arises when one observes a strangle quotation and it is not said explicitly whether it is the smile strangle or market strangle.

The market strangle is still the standard quoting convention amongst market participants and the simplified procedure can produce non market-consistent smiles. This is the most often occurring misconception regarding FX volatility data.

Consider the following numerical example. Sample data is summarized in Table 7 where we have used the calibrated smile function $\overline{\sigma}(K)$ to calculate the smile strangles σ_{25-S-S}. Given σ_{25-S-S}, σ_{ATM} and σ_{25-RR}, one can calculate the EUR-USD out-of-the-money volatilities of the call and put via the simplified Formula (21) as

$$\sigma_{25C} = 21.6215\% - \frac{1}{2}0.5\% + 0.7377\% = 22.1092\%,$$
$$\sigma_{25P} = 21.6215\% + \frac{1}{2}0.5\% + 0.7377\% = 22.6092\%,$$

which is consistent with the volatilities $\overline{\sigma}(K_{25C})$ and $\overline{\sigma}(K_{25P})$ in Table 7. Note that the market strangle volatility is very close to the smile strangle volatility in the EUR-USD case. This is due to the risk reversal of the EUR-USD smile having a value close to zero. Calculating the 25Δ volatilities via the original simplified Formula (17) would yield a call volatility of 22.109% and a put volatility of 22.609% which are approximately the 25Δ volatilities of Table 7. This confirms that the difference between the simplified and market consistent setup is not significant in the case of a small risk reversal (i.e. a risk reversal <1%). However, the smile strangle and quoted strangle volatilities differ significantly for the skewed USD-JPY smile. Using the original Formula (17) in this case would result in 18.534% and 23.834% for the $\pm 25\Delta$ call and put volatilities. These volatilities differ from the market consistent 25Δ volatilities given in Table 7.

Remark 3 (A Negative Smile Strangle?). As with the broker strangle, a smile strangle can be negative too. Firstly, a volatility smile can be non convex even if standard no-arbitrage conditions are not violated. As the smile strangle reflects the smile convexity, it can be negative. However, a more common cause for a negative smile strangle occurs when the zero delta straddle at-the-money volatility is used. For example, the at-the-money strike for a zero spot or forward delta straddle is

$$K_{ATM} = fe^{\frac{1}{2}\sigma_{ATM}\tau^2}. \tag{22}$$

The larger the time to maturity, the larger is the strike[8]. For some maturities and currency pairs this can imply that the at-the-money strike is located to the right of the 25 delta call strike and not between the call and the put strike (see also Beier and Renner (2010)). Depending on the shape of the volatility smile this can also mean that the at-the-money volatility is larger than the average of the 25 delta call and put volatility which means that the term

$$\sigma_{25-S-S} = \frac{\overline{\sigma}(K_{25C}) + \overline{\sigma}(K_{25P})}{2} - \sigma_{ATM}.$$

is negative.

Table 7: Smile strangle data.

	EUR-USD	USD-JPY
$\overline{\sigma}(K_{25C})$	22.1092%	18.7693%
$\overline{\sigma}(K_{25P})$	22.6092%	24.0693%
σ_{ATM}	21.6215%	21.00%
σ_{25-RR}	−0.5%	−5.3%
σ_{25-S-S}	0.7377%	0.419%
σ_{25-S-Q}	0.7375%	0.184%

Remark 4 (A Market Risk Reversal?). After writing a previous working version of this paper, we were contacted by some practitioners which raised the question of the existence of a market risk reversal. In this case, the risk reversal volatility would be used in a similar way to the strangle volatility and a concrete price for the risk reversal option position could be calculated (without even calibrating the smile). This would be contradictory to our previous statement that the risk reversal is a difference of two volatilities and that the risk reversal option position (long call, short put) can only be valued after the calibration to the at-the-money volatility, market strangle and risk reversal volatility. However, we claim that our definition is the market convention and that potential confusion arises because market participants confuse smile and market strangles. If it is standard within the company to work with smile strangles, one could indeed use a simplified way to calculate the risk reversal price. If the market strangle is the standard, this can not be done.

3 Simplified Parabolic Interpolation

Various different interpolation methods can be considered as basic tools for the calibration procedure. Potential candidates are the SABR model introduced by Hagan et al. (2002), or the Vanna Volga method introduced by Castagna and Mercurio (2006). It is crucial to find a model which calibrates quickly and robustly for a wide range of currency pairs. Models with closed form solutions (or approximations thereof) are beneficial. In this work, we introduce a new method for the smile construction. In Reiswich (2011), the empirical calibration robustness of this method is compared with other methods.

In Malz (1997), the mapping forward delta against volatility is constructed as a polynomial of degree two. This polynomial is constructed such that the at-the-money and risk reversal delta volatilities are matched. Malz derives the following functional relationship

$$\sigma(\Delta_f) = \sigma_{ATM} - 2\sigma_{25-RR}(\Delta_f - 0.5) + 16\sigma_{25-S-Q}(\Delta_f - 0.5)^2 \tag{23}$$

where Δ_f is a call forward delta[9]. This is a parabola centered at 0.5. The use of this functional relationship can be problematic due to the following set of problems:

- the interpolation is not a well defined volatility function since it is not always positive,
- the representation is only valid for forward deltas, although the author incorrectly uses the spot delta in his derivation (see Equation (7) and Equation (18) in Malz [1997]),

- the formula is only valid for the forward delta neutral at-the-money quotation,
- the formula is only valid for risk reversal and strangle quotes associated with a delta of 0.25,
- the matching of the market strangle restriction(9) is guaranteed for small risk reversals only.

The last point is crucial! If the risk reversal σ_{25-RR} is close to zero, the formula will yield $\sigma_{ATM} + \sigma_{25-S-Q}$ as the volatility for the ±0.25 call and put delta. This is consistent with restriction (9). However, a significant risk reversal will lead to a failure of the formula. We will fix most of the problems by deriving a new, more generalized formula with a similar structure. The problem that the formula is restricted to a specific delta and at-the-money convention can be fixed easily. The matching of the market strangle will be employed by a suitable calibration procedure. The resulting equation will be denoted as the **simplified parabolic formula**.

The simplified parabolic formula is constructed in delta space. Let a general delta function $\Delta(K,\sigma,\phi)$ be given and K_{ATM} be the at-the-money strike associated with the given at-the-money volatility σ_{ATM}. Let the risk reversal volatility quote corresponding to a general delta of $\tilde{\Delta} > 0$ be given by $\sigma_{\tilde{\Delta}-RR}$. For the sake of a compact notation of the formula we will use σ_R instead of $\sigma_{\tilde{\Delta}-RR}$. Furthermore, we parametrize the smile by using a convexity parameter called **smile strangle** which is denoted as σ_S. This parameter has been discussed before in the simplified formula section. The following theorem can be stated.

Theorem 1. *Let Δ_{ATM} denote the call delta implied by the at-the-money strike*

$$\Delta_{ATM} = \Delta(K_{ATM}, \sigma_{ATM}, 1).$$

*Furthermore, we define a variable **a** which is the difference of a call delta, corresponding to a $-\tilde{\Delta}$ put delta, and the $-\tilde{\Delta}$ put delta for any delta type and is given by*

$$a := \Delta(K_{\tilde{\Delta}P}, \sigma, 1) - \Delta(K_{\tilde{\Delta}P}, \sigma, -1).$$

Given a call delta $\tilde{\Delta}$, the parabolic mapping

$$(\overline{\Delta}, \sigma_S) \mapsto \sigma(\overline{\Delta}, \sigma_S)$$

which matches σ_{ATM} and the $\sigma_{\tilde{\Delta}-RR}$ risk reversal quote by default is

$$\sigma(\overline{\Delta}, \sigma_S) = \sigma_{ATM} + c_1(\overline{\Delta} - \Delta_{ATM}) + c_2(\overline{\Delta} - \Delta_{ATM})^2 \tag{24}$$

with

$$c_1 = \frac{a^2(2\sigma_S + \sigma_R) - 2a(2\sigma_S + \sigma_R)(\tilde{\Delta} + \Delta_{ATM}) + 2(\tilde{\Delta}^2\sigma_R + 4\sigma_S\tilde{\Delta}\Delta_{ATM} + \sigma_R\Delta_{ATM}^2)}{2(2\tilde{\Delta} - a)(\tilde{\Delta} - \Delta_{ATM})(\tilde{\Delta} - a + \Delta_{ATM})}$$

$$c_2 = \frac{4\tilde{\Delta}\sigma_S - a(2\sigma_S + \sigma_R) + 2\sigma_R\Delta_{ATM}}{2(2\tilde{\Delta} - a)(\tilde{\Delta} - \Delta_{ATM})(\tilde{\Delta} - a + \Delta_{ATM})} \tag{25}$$

assuming that the denominator of c_1 (and thus c_2) is not zero. A volatility for a put delta can be calculated via the transformation of the put delta to a call delta.

Proof: See Appendix.

We will present $\sigma(\Delta, \sigma_s)$ as a function depending on two parameters only, although of course more parameters are needed for the input. We consider σ_s explicitly, since this is the only parameter not observable in the market. This parameter will be the crucial object in the calibration procedure. Setting $\tilde{\Delta} = 0.25$, $\Delta_{ATM} = 0.5$ and $a = 1$ as in the forward delta case, yields the original Malz formula if $\sigma_s = \sigma_{25-S-Q}$. The generalized formula can handle any delta (e.g., $\tilde{\Delta} = 0.10$), any delta type and any at-the-money convention. The formula automatically matches the at-the-money volatility, since

$$\sigma(\Delta_{ATM}, \sigma_S) = \sigma_{ATM}$$

Furthermore, the risk reversal is matched since

$$\sigma(\tilde{\Delta}_C, \sigma_S) - \sigma(a + \tilde{\Delta}_P, \sigma_S) = \sigma_{\tilde{\Delta}-RR}$$

where $\tilde{\Delta}_C$ denotes the call delta and $\tilde{\Delta}_P$ the put delta[10].

We have plotted the calibrated strike vs. volatility function in Figure 3 to show the influence of the parameters σ_{ATM}, σ_R, σ_S on the simplified parabolic volatility smile in the strike space. We will explain later how to move from the delta to the strike space. Increasing σ_{ATM} shifts the smile curve up parallelly. Increasing σ_{25RR} yields a more skewed curve. A risk reversal of zero implies a symmetric smile. Increasing the strangle σ_S increases the at-the-money smile convexity. Our final goal will be the adjustment of the smile convexity by changing σ_S until condition (9) is met. The other conditions are fulfilled by construction, independent of the choice of σ_S.

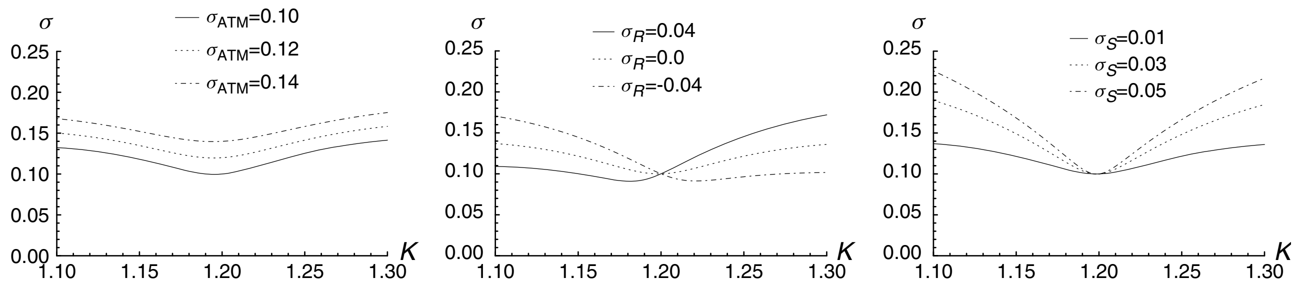
We note that the simplified parabolic formula follows the sticky-delta rule. This implies, that the smile does not move in the delta space, if the spot changes (see Balland (2002), Daglish et al. (2007), Derman (1999)). In the strike space, the smile performs a move to the right in case of an increasing spot, see Figure 4 .

Remark 5 (The parameter *a* in the Simplified Parabolic Formula). The implementation of the formula has a subtle issue regarding the parameter a which is defined as

$$a = \Delta(K_{\tilde{\Delta}P}, \sigma, 1) - \Delta(K_{\tilde{\Delta}P}, \sigma, -1).$$

In case of the non-premium adjusted versions, the implementation is straightforward as this parameter is either the foreign discount factor or 1.0. However, for the premium adjusted spot version the parameter a is

Figure 3: Simplified Parabolic σ_{ATM}, σ_R, σ_S, spot delta scenarios with $\tau = \frac{35}{365}$, $S_0 = 1.2$, $r_d = 0.03$, $r_f = 0.07$, $\tilde{\Delta} = 0.25$. Initial parameters $\sigma_{ATM} = 10.0\%$, $\sigma_R = 0.6\%$, $\sigma_S = 1.0\%$.



$$a = e^{-r_f\tau}\frac{K_{\tilde{\Delta}P}}{f}.$$

There seems to be a circular argumentation in this:

- the volatility for the strike $K_{\tilde{\Delta}P}$ is not known until the calibration is finished,
- the volatility is needed for the calculation of the strike,
- without the strike one can not determine the parameter a.

However, recall that the volatility for a put delta of $-\tilde{\Delta}$ according to Equation (21) is

$$\sigma_{\tilde{\Delta}P} = \sigma_{ATM} - \frac{1}{2}\sigma_R + \sigma_S.$$

Given the smile strangle at each step of the calibration (starting with some initial value), we can calculate the volatility at any time and consequently the strike $K_{\tilde{\Delta}P}$ and the parameter a.

Market Calibration

The advantage of Formula (24) is that it matches the at-the-money and risk reversal conditions of Equations (2) and (3) by construction. The only remaining challenge is matching the market strangle. The simplified parabolic function can be transformed from a delta-volatility to a strike-volatility space (which will be discussed later) such that a function

$$\sigma(K, \sigma_S)$$

is available. Using the variable σ_S as the free parameter, the calibration problem can be reduced to a search for a variable x such that the following holds

$$\begin{aligned} v_{\tilde{\Delta}-S-M} &= v(K_{\tilde{\Delta}C-S-M}, \sigma(K_{\tilde{\Delta}C-S-M}, x), 1) \\ &\quad + v(K_{\tilde{\Delta}P-S-M}, \sigma(K_{\tilde{\Delta}P-S-M}, x), -1). \end{aligned} \tag{26}$$

This leads to the following root search problem:

Problem Type:	Root search.
Given parameters:	$v_{\tilde{\Delta}-S-M}, K_{\tilde{\Delta}C-S-M}, K_{\tilde{\Delta}P-S-M}$ and market data.
Target parameter:	x (set x initially to $\sigma_{\tilde{\Delta}-S-Q}$)

Objective function:

$$f(x) = v(K_{\tilde{\Delta}C-S-M}, \sigma(K_{\tilde{\Delta}C-S-M}, x), 1) + v(K_{\tilde{\Delta}P-S-M}, \sigma(K_{\tilde{\Delta}P-S-M}, x), -1) - v_{\tilde{\Delta}-S-M}$$

The procedure will yield a smile strangle which can be used in the simplified parabolic formula to construct a full smile in the delta space. It is natural to ask, how well defined the problem above is and whether a solution exists and we refer the reader to Reiswich (2011) and Reiswich (2010) for relevant analyses. Does a solution always exist? No, one may observe a market situation where the smile can not be calibrated successfully. Some models are more robust than others, but all models will have a limitation with respect to admissible parameters. In this case, traders are required to adjust one of the market parameters such that the model can calibrate to the new set. For example, given an extreme risk reversal which leads to a failure of the calibration may force a trader to adjust the strangle volatility until the model calibrates successfully. The marking to market is then driven by the model.

Performing the calibration on the currency data in Table 3 yields the parameters summarized in Table 5 for the root search problem. The final calibrated smile for the USD-JPY case is illustrated in Figure 5. One can use the procedure for every time to maturity slice separately and interpolate in the time space (i.e. linear in total variance). The result is a market consistent volatility surface as shown in Figure 6 for the EUR-USD and USD-JPY market data.

Retrieving a Volatility for a Given Strike

Formula (24) returns the volatility for a given delta. However, the calibration procedure requires a mapping

Figure 4: Moving spot scenario for calibrated simplified parabolic formula in strike space. Based on market data in Table 3.

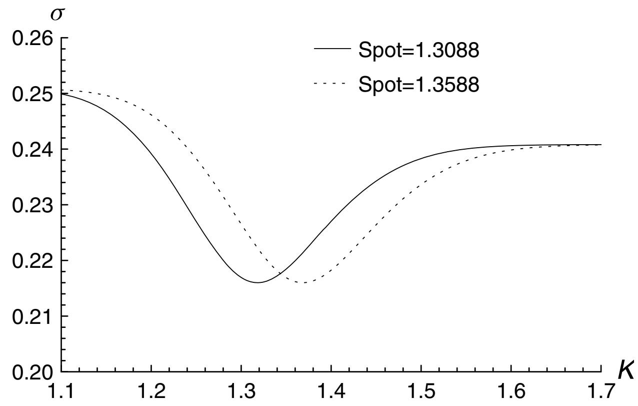


Table 8: Simplified Parabolic Calibration Results.

	EUR-USD Sample	USD-JPY Sample
σ_S	0.007377	0.00419

Figure 5: USD-JPY smile for the market data in Exhibit 3. Filled circles indicate K_{25P}, K_{25C} strikes. Unfilled circles indicate market strangle strikes $K_{25P-S-M}$, $K_{25C-S-M}$. Rectangle indicates K_{ATM}.

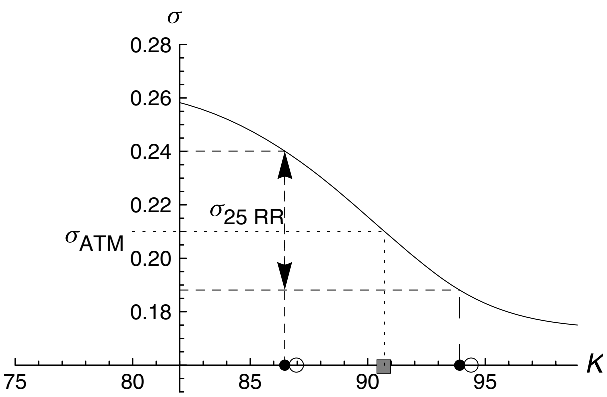
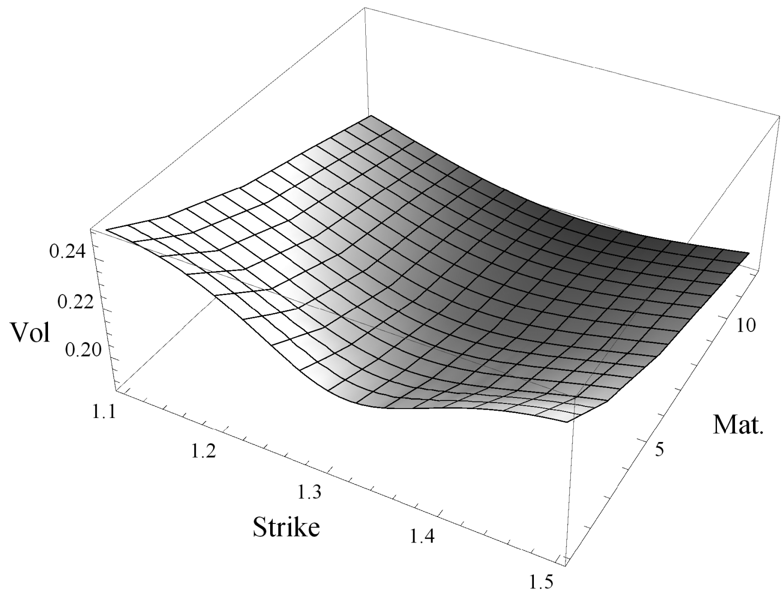
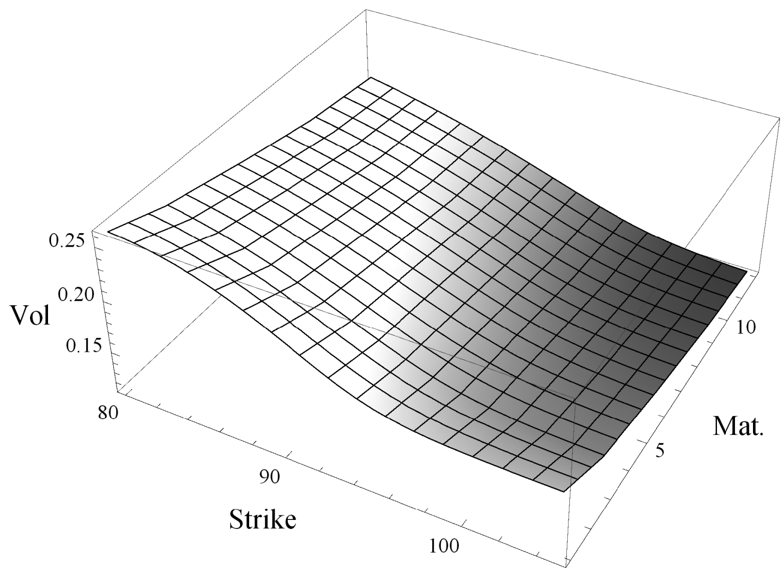


Figure 6: EUR-USD (left chart) and USD-JPY (right chart) volatility surfaces on January, 20th 2009. Time to maturity shown in months.





$$K \mapsto \sigma(K, \sigma_S)$$

since it needs a volatility corresponding to the market strangle strikes. The transformation to $\sigma(K, \sigma_s)$ can be deduced by recalling that $\sigma = \sigma(\tilde{\Delta}, \sigma_s)$ is the volatility corresponding to the delta $\tilde{\Delta}$. To be more precise, given that σ is assigned to a fixed delta $\tilde{\Delta}$ implies that $\tilde{\Delta} = \Delta(K, \sigma, \phi)$ for some strike K. Consequently, Formula (24) can be stated as

$$\sigma = \sigma_{ATM} + c_1(\Delta(K, \sigma, 1) - \Delta_{ATM}) + c_2(\Delta(K, \sigma, 1) - \Delta_{ATM})^2. \tag{27}$$

Given a strike K, it is thus possible to retrieve the corresponding volatility by searching for a σ which fulfills Equation (27). This can be achieved by using a root searcher. We recommend the method introduced by Brent (2002). The question arises, if such a volatility vs. strike function exists and how smooth it is. The answer can be given by using the implicit function theorem. In the following discussion we will avoid the explicit dependence of all variables on $(K, \sigma(K, \sigma_s))$. For example, we write

$$\frac{\partial \Delta}{\partial K} \text{ instead of } \frac{\partial \Delta}{\partial K}(x, y)|_{x=K, y=\sigma(K, \sigma_S)}$$

With this compact notation, we can state the following.

Theorem 2. *Given the volatility vs. delta mapping (24), assume that the following holds*

$$c_1 \frac{\partial \Delta}{\partial \sigma}(K_{ATM}, \sigma_{ATM}) \neq 1$$

*Then there exists a function $\sigma: U \to W$ with open sets $U, W \subseteq \mathbb{R}^+$ such that $K_{ATM} \in U$ and $\sigma_{ATM} \in W$ which maps the strike implicit in Δ against the corresponding volatility. The function is **differentiable** and has the following first- and second-order derivatives on U*

$$\frac{\partial \sigma}{\partial K} = \frac{\frac{\partial \Delta}{\partial K} A}{1 - \frac{\partial \Delta}{\partial \sigma} A} \tag{28}$$

$$\frac{\partial^2 \sigma}{\partial K^2} = \frac{\left[\left(\frac{\partial^2 \Delta}{\partial K^2} + \frac{\partial^2 \Delta}{\partial K \partial \sigma}\frac{\partial \sigma}{\partial K}\right)A + \frac{\partial \Delta}{\partial K}\frac{\partial A}{\partial K}\right]\left(1 - \frac{\partial \Delta}{\partial \sigma} A\right)}{\left(1 - \frac{\partial \Delta}{\partial \sigma} A\right)^2} + \frac{\frac{\partial \Delta}{\partial K} A\left(\left(\frac{\partial \Delta}{\partial \sigma \partial K} + \frac{\partial^2 \Delta}{\partial \sigma^2}\frac{\partial \sigma}{\partial K}\right)A + \frac{\partial \Delta}{\partial \sigma}\frac{\partial A}{\partial K}\right)}{\left(1 - \frac{\partial \Delta}{\partial \sigma} A\right)^2} \tag{29}$$

with

$$A := c_1 + 2c_2(\Delta - \Delta_{ATM}) \text{ and } \frac{\partial A}{\partial K} = 2c_2\left(\frac{\partial \Delta}{\partial K} + \frac{\partial \Delta}{\partial \sigma}\frac{\partial \sigma}{\partial K}\right)$$

Proof. See Appendix.

Note that Equations (28) and (29) require the values $\sigma(K, \sigma_s)$. In fact, Equation (28) can be seen as an non-autonomous non-linear ordinary differential equation for $\sigma(K, \sigma_s)$. However, given $\sigma(K, \sigma_s)$ as a root of Equation (27), we can analytically calculate both derivatives. Differentiability is very important for calibration procedures of the well known local volatility models (see Dupire (1994), Derman and Kani (1994), Lee (2001)), which need a smooth volatility vs. strike function. The derivatives with respect to the strike can be very problematic if calculated numerically from an interpolation function. In our case, the derivatives can be stated explicitly, similar to [Hakala and Wystup, 2002, page 254] for the kernel interpolation case. In addition, the formulas are very useful to test for arbitrage, where restrictions on the slope and convexity of $\sigma(K)$ are imposed (see for example Lee (2005)).

We summarize explicit formulas for all derivatives occurring in Equations (28) and (29) in Tables 6 and 7 in the Appendix. They can be used for derivations of analytical formulas for the strike derivatives for all delta types.

Calibration Robustness and No-Arbitrage Conditions

In Reiswich (2011), the calibration robustness of the simplified parabolic function is compared against other methods. In addition, the violation of no-arbitrage conditions is analyzed. Both analysis are based on empirical market data and show that the simplified parabolic function is robust with respect to both criteria.

>

4 Conclusion

We have introduced various delta and at-the-money quotations commonly used in FX option markets. The delta types are FX-specific, since the option can be traded in both currencies. The various at-the-money quotations have been designed to account for large interest rate differentials or to enforce an efficient trading of positions with a pure vega exposure. We have then introduced the liquid market instruments that parametrize the market and have shown which information they imply. Finally, we derived a new formula that accounts for FX specific market information and can be used to employ an efficient market calibration.

Acknowledgments

We would like to thank Travis Fisher, Boris Borowski, Andreas Weber, Jürgen Hakala and Iain Clark for their helpful comments.

5 Appendix

To reduce the notation, we will drop the dependence of $\sigma(\Delta, \sigma_s)$ on σ_s in the following proofs and write $\sigma(\Delta)$ instead.

Proof (Simplified Parabolic Formula). We will construct a parabola in the call delta space such that the following restrictions are met

$$\begin{aligned} \sigma(\Delta_{ATM}) &= \sigma_{ATM}, \\ \sigma(\tilde{\Delta}) &= \sigma_{ATM} + \frac{1}{2}\sigma_R + \sigma_S, \\ \sigma(a - \tilde{\Delta}) &= \sigma_{ATM} - \frac{1}{2}\sigma_R + \sigma_S. \end{aligned} \tag{30}$$

For example, in the forward delta case we would have $a = 1$. Given $\tilde{\Delta} = 0.25$, the call delta corresponding to a put delta of -0.25 would be $1 - 0.25 = 0.75$. The equation system is set up such that

$$\sigma_S = \frac{\sigma(\tilde{\Delta}) + \sigma(a - \tilde{\Delta})}{2} - \sigma_{ATM}.$$

One can see that σ_s measures the smile convexity, as it is the difference of the average of the out-of-the-money and in-the-money volatilities compared to the at-the-money volatility. The restriction set (30) ensures that

$$\sigma(\tilde{\Delta}) - \sigma(a - \tilde{\Delta}) = \sigma_R \tag{31}$$

is fulfilled by construction. Given the parabolic setup

$$\sigma(\overline{\Delta}) = \sigma_{ATM} + c_1(\overline{\Delta} - \Delta_{ATM}) + c_2(\overline{\Delta} - \Delta_{ATM})^2,$$

one can solve for c_1, c_2 such that Equation system (30) is fulfilled. This is a well defined problem: a system of two linear equations in two unknowns. □

Proof (Existence of a Volatility vs Strike Function). The simplified parabolic function has the following form

$$\sigma(\Delta, \sigma_S) = \sigma_{ATM} + c_1(\Delta - \Delta_{ATM}) + c_2(\Delta - \Delta_{ATM})^2. \tag{32}$$

First of all, note that $\Delta(K, \sigma)$ is continuously differentiable with respect to both variables for all delta types. Define $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ to be

$$F(K, \sigma) = \sigma_{ATM} + c_1(\Delta(K, \sigma) - \Delta_{ATM}) + c_2(\Delta(K, \sigma) - \Delta_{ATM})^2 - \sigma \tag{33}$$

Table 9: Partial Delta Derivatives I.

	∂K	$\partial \sigma$	∂K^2
Δ_S	$-\frac{e^{-r_f\tau} n(d_+)}{\sigma\sqrt{\tau}K}$	$-\frac{e^{-r_f\tau} n(d_+)d_-}{\sigma}$	$\frac{e^{-r_f\tau} n(d_+)}{\sigma\sqrt{\tau}K^2} - \frac{e^{-r_f\tau} n(d_+)d_+}{\sigma^2\tau K^2}$
$\Delta_{S,pa}$	$\frac{\phi e^{-r_f\tau} N(\phi d_-)}{f} - \frac{e^{-r_f\tau} n(d_-)}{f\sigma\sqrt{\tau}}$	$-\frac{e^{-r_f\tau} Kn(d_-)d_+}{f\sigma}$	$-\frac{e^{-r_f\tau} n(d_-)}{f\sigma\sqrt{\tau}K} - \frac{e^{-r_f\tau} n(d_-)d_-}{fK\sigma^2\tau}$
Δ_f	$-\frac{n(d_+)}{\sigma\sqrt{\tau}K}$	$-\frac{n(d_+)d_-}{\sigma}$	$\frac{n(d_+)}{\sigma\sqrt{\tau}K^2} - \frac{n(d_+)d_+}{\sigma^2\tau K^2}$
$\Delta_{f,pa}$	$\frac{\phi N(\phi d_-)}{f} - \frac{n(d_-)}{f\sigma\sqrt{\tau}}$	$-\frac{Kn(d_-)d_+}{f\sigma}$	$-\frac{n(d_-)}{f\sigma\sqrt{\tau}K} - \frac{n(d_-)d_-}{fK\sigma^2\tau}$

Table 10: Partial Delta Derivatives II.

	$\partial K \partial \sigma$	$\partial \sigma^2$
Δ_S	$\frac{e^{-r_f\tau} n(d_+)\left(1-d_+d_-\right)}{\sigma^2\sqrt{\tau}K}$	$\frac{e^{-r_f\tau} n(d_+)(d_- - d_+d_-d_- + d_+)}{\sigma^2}$
$\Delta_{S,pa}$	$\frac{e^{-r_f\tau} n(d_-)\left(-d_+\sigma\sqrt{\tau}+1-d_-d_+\right)}{f\sigma^2\sqrt{\tau}}$	$\frac{e^{-r_f\tau} Kn(d_-)(d_+ - d_-d_+d_+ + d_-)}{f\sigma^2}$
Δ_f	$\frac{n(d_+)\left(1-d_+d_-\right)}{\sigma^2\sqrt{\tau}K}$	$\frac{n(d_+)(d_- - d_+d_-d_- + d_+)}{\sigma^2}$
$\Delta_{f,pa}$	$\frac{n(d_-)\left(-d_+\sigma\sqrt{\tau}+1-d_-d_+\right)}{f\sigma^2\sqrt{\tau}}$	$\frac{Kn(d_-)(d_+ - d_-d_+d_+ + d_-)}{f\sigma^2}$

with $\Delta(K, \sigma)$ being one of the four deltas introduced before. The proof is a straightforward application of the implicit function theorem. Note that $F(K_{ATM}, \sigma_{ATM}) = 0$ is given by default. As already stated, the function F is differentiable with respect to the strike and volatility. Deriving with respect to volatility yields

$$\frac{\partial F}{\partial \sigma} = c_1 \frac{\partial \Delta}{\partial \sigma} + 2c_2(\Delta - \Delta_{ATM})\frac{\partial \Delta}{\partial \sigma} - 1. \tag{34}$$

From this derivation we have

$$\frac{\partial F}{\partial \sigma}(K_{ATM}, \sigma_{ATM}) = c_1 \frac{\partial \Delta}{\partial \sigma}(K_{ATM}, \sigma_{ATM}) - 1, \tag{35}$$

which is different from zero by assumption of the theorem. Consequently, the implicit function theorem implies the existence of a differentiable function f and an open neighborhood $U \times W \subseteq \mathbb{R}^+ \times \mathbb{R}^+$ with $K_{ATM} \in U$, $\sigma_{ATM} \in W$ such that

$$F(K, \sigma) = 0 \Leftrightarrow \sigma = f(K) \text{ for } (K, \sigma) \in U \times W.$$

The first derivative is defined on U and given by

$$\frac{\partial f}{\partial K} = -\frac{\frac{\partial F}{\partial K}}{\frac{\partial F}{\partial \sigma}} \text{ for } K \in U,$$

which can be calculated in a straightforward way. The function $f(K)$ is denoted as $\sigma(K)$ in the theorem. The second derivative can be derived in a straightforward way by remembering, that the volatility depends on the strike. This completes the proof. □

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ENDNOTES

1. We will take a delta of 0.25 as an example, although any choice is possible, e.g. 0.10.
2. This can be achieved by using a standard root search algorithm.
3. Sometimes the quoted strangle is also referred to as the broker strangle as in Bossens et al. (2009), or as the vega weighted butterfly as in Castagna (2010). In addition, the term butterfly is commonly used instead of the term strangle despite the difference of the option positions.
4. The expansion uses some simplifications and approximations. For example, we neglect the derivative of the calibrated smile function $\sigma(K)$ with respect to σ_{25-S-M}.
5. The volga is the derivative of the vega with respect to volatility.
6. Here, Δ_{ATM} is the at-the-money delta. The description is exact if we consider the forward delta case with the delta-neutral at-the-money quotation. In other cases, this is an approximation.
7. The smile strangle is denoted as $\sigma_{BF25(2vol)}$ in Bossens et al. (2009) and as the equivalent vega weighted butterfly vwb_{ε} in Castagna (2010).
8. Here we assume, that the at-the-money and forward term structure is ''regular".
9. A put volatility can be calculated by transforming the put to a call delta using the put call parity.
10. $a + \tilde{\Delta}_P$ is the call delta corresponding to a put delta of $\tilde{\Delta}_P$. In the forward delta case a = 1. If $\tilde{\Delta}_P = -0.25$, the equivalent call delta which enters the simplified parabolic formula is $a + \tilde{\Delta}_P = 0.75$.

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