Frankfurt School of Finance & Management
Bankakademie | HfB

Centre for Practical Quantitative Finance

CPQF Working Paper Series
No. 20

FX Volatility Smile Construction

Dimitri Reiswich, Uwe Wystup

September 2009

Authors: *Dimitri Reiswich*
Research Associate
Frankfurt School of
Finance & Management
Frankfurt/Main
d.reiswich@frankfurt-school.de

Uwe Wystup
Professor of Quantitative Finance
Frankfurt School of
Finance & Management
Frankfurt/Main
u.wystup@frankfurt-school.de

Publisher: Frankfurt School of Finance & Management
Phone: +49 (0) 69 154 008-0 ▪ Fax: +49 (0) 69 154 008-728
Sonnemannstr. 9-11 ▪ D-60314 Frankfurt/M. ▪ Germany

FX Volatility Smile Construction

Dimitri Reiswich, Uwe Wystup

Version: September, 8th 2009

Abstract The foreign exchange options market is one of the largest and most liquid OTC derivative markets in the world. Surprisingly, very little is known in the academic literature about the construction of the most important object in this market: The implied volatility smile. The smile construction procedure and the volatility quoting mechanisms are FX specific and differ significantly from other markets. We give a detailed overview of these quoting mechanisms and introduce the resulting smile construction problem. Furthermore, we provide a new formula which can be used for an efficient and robust FX smile construction.

Keywords: FX Quotations, FX Smile Construction, Risk Reversal, Butterfly, Strangle, Delta Conventions, Malz Formula

1 Delta– and ATM–Conventions in FX-Markets

1.1 Introduction

In financial markets the value of a vanilla option is generally quoted in terms of its implied volatility, i.e., the "wrong volatility to plug into the wrong (Black-Scholes) formula to get the right price". Observed market prices show that implied volatility

Dimitri Reiswich
Frankfurt School of Finance & Management, Centre for Practical Quantitative Finance, e-mail: d.reiswich@frankfurt-school.de

Uwe Wystup
Frankfurt School of Finance & Management, Centre for Practical Quantitative Finance, e-mail: uwe.wystup@mathfinance.com

is a function of the option's moneyness, an effect we call the *volatility smile*. The volatility smile is the crucial object in pricing and risk management procedures since it is used to price vanilla, as well as exotic option books. Market participants entering the FX OTC derivative market are confronted with the fact that the volatility smile is usually not directly observable in the market. This is in opposite to the equity market, where strike-price or strike-volatility pairs can be observed. In foreign exchange OTC derivative markets it is common to publish currency pair specific risk reversal σ_{RR}, strangle σ_{STR} and at-the-money volatility σ_{ATM} quotes as given in the market sample in Table 1. These quotes can be used to construct a complete volatility smile from which one can extract the volatility for any strike. In the next section we will introduce the basic FX terminology which is necessary to understand the following sections. We will then explain the market implied information for quotes such as those given in Table 1. Finally, we will propose an implied volatility function which accounts for this information.

Table 1: FX Market data for a maturity of 1 month, as of January, 20th 2009

	EURUSD	USDJPY
σ_{ATM}	21.6215%	21.00%
σ_{RR}	−0.5%	−5.3%
σ_{STR}	0.7375%	0.184%

1.2 Spot, Forward and Vanilla Options

FX Spot Rate S_t

The FX spot rate S_t =FOR-DOM is the **current** exchange rate at the present time t ("today", "now","horizon") representing the number of units of domestic currency needed to buy one unit of foreign currency. For example, EUR-USD= 1.3900 means that one EUR is worth 1.3900 USD. In this case, EUR is the foreign currency and USD is the domestic one. The EUR-USD= 1.3900 quote is equivalent to USD-EUR 0.7194. We will refer to the "domestic" currency in the sense of a base currency in relation to which "foreign" amounts of money are measured (see also Wystup (2006)). By definition, an amount x in foreign currency is equivalent to $x \cdot S_t$ units of domestic currency at time t. The term "domestic" does *not* refer to any geographical region.

FX Forward Rate $f(t,T)$

Similarly, the FX forward rate $f(t,T)$ is the exchange rate between the domestic and the foreign currency at some **future** point of time T as observed at the present time t $(t < T)$.

Spot-Rates Parity

Spot and forward FX rates are related by:

$$f(t,T) = S_t \cdot e^{(r_d - r_f)\tau} \ , \tag{1}$$

where

r_f is the foreign interest rate (continuously compounded),
r_d is the domestic interest rate (continuously compounded),
τ is the time to maturity, equal to $T - t$.

Value of FX Forward Contracts

When two parties agree on an FX forward contract at time s, they agree on the exchange of an amount of money in foreign currency at an agreed exchange rate K against an amount of money in domestic currency at time $T > s$. The default is the *outright forward contract*, which means taking the *forward rate* as the strike $K = f(s,T)$, so the value of the FX forward contract is zero at inception s.

As in general the forward exchange rate changes over time, at some time t $(s < t < T)$, the FX forward contract will have a non-zero value (in domestic currency) given by

$$v_f(t,T) = e^{-r_d\tau} \ (f(t,T) - K) = S_t e^{-r_f\tau} - K e^{-r_d\tau}. \tag{2}$$

Value of FX Options

The holder of an FX option obtains the right to exchange a specified amount of money in domestic currency for a specified amount of money in foreign currency at an agreed exchange rate K at maturity time T. By default, FX vanilla options are of European style, since their American style counterparts are too expensive (or not worth more). Assuming non-stochastic interest rates and the standard log-normal dynamics for the spot exchange rate, at time t the domestic currency value of a vanilla option with strike K and expiry time T is given by the Black-Scholes formula

$$
\begin{aligned}
v(S_t,K,\sigma,\phi) &= v(S_t,r_d,r_f,K,\sigma,t,T,\phi) \\
&= \phi[e^{-r_f\tau}\, S_t\, N(\phi d_+) \,-\, e^{-r_d\tau}\, K\, N(\phi d_-)] \\
&= \phi e^{-r_d\tau}\, [f(t,T)\, N(\phi d_+) \,-\, K\, N(\phi d_-)],
\end{aligned} \tag{3}
$$

where

$$
\begin{aligned}
d_\pm &= \frac{\ln\left(\frac{f(t,T)}{K}\right) \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \\
K &: \text{ strike of the FX option,} \\
\sigma &: \text{ Black-Scholes volatility,} \\
\phi &= +1 \text{ for a call }, \phi = -1 \text{ for a put,} \\
N(x) &: \text{ cumulative normal distribution function.}
\end{aligned}
$$

We may drop some of the variables of the function v depending on context. Note that v is in *domestic* currency. The option position, however, may also be held in foreign currency. We will call the currency in which an option's value is measured its **premium currency**. The **notional** is the amount of currency which the holder of an option is entitled to exchange. The value formula applies by default to one unit of foreign notional (corresponding to one share of stock in equity markets), with a value in units of domestic currency. Consider a EUR-USD call with a spot of $S_0 = 1.3900$, a strike of $K = 1.3500$ and a price of 0.1024 USD. If a notional of $1,000,000$ EUR is specified, the holder of the option will receive $1,000,000$ EUR and pay $1,350,000$ USD at maturity and the option's current price is $102,400$ USD ($73,669$ EUR).

1.3 Delta Types

The delta of an option is the percentage of the foreign notional one must buy when selling the option to hold a hedged position (equivalent to buying stock). For instance, a delta of $0.35 = 35\%$ indicates buying 35% of the foreign notional to delta-hedge a short option. In foreign exchange markets we distinguish the cases *spot delta* for a hedge in the spot market and *forward delta* for a hedge in the FX forward market. Furthermore, the standard delta is a quantity in percent of foreign currency. The actual hedge quantity must be changed if the premium is paid in foreign currency, which would be equivalent to paying for stock options in shares of stock. We call this type of delta the *premium-adjusted delta*. In the previous example the value of an option with a notional of $1,000,000$ EUR was calculated as $73,669$ EUR. Assuming a short position with a delta of 60% means, that buying $600,000$ EUR is necessary to hedge. However the final hedge quantity will be $526,331$ EUR which is the delta quantity reduced by the received premium in EUR. Consequently, the premium-adjusted delta would be 52.63%. The following sections will introduce the formulas for the different delta types. Related work can be found in Beier and Renner (forthcoming).

Unadjusted Deltas

Definition 1. For FX options the spot delta is defined as the derivative of the option value v with respect to the FX spot rate S,

$$\Delta_S(K,\sigma,\phi) \triangleq \frac{\partial v}{\partial S} = v_S. \tag{4}$$

The spot delta is the "usual" delta sensitivity, i.e. the percentage of foreign notional to buy in order to delta-hedge a short option paid in domestic currency in the spot market. We obtain

$$\text{Vanilla option:} \quad \Delta_S(K,\sigma,\phi) = \phi e^{-r_f\tau} N(\phi d_+), \tag{5}$$

$$\text{Put-call delta parity:} \quad \Delta_S(K,\sigma,+1) - \Delta_S(K,\sigma,-1) = e^{-r_f\tau}. \tag{6}$$

Note that the absolute value of delta is a number between zero and a discount factor $e^{-r_f\tau} < 100\%$. Therefore, 50% is not the center value for the delta range.

Definition 2. The forward delta Δ_f of an FX option is defined as the rate of change of the option's value with respect to the value of a long forward contract v_f (where the forward price of the FX forward contract equals the strike of the FX option),

$$\Delta_f(K,\sigma,\phi) \triangleq \frac{\partial v}{\partial v_f} = \frac{\partial v}{\partial S}\frac{\partial S}{\partial v_f} = \frac{\partial v}{\partial S}\left(\frac{\partial v_f}{\partial S}\right)^{-1} = \phi N(\phi d_+), \tag{7}$$

$$\text{Put-call delta parity:} \quad \Delta_f(K,\sigma,+1) - \Delta_f(K,\sigma,-1) = 100\%. \tag{8}$$

The forward delta is not simply the derivative of the option price formula with respect to the forward FX rate f, but with respect to the *value of* the forward contract. The above definition is motivated by the construction of a hedge portfolio using FX forward contracts as hedge instruments. The forward delta gives the number of forward contracts that an investor needs to enter to completely delta-hedge his FX option position; Δ_f, therefore, is a number in percent of foreign notional.

The forward delta is often used in FX options smile tables, because of the fact that the delta of a call and the (absolute value of the) delta of the corresponding put add up to 100%, i.e. a 25-delta call must have the same volatility as a 75-delta put. This symmetry only works for forward deltas.

Premium Adjusted Deltas

Definition 3. The premium-adjusted spot delta is defined as

$$\Delta_{S,\,pa} \stackrel{\Delta}{=} \Delta_S - \frac{v}{S}. \tag{9}$$

The definition of the premium-adjusted spot delta takes care of the correction induced by payment of the premium in foreign currency, which is the amount by which the delta hedge in foreign currency has to be corrected.

While v is the option's value in domestic currency, $\frac{v}{S}$ is the option's value in foreign currency. Equation (9) can also be interpreted as follows: Both quantities $v_S = \Delta_S$ and v/S are percentage foreign units to buy. This is equivalent to the domestic units to sell. To quantify the hedge in domestic currency we need to flip around the quotation and compute the *dual delta*

$$\begin{aligned}
&\frac{\partial \frac{v}{S} \text{ in FOR}}{\partial \frac{1}{S} \text{ in FOR per DOM}} = \text{ DOM to buy}\\
&= \frac{\partial \frac{v}{S}}{\partial S} \cdot \frac{\partial S}{\partial \frac{1}{S}}\\
&= \frac{Sv_S - v}{S^2} \cdot \left(\frac{\partial \frac{1}{S}}{\partial S}\right)^{-1} = \frac{Sv_S - v}{S^2} \cdot \left(-\frac{1}{S^2}\right)^{-1}\\
&= -(Sv_S - v) \text{ DOM to buy} = Sv_S - v \quad \text{DOM to sell} = v_S - \frac{v}{S} \quad \text{FOR to buy,}
\end{aligned}$$

which confirms the definition of the premium-adjusted delta in Equation (9). We find

$$\text{Vanilla option:} \quad \Delta_{S,\,pa}(K,\sigma,\phi) = \phi e^{-r_f \tau}\frac{K}{f}N(\phi d_-), \tag{10}$$

$$\text{Put-call delta parity:} \quad \Delta_{S,\,pa}(K,\sigma,+1) - \Delta_{S,\,pa}(K,\sigma,-1) = e^{-r_f \tau}\frac{K}{f}. \tag{11}$$

Note that

$$\begin{aligned}
\phi e^{-r_f \tau}\frac{K}{f}N(\phi d_-) &= \text{ FOR to buy per 1 FOR}\\
\phi e^{-r_d \tau}KN(\phi d_-) &= \text{ DOM to sell per 1 FOR}\\
-\phi e^{-r_d \tau}KN(\phi d_-) &= \text{ DOM to buy per 1 FOR}\\
-\phi e^{-r_d \tau}N(\phi d_-) &= \text{ DOM to buy per 1 DOM}\\
= v_K &= \text{ the dual delta,}
\end{aligned}$$

which is the strike-coefficient in the Black-Scholes Formula (3). It is now apparent that this can also be interpreted as a delta, the spot delta in reverse quotation DOM-FOR.

The defining equations for premium-adjusted deltas have interesting consequences: While put deltas are unbounded and strictly monotone functions of K, call deltas are bounded (i.e. $\Delta_S(K,\sigma,+1) \in [0,\Delta_{max}]$ with $\Delta_{max} < 1$) and are **not** monotone functions of K. Thus, the relationship between call deltas and strikes K is not one–to–one. Typical shapes of the spot and premium-adjusted deltas are plotted against the strike in Figure (1).

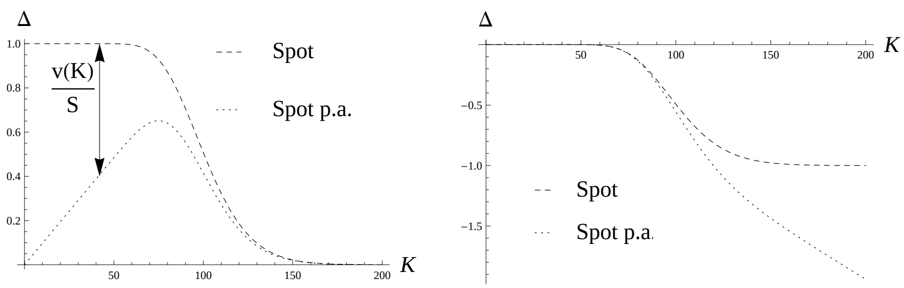

Fig. 1: Premium-adjusted and standard call (left chart) and put (right chart) spot delta, $S_t = 100$, $\tau = 1.0$, $r_d = 0.03$, $r_f = 0.0$, $\sigma = 0.2$.

Definition 4. The premium-adjusted forward delta $\Delta_{f,pa}$ is the percentage of foreign notional one needs to trade in the forward to be delta-neutral, corrected by the value of the option in foreign currency.

The derivation is similar to the one for premium-adjusted spot delta and results in

$$\text{Vanilla option:} \quad \Delta_{f,pa}(K,\sigma,\phi) = \phi \frac{K}{f} N(\phi d_-), \tag{12}$$

$$\text{Put-call delta parity:} \quad \Delta_{f,pa}(K,\sigma,+1) - \Delta_{f,pa}(K,\sigma,-1) = \frac{K}{f}. \tag{13}$$

Note again that the premium-adjusted forward delta of a call is not a monotone function of the strike.

Delta Conventions for Selected Currency Pairs

The question which of the deltas is used in practice cannot be answered systematically. A summary of current market conventions can be found in the forthcoming book by Ian Clark. Both, spot and forward deltas are used, depending on which product is used to hedge. Generally, forward hedges are popular to capture rates risk besides the spot risk. So naturally, forward hedges come up for delta-one-similar products or for long-term options. In practice, the immediate hedge executed is generally the spot-hedge, because it has to be done instantaneously with the option trade. At a later time the trader can change the spot hedge to a forward hedge using a zero-cost FX swap.

Forward delta conventions are normally used to specify implied volatilities because of the symmetry of put and call deltas adding up to 100%. Using forward deltas as a quotation standard often depends on the time to expiry T and on whether the currency pair contains at least one emerging market currency. If it does, forward deltas are the market default. However, if the currency pair contains only currencies from the OECD economies (USD, EUR, JPY, GBP, AUD, NZD, CAD, CHF, NOK, SEK, DKK), and does not include ISK, TRY, MXN, CZK, KRW, HUF, PLN, ZAR or SGD, then spot deltas are used out to and including 1Y, and forward deltas out for all longer dated tenors. For example, NZD-JPY uses spot deltas out to 1Y, but forward deltas beyond, whereas CZK-JPY uses forward deltas exclusively in the definition of the volatility smile. The premium-adjusted delta as a default is used for options in currency pairs whose premium currency is FOR. We provide examples in Table 2. Typically, the premium currency is taken to be the more commonly

Table 2: Selected currency pairs and their default premium currency determining the delta type

Currency pair	Premium ccy	delta convention
EUR-USD	USD	regular
USD-JPY	USD	premium-adjusted
EUR-JPY	EUR	premium-adjusted
USD-CHF	USD	premium-adjusted
EUR-CHF	EUR	premium-adjusted
GBP-USD	USD	regular
EUR-GBP	EUR	premium-adjusted
AUD-USD	USD	regular
AUD-JPY	AUD	premium-adjusted
USD-CAD	USD	premium-adjusted
USD-BRL	USD	premium-adjusted
USD-MXN	USD	premium-adjusted

traded currency of the two – with the exception of JPY, which is rarely the premium currency. Therefore, for virtually all currency pairs involving USD, the premium currency will be USD. For currency pairs including EUR and not including USD, it will be EUR. For currency pairs involving the Japanese Yen, it often will not be the JPY unless the currency being quoted against JPY is an emerging market currency other than CZK, PLN, TRY or MXN (all of which dominate JPY in premium terms). A basic hierarchy of which currencies dominate in premium currency terms can be stated as

$$\begin{aligned}\text{USD} &\succ \text{EUR} \succ \text{GBP} \succ \text{AUD} \succ \text{NZD} \succ \text{CAD} \succ \text{CHF}\\ &\succ \text{NOK,SEK,DKK}\\ &\succ \text{CZK,PLN,TRY,MXN} \succ \text{JPY} \succ \dots \qquad (14)\end{aligned}$$

Exceptions may occur, so in case of doubt it is advisable to check.

1.4 At–The–Money Types

Defining *at-the-money (ATM)* is by far not as obvious as one might think when first studying options. It is the attempt to specify the middle of the spot distribution in various senses. We can think of

$$\begin{aligned} \text{ATM-spot} &\quad K = S_0 \\ \text{ATM-fwd} &\quad K = f \\ \text{ATM-value-neutral} &\quad K \text{ such that call value} = \text{put value} \\ \text{ATM-}\Delta\text{-neutral} &\quad K \text{ such that call delta} = - \text{ put delta.} \end{aligned}$$

In addition to that, the notion of ATM involving delta will have sub-categories depending on which delta convention is used. **ATM-spot** is often used in beginners' text books or on term sheets for retail investors, because the majority of market participants is familiar with it. **ATM-fwd** takes into account that the risk-neutral expectation of the future spot is the forward price (1), which is a natural way of specifying the "middle". It is very common for currency pairs with a large interest rate differential (emerging markets) or long maturity. **ATM-value-neutral** is equivalent to **ATM-fwd** because of the put-call parity. Choosing the strike in the **ATM-delta-neutral** sense ensures that a straddle with this strike has a zero spot exposure which accounts for the traders' vega-hedging needs. This ATM convention is considered as the default ATM notion for short-dated FX options. We summarize the various at-the-money definitions and the relations between all relevant quantities in Table 3.

Table 3: ATM Strike values and delta values for the different delta conventions

	ATM Δ-neutral Strike	ATM fwd Strike	ATM Δ-neutral Delta	ATM fwd Delta
Spot Delta	$f\,e^{\frac{1}{2}\sigma^2\tau}$	f	$\frac{1}{2}\phi e^{-r_f\tau}$	$\phi e^{-r_f\tau}\,N(\phi\frac{1}{2}\sigma\sqrt{\tau})$
Forward Delta	$f\,e^{\frac{1}{2}\sigma^2\tau}$	f	$\frac{1}{2}\phi$	$\phi N(\phi\frac{1}{2}\sigma\sqrt{\tau})$
Spot Delta p.a.	$f\,e^{-\frac{1}{2}\sigma^2\tau}$	f	$\frac{1}{2}\phi e^{-r_f\tau}\,e^{-\frac{1}{2}\sigma^2\tau}$	$\phi e^{-r_f\tau}\,N(-\phi\frac{1}{2}\sigma\sqrt{\tau})$
Forward Delta p.a.	$f\,e^{-\frac{1}{2}\sigma^2\tau}$	f	$\frac{1}{2}\phi e^{-\frac{1}{2}\sigma^2\tau}$	$\phi N(-\phi\frac{1}{2}\sigma\sqrt{\tau})$

1.5 Converting Deltas to Strikes

Quoting volatilities as a function of the options' deltas rather than as a function of the options' strikes brings about a problem when it comes to pricing FX options. The resulting problem is to find a strike, given a delta and a volatility. The following sections will outline the algorithms that can be used to that end. As the spot and forward deltas differ only by constant discount factors, we will restrict the presentation to the forward versions of the adjusted and unadjusted deltas.

Conversion of Forward Delta to Strike

The conversion of a non-premium-adjusted delta to a strike is straightforward. With Δ_f and σ given, we can directly solve Equation (7),

$$\Delta_f(K,\phi) \;=\; \phi N(\phi d_+) \;=\; \phi N\left(\phi \frac{\ln\left(\frac{f}{K}\right)+\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right),$$

for the strike K. We get

$$K \;=\; f\exp\left\{-\phi\sigma\sqrt{\tau}N^{-1}\left(\phi\Delta_f\right)+\frac{1}{2}\sigma^2\tau\right\}. \tag{15}$$

Conversion of a Premium-Adjusted Forward Delta to Strike

The conversion from a premium-adjusted forward delta to a strike is more complicated than in the case of an unadjusted delta and cannot be formulated in a closed–form expression. The reason is that in Equation (12) for the premium-adjusted call delta,

$$\Delta_{f,pa}(K,\sigma,\phi) \;=\; \phi\frac{K}{f}N(\phi d_-) \;=\; \phi\frac{K}{f}N\left(\phi \frac{\ln\left(\frac{f}{K}\right)-\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right),$$

the strike K appears inside and outside the cumulative normal distribution function so that one cannot solve directly for K. Even though both $\Delta_{f,pa}$ and σ are given, the problem has to be solved numerically.

So when converting a given call delta $\Delta_{f,pa}(K,\sigma,\phi)$, a root finder has to be used to solve for the correspondent strike K. This is a straightforward procedure for the put delta, which is monotone in strike. This is not the case for the premium-adjusted call delta, as illustrated in Figure (1). Here, two strikes can be obtained for a given premium-adjusted call delta (for example for $\Delta_{S,pa}=0.2$). It is common to search for strikes corresponding to deltas which are on the right hand side of the delta maximum. This is illustrated as a shadowed area in the left chart of Figure (2).

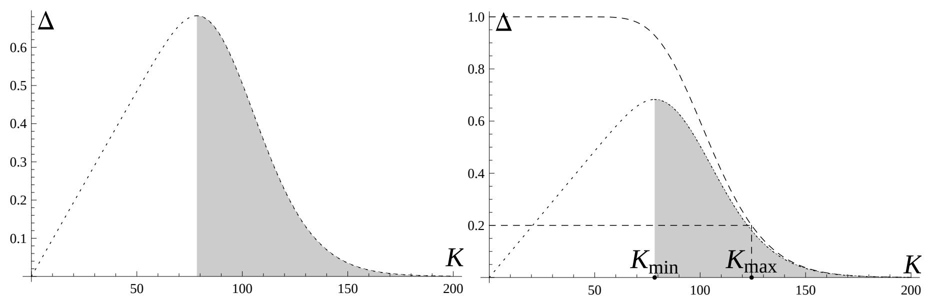

Fig. 2: Strike region for given premium-adjusted delta. $S_t = 100$

Consequently, we recommend to use Brent's root searcher (see Brent (2002)) to search for $K \in [K_{min}, K_{max}]$. The right limit K_{max} can be chosen as the strike corresponding to the non premium-adjusted delta, since the premium-adjusted delta for a strike K is always smaller than the simple delta corresponding to the same strike. For example, if we are looking for a strike corresponding to a premium-adjusted forward delta of 0.20, we can choose K_{max} to be the strike corresponding to a simple forward delta of 0.20. The last strike can be calculated analytically using Equation (15). It is easy to see that the premium-adjusted delta is always below the non-premium-adjusted one. This follows from

$$\begin{aligned}\Delta_S(K,\sigma,\phi) - \Delta_{S,pa}(K,\sigma,\phi) = e^{-r_f\tau}\phi N(\phi d_+) - \phi e^{-r_f\tau}\frac{K}{f}N(\phi d_-) \geq 0 \\ \Leftrightarrow \phi f N(\phi d_+) - \phi K N(\phi d_-) \geq 0.\end{aligned}$$

Discounting the last inequality yields the Black-Scholes formula, which is always positive. The maximum for both, the premium-adjusted spot and premium-adjusted forward delta, is given implicitly by the equation

$$\sigma\sqrt{\tau}N(d_-) = n(d_-),$$

with $n(x)$ being the normal density at x. One can solve this implicit equation numerically for K_{min} and then use Brent's method to search for the strike in $[K_{min}, K_{max}]$. The resulting interval is illustrated in the right hand side of Figure (2).

Construction of Implied Volatility Smiles

The previous section introduced the FX specific delta and ATM conventions. This knowledge is crucial to understanding the volatility construction procedure in FX

markets. In FX option markets it is common to use the delta to measure the degree of moneyness. Consequently, volatilities are assigned to deltas (for any delta type), rather than strikes. For example, it is common to quote the volatility for an option which has a premium-adjusted delta of 0.25. These quotes are often provided by market data vendors to their customers. However, the volatility-delta version of the smile is translated by the vendors after using the smile construction procedure discussed below. Other vendors do not provide delta-volatility quotes. In this case, the customers have to employ the smile construction procedure.
Unlike in other markets, the FX smile is given implicitly as a set of restrictions implied by market instruments. This is FX-specific, as other markets quote volatility versus strike directly. A consequence is that one has to employ a calibration procedure to construct a volatility vs. delta or volatility vs. strike smile. This section introduces the set of restrictions implied by market instruments and proposes a new method which allows an efficient and robust calibration.

Suppose the mapping of a strike to the corresponding implied volatility

$$K \mapsto \sigma(K)$$

is given. We will not specify $\sigma(K)$ here but treat it as a general smile function for the moment. The crucial point in the construction of the FX volatility smile is to build $\sigma(K)$ such that it matches the volatilities and prices implied by market quotes. The FX market uses three volatility quotes for a given delta such as $\Delta = \pm 0.25$ [1]:

- an at-the-money volatility σ_{ATM},
- a risk reversal volatility σ_{25-RR},
- a quoted strangle volatility σ_{25-S-Q}.

A sample of market quotes for the EURUSD and USDJPY currency pairs is given in Table 4. Before starting the smile construction it is important to analyze the exact

Table 4: Market data for a maturity of 1 month, as of January, 20th 2009

	EURUSD	USDJPY
S_0	1.3088	90.68
r_d	0.3525%	0.42875%
r_f	2.0113%	0.3525%
σ_{ATM}	21.6215%	21.00%
σ_{25-RR}	−0.5%	−5.3%
σ_{25-S-Q}	0.7375%	0.184%

characteristics of the quotes in Table 4. In particular, one has to identify first

[1] We will take a delta of 0.25 as an example, although any choice is possible.

- which at-the-money convention is used,
- which delta type is used.

For example, Figure (3) shows two market consistent smiles based on the EURUSD market data from Table 4, assuming that this data refers to different deltas, a simple **or** premium-adjusted one. It is obvious, that the smiles can have very different shapes, in particular for out-of-the-money and in-the-money options. Misunderstanding the delta type which the market data refers to would lead to a wrong pricing of vanilla options. The quotes in the given market sample refer to a spot delta for

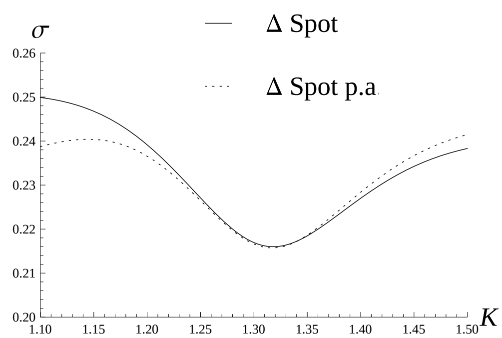

Fig. 3: Smile construction with EURUSD market data from Table 4, assuming different delta types.

the currency pair EURUSD and a premium-adjusted spot delta for the currency pair USDJPY. Both currency pairs use the forward delta neutral at-the-money quotation. The next subsections explain which information these quotes contain.

At-the-Money Volatility

After identifying the at-the-money type, we can extract the at-the-money strike K_{ATM} as summarized in Table 3. For the market sample data in Table 4 the corresponding strikes are summarized in Table 5. Independent of the choice of $\sigma(K)$,

Table 5: At-the-money strikes for market sample

	EURUSD	USDJPY
K_{ATM}	1.3096	90.86

it has to be ensured that the volatility for the at-the-money strike is σ_{ATM}. Consequently, the construction procedure for $\sigma(K)$ has to guarantee that the following Equation

$$\sigma(K_{ATM}) = \sigma_{ATM} \tag{16}$$

holds. A market consistent smile function $\overline{\sigma}(K)$ for the EURUSD currency pair thus has to yield

$$\overline{\sigma}(1.3096) = 21.6215\%$$

for the market data in Table 4. We will show later how to calibrate $\sigma(K)$ to retrieve $\overline{\sigma}(K)$, so assume for the moment that the calibrated, market consistent smile function $\overline{\sigma}(K)$ is given.

Risk Reversal

The risk reversal quotation σ_{25-RR} is the difference between two volatilities:

- the implied volatility of a call with a delta of 0.25 and
- the implied volatility of a put with a delta of -0.25.

It measures the skewness of the smile, the extra volatility which is added to the 0.25Δ put volatility compared to a call volatility which has the same absolute delta. Clearly, the delta type has to be specified in advance. For example, the implied volatility of a USD call JPY put with a premium-adjusted spot delta of 0.25 could be considered. Given $\sigma(K)$, it is possible to extract strike-volatility pairs[2] for a call and a put

$$\Big(K_{25C}, \sigma(K_{25C})\Big) \qquad \Big(K_{25P}, \sigma(K_{25P})\Big)$$

which yield a delta of 0.25 and -0.25 respectively:

$$\begin{aligned} \Delta\ (K_{25C}, \sigma(K_{25C}), 1) &= 0.25 \\ \Delta\ (K_{25P}, \sigma(K_{25P}), -1) &= -0.25 \end{aligned}$$

In the equation system above, Δ denotes a general delta which has to be specified to $\Delta_S, \Delta_{S,pa}$ or $\Delta_f, \Delta_{f,pa}$. The market consistent smile function $\overline{\sigma}(K)$ has to match the information implied in the risk reversal. Consequently, it has to fulfill

$$\overline{\sigma}(K_{25C}) - \overline{\sigma}(K_{25P}) = \sigma_{25-RR}. \tag{17}$$

Examples of such 0.25 Δ strike-volatility pairs for the market data in Table 4 and a calibrated smile function $\overline{\sigma}(K)$ are given in Table 6.

For the currency pair EURUSD we can calculate the difference of the 0.25 Δ call and put volatilities as

[2] This can be achieved by using a standard root search algorithm.

Table 6: 0.25 Δ strikes

	EURUSD	USDJPY
K_{25C}	1.3677	94.10
K_{25P}	1.2530	86.51
$\overline{\sigma}(K_{25C})$	22.1092%	18.7693%
$\overline{\sigma}(K_{25P})$	22.6092%	24.0693%

$$\overline{\sigma}(1.3677) - \overline{\sigma}(1.2530) = 22.1092\% - 22.6092\% = -0.5\%$$

which is consistent with the risk reversal quotation in Table 4. It can also be verified that

$$\Delta_S\ (1.3677, 22.1092\%, 1) = 0.25 \text{ and } \Delta_S(1.2530, 22.6092\%, -1) = -0.25.$$

Market Strangle

The strangle is the third restriction on the function $\sigma(K)$. Define the market strangle volatility σ_{25-S-M} as

$$\sigma_{25-S-M} = \sigma_{ATM} + \sigma_{25-S-Q}. \tag{18}$$

For the market sample from Table 4 and the USDJPY case this would correspond to

$$\sigma_{25-S-M} = 21.00\% + 0.184\% = 21.184\%.$$

Given this single volatility, we can extract a call strike $K_{25C-S-M}$ and a put strike $K_{25P-S-M}$ which - using σ_{25-S-M} as the volatility - yield a delta of 0.25 and -0.25 respectively. The procedure to extract a strike given a delta and volatility has been introduced in Section 1.5. The resulting strikes will then fulfill

$$\Delta\ (K_{25C-S-M}, \sigma_{25-S-M}, 1) = 0.25 \tag{19}$$

$$\Delta\ (K_{25P-S-M}, \sigma_{25-S-M}, -1) = -0.25. \tag{20}$$

The strikes corresponding to the market sample are summarized in Table 7. For the USDJPY case the strike volatility combinations given in Table 7 fulfill

$$\Delta_{S,pa}(94.55, 21.184\%, 1) = 0.25 \tag{21}$$

$$\Delta_{S,pa}(87.00, 21.184\%, -1) = -0.25 \tag{22}$$

where $\Delta_{S,pa}(K, \sigma, \phi)$ is the premium-adjusted spot delta. Given the strikes $K_{25C-S-M}$, $K_{25P-S-M}$ and the volatility σ_{25-S-M}, one can calculate the price of an option position of a long call with a strike of $K_{25C-S-M}$ and a volatility of σ_{25-S-M} and a long put with a strike of $K_{25P-S-M}$ and the same volatility. The resulting price v_{25-S-M} is

$$v_{25-S-M} = v(K_{25C-S-M}, \sigma_{25-S-M}, 1) + v(K_{25P-S-M}, \sigma_{25-S-M}, -1) \quad (23)$$

and is the final variable one is interested in. This is the third information implied by the market: The sum of the call option with a strike of $K_{25C-S-M}$ and the put option with a strike of $K_{25P-S-M}$ has to be v_{25-S-M}. This information has to be incorporated by a market consistent volatility function $\overline{\sigma}(K)$ which can have different volatilities at the strikes $K_{25C-S-M}$, $K_{25P-S-M}$ but should guarantee that the corresponding option prices at these strikes add up to v_{25-S-M}. The delta of these options with the smile volatilities is not restricted to yield 0.25 or -0.25. To summarize,

$$v_{25-S-M} = v(K_{25C-S-M}, \sigma(K_{25C-S-M}), 1) + v(K_{25P-S-M}, \sigma(K_{25P-S-M}), -1) \quad (24)$$

is the last restriction on the volatility smile. Taking again the USDJPY as an example yields that the strangle price to be matched is

$$v_{25-S-M} = v(94.55, 21.184\%, 1) + v(87.00, 21.184\%, -1) = 1.67072. \quad (25)$$

The resulting price v_{25-S-M} is in the domestic currency, JPY in this case. One can then extract the volatilities from a calibrated smile $\overline{\sigma}(K)$ –as in Table 7– and calculate the strangle price with volatilities given by the calibrated smile function $\overline{\sigma}(K)$

$$v(94.55, 18.5435\%, 1) + v(87.00, 23.7778\%, -1) = 1.67072. \quad (26)$$

This is the same price as the one implied by the market in Equation (25).

Table 7: Market Strangle data

	EURUSD	USDJPY
$K_{25C-S-M}$	1.3685	94.55
$K_{25P-S-M}$	1.2535	87.00
$\overline{\sigma}(K_{25C-S-M})$	22.1216%	18.5435%
$\overline{\sigma}(K_{25P-S-M})$	22.5953%	23.7778%
v_{25-S-M}	0.0254782	1.67072

The introduced smile construction procedure is designed for a market that quotes three volatilities. This is often the case for illiquid markets. It can also be used for markets where more than three volatilities are quoted on an irregular basis, such that these illiquid quotes might not be a necessary input.

The Simplified Formula

Very often, a simplified formula is stated in the literature which allows an easy calculation of the 0.25 delta volatilities given the market quotes. Let σ_{25C} be the call volatility corresponding to a delta of 0.25 and σ_{25P} the -0.25 delta put volatility. Let K_{25C} and K_{25P} denote the corresponding strikes. The simplified formula states that

$$\begin{aligned} \sigma_{25C} &= \sigma_{ATM} + \frac{1}{2}\sigma_{25-RR} + \sigma_{25-S-Q} \\ \sigma_{25P} &= \sigma_{ATM} - \frac{1}{2}\sigma_{25-RR} + \sigma_{25-S-Q}. \end{aligned} \tag{27}$$

This would allow a simple calculation of the 0.25Δ volatilities $\sigma_{25C}, \sigma_{25P}$ with market quotes as given in Table 4. Including the at-the-money volatility would result in a smile with three anchor points which can then be interpolated in the usual way. In this case, no calibration procedure is needed. Note, that

$$\sigma_{25C} - \sigma_{25P} = \sigma_{25-RR} \tag{28}$$

such that the $0.25\ \Delta$ volatility difference automatically matches the quoted risk reversal volatility. The simplified formula can be reformulated to calculate σ_{25-S-Q}, given σ_{25C}, σ_{25P} and σ_{ATM} quotes. This yields

$$\sigma_{25-S-Q} = \frac{\sigma_{25C} + \sigma_{25P}}{2} - \sigma_{ATM}, \tag{29}$$

which presents the strangle as a convexity parameter. However, the problem arises in the matching of the market strangle as given in Equation (23), which we repeat here for convenience

$$v_{25-S-M} = v(K_{25C-S-M}, \sigma_{25-S-M}, 1) + v(K_{25P-S-M}, \sigma_{25-S-M}, -1).$$

Interpolating the smile from the three anchor points given by the simplified formula and calculating the market strangle with the corresponding volatilities at $K_{25P-S-M}$ and $K_{25C-S-M}$ does not necessary lead to the matching of v_{25-S-M}. The reason why the formula is stated very often (see for example Malz (1997)) is that the market strangle matching works for small risk reversal volatilities σ_{25-RR}. Assume that σ_{25-RR} is zero. The simplified Formula (27) then reduces to

$$\begin{aligned} \sigma_{25C} &= \sigma_{ATM} + \sigma_{25-S-Q}, \\ \sigma_{25P} &= \sigma_{ATM} + \sigma_{25-S-Q}. \end{aligned}$$

This implies, that the volatility corresponding to a delta of 0.25 is the same as the volatility corresponding to a delta of -0.25, which is the same as the market strangle volatility σ_{25-S-M} introduced in Equation (18). Assume that in case of a vanishing risk reversal the smile is built using three anchor points given by the simplified formula and one is asked to price a strangle with strikes $K_{25C-S-M}$ and $K_{25P-S-M}$. Given the volatility $\sigma_{25C} = \sigma_{ATM} + \sigma_{25-S-Q}$ and a delta of 0.25 would result in $K_{25C-S-M}$ as the corresponding strike. Consequently, we would assign $\sigma_{ATM} + \sigma_{25-S-Q}$ to the strike $K_{25C-S-M}$ if we move from delta to the strike space. Similarly, a volatility of $\sigma_{ATM} + \sigma_{25-S-Q}$ would be assigned to $K_{25P-S-M}$. The resulting strangle from the three anchor smile would be

$$v(K_{25C-S-M}, \sigma_{ATM} + \sigma_{25-S-Q}, 1) + v(K_{25P-S-M}, \sigma_{ATM} + \sigma_{25-S-Q}, -1)$$

which is exactly the market strangle price v_{25-S-M}. In this particular case, we have

$$\begin{aligned} K_{25C-S-M} &= K_{25C}, \\ K_{25P-S-M} &= K_{25P}. \end{aligned}$$

Using the simplified smile construction procedure yields a market strangle consistent smile setup in case of a zero risk reversal. The other market matching requirements are met by default. In any other case, the strangle price might not be matched which leads to a non market consistent setup of the volatility smile.

The simplified formula can still be useful, even for large risk reversals, if σ_{25-S-Q} is replaced by some other parameter introduced below. This parameter can be extracted after finishing the market consistent smile construction and is calculated in a way which is similar to Equation (29). Assume that the 0.25 delta volatilities $\sigma_{25C} = \overline{\sigma}(K_{25C})$ and $\sigma_{25P} = \overline{\sigma}(K_{25P})$ are given by the **calibrated** smile function $\overline{\sigma}(K)$. We can then calculate another strangle, called the smile strangle via

$$\sigma_{25-S-S} = \frac{\overline{\sigma}(K_{25C}) + \overline{\sigma}(K_{25P})}{2} - \sigma_{ATM}. \tag{30}$$

The smile strangle measures the convexity of the calibrated smile function and is plotted in Figure (4). It is approximately the difference between a straight line between the 25Δ put and call volatilities and the at-the-money volatility, evaluated at Δ_{ATM}. [3] This is equivalent to Equation (29), but in this case we are using out-of-the-money volatilities obtained from the calibrated smile and not from the simplified formula. Given σ_{25-S-S}, the simplified Equation (27) can still be used if the quoted strangle volatility σ_{25-S-Q} is replaced by the smile strangle volatility σ_{25-S-S}. Clearly, σ_{25-S-S} is not known a priori but is obtained after finishing the calibration. Thus, one obtains a correct simplified formula as

$$\begin{aligned} \sigma_{25C} &= \sigma_{ATM} + \frac{1}{2}\sigma_{25-RR} + \sigma_{25-S-S}, \\ \sigma_{25P} &= \sigma_{ATM} - \frac{1}{2}\sigma_{25-RR} + \sigma_{25-S-S}. \end{aligned} \tag{31}$$

[3] Here, Δ_{ATM} is the at-the-money delta. The description is exact if we consider the forward delta case with the delta-neutral at-the-money quotation. In other cases, this is an approximation.

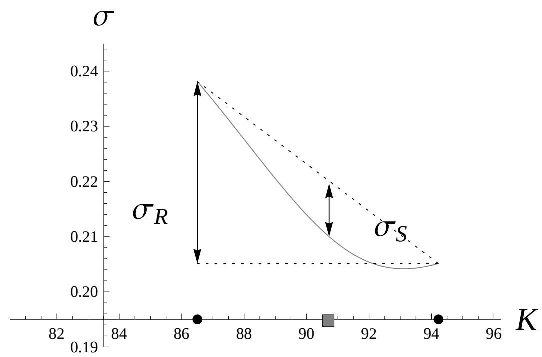

Fig. 4: Smile strangle for random market data. Filled circles indicate K_{25P}, K_{25C} strikes. Rectangle indicates K_{ATM}.

A sample data example is summarized in Table 8 where we have used the calibrated smile function $\overline{\sigma}(K)$ to calculate the smile strangles σ_{25-S-S}. Given σ_{25-S-S} , σ_{ATM}

Table 8: Smile strangle data

	EURUSD	USDJPY
$\overline{\sigma}(K_{25C})$	22.1092%	18.7693%
$\overline{\sigma}(K_{25P})$	22.6092%	24.0693%
σ_{ATM}	21.6215%	21.00%
σ_{25-RR}	−0.5%	−5.3%
σ_{25-S-S}	0.7377%	0.419%
σ_{25-S-Q}	0.7375%	0.184%

and σ_{25-RR}, we can calculate the EURUSD out-of-the-money volatilities of the call and put via the simplified Formula (31) as

$$\sigma_{25C} = 21.6215\% - \frac{1}{2}0.5\% + 0.7377\% = 22.1092\%,$$
$$\sigma_{25P} = 21.6215\% + \frac{1}{2}0.5\% + 0.7377\% = 22.6092\%,$$

which is consistent with the volatilities $\overline{\sigma}(K_{25C})$ and $\overline{\sigma}(K_{25P})$ in Table 8. Note that the market strangle volatility is very close to the smile strangle volatility in the EURUSD case. This is due to the small risk reversal of the EURUSD smile. Calculating the 25Δ volatilities via the original simplified Formula (27) would yield a call volatility of 22.109% and a put volatility of 22.609% which are approximately the 0.25Δ volatilities of Table 8. However, the smile strangle and quoted strangle volatilities differ significantly for the skewed JPYUSD smile. Using the original Formula (27) in this case would result in 18.534% and 23.834% for the 25Δ call and put volatilities. These volatilities differ from the market consistent 25Δ volatilities given in Table 8.

Simplified Parabolic Interpolation

Various different interpolation methods can be considered as basic tools for the calibration procedure. Potential candidates are the SABR model introduced by Hagan et al. (2002), or the Vanna Volga method introduced by Castagna and Mercurio (2007). In this work, we introduce a new method for the smile construction. In a proceeding paper, we will compare all methods and analyze their calibration robustness empirically. The method introduced below turns out to be the most robust method.

In Malz (1997), the mapping forward delta against volatility is constructed as a polynomial of degree 2. This polynomial is constructed such that the at-the-money and risk reversal delta volatilities are matched. Malz derives the following functional relationship

$$\sigma(\Delta_f) = \sigma_{ATM} - 2\sigma_{25-RR}(\Delta_f - 0.5) + 16\sigma_{25-S-Q}(\Delta_f - 0.5)^2 \tag{32}$$

where Δ_f is a call forward delta[4]. This is a parabola centered at 0.5. The use of this functional relationship can be problematic due to the following set of problems:

- the interpolation is not a well defined volatility function since it is not always positive,
- the representation is only valid for forward deltas, although the author incorrectly uses the spot delta in his derivation (see Equation (7) and Equation (18) in Malz [1997]),
- the formula is only valid for the forward delta neutral at-the-money quotation,
- the formula is only valid for risk reversal and strangle quotes associated with a delta of 0.25,
- the matching of the market strangle restriction (24) is guaranteed for small risk reversals only.

The last point is crucial! If the risk reversal σ_{25-RR} is close to zero, the formula will yield $\sigma_{ATM} + \sigma_{25-S-Q}$ as the volatility for the ± 0.25 call and put delta. This is consistent with restriction (24). However, a significant risk reversal will lead to a failure of the formula. We will fix most of the problems by deriving a new, more generalized formula with a similar structure. The problem that the formula is restricted to a specific delta and at-the-money convention can be fixed easily. The matching of the market strangle will be employed by a suitable calibration procedure. The resulting equation will be denoted as the simplified parabolic formula.

The simplified parabolic formula is constructed in delta space. Let a general delta

[4] A put volatility can be calculated by transforming the put to a call delta using the put call parity.

function $\Delta(K,\sigma,\phi)$ be given and K_{ATM} be the at-the-money strike associated with the given at-the-money volatility σ_{ATM}. Let the risk reversal volatility quote corresponding to a general delta of $\tilde{\Delta} > 0$ be given by $\sigma_{\tilde{\Delta}-RR}$. For the sake of a compact notation of the formula we will use σ_R instead of $\sigma_{\tilde{\Delta}-RR}$. Furthermore, we parametrize the smile by using a convexity parameter called **smile strangle** which is denoted as σ_S. This parameter has been discussed before in the simplified formula section. The following theorem can be stated.

Theorem 1. *Let Δ_{ATM} denote the call delta implied by the at-the-money strike*

$$\Delta_{ATM} = \Delta(K_{ATM}, \sigma_{ATM}, 1).$$

*Furthermore, we define a variable **a** which is the difference of a call delta, corresponding to a $-\tilde{\Delta}$ put delta, and the $-\tilde{\Delta}$ put delta for any delta type and is given by*

$$a := \Delta(K_{\tilde{\Delta}P}, \sigma, 1) - \Delta(K_{\tilde{\Delta}P}, \sigma, -1).$$

Given a call delta $\overline{\Delta}$, the parabolic mapping

$$(\overline{\Delta}, \sigma_S) \mapsto \sigma(\overline{\Delta}, \sigma_S)$$

which matches σ_{ATM} and the $\sigma_{\tilde{\Delta}-RR}$ risk reversal quote by default is

$$\sigma(\overline{\Delta}, \sigma_S) = \sigma_{ATM} + c_1(\overline{\Delta} - \Delta_{ATM}) + c_2(\overline{\Delta} - \Delta_{ATM})^2 \tag{33}$$

with

$$c_1 = \frac{a^2(2\sigma_S + \sigma_R) - 2a(2\sigma_S + \sigma_R)(\tilde{\Delta} + \Delta_{ATM}) + 2(\tilde{\Delta}^2\sigma_R + 4\sigma_S\tilde{\Delta}\Delta_{ATM} + \sigma_R\Delta_{ATM}^2)}{2(2\tilde{\Delta} - a)(\tilde{\Delta} - \Delta_{ATM})(\tilde{\Delta} - a + \Delta_{ATM})}$$

$$c_2 = \frac{4\tilde{\Delta}\sigma_S - a(2\sigma_S + \sigma_R) + 2\sigma_R\Delta_{ATM}}{2(2\tilde{\Delta} - a)(\tilde{\Delta} - \Delta_{ATM})(\tilde{\Delta} - a + \Delta_{ATM})} \tag{34}$$

assuming that the denominator of c_1 (and thus c_2) is not zero. A volatility for a put delta can be calculated via the transformation of the put delta to a call delta.

Proof: See Appendix.

We will present $\sigma(\Delta, \sigma_S)$ as a function depending on two parameters only, although of course more parameters are needed for the input. We consider σ_S explicitly, since this is the only parameter not observable in the market. This parameter will be the crucial object in the calibration procedure. Setting $\tilde{\Delta} = 0.25$, $\Delta_{ATM} = 0.5$ and $a = 1$ as in the forward delta case, yields the original Malz formula if $\sigma_S = \sigma_{25-S-Q}$. The generalized formula can handle any delta (e.g $\tilde{\Delta} = 0.10$), any delta type and any at-the-money convention. The formula automatically matches the at-the-money volatility, since

$$\sigma(\Delta_{ATM}, \sigma_S) = \sigma_{ATM}$$

Furthermore, the risk reversal is matched since

$$\sigma(\tilde{\Delta}_C, \sigma_S) - \sigma(a + \tilde{\Delta}_P, \sigma_S) = \sigma_{\tilde{\Delta}-RR}$$

where $\tilde{\Delta}_C$ denotes the call delta and $\tilde{\Delta}_P$ the put delta[5].

We have plotted the calibrated strike vs. volatility function in Figure (5) to show the influence of the parameters $\sigma_{ATM}, \sigma_R, \sigma_S$ on the simplified parabolic volatility smile in the strike space. We will explain later how to move from the delta to the strike space. Increasing σ_{ATM} leads to a parallel upper shift of the smile. Increasing σ_{25RR} yields to a more skewed curve. A risk reversal of zero implies a symmetric smile. Increasing the strangle σ_S increases the at-the-money smile convexity. Our final goal

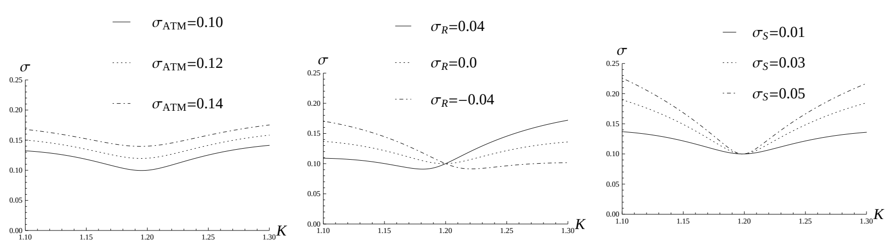

Fig. 5: Simplified Parabolic $\sigma_{ATM}, \sigma_R, \sigma_S$ spot delta scenarios with $\tau = \frac{35}{365}, S_0 = 1.2, r_d = 0.03, r_f = 0.01, \tilde{\Delta} = 0.25$. Initial parameters $\sigma_{ATM} = 10.0\%$, $\sigma_R = 0.6\%$, $\sigma_S = 1.0\%$.

will be the adjustment of the smile convexity by changing σ_S until condition (24) is met. The other conditions are fulfilled by default, independent of the choice of σ_S.

We note that the simplified parabolic formula follows the sticky-delta rule. This implies, that the smile does not move in the delta space, if the spot changes (see Balland (2002), Daglish et al. (2007), Derman (1999)). In the strike space, the smile performs a move to the right in case of an increasing spot, see Figure (6) .

Market Calibration

The advantage of Formula (33) is that it matches the at-the-money and risk reversal conditions of Equations (16) and (17) by default. The only remaining challenge is matching the market strangle. The simplified parabolic function can be transformed from a delta-volatility to a strike-volatility space (which will be discussed later) such that a function

$$\sigma(K, \sigma_S)$$

is available. Using the variable σ_S as the free parameter, the calibration problem can be reduced to a search for a variable x such that the following holds

---

[5] $a + \tilde{\Delta}_P$ is the call delta corresponding to a put delta of $\tilde{\Delta}_P$. In the forward delta case $a = 1$. If $\tilde{\Delta}_P = -0.25$, the equivalent call delta which enters the simplified parabolic formula is $a + \tilde{\Delta}_P = 0.75$.

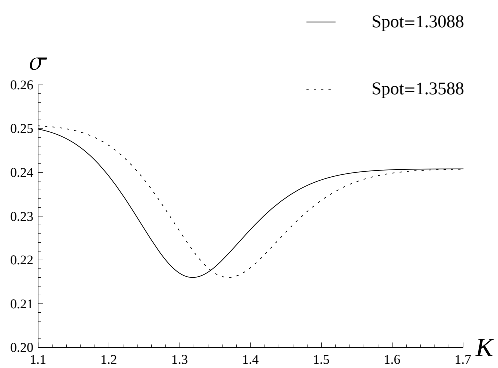

Fig. 6: Moving spot scenario for calibrated simplified parabolic formula in strike space. Based on market data in Table 3.

$$v_{\tilde{\Delta}-S-M} = v(K_{\tilde{\Delta}C-S-M}, \sigma(K_{\tilde{\Delta}C-S-M}, x), 1) \\ + v(K_{\tilde{\Delta}P-S-M}, \sigma(K_{\tilde{\Delta}P-S-M}, x), -1). \tag{35}$$

This leads to the following root search problem:

`Problem Type:`	Root search.
`Given parameters:`	$v_{\tilde{\Delta}-S-M}, K_{\tilde{\Delta}C-S-M}, K_{\tilde{\Delta}P-S-M}$ and market data.
`Target parameter:`	x (set x initially to $\sigma_{\tilde{\Delta}-S-Q}$)

`Objective function:`

$$f(x) = v(K_{\tilde{\Delta}C-S-M}, \sigma(K_{\tilde{\Delta}C-S-M}, x), 1) + v(K_{\tilde{\Delta}P-S-M}, \sigma(K_{\tilde{\Delta}P-S-M}, x), -1) - v_{\tilde{\Delta}-S-M}$$

The procedure will yield a smile strangle which can be used in the simplified parabolic formula to construct a full smile in the delta space. It is natural to ask, how well defined the problem above is and whether a solution exists. We will not present a rigorous analysis of this problem here, but it will be presented in follow-up research. We will show that a solution exists in a neighborhood of $\sigma_R = 0$ assuming that a weak condition is fulfilled. However, the neighborhood might be very small such that no solution for large risk-reversals might be available. The empirical tests in the following section will show, that the non-existence of such a solution has occurred in the past in very extreme market scenarios.
Performing the calibration on the currency data in Table 4 yields the parameters summarized in Table 7 for the root search problem. The final calibrated smile for the JPYUSD case is illustrated in Figure (8).

Table 7: Simplified Parabolic Calibration Results

	EURUSD Sample	USDJPY Sample
σ_S	0.007377	0.00419

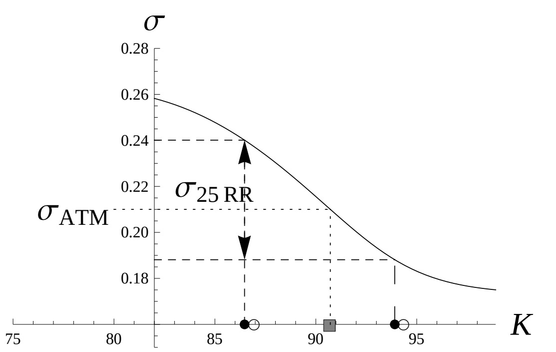

Fig. 8: JPYUSD smile for the market data in Exhibit 4. Filled circles indicate K_{25P}, K_{25C} strikes. Unfilled circles indicate market strangle strikes $K_{25P-S-M}, K_{25C-S-M}$. Rectangle indicates K_{ATM}.

Retrieving a Volatility for a Given Strike

Formula (33) returns the volatility for a given delta. However, the calibration procedure requires a mapping

$$K \mapsto \sigma(K, \sigma_S)$$

since it needs a volatility corresponding to the market strangle strikes. The transformation to $\sigma(K, \sigma_S)$ can be deduced by recalling that $\sigma = \sigma(\overline{\Delta}, \sigma_S)$ is the volatility corresponding to the delta $\overline{\Delta}$. To be more precise, given that σ is assigned to delta $\overline{\Delta}$ implies that $\overline{\Delta} = \Delta(K, \sigma, \phi)$ for some strike K. Consequently, Formula (33) can be stated as

$$\sigma = \sigma_{ATM} + c_1(\Delta(K, \sigma, 1) - \Delta_{ATM}) + c_2(\Delta(K, \sigma, 1) - \Delta_{ATM})^2. \tag{36}$$

Given a strike K, it is thus possible to retrieve the corresponding volatility by searching for a σ which fulfills Equation (36). This can be achieved by using a root searcher. We recommend the method introduced by Brent (2002). The question arises, if such a volatility vs. strike function exists and how smooth it is. The answer can be given by using the implicit function theorem. In the following discussion we will avoid the explicit dependence of all variables on $(K, \sigma(K, \sigma_S))$. For example, we write

$$\frac{\partial \Delta}{\partial K} \text{ instead of } \frac{\partial \Delta}{\partial K}(x, y)|_{x=K, y=\sigma(K, \sigma_S)}$$

With this compact notation, we can state the following.

Theorem 2. *Given the volatility vs. delta mapping (33), assume that the following holds*

$$c_1 \frac{\partial \Delta}{\partial \sigma}(K_{ATM}, \sigma_{ATM}) \neq 1$$

*Then there exists a function $\sigma : U \to W$ with open sets $U, W \subseteq \mathbb{R}^+$ such that $K_{ATM} \in U$ and $\sigma_{ATM} \in W$ which maps the strike implicit in Δ against the corresponding volatility. The function is **differentiable** and has the following first- and second-order derivatives on U*

$$\frac{\partial \sigma}{\partial K} = \frac{\frac{\partial \Delta}{\partial K} A}{1 - \frac{\partial \Delta}{\partial \sigma} A} \tag{37}$$

$$\frac{\partial^2 \sigma}{\partial K^2} = \frac{\left[\left(\frac{\partial^2 \Delta}{\partial K^2} + \frac{\partial^2 \Delta}{\partial K \partial \sigma}\frac{\partial \sigma}{\partial K}\right)A + \frac{\partial \Delta}{\partial K}\frac{\partial A}{\partial K}\right]\left(1 - \frac{\partial \Delta}{\partial \sigma} A\right)}{\left(1 - \frac{\partial \Delta}{\partial \sigma} A\right)^2} + \frac{\frac{\partial \Delta}{\partial K} A \left(\left(\frac{\partial \Delta}{\partial \sigma \partial K} + \frac{\partial^2 \Delta}{\partial \sigma^2}\frac{\partial \sigma}{\partial K}\right)A + \frac{\partial \Delta}{\partial \sigma}\frac{\partial A}{\partial K}\right)}{\left(1 - \frac{\partial \Delta}{\partial \sigma} A\right)^2} \tag{38}$$

with

$$A := c_1 + 2c_2(\Delta - \Delta_{ATM}) \text{ and } \frac{\partial A}{\partial K} = 2c_2\left(\frac{\partial \Delta}{\partial K} + \frac{\partial \Delta}{\partial \sigma}\frac{\partial \sigma}{\partial K}\right)$$

Proof. See Appendix.

Note that Equations (37) and (38) require the values $\sigma(K, \sigma_S)$. In fact, Equation (37) can be seen as an non-autonomous non-linear ordinary differential equation for $\sigma(K, \sigma_S)$. However, given $\sigma(K, \sigma_S)$ as a root of Equation (36), we can analytically calculate both derivatives. Differentiability is very important for calibration procedures of the well known local volatility models (see Dupire (1994), Derman and Kani (1994), Lee (2001)), which need a smooth volatility vs. strike function. To be more precise, given the local volatility SDE

$$dS_t = (r_d - r_f)S_t dt + \sigma(S_t, t)dW_t$$

the function $\sigma(K, t)$ can be stated in terms of the implied volatility (see Andersen and Brotherton-Ratcliffe (1998), Dempster and Richards (2000)) as

$$\sigma^2(K, T) = \frac{2\frac{\partial \sigma}{\partial T} + \frac{\sigma}{T-t} + 2K(r_d - r_f)\frac{\partial \sigma}{\partial K}}{K^2\left[\frac{\partial^2 \sigma}{\partial K^2} - d_+\sqrt{T-t}\left(\frac{\partial \sigma}{\partial K}\right)^2 + \frac{1}{\sigma}\left(\frac{1}{K\sqrt{T-t}} + d_+\frac{\partial \sigma}{\partial K}\right)^2\right]}.$$

The derivatives with respect to the strike can be very problematic if calculated numerically from an interpolation function. In our case, the derivatives can be stated explicitly, similar to (Hakala and Wystup, 2002, page 254) for the kernel interpolation case. In addition, the formulas are very useful to test for arbitrage, where restrictions on the slope and convexity of $\sigma(K)$ are imposed (see for example Lee (2005)).

We summarize explicit formulas for all derivatives occurring in Equations (37) and (38) in Tables 10 and 11 in the Appendix. They can be used for derivations of analytical formulas for the strike derivatives for all delta types.

Extreme Strike Behaviour

Lee (2004) published a very general result about the extreme strike behaviour of any implied volatility function. Work in this area has been continued by Benaim, Friz and Lee in Benaim et al. (2009), Benaim and Friz (2009). The basic idea of Lee is the following. Let

$$x := \ln\left(\frac{K}{f}\right)$$

be the log-moneyness and $I^2(x)$ the implied variance for a given moneyness x. Independent of the underlying model for the asset S there exists a $\beta_R \in [0,2]$ such that

$$\beta_R := \limsup_{x\to\infty} \frac{I^2(x)}{|x|/T}.$$

A very important result is that the number β_R is directly related to the highest finite moment of the underlying S at time T such that β_R can be stated more explicitly depending on the model. Define

$$\tilde{p} := \sup\{p : E(S_T^{1+p}) < \infty.\}$$

then we have

$$\beta_R = 2 - 4(\sqrt{\tilde{p}^2 + \tilde{p}} - \tilde{p}),$$

where the right hand expression is to be read as zero in the case $\tilde{p} = \infty$. A similar expression can be obtained for $x \to -\infty$. Consequently, the modeling of the implied volatility function in the delta space can not be arbitrarily, since Lee's extreme strike behavior has to be fulfilled. In the Appendix, we prove the following extreme strike behavior for the simplified parabolic formula:

$$\lim_{x\to\infty} \sigma(\Delta_S(x), \sigma_S) = \sigma_{ATM} - c_1\Delta_{ATM} + c_2\Delta_{ATM}^2, \tag{39}$$

which is a constant. Similarly,

$$\lim_{x\to-\infty} \sigma(\Delta_S(x), \sigma_S) = \sigma_{ATM} + c_1(e^{-r_f\tau} - \Delta_{ATM}) + c_2(e^{-r_f\tau} - \Delta_{ATM})^2, \tag{40}$$

which is again a constant. Equivalent results can be derived for the forward delta and the premium-adjusted versions. Consequently, the simplified formula implies a constant extrapolation, which is consistent with Lee's moment formula. The constant extrapolation implies that

$$\lim_{x\to\infty} \frac{I(x)}{\sqrt{|x|/T}} = 0 = \lim_{x\to-\infty} \frac{I(x)}{\sqrt{|x|/T}}.$$

This is only consistent, if

$$\sup\{p : E(S_T^{p+1}) < \infty\} = \infty,$$

e.g. all moments of the underlying at time T are finite. Although the simplified parabolic formula has been derived with a rather heuristic argumentation, it is only consistent if the underlying that generates such a volatility smile has finite moments of all orders.

Potential Problems

Potential numerical issues may arise due to the following:

1. Formula (33) is not restricted to yield positive values.
2. A root for Equation (36) might not exist. We do not know how large U, W are and whether a volatility can be found for any strike K.
3. The denominator in equation system (34) can be zero.
4. A root for Equation (35) might not exist.

The question arises, how often these problems occur in the daily market calibration. We have analyzed the occurrence of the problems above based on market data published on Bloomberg, where $\sigma_{ATM}, \sigma_{10-RR}, \sigma_{25-RR}$ and $\sigma_{10-S-Q}, \sigma_{25-S-Q}$ volatilities are quoted. We have considered the currencies EUR, GBP, JPY, CHF, CAD and AUD, which account for 88% of the worldwide traded OTC derivative notionals[6]. The data is summarized in Figure (9). The volatilities are quoted for maturities of $1, 3, 6, 9$ and 12 months. The delta types for all maturities below 9 months are spot deltas for the currency pairs EURUSD, GBPUSD, AUDUSD and premium-adjusted spot deltas for the currency pairs USDJPY, USDCHF, USDCAD. For the 12 month maturity, the first currency group uses forward deltas, while the second one uses premium-adjusted forward deltas. All currencies use the forward delta neutral straddle as the at-the-money convention. We have performed a daily calibration to market data for all maturities and currencies. The calibrations were performed to the 0.25Δ and 0.10Δ quotes separately. Then we have tested for problems occurring within a $\pm 0.10\Delta$ range. A check for a zero denominator in equation system (34) has been performed. Finally, we checked the existence of a root for the implied problem

[6] Based on data as of December 2008, published by the Bank for International Settlements on www.bis.org/publ/qtrpdf/r_qa0906.pdf

Table 9: FX Data Summary

	EURUSD	GBPUSD	USDJPY	USDCHF	USDCAD	AUDUSD
Begin Date	03.10.2003	03.10.2003	03.10.2003	05.01.2006	03.10.2003	03.10.2003
End Date	20.01.2009	20.01.2009	20.01.2009	20.01.2009	20.01.2009	20.01.2009
Data Sets	5834	5780	6126	3849	5775	5961

(36). In **none** of the more than $30,000$ calibrations did we observe any of the first three problems. We thus conclude, that the method is very robust in the daily calibration.

However, the calibration failed 6 times (in more than $30,000$ calibrations) in the root searching procedure for Equation (35). This happened for the 0.10Δ case for the extremely skewed currency pair JPYUSD, where risk reversals of 19% and more were observed in the extreme market scenarios following the financial crisis. The calibration procedure is more robust than other methods which have shown more than 300 failures in some cases. Also, it is not obvious whether any smile function can match the market quotes in these extreme scenarios. These issues will be covered in future research.

2 Conclusion

We have introduced various delta and at-the-money quotations commonly used in FX option markets. The delta types are FX-specific, since the option can be traded in both currencies. The various at-the-money quotations have been designed to account for large interest rate differentials or to enforce an efficient trading of positions with a pure vega exposure. We have then introduced the liquid market instruments that parametrize the market and have shown which information they imply. Finally, we derived a new formula that accounts for FX specific market information and can be used to employ an efficient market calibration.

Follow-up research will compare the robustness and potential problems of different smile calibration procedures by using empirical data. Also, potential calibration problems in extreme market scenarios will be analyzed.

Acknowledgments

We would like to thank Travis Fisher, Boris Borowski, Andreas Weber and Jürgen Hakala for their helpful comments.

3 Appendix

To reduce the notation, we will drop the dependence of $\sigma(\Delta, \sigma_S)$ on σ_S in the following proofs and write $\sigma(\Delta)$ instead.

Proof (Simplified Parabolic Formula). We will construct a parabola in the call delta space such that the following restrictions are met

$$\begin{aligned} \sigma(\Delta_{ATM}) &= \sigma_{ATM}, \\ \sigma(\tilde{\Delta}) &= \sigma_{ATM} + \frac{1}{2}\sigma_R + \sigma_S, \\ \sigma(a - \tilde{\Delta}) &= \sigma_{ATM} - \frac{1}{2}\sigma_R + \sigma_S. \end{aligned} \tag{41}$$

For example, in the forward delta case we would have $a = 1$. Given $\tilde{\Delta} = 0.25$, the call delta corresponding to a put delta of -0.25 would be $1 - 0.25 = 0.75$. The equation system is set up such that

$$\sigma_S = \frac{\sigma(\tilde{\Delta}) + \sigma(a - \tilde{\Delta})}{2} - \sigma_{ATM}.$$

One can see that σ_S measures the smile convexity, as it is the difference of the average of the out-of-the-money and in-the-money volatilities compared to the at-the-money volatility. The restriction set (41) ensures that

$$\sigma(\tilde{\Delta}) - \sigma(a - \tilde{\Delta}) = \sigma_R \tag{42}$$

is fulfilled by default. Given the parabolic setup

$$\sigma(\overline{\Delta}) = \sigma_{ATM} + c_1(\overline{\Delta} - \Delta_{ATM}) + c_2(\overline{\Delta} - \Delta_{ATM})^2,$$

one can solve for c_1, c_2 such that Equation system (41) is fulfilled. This is a well defined problem: a system of two linear equations in two unknowns. □

Proof (Existence of a Volatility vs Strike Function). The simplified parabolic function has the following form

$$\sigma(\Delta, \sigma_S) = \sigma_{ATM} + c_1(\Delta - \Delta_{ATM}) + c_2(\Delta - \Delta_{ATM})^2. \tag{43}$$

First of all, note that $\Delta(K, \sigma)$ is continuously differentiable with respect to both variables for all delta types. Define $F : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ to be

$$F(K,\sigma) = \sigma_{ATM} + c_1(\Delta(K,\sigma) - \Delta_{ATM}) + c_2(\Delta(K,\sigma) - \Delta_{ATM})^2 - \sigma \tag{44}$$

with $\Delta(K,\sigma)$ being one of the four deltas introduced before. The proof is a straightforward application of the implicit function theorem. Note that $F(K_{ATM},\sigma_{ATM}) = 0$ is given by default. As already stated, the function F is differentiable with respect to the strike and volatility. Deriving with respect to volatility yields

$$\frac{\partial F}{\partial \sigma} = c_1 \frac{\partial \Delta}{\partial \sigma} + 2c_2(\Delta - \Delta_{ATM}) \frac{\partial \Delta}{\partial \sigma} - 1. \tag{45}$$

From this derivation we have

$$\frac{\partial F}{\partial \sigma}(K_{ATM},\sigma_{ATM}) = c_1 \frac{\partial \Delta}{\partial \sigma}(K_{ATM},\sigma_{ATM}) - 1, \tag{46}$$

which is different from zero by assumption of the theorem. Consequently, the implicit function theorem implies the existence of a differentiable function f and an open neighborhood $U \times W \subseteq \mathbb{R}^+ \times \mathbb{R}^+$ with $K_{ATM} \in U$, $\sigma_{ATM} \in W$ such that

$$F(K,\sigma) = 0 \Leftrightarrow \sigma = f(K) \text{ for } (K,\sigma) \in U \times W.$$

The first derivative is defined on U and given by

$$\frac{\partial f}{\partial K} = -\frac{\frac{\partial F}{\partial K}}{\frac{\partial F}{\partial \sigma}} \text{ for } K \in U,$$

which can be calculated in a straightforward way. The function $f(K)$ is denoted as $\sigma(K)$ in the theorem. The second derivative can be derived in a straightforward way by remembering, that the volatility depends on the strike. This completes the proof. □

Proof (Extreme Strike Behaviour of Simplified Parabolic Interpolation). Let

$$x := \log\left(\frac{K}{f}\right)$$

be the log moneyness. The terms $d_\pm$ can be rewritten as

$$d_\pm(x) := \frac{-x \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}.$$

We then have:

$$\lim_{x\to\infty} N(d_\pm(x)) = 0, \tag{47}$$

$$\lim_{x\to-\infty} N(d_\pm(x)) = 1. \tag{48}$$

The c_1, c_2 terms are constants. Consequently, for the spot delta we derive:

$$\lim_{x\to\infty} \sigma(\Delta(x), \sigma_S) = \sigma_{ATM} - c_1\Delta_{ATM} + c_2\Delta_{ATM}^2, \tag{49}$$

which is a constant. Similarly,

$$\lim_{x\to-\infty} \sigma(\Delta(x), \sigma_S) = \sigma_{ATM} + c_1(e^{-r_f\tau} - \Delta_{ATM}) + c_2(e^{-r_f\tau} - \Delta_{ATM})^2, \tag{50}$$

which is again a constant. Equivalent results can be derived for the forward delta. The next analysis discusses the premium adjusted forward delta case; the spot premium adjusted case is similar. Rewriting the premium adjusted forward delta in terms of the log moneyness x yields

$$\Delta_{fpa} = e^x N(d_-(x)) = e^x N\Big(-\Big[\frac{x+\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\Big]\Big) = e^x - e^x N\Big(\frac{x+\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\Big).$$

Consequently, we have

$$\lim_{x\to\infty} \Delta_{f,pa}(x) = 0 = \lim_{x\to-\infty} \Delta_{f,pa}(x).$$

This implies that

$$\lim_{x\to\infty} \sigma(\Delta_{f,pa}(x), \sigma_S) = \sigma_{ATM} - c_1\Delta_{ATM} + c_2\Delta_{ATM}^2 = \lim_{x\to-\infty} \sigma(\Delta_{f,pa}(x), \sigma_S). \tag{51}$$

Note, that this limit differs from the spot delta case, since the terms a and Δ_{ATM} are different. □

	∂K	$\partial\sigma$	∂K^2
Δ_S	$-\frac{e^{-r_f\tau}n(d_+)}{\sigma\sqrt{\tau}K}$	$-\frac{e^{-r_f\tau}n(d_+)d_-}{\sigma}$	$\frac{e^{-r_f\tau}n(d_+)}{\sigma\sqrt{\tau}K^2} - \frac{e^{-r_f\tau}n(d_+)d_+}{\sigma^2\tau K^2}$
$\Delta_{S,pa}$	$\frac{\phi e^{-r_f\tau}N(\phi d_-)}{f} - \frac{e^{-r_f\tau}n(d_-)}{f\sigma\sqrt{\tau}}$	$-\frac{e^{-r_f\tau}Kn(d_-)d_+}{f\sigma}$	$-\frac{e^{-r_f\tau}n(d_-)}{f\sigma\sqrt{\tau}K} - \frac{e^{-r_f\tau}n(d_-)d_-}{fK\sigma^2\tau}$
Δ_f	$-\frac{n(d_+)}{\sigma\sqrt{\tau}K}$	$-\frac{n(d_+)d_-}{\sigma}$	$\frac{n(d_+)}{\sigma\sqrt{\tau}K^2} - \frac{n(d_+)d_+}{\sigma^2\tau K^2}$
$\Delta_{f,pa}$	$\frac{\phi N(\phi d_-)}{f} - \frac{n(d_-)}{f\sigma\sqrt{\tau}}$	$-\frac{Kn(d_-)d_+}{f\sigma}$	$-\frac{n(d_-)}{f\sigma\sqrt{\tau}K} - \frac{n(d_-)d_-}{fK\sigma^2\tau}$

Table 10: Partial Delta Derivatives I

	$\partial K \partial \sigma$	$\partial \sigma^2$
Δ_S	$\frac{e^{-r_f \tau} n(d_+) \left(1 - d_+ d_-\right)}{\sigma^2 \sqrt{\tau} K}$	$\frac{e^{-r_f \tau} n(d_+)(d_- - d_+ d_- d_- + d_+)}{\sigma^2}$
$\Delta_{S,pa}$	$\frac{e^{-r_f \tau} n(d_-) \left(-d_+ \sigma \sqrt{\tau} + 1 - d_- d_+\right)}{f \sigma^2 \sqrt{\tau}}$	$\frac{e^{-r_f \tau} K n(d_-)(d_+ - d_- d_+ d_+ + d_-)}{f \sigma^2}$
Δ_f	$\frac{n(d_+) \left(1 - d_+ d_-\right)}{\sigma^2 \sqrt{\tau} K}$	$\frac{n(d_+)(d_- - d_+ d_- d_- + d_+)}{\sigma^2}$
$\Delta_{f,pa}$	$\frac{n(d_-) \left(-d_+ \sigma \sqrt{\tau} + 1 - d_- d_+\right)}{f \sigma^2 \sqrt{\tau}}$	$\frac{K n(d_-)(d_+ - d_- d_+ d_+ + d_-)}{f \sigma^2}$

Table 11: Partial Delta Derivatives II

References

Andersen, L. and R. Brotherton-Ratcliffe, "The equity option volatility smile: an implicit finite-difference approach," *Journal of Computational Finance*, 1998, *1* (2), 5–37.

Balland, P., "Deterministic implied volatility models," *Quantitative Finance*, 2002, *2* (1), 31–44.

Beier, C.C. and C. Renner, *Encyclopedia of Quantitative Finance (Rama Cont and Peter Tankov Eds.)*, Wiley, forthcoming.

Benaim, S. and P. Friz, "Regular variation and smile asymptotics," *Mathematical Finance*, 2009, *19* (1), 1–12.

_ , _ , and R. Lee, "The Black-Scholes Implied Volatility at Extreme Strikes," *Frontiers in Quantitative Finance: Volatility and Credit Risk Modeling*, 2009, pp. 18–46.

Brent, R.P., *Algorithms for minimization without derivatives*, Courier Dover Publications, 2002.

Castagna, A. and F. Mercurio, "The Vanna-Volga Method for Implied Volatilities," *Risk*, 2007, *20* (1), 106.

Daglish, T., J. Hull, and W. Suo, "Volatility surfaces: theory, rules of thumb, and empirical evidence," *Quantitative Finance*, 2007, *7* (5), 507.

Dempster, M.A.H. and D.G. Richards, "Pricing American options fitting the smile," *Mathematical Finance*, 2000, *10* (2), 157–177.

Derman, E., "Regimes of volatility," *Risk*, 1999, *4*, 55–59.

_ and I. Kani, "Riding on a smile," *Risk*, 1994, *7* (2), 32–39.

Dupire, B., "Pricing with a smile," *Risk*, 1994, *7* (1), 18–20.

Hagan, P.S., D. Kumar, A.S. Lesniewski, and D.E. Woodward, "Managing smile risk," *Wilmott Magazine*, 2002, *1*, 84–108.

Hakala, J. and U. Wystup, *Foreign Exchange Risk*, Risk Books, 2002.

Lee, R.W., "Implied and local volatilities under stochastic volatility," *International Journal of Theoretical and Applied Finance*, 2001, *4* (1), 45–89.

_ , "The moment formula for implied volatility at extreme strikes," *Mathematical Finance*, 2004, *14* (3), 469–480.

_ , "Implied volatility: Statics, dynamics, and probabilistic interpretation," *Recent Advances in Applied Probability*, 2005.

Malz, A.M., "Option-Implied Probability Distributions and Currency Excess Returns," *SSRN eLibrary*, 1997.

Wystup, U., *FX Options and Structured Products*, Wiley, 2006.

FRANKFURT SCHOOL / HFB – WORKING PAPER SERIES

No.	Author/Title	Year
127.	Cremers, Heinz / Walzner, Jens Modellierung des Kreditrisikos im Portfoliofall	2009
126.	Cremers, Heinz / Walzner, Jens Modellierung des Kreditrisikos im Einwertpapierfall	2009
125.	Heidorn, Thomas / Schmaltz, Christian Interne Transferpreise für Liquidität	2009
124.	Bannier, Christina E. / Hirsch, Christian The economic function of credit rating agencies - What does the watchlist tell us?	2009
123.	Herrmann-Pillath, Carsten A Neurolinguistic Approach to Performativity in Economics	2009
122.	Winkler, Adalbert / Vogel, Ursula Finanzierungsstrukturen und makroökonomische Stabilität in den Ländern Südosteuropas, der Türkei und in den GUS-Staaten	2009
121.	Heidorn, Thomas / Rupprecht, Stephan Einführung in das Kapitalstrukturmanagement bei Banken	2009
120.	Rossbach, Peter Die Rolle des Internets als Informationsbeschaffungsmedium in Banken	2009
119.	Herrmann-Pillath, Carsten Diversity Management und diversi-tätsbasiertes Controlling: Von der „Diversity Scorecard" zur „Open Balanced Scorecard	2009
118.	Hölscher, Luise / Clasen, Sven Erfolgsfaktoren von Private Equity Fonds	2009
117.	Bannier, Christina E. Is there a hold-up benefit in heterogeneous multiple bank financing?	2009
116.	Roßbach, Peter / Gießamer, Dirk Ein eLearning-System zur Unterstützung der Wissensvermittlung von Web-Entwicklern in Sicherheitsthemen	2009
115.	Herrmann-Pillath, Carsten Kulturelle Hybridisierung und Wirtschaftstransformation in China	2009
114.	Schalast, Christoph: Staatsfonds – „neue" Akteure an den Finanzmärkten?	2009
113.	Schalast, Christoph / Alram, Johannes Konstruktion einer Anleihe mit hypothekarischer Besicherung	2009
112.	Schalast, Christoph / Bolder, Markus / Radünz, Claus / Siepmann, Stephanie / Weber, Thorsten Transaktionen und Servicing in der Finanzkrise: Berichte und Referate des Frankfurt School NPL Forums 2008	2009
111.	Werner, Karl / Moormann, Jürgen Efficiency and Profitability of European Banks – How Important Is Operational Efficiency?	2009
110.	Herrmann-Pillath, Carsten Moralische Gefühle als Grundlage einer wohlstandschaffenden Wettbewerbsordnung: Ein neuer Ansatz zur erforschung von Sozialkapital und seine Anwendung auf China	2009
109.	Heidorn, Thomas / Kaiser, Dieter G. / Roder, Christoph Empirische Analyse der Drawdowns von Dach-Hedgefonds	2009
108.	Herrmann-Pillath, Carsten Neuroeconomics, Naturalism and Language	2008
107.	Schalast, Christoph / Benita, Barten Private Equity und Familienunternehmen – eine Untersuchung unter besonderer Berücksichtigung deutscher Maschinen- und Anlagenbauunternehmen	2008
106.	Bannier, Christina E. / Grote, Michael H. Equity Gap? – Which Equity Gap? On the Financing Structure of Germany's Mittelstand	2008
105.	Herrmann-Pillath, Carsten The Naturalistic Turn in Economics: Implications for the Theory of Finance	2008
104.	Schalast, Christoph (Hrgs.) / Schanz, Kay-Michael / Scholl, Wolfgang Aktionärsschutz in der AG falsch verstanden? Die Leica-Entscheidung des LG Frankfurt am Main	2008
103.	Bannier, Christina E./ Müsch, Stefan Die Auswirkungen der Subprime-Krise auf den deutschen LBO-Markt für Small- und MidCaps	2008

102. Cremers, Heinz / Vetter, Michael
Das IRB-Modell des Kreditrisikos im Vergleich zum Modell einer logarithmisch normalverteilten Verlustfunktion 2008

101. Heidorn, Thomas / Pleißner, Mathias
Determinanten Europäischer CMBS Spreads. Ein empirisches Modell zur Bestimmung der Risikoaufschläge von Commercial Mortgage-Backed Securities (CMBS) 2008

100. Schalast, Christoph (Hrsg.) / Schanz, Kay-Michael
Schaeffler KG/Continental AG im Lichte der CSX Corp.-Entscheidung des US District Court for the Southern District of New York 2008

99. Hölscher, Luise / Haug, Michael / Schweinberger, Andreas
Analyse von Steueramnestiedaten 2008

98. Heimer, Thomas / Arend, Sebastian
The Genesis of the Black-Scholes Option Pricing Formula 2008

97. Heimer, Thomas / Hölscher, Luise / Werner, Matthias Ralf
Access to Finance and Venture Capital for Industrial SMEs 2008

96. Böttger, Marc / Guthoff, Anja / Heidorn, Thomas
Loss Given Default Modelle zur Schätzung von Recovery Rates 2008

95. Almer, Thomas / Heidorn, Thomas / Schmaltz, Christian
The Dynamics of Short- and Long-Term CDS-spreads of Banks 2008

94. Barthel, Erich / Wollersheim, Jutta
Kulturunterschiede bei Mergers & Acquisitions: Entwicklung eines Konzeptes zur Durchführung einer Cultural Due Diligence 2008

93. Heidorn, Thomas / Kunze, Wolfgang / Schmaltz, Christian
Liquiditätsmodellierung von Kreditzusagen (Term Facilities and Revolver) 2008

92. Burger, Andreas
Produktivität und Effizienz in Banken – Terminologie, Methoden und Status quo 2008

91. Löchel, Horst / Pecher, Florian
The Strategic Value of Investments in Chinese Banks by Foreign Financial Insitutions 2008

90. Schalast, Christoph / Morgenschweis, Bernd / Sprengetter, Hans Otto / Ockens, Klaas / Stachuletz, Rainer / Safran, Robert
Der deutsche NPL Markt 2007: Aktuelle Entwicklungen, Verkauf und Bewertung – Berichte und Referate des NPL Forums 2007 2008

89. Schalast, Christoph / Stralkowski, Ingo
10 Jahre deutsche Buyouts 2008

88. Bannier, Christina E./ Hirsch, Christian
The Economics of Rating Watchlists: Evidence from Rating Changes 2007

87. Demidova-Menzel, Nadeshda / Heidorn, Thomas
Gold in the Investment Portfolio 2007

86. Hölscher, Luise / Rosenthal, Johannes
Leistungsmessung der Internen Revision 2007

85. Bannier, Christina / Hänsel, Dennis
Determinants of banks' engagement in loan securitization 2007

84. Bannier, Christina
“Smoothing“ versus “Timeliness“ - Wann sind stabile Ratings optimal und welche Anforderungen sind an optimale Berichtsregeln zu stellen? 2007

83. Bannier, Christina E.
Heterogeneous Multiple Bank Financing: Does it Reduce Inefficient Credit-Renegotiation Incidences? 2007

82. Cremers, Heinz / Löhr, Andreas
Deskription und Bewertung strukturierter Produkte unter besonderer Berücksichtigung verschiedener Marktszenarien 2007

81. Demidova-Menzel, Nadeshda / Heidorn, Thomas
Commodities in Asset Management 2007

80. Cremers, Heinz / Walzner, Jens
Risikosteuerung mit Kreditderivaten unter besonderer Berücksichtigung von Credit Default Swaps 2007

79. Cremers, Heinz / Traughber, Patrick
Handlungsalternativen einer Genossenschaftsbank im Investmentprozess unter Berücksichtigung der Risikotragfähigkeit 2007

78. Gerdesmeier, Dieter / Roffia, Barbara
Monetary Analysis: A VAR Perspective 2007

77. Heidorn, Thomas / Kaiser, Dieter G. / Muschiol, Andrea
Portfoliooptimierung mit Hedgefonds unter Berücksichtigung höherer Momente der Verteilung 2007

76. Jobe, Clemens J. / Ockens, Klaas / Safran, Robert / Schalast, Christoph
Work-Out und Servicing von notleidenden Krediten – Berichte und Referate des HfB-NPL Servicing Forums 2006 2006

75. Abrar, Kamyar / Schalast, Christoph
Fusionskontrolle in dynamischen Netzsektoren am Beispiel des Breitbandkabelsektors 2006

74. Schalast, Christoph / Schanz, Kay-Michael
Wertpapierprospekte: Markteinführungspublizität nach EU-Prospektverordnung und Wertpapierprospektgesetz 2005 2006

73. Dickler, Robert A. / Schalast, Christoph
Distressed Debt in Germany: What´s Next? Possible Innovative Exit Strategies 2006

72. Belke, Ansgar / Polleit, Thorsten
How the ECB and the US Fed set interest rates 2006

71. Heidorn, Thomas / Hoppe, Christian / Kaiser, Dieter G.
Heterogenität von Hedgefondsindizes 2006

70. Baumann, Stefan / Löchel, Horst
The Endogeneity Approach of the Theory of Optimum Currency Areas - What does it mean for ASEAN + 3? 2006

69. Heidorn, Thomas / Trautmann, Alexandra
Niederschlagsderivate 2005

68. Heidorn, Thomas / Hoppe, Christian / Kaiser, Dieter G.
Möglichkeiten der Strukturierung von Hedgefondsportfolios 2005

67. Belke, Ansgar / Polleit, Thorsten
(How) Do Stock Market Returns React to Monetary Policy ? An ARDL Cointegration Analysis for Germany 2005

66. Daynes, Christian / Schalast, Christoph
Aktuelle Rechtsfragen des Bank- und Kapitalmarktsrechts II: Distressed Debt - Investing in Deutschland 2005

65. Gerdesmeier, Dieter / Polleit, Thorsten
Measures of excess liquidity 2005

64. Becker, Gernot M. / Harding, Perham / Hölscher, Luise
Financing the Embedded Value of Life Insurance Portfolios 2005

63. Schalast, Christoph
Modernisierung der Wasserwirtschaft im Spannungsfeld von Umweltschutz und Wettbewerb – Braucht Deutschland eine Rechtsgrundlage für die Vergabe von Wasserversorgungskonzessionen? – 2005

62. Bayer, Marcus / Cremers, Heinz / Kluß, Norbert
Wertsicherungsstrategien für das Asset Management 2005

61. Löchel, Horst / Polleit, Thorsten
A case for money in the ECB monetary policy strategy 2005

60. Richard, Jörg / Schalast, Christoph / Schanz, Kay-Michael
Unternehmen im Prime Standard - „Staying Public“ oder „Going Private“? - Nutzenanalyse der Börsennotiz - 2004

59. Heun, Michael / Schlink, Torsten
Early Warning Systems of Financial Crises - Implementation of a currency crisis model for Uganda 2004

58. Heimer, Thomas / Köhler, Thomas
Auswirkungen des Basel II Akkords auf österreichische KMU 2004

57. Heidorn, Thomas / Meyer, Bernd / Pietrowiak, Alexander
Performanceeffekte nach Directors´Dealings in Deutschland, Italien und den Niederlanden 2004

56. Gerdesmeier, Dieter / Roffia, Barbara
The Relevance of real-time data in estimating reaction functions for the euro area 2004

55. Barthel, Erich / Gierig, Rauno / Kühn, Ilmhart-Wolfram
Unterschiedliche Ansätze zur Messung des Humankapitals 2004

54. Anders, Dietmar / Binder, Andreas / Hesdahl, Ralf / Schalast, Christoph / Thöne, Thomas
Aktuelle Rechtsfragen des Bank- und Kapitalmarktrechts I :
Non-Performing-Loans / Faule Kredite - Handel, Work-Out, Outsourcing und Securitisation 2004

53. Polleit, Thorsten
The Slowdown in German Bank Lending – Revisited 2004

52. Heidorn, Thomas / Siragusano, Tindaro
Die Anwendbarkeit der Behavioral Finance im Devisenmarkt 2004

51. Schütze, Daniel / Schalast, Christoph (Hrsg.)
Wider die Verschleuderung von Unternehmen durch Pfandversteigerung 2004

50. Gerhold, Mirko / Heidorn, Thomas
Investitionen und Emissionen von Convertible Bonds (Wandelanleihen) 2004

49. Chevalier, Pierre / Heidorn, Thomas / Krieger, Christian
Temperaturderivate zur strategischen Absicherung von Beschaffungs- und Absatzrisiken 2003

48.	Becker, Gernot M. / Seeger, Norbert Internationale Cash Flow-Rechnungen aus Eigner- und Gläubigersicht	2003
47.	Boenkost, Wolfram / Schmidt, Wolfgang M. Notes on convexity and quanto adjustments for interest rates and related options	2003
46.	Hess, Dieter Determinants of the relative price impact of unanticipated Information in U.S. macroeconomic releases	2003
45.	Cremers, Heinz / Kluß, Norbert / König, Markus Incentive Fees. Erfolgsabhängige Vergütungsmodelle deutscher Publikumsfonds	2003
44.	Heidorn, Thomas / König, Lars Investitionen in Collateralized Debt Obligations	2003
43.	Kahlert, Holger / Seeger, Norbert Bilanzierung von Unternehmenszusammenschlüssen nach US-GAAP	2003
42.	Beiträge von Studierenden des Studiengangs BBA 012 unter Begleitung von Prof. Dr. Norbert Seeger Rechnungslegung im Umbruch - HGB-Bilanzierung im Wettbewerb mit den internationalen Standards nach IAS und US-GAAP	2003
41.	Overbeck, Ludger / Schmidt, Wolfgang Modeling Default Dependence with Threshold Models	2003
40.	Balthasar, Daniel / Cremers, Heinz / Schmidt, Michael Portfoliooptimierung mit Hedge Fonds unter besonderer Berücksichtigung der Risikokomponente	2002
39.	Heidorn, Thomas / Kantwill, Jens Eine empirische Analyse der Spreadunterschiede von Festsatzanleihen zu Floatern im Euroraum und deren Zusammenhang zum Preis eines Credit Default Swaps	2002
38.	Böttcher, Henner / Seeger, Norbert Bilanzierung von Finanzderivaten nach HGB, EstG, IAS und US-GAAP	2003
37.	Moormann, Jürgen Terminologie und Glossar der Bankinformatik	2002
36.	Heidorn, Thomas Bewertung von Kreditprodukten und Credit Default Swaps	2001
35.	Heidorn, Thomas / Weier, Sven Einführung in die fundamentale Aktienanalyse	2001
34.	Seeger, Norbert International Accounting Standards (IAS)	2001
33.	Moormann, Jürgen / Stehling, Frank Strategic Positioning of E-Commerce Business Models in the Portfolio of Corporate Banking	2001
32.	Sokolovsky, Zbynek / Strohhecker, Jürgen Fit für den Euro, Simulationsbasierte Euro-Maßnahmenplanung für Dresdner-Bank-Geschäftsstellen	2001
31.	Roßbach, Peter Behavioral Finance - Eine Alternative zur vorherrschenden Kapitalmarkttheorie?	2001
30.	Heidorn, Thomas / Jaster, Oliver / Willeitner, Ulrich Event Risk Covenants	2001
29.	Biswas, Rita / Löchel, Horst Recent Trends in U.S. and German Banking: Convergence or Divergence?	2001
28.	Eberle, Günter Georg / Löchel, Horst Die Auswirkungen des Übergangs zum Kapitaldeckungsverfahren in der Rentenversicherung auf die Kapitalmärkte	2001
27.	Heidorn, Thomas / Klein, Hans-Dieter / Siebrecht, Frank Economic Value Added zur Prognose der Performance europäischer Aktien	2000
26.	Cremers, Heinz Konvergenz der binomialen Optionspreismodelle gegen das Modell von Black/Scholes/Merton	2000
25.	Löchel, Horst Die ökonomischen Dimensionen der ‚New Economy'	2000
24.	Frank, Axel / Moormann, Jürgen Grenzen des Outsourcing: Eine Exploration am Beispiel von Direktbanken	2000
23.	Heidorn, Thomas / Schmidt, Peter / Seiler, Stefan Neue Möglichkeiten durch die Namensaktie	2000
22.	Böger, Andreas / Heidorn, Thomas / Graf Waldstein, Philipp Hybrides Kernkapital für Kreditinstitute	2000
21.	Heidorn, Thomas Entscheidungsorientierte Mindestmargenkalkulation	2000

20.	Wolf, Birgit Die Eigenmittelkonzeption des § 10 KWG	2000
19.	Cremers, Heinz / Robé, Sophie / Thiele, Dirk Beta als Risikomaß - Eine Untersuchung am europäischen Aktienmarkt	2000
18.	Cremers, Heinz Optionspreisbestimmung	1999
17.	Cremers, Heinz Value at Risk-Konzepte für Marktrisiken	1999
16.	Chevalier, Pierre / Heidorn, Thomas / Rütze, Merle Gründung einer deutschen Strombörse für Elektrizitätsderivate	1999
15.	Deister, Daniel / Ehrlicher, Sven / Heidorn, Thomas CatBonds	1999
14.	Jochum, Eduard Hoshin Kanri / Management by Policy (MbP)	1999
13.	Heidorn, Thomas Kreditderivate	1999
12.	Heidorn, Thomas Kreditrisiko (CreditMetrics)	1999
11.	Moormann, Jürgen Terminologie und Glossar der Bankinformatik	1999
10.	Löchel, Horst The EMU and the Theory of Optimum Currency Areas	1998
09.	Löchel, Horst Die Geldpolitik im Währungsraum des Euro	1998
08.	Heidorn, Thomas / Hund, Jürgen Die Umstellung auf die Stückaktie für deutsche Aktiengesellschaften	1998
07.	Moormann, Jürgen Stand und Perspektiven der Informationsverarbeitung in Banken	1998
06.	Heidorn, Thomas / Schmidt, Wolfgang LIBOR in Arrears	1998
05.	Jahresbericht 1997	1998
04.	Ecker, Thomas / Moormann, Jürgen Die Bank als Betreiberin einer elektronischen Shopping-Mall	1997
03.	Jahresbericht 1996	1997
02.	Cremers, Heinz / Schwarz, Willi Interpolation of Discount Factors	1996
01.	Moormann, Jürgen Lean Reporting und Führungsinformationssysteme bei deutschen Finanzdienstleistern	1995

FRANKFURT SCHOOL / HFB – WORKING PAPER SERIES
CENTRE FOR PRACTICAL QUANTITATIVE FINANCE

No.	Author/Title	Year
19.	Reiswich, Dimitri / Tompkins, Robert Potential PCA Interpretation Problems for Volatility Smile Dynamics	2009
18.	Keller-Ressel, Martin / Kilin, Fiodar Forward-Start Options in the Barndorff-Nielsen-Shephard Model	2008
17.	Griebsch, Susanne / Wystup, Uwe On the Valuation of Fader and Discrete Barrier Options in Heston's Stochastic Volatility Model	2008
16.	Veiga, Carlos / Wystup, Uwe Closed Formula for Options with Discrete Dividends and its Derivatives	2008
15.	Packham, Natalie / Schmidt, Wolfgang Latin hypercube sampling with dependence and applications in finance	2008
14.	Hakala, Jürgen / Wystup, Uwe FX Basket Options	2008

13.	Weber, Andreas / Wystup, Uwe Vergleich von Anlagestrategien bei Riesterrenten ohne Berücksichtigung von Gebühren. Eine Simulationsstudie zur Verteilung der Renditen	2008
12.	Weber, Andreas / Wystup, Uwe Riesterrente im Vergleich. Eine Simulationsstudie zur Verteilung der Renditen	2008
11.	Wystup, Uwe Vanna-Volga Pricing	2008
10.	Wystup, Uwe Foreign Exchange Quanto Options	2008
09.	Wystup, Uwe Foreign Exchange Symmetries	2008
08.	Becker, Christoph / Wystup, Uwe Was kostet eine Garantie? Ein statistischer Vergleich der Rendite von langfristigen Anlagen	2008
07.	Schmidt, Wolfgang Default Swaps and Hedging Credit Baskets	2007
06.	Kilin, Fiodor Accelerating the Calibration of Stochastic Volatility Models	2007
05.	Griebsch, Susanne/ Kühn, Christoph / Wystup, Uwe Instalment Options: A Closed-Form Solution and the Limiting Case	2007
04.	Boenkost, Wolfram / Schmidt, Wolfgang M. Interest Rate Convexity and the Volatility Smile	2006
03.	Becker, Christoph/ Wystup, Uwe On the Cost of Delayed Currency Fixing	2005
02.	Boenkost, Wolfram / Schmidt, Wolfgang M. Cross currency swap valuation	2004
01.	Wallner, Christian / Wystup, Uwe Efficient Computation of Option Price Sensitivities for Options of American Style	2004

HFB – SONDERARBEITSBERICHTE DER HFB - BUSINESS SCHOOL OF FINANCE & MANAGEMENT

No.	Author/Title	Year
01.	Nicole Kahmer / Jürgen Moormann Studie zur Ausrichtung von Banken an Kundenprozessen am Beispiel des Internet (Preis: € 120,--)	2003

Printed edition: € 25.00 + € 2.50 shipping

Download:
Working Paper: http://www.frankfurt-school.de/content/de/research/Publications/list_of_publication0.html
CPQF: http://www.frankfurt-school.de/content/de/research/quantitative_Finance/research_publications.html

Order address / contact
Frankfurt School of Finance & Management
Sonnemannstr. 9 – 11 ▪ D – 60314 Frankfurt/M. ▪ Germany
Phone: +49 (0) 69 154 008 – 734 ▪ Fax: +49 (0) 69 154 008 – 728
eMail: m.biemer@frankfurt-school.de
Further information about Frankfurt School of Finance & Management
may be obtained at: http://www.frankfurt-school.de