The Pricing and Hedging of the Range Accrual Note

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Abstract

The range accrual note is an exotic interest rate derivative that pays out a fixed rate for every day that a chosen reference rate falls within a predefined corridor. This project derives expressions for the price and Δ of the range note before considering FRAs as a hedge instrument.

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Contents

List of Figures

Chapter 1

Introduction

At initiation of the contract a range and rate are fixed, the range note pays the fixed rate for every day that the reference interest rate is in the corridor and nothing when it is not. The contract makes payments at periods equal to the tenor of the reference rate. This introduces timing issues into the pricing that make the daily replication of the note slightly more advanced than a simple spread of digital caplets. Instead the range note is replicated by daily *range contingent payoffs* which, themselves, are a spread of daily *contingent payoffs*. The contingent payoff differs from the digital caplet in that, instead of paying out 1, it pays out the present value of 1 at the actual date of payment. The range note is then fully replicated by summing over all these range contingent payoffs for all 'active' days in the future. Hence the pricing of the range note depends entirely on finding an expression for the contingent payoff.

The next chapter sets up the theory of the pricing procedure sketched in the above paragraph. After making the above more exact, we begin by considering the two cases of the single period range note- when the evaluation date is before the initiation of the period and when it falls during the period. We then go on to show how multi-period range notes are built from these two cases.

Having established that pricing the range note depends entirely on evaluating the contingent payoff, the third chapter accomplishes this in the South African market. A Black's model is assumed for JIBAR, a suitable numeraire is chosen and an integral expression for the contingent payoff is found. This expression cannot be solved analytically [1] and is, instead, approximated numerically by Taylor expanding the integrand and interchanging summation and integral. This approximation then allows us to implement the pricing of the range note in `matlab`.

[1] At least not to my knowledge

The fourth chapter is concerned with finding Δ of the range note. Again this depends entirely on finding Δ of the contingent payoff. The integral expression for the contingent payoff is differentiated using the *Leibnitz integral rule* resulting in a boundary and integral term. The integral term is then solved with exactly the same method as that in chapter three. The rest of the chapter then gives the `matlab` algorithm for finding Δ.

In the final chapter we consider the Δ-hedging of the range note with FRA's and a riskless money market account. After deriving the hedge portfolio, we obtain an expression for the associated hedge slippage.

Chapter 2

General Method For Pricing a Range Note

In this section we build up a general method for pricing a range note by considering a sequence of simpler derivatives. We will begin by considering the following instruments:

- European digital call option
- European range digital call option
- European contingent payoff call option
- European range contingent payoff call option

Before we discuss their relevance to the pricing of

- Single period range note
- Multi-period range note

2.1 European Digital Call Option

Definition: A *digital call* is an option that has unit payoff if the reference interest rate is above the strike interest rate at maturity and zero if it is below or equal to the strike rate at maturity.

Let $T, R(T, \alpha)$ and k denote the options's maturity, reference interest rate at maturity and strike rate respectively, where α is the tenor of the reference interest rate. Then the payoff at maturity T is given by

$$DC\left(T; T, \alpha, k\right) = \theta\left(R\left(T; \alpha\right) - k\right) \tag{2.1}$$

Where the *Heaviside function* $\theta\left(R\left(T; \alpha\right) - k\right)$ equals 1 on $R\left(T; \alpha\right) > k$ and

0 on $R(T;\alpha) \leq k$.

Thus with a suitable choice of numeraire ϕ and equivalent martingale measure $\mathbb{Q}$ we can price the digital call by evaluating

$$DC(t;T,\alpha,k) = \phi(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{\theta(R(T;\alpha)-k)}{\phi(T)}\right] \tag{2.2}$$

2.2 European Range Digital Call Option

Definition: A *range digital call* is an option that has unit payoff if the reference interest rate is within the range $(k_L, k_U]$ at maturity and zero outside this range at maturity. We call k_L the lower strike and k_U the upper strike.

Denoting the range digital call by RD, we can mathematically express the definition as

$$\begin{aligned} RD(T;T,\alpha,k_L,k_U) &= 1; k_L < R(T;\alpha) \leq k_U \\ & \quad\; 0; otherwise \\ &= [\theta(R(T;\alpha)-k_L) - \theta(R(T;\alpha)-k_U)] \end{aligned} \tag{2.3}$$

This payoff can be replicated with the following portfolio

- Long a digital call at the lower strike
- Short a digital call at the upper strike

Thus, by the law of one price we have that the price of a range digital call at date t $(t < T)$ is

$$RD(t;T,\alpha,k_L,k_U) = DC(t;T,\alpha,k_L) - DC(t;T,\alpha,k_U) \tag{2.4}$$

2.3 European Contingent Payoff Call Option

Definition: A *contingent payoff call* is an option that pays at maturity T the present value of one at a later time T' if the reference interest rate is greater than the strike rate and zero if it is below or equal to the strike rate.

Denoting the present value at T of one at T' by the discount factor $Z(T,T')$ and the contingent payoff call by CP, the payoff can be expressed as

$$CP\left(T;T,T',\alpha,k\right) = Z(T,T')\theta(R(T;\alpha)-k)$$

Again with a suitable choice of numeraire ϕ and equivalent martingale measure $\mathbb{Q}$ we can price the digital call by evaluating

$$CP\left(t;T,T',\alpha,k\right)=\phi(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{Z(T,T')\theta\left(R\left(T;\alpha\right)-k\right)}{\phi(T)}\right] \tag{2.5}$$

2.4 European Range Contingent Payoff Call Option

Definition: A *range contingent payoff call* is an option that has payoff $Z(T,T')$ if the reference interest rate is within the range $(k_L,k_U]$ at maturity and zero outside this range at maturity.

Denoting the range contingent payoff call by RCP, we can express the payoff at maturity as

$$RCP\left(T;T,T',\alpha,k_L,k_U\right)=Z(T,T')\left[\theta\left(R\left(T;\alpha\right)-k_L\right)-\theta\left(R\left(T;\alpha\right)-k_U\right)\right] \tag{2.6}$$

Analogously to the case of the range digital call, the range contingent payoff call is a spread of contingent payoff calls and can be priced with the expression

$$RCP\left(t;T,T',\alpha,k_L,k_U\right)=CP\left(t;T,T',\alpha,k_L\right)-CP\left(t;T,T',\alpha,k_U\right) \tag{2.7}$$

2.5 Range Accrual Note

We begin the study of the range note by considering the basic single period range note, which has two cases. In the first case the valuation date falls before the initiation of the period and in the second case, the valuation date falls during the period. The second case is split into two parts: the deterministic period between the initiation of the period and the valuation date, and a truncated single period from the valuation date to the end of the period. This treatment of the single period range note then allows an easy description of the multiperiod range note.

2.5.1 Single Period Range Note

Definition: A fixed interest rate, interest rate range and period is specified on initiation of the *single period range note*. This note entitles the holder to a payment at then end of the period calculated by multiplying the fixed interest rate by the number of days during the period that a reference interest rate fell within the specified range. At the expiry of the contract the nominal is also, of course, paid back.

Valuation Date Before Period Initiation

Denoting the initiation and end of the period by T_0 and T_1 respectively, the fixed rate as R and the range note as V, the value of the range note at T_1 is

$$V\left(T_1; T_0, T_1, R, k_L, k_U\right) = \frac{RN}{D}\sum_{i=0}^{n}\left[\theta\left(R\left(T_0+i;\alpha\right)-k_L\right)-\theta\left(R\left(T_0+i;\alpha\right)-k_U\right)\right]+N \tag{2.8}$$

Where N is the nominal, D is the number of days in the year and T_0+i is the i^{th} day after T_0 with $T_0+n=T_1$. Obviously $R\left(T_0+i;\alpha\right)$ is the reference interest rate on day T_0+i, which is known at T_1, and so the summation counts the number of days in which the reference rate fell within the specified range over the period. Comparing this to (2.3) one might naively suspect that we can use digital calls as counters for the number of days and so replicate the range note with digital calls. However the payment of the digital call occurs at the same time as the reference rate is observed, thus if $k_L < R\left(T_0+i;\alpha\right) \leq k_U$, the holder of the digital call would receive one at T_0+i whereas a holder of the range note only receives this payment at T_1. But if the holder received $Z(T_0+i,T_1)$ at T_0+i, then this payment would be worth one at T_1 and we could replicate the range note. Thus we replicate the range note with the collection of range contingent payoffs $\left\{RCP\left(t; T_0+i, T_1, \alpha, k_L, k_U\right)\right\}_{i=0}^{n}$ giving

$$V\left(t; T_0, T_1, R, k_L, k_U\right) = \frac{RN}{D}\sum_{i=0}^{n} RCP\left(t; T_0+i, T_1, \alpha, k_L, k_U\right) + Z(t,T_1)N \tag{2.9}$$

for $t < T_0$.

Valuation Date During Period

We now consider the case where $T_0 < t \leq T_1$. We split the period into two parts as below

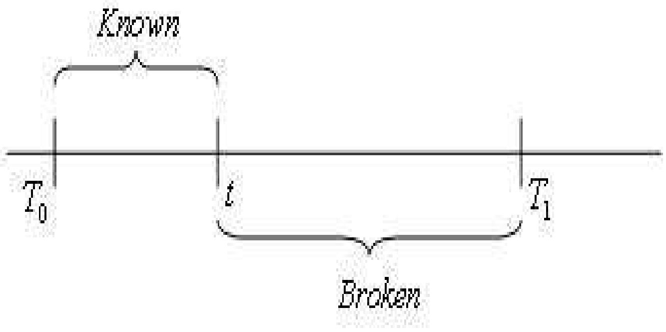

Figure 2.1: When the valuation date falls in the middle of the period, the valuation is split into two parts. The deterministic *known period* and the stochastic *broken period*

We call the period $(T_0, t]$ the *known period* as we have already observed how many days within this period will contribute to the payment at T_1, we denote this number by $H(t)$.

$$V_{Known} = \frac{RNH(t)}{D} Z(t, T_1)$$

Defining p such that $t = T_0 + p$, we then price the contribution of the *broken period* $(t, T_1]$ in exactly the same way as in(2.9) except the summation now begins from $i = p + 1$ as opposed to $i = 0$. Putting the parts together, we obtain

$$V(t; T_0, T_1, R, k_L, k_U) = \frac{RNH(t)}{D} Z(t, T_1) + \frac{RN}{D} \sum_{i=p+1}^{n} RCP(t; T_0 + i, T_1, \alpha, k_L, k_U) + Z(t, T_1) N \tag{2.10}$$

for $T_0 < t \leq T_1$.

2.5.2 Multi-Period Range Note

Definition:A *multi-period range note* is a successive series of single period range notes with interest being paid at the end of each period and the nominal payment occurring at the end of the final period.

We will only consider the case [1] where the fixed rate and strike range are constant throughout the life of the range note [2].

Define

[1]The generalisation is obvious

[2]*Range note* refers to the general case of the multi-period range note

$$V_j\left(t;T_{j-1},T_j,R,k_L,k_U\right)=\frac{RN}{D}\sum_{i=0}^{n_j}RCP\left(t;T_{j-1}+i,T_j,\alpha,k_L,k_U\right) \quad (2.11)$$

Where $T_j = T_{j-1}+n_j$. Then a $m-period$ range note V, with initiation date of first period T_0 and end date T_m of the m^{th} period, has value at date $t < T_0$ of

$$V\left(t;T_0,T_m,R,k_L,k_U\right)=\sum_{j=1}^{m}V_j\left(t;T_{j-1},T_j,R,k_L,k_U\right)+Z(t,T_m)N \quad (2.12)$$

Where V_j is given by (2.11), RCP by (2.7) and CP by (2.5).

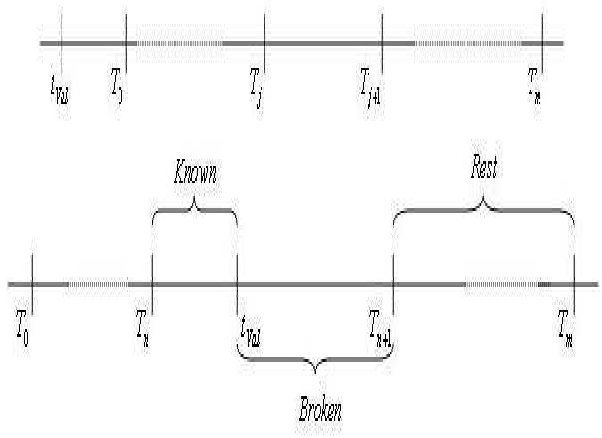

Figure 2.2: The bottom time-line illustrates the case where the evaluation date lies before the initiation of the first period while the time-line above it illustrates the case where the evaluation date falls in the middle of a period

In the case where $T_n < t \leq T_{n+1} \quad (n < m)$, we make the obvious adjustments to (2.12) and (2.10) to obtain

$$\begin{aligned}V\left(t;T_n,T_m,R,k_L,k_U\right) &= V_{n+1}\left(t;T_n,T_{n+1},R,k_L,k_U\right)+V\left(t;T_{n+1},T_m,R,k_L,k_U\right) \quad (2.13)\\ V_{n+1}\left(t;T_n,T_{n+1},R,k_L,k_U\right) &= \frac{RNH(t)}{D}Z(t,T_1)+\frac{RN}{D}\sum_{i=p+1}^{n_{n+1}}RCP\left(t;T_n+i,T_{n+1},\alpha,k_L,k_U\right) \quad (2.14)\end{aligned}$$

2.5.3 The Utility of The Range Note

To tease out the financial understanding of a range note we begin by comparing it to the most common of pure interest rate derivatives- the bond. A bond has a fixed coupon with fixed payment dates with a bullet accompanying the last coupon payment. The future cash flows of the bond are completely deterministic and the interest rate dependence enters with the present valuing of these flows [3]. This is the basis for the inverse relationship between bonds and rates. The higher the rate, the lower the discount factor and the smaller the present value of the cash flow. The differing maturity of the bonds determines the exposure of the bond to the term structure of interest rates.

A range note also entitles the holder to cash flows at known dates including a known bullet at the end. However these cash flows are stochastic depending on the daily level of some reference interest rate and are floored to be positive. This gives the holder of the range note direct exposure to the reference rate and not just to an averaging of the yield curve as in the case of bonds. In addition, the details of the stochastic dependence of the payoff are a direct play on the reference interest rate volatility where the specification of the in the money corridor allows one to make this play as fine or coarse as desired. The purchaser of an in the money range note is expecting volatility to be low. An out of the money range note pays off for high volatility and a directional move up or down. While a spread of out of the money range notes pays off under high volatility regardless of the directional move. In addition the binary, all-or-nothing nature of the daily payoff makes this a relatively cheap derivative.

[3] All pricing using the rule of present worth exposes the instrument to interest rates, it is the fact that the coupons are known which makes bonds *only* exposed to interest rates

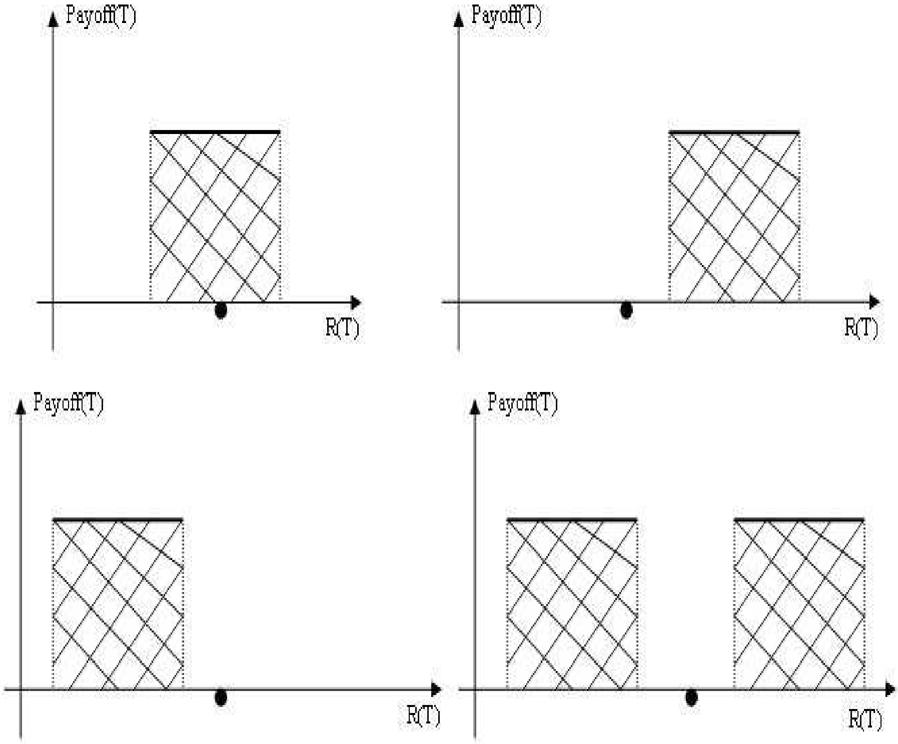

Figure 2.3: The top left graph gives the payoff for an in the money range note paying out under low volatility. The bottom right graph is a spread of out of the money range notes (graphed above and next to it) paying out under high volatility

2.6 Pricing Schematic

Below is a summary for the pricing of a range note:

$$
\begin{aligned}
&CP\left(t;T,T',\alpha,k\right)=\phi(t)\mathbb{E}^{\mathbb{Q}}\left[\frac{Z(T,T')\theta(R(T;\alpha)-k)}{\phi(T)}\right]\\
&RCP\left(t;T,T',\alpha,k_L,k_U\right)=CP\left(t;T,T',\alpha,k_L\right)-CP\left(t;T,T',\alpha,k_U\right)\\
&V_j\left(t;T_{j-1},T_j,R,k_L,k_U\right)=\frac{RN}{D}\sum_{i=0}^{n_j}RCP\left(t;T_{j-1}+i,T_j,\alpha,k_L,k_U\right)\\
&\text{For } t<T_0\\
&V\left(t;T_0,T_m,R,k_L,k_U\right)=\sum_{j=1}^{m}V_j\left(t;T_{j-1},T_j,R,k_L,k_U\right)+Z(t,T_m)N\\
&\text{And for } T_n<t\leq T_{n+1}\text{, we have}\\
&V\left(t;T_n,T_m,R,k_L,k_U\right)=V_{n+1}\left(t;T_n,T_{n+1},R,k_L,k_U\right)+V\left(t;T_{n+1},T_m,R,k_L,k_U\right)\\
&\text{Where}\\
&V_{n+1}\left(t;T_n,T_{n+1},R,k_L,k_U\right)=\frac{NH(t)}{D}Z(t,T_1)+\frac{RN}{D}\sum_{i=p+1}^{n_{n+1}}RCP\left(t;T_n+i,T_{n+1},\alpha,k_L,k_U\right)
\end{aligned}
\tag{2.15}
$$

Chapter 3

Valuing a Range Note in the South African Market

From this summary it is immediately evident that the valuation of a range note depends entirely on solving (2.5) for the contingent payoff call [1]. In this chapter we assume dynamics for forward JIBAR, choose a suitable numeraire, and evaluate (2.5).

3.1 Forward JIBAR Dynamics

We use Black's model to price the contingent payoff by assuming that forward JIBAR f_i for period t_i to $t_i + \alpha$ has marginal distribution on each day given by solving the SDE

$$df_i = f_i \sigma_i dW \tag{3.1}$$

Where dW is a brownian motion under the equivalent martingale measure and σ_i is the volatility measure.
We now solve for f_i:

$$\begin{aligned} d\left(lnf_i\right) &= \frac{1}{f_i}df - \frac{1}{2f_i^2}\left(df_i\right)^2 \\ &= -\frac{1}{2}\sigma_i^2 dt + \sigma_i dW \end{aligned}$$

Integrating from t to later time t_j $(\leq t_i)$ gives

$$f_i(t_j) = f_i(t)\exp\left(-\frac{1}{2}\sigma_i^2(t_j - t) + \sigma_i\sqrt{t_j - t}z\right); \quad z \sim N(0,1) \tag{3.2}$$

$f_i(t)$ are the forward JIBAR rates implied by the current yield curve

[1] The current yield curve is assumed to be given

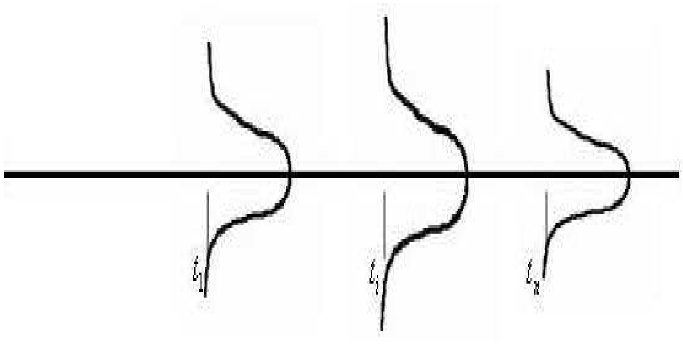

Figure 3.1: Black's model assumes that the underlying is marginally distributed according to geometric Brownian motion at discrete points in time but makes no comment about the process between these dates

$$
\begin{aligned}
Z(t, t_i)\frac{1}{(1 + \alpha f_i(t))} &= Z(t, t_i + \alpha) \\
\Rightarrow f_i(t) &= \frac{1}{\alpha}\left[\frac{Z(t, t_i)}{Z(t, t_i + \alpha)} - 1\right]
\end{aligned}
\tag{3.3}
$$

Now (3.2) holds for all points t_j where we assume f_i is marginally distributed, in our case this will just be at the payoff day t_i. Hence (3.2) will give the marginal distribution of $j(t_i; \alpha) = f_i(t_i)$, the spot JIBAR rate from t_i to $t_i + \alpha$

$$
j(t_i; \alpha) = f_i(t_i) = f_i(t) \exp\left(-\frac{1}{2}\sigma_i^2(t_i - t) + \sigma_i\sqrt{t_i - t}z\right) \tag{3.4}
$$

3.2 Evaluating a Contingent Payoff Call

We now evaluate $CP\left(t; t_i, T + \alpha, \alpha, k\right)$. We begin by rewriting the payoff in terms of the numeraire.

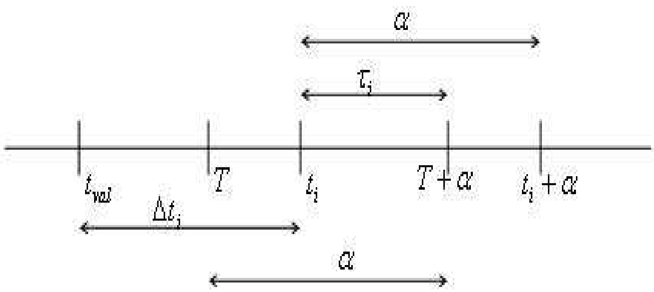

Figure 3.2: Timeline illustrating the procedure for finding the payoff as a function of the numeraire

Given $j(t_i;\alpha)$ we know that $Z(t_i, t_i+\alpha) = (1+j(t_i;\alpha)\alpha)^{-1}$. What does this imply for $Z(t_i, T+\alpha)$?
Since α and τ_i are rational numbers, we know that there exists positive integers m and n such that $n\tau_i = m\alpha$. Now if we invest n-times at the τ_i-rate and m-times at the α-rate, then be no arbitrage we must have

$$\begin{aligned}
Z(t_i, T+\alpha)^n &= Z(t_i, t_i+\alpha)^m \\
\Rightarrow Z(t_i, T+\alpha) &= Z(t_i, t_i+\alpha)^{\frac{m}{n}} \\
\Rightarrow Z(t_i, T+\alpha) &= Z(t_i, t_i+\alpha)^{\frac{\tau_i}{\alpha}} \\
\Rightarrow Z(t_i, T+\alpha) &= (1+j(t_i;\alpha)\alpha)^{-\frac{\tau_i}{\alpha}} \qquad (3.5) \\
\Rightarrow Z(t_i, T+\alpha) &= \left(1+\alpha f_i(t)\exp\left(-\frac{1}{2}\sigma_i^2(\Delta t_i) + \sigma_i\sqrt{\Delta t_i}z\right)\right)^{-\frac{\tau_i}{\alpha}} \qquad (3.6)
\end{aligned}$$

We now choose the discount factor $Z(t, t_i+\alpha)$ with fixed maturity at $t_i+\alpha$ as the numeraire. Resultantly (2.5) becomes

$$\begin{aligned}
CP\left(t; t_i, T+\alpha, \alpha, k\right) &= Z(t, t_i+\alpha)\mathbb{E}^{\mathbb{Q}}\left[\frac{Z(t_i, T+\alpha)}{Z(t_i, t_i+\alpha)}\theta\left(j(t_i;\alpha)-k\right)\right] \\
&= Z(t, t_i+\alpha)\mathbb{E}^{\mathbb{Q}}\left[(1+j(t_i;\alpha)\alpha)^{\left(1-\frac{\tau_i}{\alpha}\right)}\theta\left(j(t_i;\alpha)-k\right)\right] \\
&= Z(t, t_i+\alpha)CP_{\mathbb{E}}\left(t; t_i, T+\alpha, \alpha, k\right) \qquad (3.7)
\end{aligned}$$

Now the discount factor will be exogenously taken off the yield curve at t and, so, the challenge becomes to solve for

$$CP_{\mathbb{E}}\left(t; t_i, T+\alpha, \alpha, k\right) = \mathbb{E}^{\mathbb{Q}}\left[(1+j(t_i;\alpha)\alpha)^{\left(1-\frac{\tau_i}{\alpha}\right)}\theta\left(j(t_i;\alpha)-k\right)\right] \qquad (3.8)$$

The strategy will be to Taylor expand $(1 + j(t_i;\alpha)\alpha)^{(1-\frac{\tau_i}{\alpha})}$, interchange integral and summation before finding a series solution for the expectation. In order to implement this strategy, we require the following theorems.

3.2.1 Some Useful Theorems

Theorem (Binomial Series): *For* $|x| < 1$ *and* $0 < |\beta| < 1$*, we have*

$$(1+x)^{\beta} = \sum_{n=0}^{\infty} \frac{(-\beta)_n}{n!}(-x)^n$$

where $(-\beta)_n$ *is the pochammer number defined in terms of the gamma function as*

$$(-\beta)_n = \frac{\Gamma(n-\beta)}{\Gamma(-\beta)}$$

Theorem (Weierstrass M-test): *For each* $n \geq 0$ *let* $f_n : E \to \mathbb{R}$ *be a continuous function. Suppose that for each* $n \in \mathbb{N}$ $\exists$ $M_n \geq 0$ *such that* $|f_n(x)| \leq M_n$ $\forall x \in E$*, and also* $\sum M_n < \infty$*, then* $\sum f_n(x)$ *converges absolutely and uniformly on* E *to some continuous function* F

Theorem (Term-By-Term Integration): *Suppose for each* $n \geq 0$ *we have that* $f_n : (a, b) \to \mathbb{R}$ *is integrable over* (a, b) *and that* $\sum f_n$ *converges uniformly on (a,b) to some function* $F : (a, b) \to \mathbb{R}$*. Then* F *is integrable over* (a, b) *too and*

$$\int_a^b F(x)dx = \int_a^b \left(\sum_{n=0}^{\infty} f_n(x)\right) dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)dx$$

3.2.2 Approximating $CP_{\mathbb{E}}$

Setting $\beta_i = (1-\frac{\tau i}{\alpha})$ we see that $0 < \beta_i < 1$ as long as $\tau_i \neq 0, \alpha$. Additionally, in order to use the Binomial series we require

$$\begin{aligned} j(t_i;\alpha)\alpha &< 1 \\ \Rightarrow j(t_i;\alpha) &< \frac{1}{\alpha} \\ \Rightarrow z &< \frac{1}{\sigma_i\sqrt{\Delta t_i}}\left[-\ln(f_i(t)\alpha) + \frac{1}{2}\sigma_i^2\Delta t_i\right] = D_u \end{aligned}$$

Where we used (3.4) in the last line. Again using (3.4) we note that

$$\theta\left(j(t_i;\alpha) - k\right) = \theta\left(z - D_L\right) \tag{3.9}$$

$$\text{where} \quad D_L = \frac{1}{\sigma_i\sqrt{\Delta t_i}}\left[\ln\left(\frac{k}{f_i(t)}\right) + \frac{1}{2}\sigma_i^2\Delta t_i\right] \tag{3.10}$$

As a quick check on the limits, we note that

$$\begin{aligned} D_U &= D_L - \frac{1}{\sigma_i\sqrt{\Delta t_i}}\ln(k\alpha) & (3.11)\\ \Rightarrow \qquad D_L &< D_U \Longleftrightarrow k < \frac{1}{\alpha} & (3.12)\end{aligned}$$

Using the Binomial series we can rewrite (3.8) as

$$\begin{aligned} CP_{\mathbb{E}}\left(t; t_i, k\right) \quad = \quad & \underbrace{\int_{D_L}^{D_U} \sum_{n=0}^{\infty} \frac{(-\beta_i)_n}{\sqrt{2\pi} n!} \left(-\alpha f_i(t) \exp\left(-\frac{1}{2}\sigma_i^2(\Delta t_i) + \sigma_i\sqrt{\Delta t_i} z\right)\right)^n \exp\left(-\frac{1}{2}z^2\right) dz}_{\mathrm{I}_1} \\ & + \underbrace{\int_{D_U}^{\infty} (1 + j(t_i;\alpha)\alpha)^{\beta_i} \exp\left(-\frac{1}{2}z^2\right) \frac{dz}{\sqrt{2\pi}}}_{\text{Error Term}} \qquad (3.13)\end{aligned}$$

In order to approximate $CP_{\mathbb{E}}$ by I_1 we have to demonstrate that the error term is negligible. JIBAR is a quarterly rate implying that $\alpha = \frac{1}{4}$. Thus the exclusion of the error term involves ignoring the expectation over the range $j(t_i; 1/4) > 4$. Both history and common sense would indicate that these events truly are negligible. From an economic standpoint, before interest rates reach anywhere near 4 the mispricing of the derivative is the last of the writer's problems. This is indeed evident in the mathematical framework of the model as shown in the table below where $\mathbb{P}[z \geq D_U]$ is calculated for various choices of parameters. Even for long dated derivatives with current forward rate at 1 and high vol, we only have a 0.05 probability of the spot rate reaching 4. On the technical side this table also illustrates that the strike will, with almost certainty [2] , also be below 4 showing by (3.11) that $D_L \leq D_U$.

[2] In fact, since the strike is agreed upon before initiating deal, the writer can choose to never enter into a deal where this restriction is violated

Time(years)	Forward Rates	Volatility	Probability
1.5	8%	20%	1.42271E-58
5	8%	50%	0.0025%
1.5	30%	80%	0.0863%
5	30%	100%	1.1410%
1.5	100%	100%	4.0555%
5	100%	100%	4.1105%

Figure 3.3: This table illustrates how negligible the error term is in the approximation for $CP_{\mathbb{E}}$

Hence we will use the approximation

$$
\begin{aligned}
CP_{\mathbb{E}}\left(t; t_i, k\right) &\approx \int_{D_L}^{D_U} \sum_{n=0}^{\infty} \frac{(-\beta_i)_n}{\sqrt{2\pi}n!} \left(-\alpha f_i(t) \exp\left(-\frac{1}{2}\sigma_i^2(\Delta t_i) + \sigma_i\sqrt{\Delta t_i} z\right)\right)^n \exp\left(-\frac{1}{2}z^2\right) dz \\
&= \mathrm{I}_1
\end{aligned}
\tag{3.14}
$$

In order to interchange integral and summation, we set $f_n(z) = \frac{(-\beta_i)_n}{n!}\left(-\alpha f_i(t) \exp\left(-\frac{1}{2}\sigma_i^2(\Delta t_i) + \sigma_i\sqrt{\Delta t_i} z\right)\right)^n \exp\left(-\frac{1}{2}z^2\right)$ and, with the aid of the Weierstrass M-test, check that the conditions set out in the theorem on term -by-term integration are satisfied:

- $\Gamma(x)$ is an increasing function for $x > 2$ implying that for $n \geq 3$, $\left|\frac{(-\beta_i)_n}{n!}\right| = \left|\frac{\Gamma(n-\beta_i)}{\Gamma(-\beta_i)n!}\right|$ is a decreasing sequence in n since $\Gamma(n - \beta_i) < \Gamma(n) = (n-1)!$ and $\Gamma(-\beta_i)$ is finite for $0 < \beta_i < 1$. This together with fact that $\left|\frac{(-\beta_i)_n}{n!}\right|$ is finite for all n implies that $\left|\frac{(-\beta_i)_n}{n!}\right| \leq \gamma^* = max\{\left|\frac{(-\beta_i)_n}{n!}\right| |n = 0, 1, 2, 3\}$. Thus $\sum |f_n(z)| < \sum \gamma^* \left(\alpha f_i(t) \exp\left(-\frac{1}{2}\sigma_i^2(\Delta t_i) + \sigma_i\sqrt{\Delta t_i} z\right)\right)^n = \sum \gamma^* \nu^n < \infty$ on $z \in (D_L, D_U)$ since $\nu < 1$ on this domain. Thus by the Weierstrass M-test we can conclude that $f_n(z)$ is uniformly convergent on (D_L, D_U).

- $\int_{D_L}^{D_U} |f_n(z)| \frac{dz}{\sqrt{2\pi}} < \int_{D_L}^{D_U} \exp\left(-\frac{1}{2}z^2\right) \frac{dz}{\sqrt{2\pi}} < \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}z^2\right) \frac{dz}{\sqrt{2\pi}} < \infty$

Hence we can write

$$\mathrm{I}_1 = \sum_{n=0}^{\infty} \frac{(-\beta_i)_n}{n!} \left(-\alpha f_i(t) \exp\left(-\frac{1}{2}\sigma_i^2(\Delta t_i) \right) \right)^n \underbrace{\int_{D_L}^{D_U} \exp\left(-\frac{1}{2}z^2 + \sigma_i\sqrt{\Delta t_i}z \right) \frac{dz}{\sqrt{2\pi}}}_{\mathrm{I}^*} \tag{3.15}$$

Solving for I^* gives

$$\begin{aligned}
\mathrm{I}^* &= \exp\left(\frac{1}{2}n^2\sigma_i^2\Delta t_i\right) \int_{D_L}^{D_U} \exp\left(-\frac{1}{2}\left(z - n\sigma_i\sqrt{\Delta t_i}\right)^2\right) \frac{dz}{\sqrt{2\pi}} \\
&= \exp\left(\frac{1}{2}n^2\sigma_i^2\Delta t_i\right) \int_{D_L - n\sigma_i\sqrt{\Delta t_i}}^{D_U - n\sigma_i\sqrt{\Delta t_i}} \exp\left(-\frac{1}{2}z^2\right) \frac{dz}{\sqrt{2\pi}} \\
&= \exp\left(\frac{1}{2}n^2\sigma_i^2\Delta t_i\right) \int_{D_L^n}^{D_U^n} \exp\left(-\frac{1}{2}z^2\right) \frac{dz}{\sqrt{2\pi}} \\
&= \exp\left(\frac{1}{2}n^2\sigma_i^2\Delta t_i\right) [N(D_U^n) - N(D_L^n)] \qquad (3.16)
\end{aligned}$$

Where as usual $N(x)$ denotes the cumulative normal function. Putting (3.16), (3.15) and (3.14) together, we have for $0 < \beta_i < 1$

$$CP_{\mathbb{E}}(t; t_i, T + \alpha, \alpha, k) \approx \sum_{n=0}^{\infty} \frac{\Gamma(n - \beta_i)}{\Gamma(-\beta_i)n!} \left(-\alpha f_i(t) \exp\left(-\frac{1}{2}\sigma_i^2(\Delta t_i(1-n)) \right) \right)^n [N(D_U^n) - N(D_L^n)] \tag{3.17}$$

Where

$$\begin{aligned}
D_L &= \frac{1}{\sigma_i\sqrt{\Delta t_i}} \left[\ln\left(\frac{k}{j_t}\right) + \frac{1}{2}\sigma_i^2\Delta t_i \right] \\
D_U &= D_L - \frac{1}{\sigma_i\sqrt{\Delta t_i}} \ln(k\alpha) \\
D_L^n &= D_L - \sigma_i\sqrt{\Delta t_i} \\
D_U^n &= D_L^n - \frac{1}{\sigma_i\sqrt{\Delta t_i}} \ln(k\alpha)
\end{aligned}$$

We now consider the case $\beta_i = 1$. In this case we can solve (3.8) explicitly

$\bullet \beta_i = 1 \Longleftrightarrow \tau_i = 0$:

$$\begin{aligned}
&CP_{\mathbb{E}}(t; T + \alpha, T + \alpha, \alpha, k) = \mathbb{E}^{\mathbb{Q}}\left[(1 + j(t_i;\alpha)\alpha)\,\theta\,(j(t_i;\alpha) - k)\right] \\
&= \int_{D_L}^{\infty} \left(1 + \alpha f_i(t) \exp\left(-\frac{1}{2}\sigma_i^2(T + \alpha - t) + \sigma_i\sqrt{T + \alpha - t}z \right) \right) \exp\left(-\frac{1}{2}z^2\right) \frac{dz}{\sqrt{2\pi}} \\
&= N(-D_L) + \alpha f_i(t) \int_{D_L}^{\infty} \exp\left(-\frac{1}{2}\left(z - \sigma_i\sqrt{T + \alpha - t}\right)^2\right) \frac{dz}{\sqrt{2\pi}} \\
&= N(-D_L) + \alpha f_i(t) N(\sigma_i\sqrt{T + \alpha - t} - D_L) \qquad (3.18)
\end{aligned}$$

Where D_L is given by (3.10).

3.3 Matlab Implementation of Derived Scheme

In this section we implement the scheme set out in (2.15) with (3.8) as the expression for the contingent payoff. Note that the code assumes a flat volatility structure. If one were to generalise this, one would include a volatility curve in the body of the code and use it in an analogous fashion to the code for the term structure of forward rates taken off the discount curve.

3.3.1 The Discount Factor Function

Before we can implement the scheme proper we require a current discount curve in order to obtain

- the discounting of the expected payoff given by $CP_{\mathbb{E}}$.
- the current forward JIBAR rates which are inputs in $CP_{\mathbb{E}}$

The program implementing this will generally involve bootstrapping to obtain node points on a yield curve which will then be generated in full by some interpolation scheme. Since this is not a focus of the project, my generation of discount factors is decidedly more sloppy. My function has two matrix inputs consisting of the node discount factors and their respective dates. Arbitrary discount factors are then obtained by linearly interpolating between these points. In the function below I have used node points [3] as on Monday 3 October 2005:

```
function William=currentdiscount(T_start,T_maturity) df= [1.000000
0.999812.................... 0.342328]; T=[0............
0.002739726 13.26027397];
Znow1=interp1(T,df,T_maturity);
Znow2=interp1(T,df,T_start);
William=Znow1./Znow2;
```

3.3.2 Evaluating $CP_{\mathbb{E}}$

```
%CPe for 0<beta<1
function Martin=CPe(alpha,tau,t,vol,f, strike,N)
beta=1-tau./alpha;
Dlow=(log(strike./f)+0.5.*vol.^2.*t)./(vol.*sqrt(t));
Dup=(-log(f.*alpha)+0.5.*vol.^2.*t)./(vol.*sqrt(t));
b=0.5.*vol.^2.*t;
```

[3] As given to me by my supervisor Glenn Brickhill

```
c=vol.*sqrt(t);
a=alpha.*f;
g=gamma(-beta);
A=0;
for n=0:N
    B=-b.*(1-n);
    C= n.*c;
    A=A+ gamma(n-beta)./(g.*factorial(n))
                              .*(-a.*exp(B)).^n.*(cdf(Dup-C)-cdf(Dlow-C));
end
Martin=A;

%CpeEnd- Cpe for beta=1
function Malcolm=CPeEnd(alpha,t,vol,f, strike)
Dlow=(log(strike./f)+0.5*vol^2*t)./(vol.*sqrt(t));
Malcolm=cdf(-Dlow)+alpha.*f.*cdf(vol.*sqrt(t)-Dlow);
```

3.3.3 Evaluating $RCP_{\mathbb{E}}$

```
%RCPe for 0<beta<1
function Morris=RCPe(alpha,tau,t,vol,f, StrikeLow,StrikeUp,N)
Morris=CPe(alpha,tau,t,vol,f,StrikeLow,N)-CPe(alpha,tau,t,vol,f,StrikeUp,N);

%RCpeEnd- RCpe for beta=1
function Rachel=RCPeEnd(alpha,t,vol,f, StrikeLow,StrikeUp)
Rachel=CPeEnd(alpha,t,vol,f,StrikeLow)-CPeEnd(alpha,t,vol,f,StrikeUp);
```

3.3.4 Evaluating V_j

Denote V_j by $Vsingle$.

```
function
Arthur=Vsingle(To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N)

t_i=[To+1/360:1/360:To+alpha-1/360]; Zend=t_i+alpha;
tau=To+alpha-t_i; t=t_i-tvalue;
f=(DiscountFactor(t_i)./DiscountFactor(Zend)-1)./alpha;
fend=(DiscountFactor(To+alpha)./DiscountFactor(To+2.*alpha)-1)./alpha;
Z=DiscountFactor(Zend);
RCP=RCPe(alpha,tau,t,vol,f,StrikeLow,StrikeUp,N); RN=Z.*RCP;

Arthur=(Nominal.*rate/360).*(sum(RN)+
                DiscountFactor(To+alpha).*
                     RCPeEnd(alpha,To+alpha-tvalue,vol,fend,StrikeLow,StrikeUp));
```

3.3.5 Evaluating V for $t < T_0$

```
function
Timothy=VBefore(m,To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N)

if tvalue>To
    disp(['Error:tvalue>To. Use function V2Mid'])
else
PeriodValue=0;
for j=0:m-1
T=To+j.*alpha;
PeriodValue=PeriodValue +
                Vsingle(T,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N);
end
Timothy=PeriodValue+DiscountFactor(To+m.*alpha).*Nominal;
end
```

3.3.6 Evaluating V for $T_n < t \leq T_{n+1}$

We first calculate the value of the broken period

```
function
Allan=VBroken(To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N)

t_i=[tvalue+1/360:1/360:To+alpha-1/360]; Zend=t_i+alpha;
tau=To+alpha-t_i; t=t_i-tvalue; Z=DiscountFactor(Zend);
f=(DiscountFactor(t_i)./DiscountFactor(Zend)-1)./alpha;
fend=(DiscountFactor(To+alpha)./DiscountFactor(To+2.*alpha)-1)./alpha;
RCP=RCPe(alpha,tau,t,vol,f,StrikeLow,StrikeUp,N); RN=Z.*RCP;

Allan=(Nominal.*rate/360).*(sum(RN)+DiscountFactor(To+alpha).*
                    RCPeEnd(alpha,To+alpha-tvalue,vol,fend,StrikeLow,StrikeUp));
```

Which is used in the calculation of $Vmid$

```
function
Walter=VMid(m,Days,To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N)

if tvalue<=To
     disp(['Error: tvalue<=To. Use function V'])
else
Nodes=[To:alpha:To+(m-1)*alpha];
NodeStart=max(Nodes.*(tvalue>=Nodes));
n=(NodeStart-To)./alpha;
```

```
Broken=VBroken(NodeStart,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N);
Known=DiscountFactor(NodeStart+alpha).*Days.*rate.*Nominal/360;
Rest=VBefore(m-n-1,NodeStart+alpha,tvalue,alpha,vol
                                          ,StrikeLow,StrikeUp,Nominal,rate,N);

Walter=Broken+Known+Rest;
end
```

3.4 Graphing a Value Surface: Flat Yield Curve

The underlyings of the range note are the MANY forward rates for the reference rate on the remaining 'active' days of the note. As such the pricing of the note in a market with any realistic structure is not amenable to graphical representation. Instead we use the current yield curve for the discounting and a flat yield curve to obtain the single forward rate. The value of the range note is then plotted against this forward rate and time to obtain a surface that illustrates the fundamentals of the pricing surprisingly well.

The `matlab` code for this involved some tampering with the previous code:

• `CPe` and `RCPe` remain unchanged

• The old `DiscountFactor` is renamed `currentdiscount`

• The function `DiscountFactor` now has the flat yield as an extra argument and gives the corresponding discount curve

• `VSingle`, `VBroken`, `VBefore` and `VMid` now have this flat yield as an extra input

• The forward rates f and $fend$ appearing in `VSingle` and `VBroken` are now obtained from the new function in `DiscountFactor`

This code together with that for **section 4.3** is given in appendix A. The surface generated by `ValueSurf` is given below:

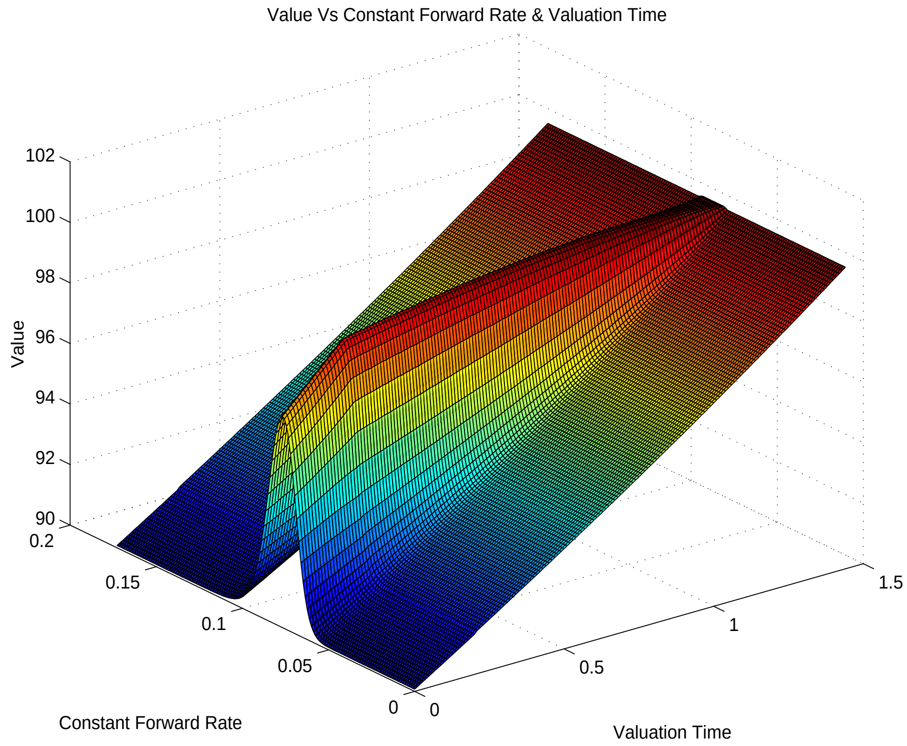

Figure 3.4: Value surface of a 5-period range note as a function of time and constant forward rate for a flat yield curve with initiation time 0.2, period 0.25, flat vol 0.1, upper strike 0.085, lower strike 0.07, nominal 100, fixed rate 0.08 and $Days$ 2

The first thing one notices is the jump at a day after the initiation date of the first period $time = 0.201$. This arises due to the fact that at this date we swap from using $VBefore$ to $VMid$ which now has a deterministic input representing the number of in the money days we have observed in the current period. As such it is natural to distinguish between these two surfaces in the analysis.

The top of the $VMid$ surface jumps above that of $VBefore$ since in the plot I set $Days$ equal to 2 throughout the calculation. For the first few days of the first period this input of 2 is greater than the probability of being in the money as predicted in $VBefore$. Accruing a payment for 2 days after the first day of the period will certainly push the surface up here! This constant input also has the effect of making the $VMid$ surface very smooth. In practice, the discrete, changing nature of $Days$ will create an effect very similar to that at $time = 0.201$ throughout the surface.

The lines of constant time for both surfaces look like a normal density function with mean near the middle of the in the money corridor flattening out very quickly at the limits of the corridor. The current forward value is the expected value of the spot at the 'pay day' meaning the closer to this mean the forward rate is, the more area of its marginal distribution will fall in the money and vice versa for the tails at the edge of the corridor limits. This example is plotted with a low volatility and so the tails are quite thin, however higher volatility inputs will result in fatter tails. The non-zero constant value the tails peteer off to is the present value of the bullet.

For lines of constant forward rate, we distinguish between the evolution of the flat base and that of the peaks. The base moves up quite linearly with time, this is just the fact that the present value of the bullet becomes greater having the net effect of adding a positive constant to the minimum value of the note to the time period before. This is true for both surfaces. Additionally, the peaks of these lines narrow as one moves forward in time. This is also true of both surfaces and occurs because the volatility of the forward rates decrease with time to maturity as in any theory with a geometric brownian motion assumption. Now the lines of constant f appearing on the peaks of both surface differ quite noticeably. They are increasing for $VBefore$ and decreasing for $VMid$. For the time period before the initiation of the first period, there are an equal number of days from which we might receive payment and the present value of these possible payments increase in time in such a way as they dominate the fluctuations in the 'possibility' of these payments. However on the $VMid$ surface these lines decrease for two reasons. During the periods, they decrease because we have a constant input of 2 for $Days$ meaning after the first two days there are no more observed in the money days. Between periods this peak will always drop as there are fewer possible payments left as compared to the period before.

Most of these properties elucidate on what we can expect for the Δ surface of the range note. This surface and its properties appear in **section 4.3**.

Chapter 4

Delta of the Range Note

In this section we solve for Δ in an analogous fashion to the numerical scheme set out for the pricing.

4.1 Evaluating Delta

The underlyings of the range note are the many forward rates $f_i(T)$. Delta is obtained by summing the partial derivatives of the range note with respect to each of these forward rates. Of course differentiating the numerical approximation for the value of the range note will not result in an accurate approximation for the first derivative. Instead we use the Leibnitz integral rule to differentiate the integral expression for the range note and then employ the same procedure as in the previous section for approximating the integrals appearing in the expression for Δ for the range note. The only dependence on $f_i(T)$ for the value of the range note is contained in the integral $CP_{\mathbb{E}}$.

Before this evaluation, we use (2.15) and (3.7) to set up the following summary for finding Δ of the range note

$$
\begin{array}{|l|}
\hline
\frac{\partial}{\partial f_i}CP\left(t;T,T',\alpha,k\right) = Z(t,t_i+\alpha)\frac{\partial}{\partial f_i}CP_{\mathbb{E}}\left(t;t_i,T+\alpha,\alpha,k\right) \\
\Delta RCP\left(t;T,T',\alpha,k_L,k_U\right) = \frac{\partial}{\partial f_i}CP\left(t;T,T',\alpha,k_L\right) - \frac{\partial}{\partial f_i}CP\left(t;T,T',\alpha,k_U\right) \\
\Delta V_j\left(t;T_{j-1},T_j,R,k_L,k_U\right) = \frac{RN}{D}\sum_{i=0}^{n_j}\Delta RCP\left(t;T_{j-1}+i,T_j,\alpha,k_L,k_U\right) \\
\text{For } t < T_0 \\
\Delta V\left(t;T_0,T_m,R,k_L,k_U\right) = \sum_{j=1}^{m}\Delta V_j\left(t;T_{j-1},T_j,R,k_L,k_U\right) \\
\text{And for } T_n < t \leq T_{n+1}\text{, we have} \\
\Delta V\left(t;T_n,T_m,R,k_L,k_U\right) = \sum_{j=n+1}^{m}\Delta V_j\left(t;T_{j-1},T_j,R,k_L,k_U\right) \\
\text{Where} \\
\Delta V_{n+1}\left(t;T_n,T_{n+1},R,k_L,k_U\right) = \frac{RN}{D}\sum_{i=p+1}^{n_{n+1}}\Delta RCP\left(t;T_n+i,T_{n+1},\alpha,k_L,k_U\right) \\
\hline
\end{array}
\tag{4.1}
$$

Hence in order to solve for Δ we need to approximate $\frac{\partial CP_{\mathbb{E}}}{\partial f_i}$.We begin by stating the *Leibnitz integral rule*:

Theorem (Leibnitz Integral Rule): For a Riemann-integrable function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ and differentiable functions a(y) and b(y), we have

$$
\frac{\partial}{\partial y}\int_{a(y)}^{b(y)} f(y,z)dz = \int_{a(y)}^{b(y)} \frac{\partial f(y,z)}{\partial y}dz + \frac{\partial f}{\partial y}f(y,a(y)) - \frac{\partial f}{\partial y}f(y,b(y))
$$

Applying the Leibnitz integral rule to (3.8) gives

$$
\begin{aligned}
\frac{\partial}{\partial f_i}CP_{\mathbb{E}}\left(t;t_i,T+\alpha,\alpha,k\right) &= \mathbb{E}^{\mathbb{Q}}\left[\frac{\partial}{\partial f_i}g(f_i,z)\theta\left(j(t_i;\alpha)-k\right)\right] \\
&\quad -\frac{\partial D_L}{\partial f_i}g(f_i,D_L) \qquad (4.2) \\
\text{Where} \qquad g(f_i,z) &= \left(1+j(t_i;\alpha)\alpha\right)^{\left(1-\frac{\tau_i}{\alpha}\right)}\frac{\exp\left(-\frac{1}{2}z^2\right)}{\sqrt{2\pi}} \\
D_L &= \frac{1}{\sigma_i\sqrt{\Delta t_i}}\left[\ln\left(\frac{k}{f_i(t)}\right)+\frac{1}{2}\sigma_i^2\Delta t_i\right]
\end{aligned}
$$

4.1.1 Solving for the Boundary Term

We first do the easy bit and solve for the boundary term

$$\begin{aligned}\frac{\partial D_L}{\partial f_i} &= -\frac{1}{f_i(t)\sigma_i\sqrt{\Delta t_i}} \qquad (4.3)\\ g(f_i, D_L) &= \left(1 + f_i(t)\alpha \exp\left(-\frac{1}{2}\sigma_i^2\Delta t_i + \sigma_i\sqrt{\Delta t_i}D_L\right)\right)^{(1-\frac{\tau_i}{\alpha})} \frac{\exp\left(-\frac{1}{2}D_L^2\right)}{\sqrt{2\pi}}\\ &= (1 + k\alpha)^{1-\frac{\tau}{\alpha}} \frac{\exp\left(-\frac{1}{2}D_L^2\right)}{\sqrt{2\pi}}\end{aligned}$$

Where the last step follows from the fact that

$$\exp\left(\sigma_i\sqrt{\Delta t}\right) = \exp\left((\ln\left(\frac{k}{f_i(t)}\right) + \frac{1}{2}\sigma_i^2\Delta t\right) = \frac{k}{f_i(t)}\exp\left(\frac{1}{2}\sigma_i^2\Delta t\right)$$

Thus

$$\frac{\partial g}{\partial f_i}g(f_i, D_L) = \frac{1}{f_i(t)\sigma_i\sqrt{\Delta t_i}}(1 + k\alpha)^{1-\frac{\tau}{\alpha}} \frac{\exp\left(-\frac{1}{2}D_L^2\right)}{\sqrt{2\pi}} \qquad (4.4)$$

4.1.2 Solving for the Integral Term

We now begin the task of approximating the expectation in (4.2). Define

$$\Delta_{CP_{\mathbb{E}}} = \mathbb{E}^{\mathbb{Q}}\left[\frac{\partial}{\partial f_i}g(f_i, z)\theta\left(j(t_i;\alpha) - k\right)\right] \qquad (4.5)$$

Differentiating g gives

$$\frac{\partial g}{\partial f_i} = \alpha\left(1 - \frac{\tau}{\alpha}\right)\left(1 + f_i(t)\alpha\exp\left(-\frac{1}{2}\sigma_i^2\Delta t_i + \sigma_i\sqrt{\Delta t_i}z\right)\right)^{-\frac{\tau}{\alpha}} \frac{\exp\left(-\frac{1}{2}\left(z - \sigma_i\sqrt{t_i}\right)^2\right)}{\sqrt{2\pi}}$$

Showing

$$\begin{aligned}\Delta_{CP_{\mathbb{E}}} &= (\alpha - \tau)\int_{D_L}^{\infty}\left(1 + f_i(t)\alpha\exp\left(-\frac{1}{2}\sigma_i^2\Delta t_i + \sigma_i\sqrt{\Delta t_i}z\right)\right)^{-\frac{\tau}{\alpha}} \frac{\exp\left(-\frac{1}{2}z^2\right)}{\sqrt{2\pi}}dz\\ &\approx \frac{(\alpha - \tau)}{\sqrt{2\pi}}\int_{D_L}^{D_u}\sum_{n=0}^{\infty}\frac{\left(\frac{\tau}{\alpha}\right)_n}{n!}\left(-f_i\alpha\exp\left(-\frac{1}{2}\sigma_i^2\Delta t_i\right)\right)^n\\ &\qquad \exp\left(\sigma_i\sqrt{\Delta t_i}zn - \frac{1}{2}\left(z - \sigma_i\sqrt{\Delta t_i}\right)^2\right)dz\\ &= \frac{(\alpha - \tau)}{\sqrt{2\pi}}\int_{D_L}^{D_u}\sum_{n=0}^{\infty}\frac{\left(\frac{\tau}{\alpha}\right)_n}{n!}(-f_i\alpha)^n\exp\left(-\frac{1}{2}\left(z - \sigma_i\sqrt{t_i}(n+1)\right)^2\right)dz \qquad (4.6)\end{aligned}$$

Where the last line follows from grouping the exponentials and then completing the square. The approximation follows from the truncation of the integral to D_u as given in (3.11) in order for the binomial series to be convergent over the integral. The justification for neglecting the rest of the domain is exactly the same as the discussion preceding **figure 3.3**. In order to interchange summation and integral we check that the conditions in the theorem on term-by-term integration are satisfied. To this end we set $h_n(z) = \frac{\left(\frac{\tau}{\alpha}\right)_n}{n!}(-f_i\alpha)^n \exp\left(-\frac{1}{2}\left(z - \sigma_i\sqrt{t_i}(n+1)\right)^2\right)$ and note

• $\Gamma(x)$ is convex on $(0,1)$ implying that $\Gamma(\tau/\alpha)$ has a minimum, call it Γ^*. In addition $\Gamma(x) \leq 1$ on $[1,2]$ and increasing on $(2,\infty)$ showing that for $n \geq 1$ we have $\Gamma(n+\tau/\alpha) \leq \Gamma(n+1) = n!$ since $\tau/\alpha < 1$. Thus for $n \geq 1$, $\frac{(\tau/\alpha)_n}{n!} = \frac{\Gamma(n+\tau/\alpha)}{\Gamma(\tau/\alpha)n!} \leq \frac{1}{\Gamma^*}$. Also $\frac{(\tau/\alpha)_n}{n!} = 1$ for n=1. Now set $\gamma^* = max\{\frac{1}{\Gamma^*}, 1\}$, then $|h_n(z)| \leq \gamma^*(f_i\alpha)^n \exp\left(-\frac{1}{2}(z - \sigma_i\sqrt{\Delta t_i}(n+1))\right) \leq \varphi^n < \infty$ on $z \in (D_L, D_U)$ since $\varphi < 1$ on this domain. Thus by the Weierstrass M-test we can conclude that $h_n(z)$ is uniformly convergent on (D_L, D_U).

• $\int_{D_L}^{D_U} |h_n(z)| \frac{dz}{\sqrt{2\pi}} < \gamma^* \int_{D_L}^{D_U} \exp\left(-\frac{1}{2}z^2\right)\frac{dz}{\sqrt{2\pi}} < \gamma^* \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2}z^2\right)\frac{dz}{\sqrt{2\pi}} < \infty \quad (\text{for } f_i\alpha < 1)$

Hence

$$\begin{aligned}
\Delta_{CP_{\mathbb{E}}} &\approx \frac{(\alpha-\tau)}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{\left(\frac{\tau}{\alpha}\right)_n}{n!}(-f_i\alpha)^n \int_{D_L}^{D_u} \exp\left(-\frac{1}{2}\left(z - \sigma_i\sqrt{\Delta t_i}(n+1)\right)^2\right) dz \\
&= \frac{(\alpha-\tau)}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{\left(\frac{\tau}{\alpha}\right)_n}{n!}(-f_i\alpha)^n \int_{D_L-\sigma_i\sqrt{\Delta t_i}(n+1)}^{D_u-\sigma_i\sqrt{\Delta t_i}(n+1)} \exp\left(-\frac{1}{2}z^2\right) dz \\
&= (\alpha-\tau) \sum_{n=0}^{\infty} \frac{\left(\frac{\tau}{\alpha}\right)_n}{n!}(-f_i\alpha)^n \left[N(D_U^{n'}) - N(D_L^{n'})\right] \\
&= (\alpha-\tau) \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{\tau}{\alpha}\right)}{\Gamma\left(\frac{\tau}{\alpha}\right)n!}(-f_i\alpha)^n \left[N(D_U^{n'}) - N(D_L^{n'})\right] \qquad (4.7)
\end{aligned}$$

Thus putting (4.4) and (4.7) together, we obtain for $t_i \neq T + \alpha$

$$\begin{aligned}
\frac{\partial}{\partial f_i} CP_{\mathbb{E}}(t; t_i, T+\alpha, \alpha, k) &\approx (\alpha-\tau) \sum_{n=0}^{\infty} \frac{\Gamma\left(n+\frac{\tau}{\alpha}\right)}{\Gamma\left(\frac{\tau}{\alpha}\right)n!}(-f_i\alpha)^n \left[N(D_U^{n'}) - N(D_L^{n'})\right] \\
&\quad - \frac{1}{f_i(t)\sigma_i\sqrt{\Delta t_i}}(1+k\alpha)^{1-\frac{\tau}{\alpha}} \frac{\exp\left(-\frac{1}{2}D_L^2\right)}{\sqrt{2\pi}} \qquad (4.8)
\end{aligned}$$

Where

$$\begin{aligned}
D_L^{n'} &= D_L - \sigma_i\sqrt{\Delta t_i}(n+1) = D_L^n - \sigma_i\sqrt{\Delta t_i} \qquad (4.9) \\
D_U^{n'} &= D_U - \sigma_i\sqrt{\Delta t_i}(n+1) = D_U^n - \sigma_i\sqrt{\Delta t_i} \qquad (4.10)
\end{aligned}$$

For the case where $\tau = 0 \Leftrightarrow t_i = T + \alpha$, we have an exact solution for $CP_{\mathbb{E}}$ and differentiating (3.18) gives

$$\begin{aligned}
\frac{\partial}{\partial f_i} CP_{\mathbb{E}}(t; T+\alpha, T+\alpha, \alpha, k) &= \frac{\partial D_L}{\partial f_i}\Big(\alpha f_i(t) N'(\sigma_i\sqrt{T+\alpha-t} - D_L) + N'(-D_L)\Big) \\
&\quad + \alpha N(\sigma_i\sqrt{T+\alpha-t}) \\
&= \frac{1}{f_i(t)\sigma_i\sqrt{\Delta t_i}}\Big(\alpha f_i(t) N'(\sigma_i\sqrt{T+\alpha-t} - D_L) + N'(-D_L)\Big) \\
&= \frac{\exp\left(-\frac{1}{2}D_L^2\right)}{f_i(t)\sigma_i\sqrt{2\pi\Delta t_i}} \\
&\quad \left[1 + \alpha f_i(t)\exp\left(-\frac{1}{2}\sigma_i^2(T+\alpha-t) - \sigma_i\sqrt{T+\alpha-t}D_L\right)\right]
\end{aligned} \tag{4.11}$$

Where the first equality follows from (4.3) and the last equality follows from rewriting $N'(\sigma_i\sqrt{T+\alpha-t} - D_L)$ in terms of $N'(-D_L)$

4.2 Matlab Implementation for Finding Δ

The above expression together with Δ-schematic (4.1) gives an algorithm for finding Δ of the range note. This section contains the implementation of this scheme in `matlab`.

4.2.1 Evaluating $\frac{\partial}{\partial f_i} CP_{\mathbb{E}}$

```
%DeltaCPe for 0<tau<1
function Meredith=DeltaCPe(alpha,tau,t,vol,f,strike,N)
Dlow=(log(strike./f)+0.5.*vol.^2.*t)./(vol.*sqrt(t));
a=tau./alpha; b=vol.*sqrt(t); c=log(strike./alpha); D=Dlow-c./b;
d=f.*alpha; g=gamma(a);
A=0;
for n=0:N
  A=A+gamma(n+a)./(g.*factorial(n)).*(-d).^n.*(cdf(D-b.*(n+1))-cdf(Dlow-b.*(n+1)));
end

Meredith=(alpha-tau).*A
    +(1./(f.*b.*sqrt(2*pi))).*(1+strike.*alpha).^(1-tau/alpha).*exp(-0.5.*Dlow.^2);

%DeltaCPe for tau=0
function Margeret=DeltaCPeEnd(alpha,t,vol,f,strike)
Dlow=(log(strike./f)+0.5.*vol.^2.*t)./(vol.*sqrt(t));
q=vol.*sqrt(t);
```

```
Margeret=exp(-0.5.*Dlow.^2)./(f.*q*sqrt(2*pi))
                        .*(1+alpha.*f.*exp(-0.5.*q.^2+q.*Dlow))+alpha.*cdf(q-Dlow);
```

4.2.2 Evaluating $\Delta RCP_{\mathbb{E}}$

```
%DeltaRCPe for 0<tau<1
function Kylie=DeltaRCPe(alpha,tau,t,vol,f,StrikeLow,StrikeUp,N)
Kylie=DeltaCPe(alpha,tau,t,vol,f,StrikeLow,N)
                                          -DeltaCPe(alpha,tau,t,vol,f,StrikeUp,N);

%DeltaRCPe for tau=0
function Ralph=DeltaRCPeEnd(alpha,t,vol,f, StrikeLow,StrikeUp)
Ralph=DeltaCPeEnd(alpha,t,vol,f,StrikeLow)-DeltaCPeEnd(alpha,t,vol,f,StrikeUp);
```

4.2.3 Evaluating ΔV_j

```
function
Rachel=Vsingle(To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N)

t_i=[To+1/360:1/360:To+alpha-1/360]; Zend=t_i+alpha;
tau=To+alpha-t_i; t=t_i-tvalue;
f=(DiscountFactor(tvalue,t_i)./DiscountFactor(tvalue,Zend)-1)./alpha
fend=(DiscountFactor(tvalue,To+alpha)./DiscountFactor(tvalue,To+2.*alpha)-1)./alpha
Z=DiscountFactor(tvalue,Zend)
DeltaRCP=DeltaRCPe(alpha,tau,t,vol,f,StrikeLow,StrikeUp,N)
DeltaRN=Z.*DeltaRCP

Rachel=
(Nominal.*rate/360).*(sum(DeltaRN)+DiscountFactor(tvalue,To+alpha)
              .*DeltaRCPeEnd(alpha,To+alpha-tvalue,vol,fend,StrikeLow,StrikeUp));
```

4.2.4 Evaluating ΔV for $t \leq T_0$

```
function
Tobias=DeltaVBefore(m,To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N)

if tvalue>To
    disp(['Error:tvalue>To. Use function DeltaVMid'])
else
PeriodValue=0;
for j=0:m-1
T=To+j.*alpha;
PeriodValue=PeriodValue
           +DeltaVsingle(T,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N);
```

```
end
end

Tobias=PeriodValue;
```

4.2.5 evaluating V for $T_n < t \leq T_{n+1}$

We first calculate the value of the broken period

```
function
Howard=VBroken(To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N)

t_i=[tvalue+1/360:1/360:To+alpha-1/360]; Zend=t_i+alpha;
tau=To+alpha-t_i; t=t_i-tvalue; Z=DiscountFactor(tvalue,Zend);
f=(DiscountFactor(tvalue,t_i)./DiscountFactor(tvalue,Zend)-1)./alpha;
fend=(DiscountFactor(To+alpha)./DiscountFactor(To+2.*alpha)-1)./alpha;
DeltaRCP=DeltaRCPe(alpha,tau,t,vol,f,StrikeLow,StrikeUp,N);
DeltaRN=Z.*DeltaRCP;

Howard=(Nominal.*rate/360).*(sum(DeltaRN)+DiscountFactor(tvalue,To+alpha)
               .*DeltaRCPeEnd(alpha,To+alpha-tvalue,vol,fend,StrikeLow,StrikeUp));
```

Which is used in the calculation of $\Delta Vmid$

```
function
Betty=DeltaVMid(m,Days,To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N)

if tvalue<=To
     disp(['Error: tvalue<=To. Use function DeltaVBefore'])
else
    Nodes=[To:alpha:To+(m-1)*alpha];
    NodeStart=max(Nodes.*(tvalue<=Nodes));
    n=(NodeStart-To)./alpha;

DeltaBroken=DeltaVBroken(NodeStart,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal
                                                                    ,rate,N);
DeltaRest=DeltaVBefore(m-n-1,NodeStart+alpha,tvalue,alpha,vol,StrikeLow,StrikeUp
                                                            ,Nominal,rate,N);

Betty=DeltaBroken+DeltaRest;
end
```

4.3 Graphing Δ: Flat Yield Curve

The Delta surface is obtained in the same manner as the value surface in **section 3.4** and its code is given in the same **appendix A**. In addition the function `SolSurfSeperate` plots line segments of the Delta and Value surfaces.

The `matlab` code for this involved some tampering with the code of the previous section:

• `DeltaCPe` and `DeltaRCPe` remain unchanged

• The old `DiscountFactor` is renamed `currentdiscount`

• The function `DiscountFactor` now has the flat yield as an extra argument and gives the corresponding discount curve

• `DeltaVSingle`, `DeltaVBroken`, `DeltaVBefore` and `DeltaVMid` now have this flat yield as an extra input

• The forward rates f and $fend$ appearing in `DeltaVSingle` and `DeltaVBroken` are now obtained from the new function in `DiscountFactor`

The surface given by `DeltaSurf` generates the Δ corresponding to the value surface given in **section 3.4**. Despite the look of it, the surface is quite neat and is as expected. Three lines of constant time for Δ are also given. These were generated by running `SolSurfSeperate`

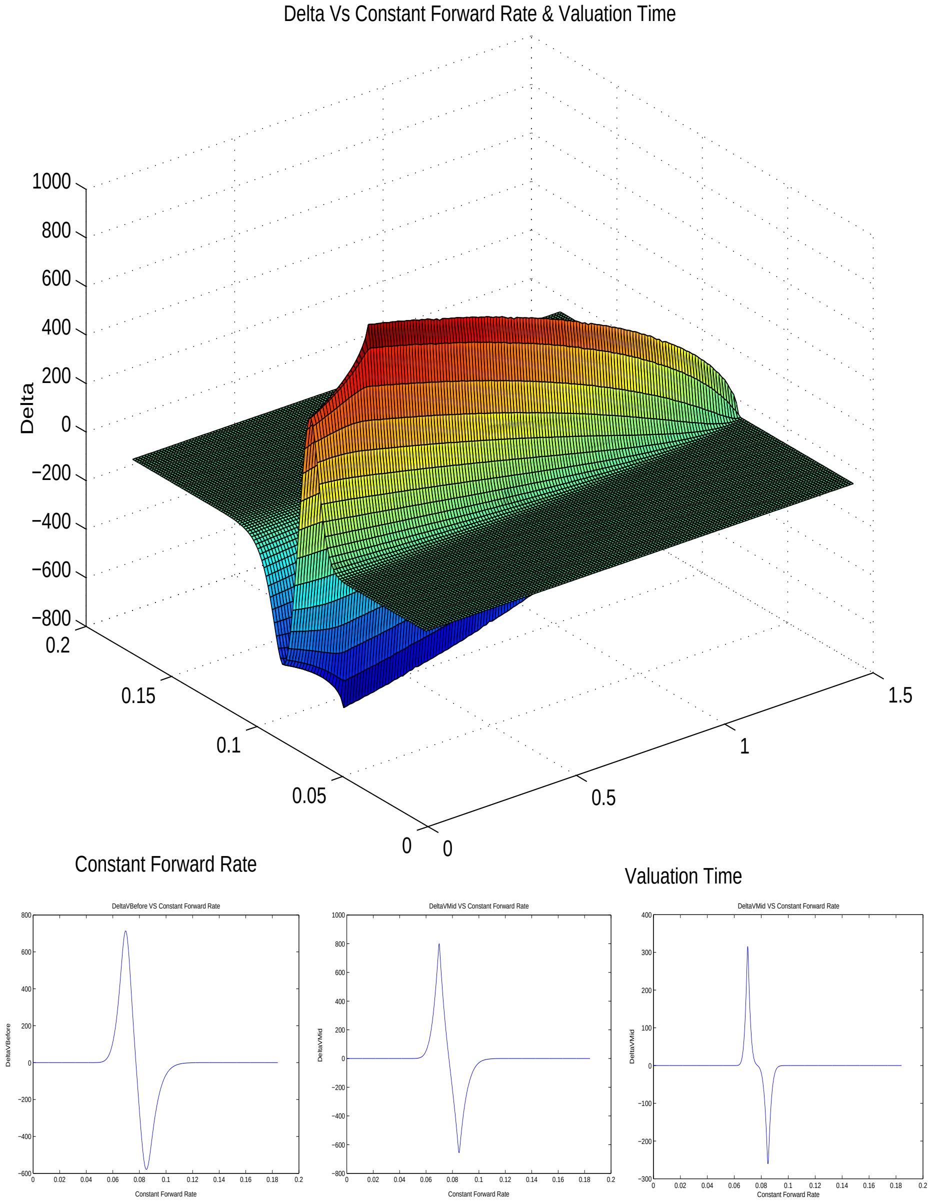

Figure 4.1: Δ-surface of a 5-period range note as a function of time and constant forward rate for a flat yield curve with initiation time 0.2, period 0.25, flat vol 0.1, upper strike 0.085, lower strike 0.07, nominal 100, fixed rate 0.08 and *Days* 2 with lines of constant time for $t = 0.12, 0.52$ and 1.3

The lines of constant t are exactly as expected. These lines on the value surface look like a normal density function. Hence we would expect the Δ of these lines to look like a derivative of the normal density function (up to some scaling factor given by the chain rule) which is exactly what we see. The additional structure can also be explained in relation to the value surface:

• The non-linear decreasing (increasing) lips of $\Delta VBefore$ can be explained by the fact that as we move forward in time the peak of the 'normal distribution' is increasing while the volatility is decreasing exaggerating the slopes on either side of the peak.

• The (symmetric) hills of the surface both narrow as time evolves. This is just a result of the fact that as we move forward in time, the 'normal distributions' narrow causing the tails to flatten earlier.

• The decreasing ridge formed by $\Delta VMid$ is an indication that as we move through time the peaks of the 'normal distributions' begin to drop in such a manner that it dominates the effect of the decreasing volatilities and, so, eases the slopes on either side of the peaks.

Chapter 5

FRAs as a Hedge Instrument for the Range Note

We begin this chapter by using FRAs to construct a hedge portfolio for the range note. We then derive an expression for the hedge slippage involved in this replication.

5.1 Creating the Hedge Portfolio

In this section we derive a hedge portfolio for the range note. As is explicit in the pricing, the range note can be decomposed into daily range contingent payoffs for all remaining 'pay days'. The forward rates beginning on each of these days over a period α are the stochastic drivers of these derivatives. Hence it seems to natural to use FRAs as the hedge instrument. The total hedge portfolio will consist of a position in each of these FRAs and a position in a riskless money market account. That is each range contingent payoff $RCP\left(t;t_i,T+\alpha,\alpha,k_L,k_U\right)$ will be hedged with a position in the FRA $U(t;t_i,t_i+\alpha)$ struck at R_i for the period t_i to $t_i+\alpha$ and a position in the money market account $M(t)=\exp(r(t-t_0)$ where t_0 is the initiation date of the hedge and r is the risk free rate. To be Δ-hedged at every moment t we require:

$$V(t) = \frac{RN}{D}\sum_i RCP\left(t;t_i,\alpha,k_L,k_U\right) = \sum_i \phi_i(t)U(t;t_i,t_i+\alpha)+\mu(t)M(t) \tag{5.1}$$

$$\Delta V(t) = \frac{RN}{D}\sum_i \Delta RCP\left(t;t_i,\alpha,k_L,k_U\right) = \sum_i \phi_i(t)\frac{\partial}{\partial f_i}U(t;t_i,t_i+\alpha) \tag{5.2}$$

Where the summation over i denotes summing over all remaining 'pay days' of the range note. $\phi^i(t)$ and $\mu(t)$ denote the position at t in the FRA and money account respectively and are the variables we wish to solve for. We begin by finding $U(t; t_i, t_i + \alpha)$ and $\frac{\partial}{\partial f_i} U(t; t_i, t_i + \alpha)$.

The value of the FRA at time t is given by

$$\begin{aligned} U(t; t_i, t_i + \alpha) &= \underbrace{Z(t, t_i) - Z(t, t_i + \alpha)}_{\text{floating leg}} - \underbrace{R_i \alpha Z(t, t_i + \alpha)}_{\text{fixed leg}} \\ &= Z(t, t_i) - Z(t, t_i) \frac{1 + R_i \alpha}{1 + \alpha f_i} \\ &= \frac{Z(t, t_i)}{1 + \alpha f_i} [1 + \alpha f_i - 1 - R_i \alpha] \\ &= \alpha \frac{Z(t, t_i)}{1 + \alpha f_i} [f_i - R_i] \end{aligned} \tag{5.3}$$

and so

$$\begin{aligned} \frac{\partial}{\partial f_i} U(t; t_i, t_i + \alpha) &= -\alpha^2 \frac{Z(t, t_i)}{(1 + \alpha f_i)^2} [f_i - R_i] + \alpha \frac{Z(t, t_i)}{1 + \alpha f_i} \\ &= \alpha \frac{Z(t, t_i)}{(1 + \alpha f_i)^2} [1 + \alpha f_i - \alpha f_i + \alpha R_i] \\ &= \alpha \frac{Z(t, t_i)}{(1 + \alpha f_i)^2} [1 + \alpha R_i] \end{aligned} \tag{5.4}$$

Putting (5.4) and (5.2) together and comparing term by term gives:

$$\begin{aligned} \phi_i(t) &= \frac{RN}{D} \frac{\Delta RCP\,(t; t_i, \alpha, k_L, k_U)}{\frac{\partial}{\partial f_i} U(t; t_i, t_i + \alpha)} \\ &= \frac{RN}{D} \frac{(1 + \alpha f_i)^2}{\alpha Z(t, t_i)[1 + \alpha R_i]} \Delta RCP\,(t; t_i, \alpha, k_L, k_U) \end{aligned} \tag{5.5}$$

Which together with (5.1) gives

$$\mu(t) = \exp(-r(t - t_0)) \left[V\,(t; T_0, T_m, R, k_L, k_U) - \sum_i \phi_i(t) U(t_i, t_i + \alpha) \right] \tag{5.6}$$

Thus the hedge portfolio $\Omega(t)$ is

$$\Omega(t) = \sum_i \phi^i(t) U(t; t_i, t_i + \alpha) + \mu(t) \exp(r(t - t_0)) \tag{5.7}$$

where $\phi_i(t)$ is given by (5.5), $U(t; t_i, t_i + \alpha)$ by (5.3) and $\mu(t)$ by (5.6).

5.2 Cost of Refinancing

The first fundamental theorem of no arbitrage pricing says that a model is arbitrage free if and only if there exists an equivalent martingale measure. Hence with every martingale measure there exists a strategy that replicates the derivative. In particular the price of the derivative is the cost of replication. This theory is developed in the world of continuous time and assumes that one continuously rebalances the hedge portfolio. In practice this is not possible and one can only rebalance the hedge in discrete time. This introduces a hedge slippage with an associated cost referred to as the cost of refinancing. This cost is the difference between the derivative and the hedge portfolio just before rebalancing. Let $\Pi(jt + \delta t)$ denote the cost of refinancing associated with the period jt to $jt + \delta t$ then

$$\begin{aligned}
\Pi(jt+\delta t) &= V(jt+\delta t) - \left[\sum_i \phi^i(jt)U(jt+\delta t; t_i, t_i+\alpha) + \mu(jt)\exp(r(jt+\delta t - t_0))\right] \\
&= V(jt+\delta t) - \exp(r(jt+\delta t))V(jt) \\
&\qquad - \sum_i \phi_i(jt)\left[U(jt+\delta t; t_i, t_i+\alpha) - \exp(r(jt+\delta t))U(jt+\delta t; t_i, t_i+\alpha)\right] \\
&= \delta V(jt+\delta t) - \sum_i \phi_i(jt)\delta U(jt+\delta t; t_i, t_i+\alpha) \qquad (5.8)
\end{aligned}$$

Where the first equality is a definition and the second follows from (5.6). Now $\delta V(jt + \delta t)$ is the difference between holding the derivative or selling it and investing the cash in a bank account over δt and, similarly, $\delta U(jt + \delta t; t_i, t_i + \alpha)$ is the difference between holding the FRA or selling it and investing the cash in a bank account over δt. This makes sense as if the derivative and the underlying both grew at the risk free rate there would be no need to adjust the hedge. In fact we would just invest the original cost of the derivative in the bank.

The total cost of refinancing the range note $\Pi(t, mt+\delta t)$ from t to $mt+\delta t$ is then obtained by summing over all of the readjustment points in this period

$$\Pi(t, mt+\delta t) = \sum_{j=1}^{m} \Pi(jt+\delta t) \qquad (5.9)$$

Appendix A

Code for Sections 3.4 and 4.3

A.1 Discount Factors

```
function William=DiscountFactor(R,T_start,T_maturity)
William=exp(-R.*(T_maturity-T_start));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function earth=currentdiscount(T_start,T_maturity)

df= [1.000000 0.999812.................. 0.342328];
T=[0............ 0.002739726 13.26027397];
Znow1=interp1(T,df,T_maturity); Znow2=interp1(T,df,T_start);
earth=Znow1./Znow2;
```

A.2 Value Functions

```
function
Arthur=Vsingle(To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N,R)

t_i=[To+1/360:1/360:To+alpha-1/360];
Zend=t_i+alpha;
tau=To+alpha-t_i;
t=t_i-tvalue;
f=(DiscountFactor(R,tvalue,t_i)./DiscountFactor(R,tvalue,Zend)-1)./alpha;
fend=(DiscountFactor(R,tvalue,To+alpha)./DiscountFactor(R,tvalue,To+2.*alpha)-1)
                                                                      ./alpha;
Z=currentdiscount(tvalue,Zend);
RCP=RCPe(alpha,tau,t,vol,f,StrikeLow,StrikeUp,N); RN=Z.*RCP;
```

```
Arthur=
(Nominal.*rate/360).*(sum(RN)+currentdiscount(tvalue,To+alpha)
                    .*RCPeEnd(alpha,To+alpha-tvalue,vol,fend, StrikeLow,StrikeUp));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function
Allan=VBroken(To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N,R)

t_i=[tvalue+1/360:1/360:To+alpha-1/360];
Zend=t_i+alpha;
tau=To+alpha-t_i;
t=t_i-tvalue;
Z=currentdiscount(tvalue,Zend);
f=(DiscountFactor(R,tvalue,t_i)./DiscountFactor(R,tvalue,Zend)-1)./alpha;
fend=(DiscountFactor(R,tvalue,To+alpha)./DiscountFactor(R,tvalue,To+2.*alpha)-1)
                                                                          ./alpha;
RCP=RCPe(alpha,tau,t,vol,f,StrikeLow,StrikeUp,N);
RN=Z.*RCP;

Allan=(Nominal.*rate/360).*(sum(RN)+currentdiscount(tvalue,To+alpha)
                   .*RCPeEnd(alpha,To+alpha-tvalue,vol,fend, StrikeLow,StrikeUp));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function
Timothy=VBefore(m,To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N,R)

if tvalue>To
    disp(['Error:tvalue>To. Use function V2Mid'])
else PeriodValue=0; for j=0:m-1 T=To+j.*alpha;
VSingle(T,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N,R);
PeriodValue=PeriodValue+Vsingle(T,tvalue,alpha,vol,StrikeLow,StrikeUp
                                                      ,Nominal,rate,N,R);
end

Timothy=PeriodValue+currentdiscount(tvalue,To+m.*alpha).*Nominal;

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function
Walter=VMid(m,Days,To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N,R)

if tvalue<=To
```

```
     disp(['Error: tvalue<=To. Use function V'])
else
Nodes=[To:alpha:To+(m-1)*alpha];
NodeStart=max(Nodes.*(tvalue>=Nodes));
n=(NodeStart-To)./alpha;

Broken=VBroken(NodeStart,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal
                                                            ,rate,N,R);
Known=currentdiscount(tvalue,NodeStart+alpha).*Days.*rate.*Nominal/360;
Rest=VBefore(m-n-1,NodeStart+alpha,tvalue,alpha,vol,
                                    StrikeLow,StrikeUp,Nominal,rate,N,R);

Walter=Broken+Known+Rest;
```

A.3 Delta Functions

```
function
Rachel=Vsingle(To,tvalue,alpha,vol,StrikeLow,StrikeUp
                                                  ,Nominal,rate,N,R)

t_i=[To+1/360:1/360:To+alpha-1/360];
Zend=t_i+alpha;
tau=To+alpha-t_i; t=t_i-tvalue;
f=(DiscountFactor(R,tvalue,t_i)./DiscountFactor(R,tvalue,Zend)-1)./alpha;
fend=(DiscountFactor(R,tvalue,To+alpha)./DiscountFactor(R,tvalue,To+2.*alpha)-1)
                                                                      ./alpha;
Z=currentdiscount(tvalue,Zend);
DeltaRCP=DeltaRCPe(alpha,tau,t,vol,f,StrikeLow,StrikeUp,N);
DeltaRN=Z.*DeltaRCP;

Rachel=(Nominal.*rate/360).*(sum(DeltaRN)+currentdiscount(tvalue,To+alpha)
                .*DeltaRCPeEnd(alpha,To+alpha-tvalue,vol,fend,StrikeLow,StrikeUp));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function
Howard=DeltaVBroken(To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N,R)

t_i=[tvalue+1/360:1/360:To+alpha-1/360];
Zend=t_i+alpha;
tau=To+alpha-t_i;
t=t_i-tvalue;
```

```
Z=currentdiscount(tvalue,Zend);
f=(DiscountFactor(R,tvalue,t_i)./DiscountFactor(R,tvalue,Zend)-1)./alpha;
fend=(DiscountFactor(R,tvalue,To+alpha)./DiscountFactor(R,tvalue,To+2.*alpha)-1)
                                                                            ./alpha;
DeltaRCP=DeltaRCPe(alpha,tau,t,vol,f,StrikeLow,StrikeUp,N);
DeltaRN=Z.*DeltaRCP;

Howard=Nominal.*rate/360).*(sum(DeltaRN)+currentdiscount(tvalue,To+alpha)
                .*DeltaRCPeEnd(alpha,To+alpha-tvalue,vol,fend,StrikeLow,StrikeUp));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

function
Tobias=DeltaVBefore(m,To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N,R)

if tvalue>To
    disp(['Error:tvalue>To. Use function DeltaVMid'])
else PeriodValue=0; for j=0:m-1 T=To+j.*alpha;
PeriodValue=PeriodValue+DeltaVsingle(T,tvalue,alpha,vol,StrikeLow,StrikeUp
                                                              ,Nominal,rate,N,R);
end
end
Tobias=PeriodValue;
```

A.4 Plotting Value Surface

```
m=5; To=0.2;
alpha=0.25;
vol=0.1;
StrikeLow=0.07;
StrikeUp=0.085;
Nominal=100;
rate=0.08;
N=8;

Tend=To+m*alpha;
TMid=[To+0.01:0.01:Tend-2/360];  %time
TBefore=[0:0.01:To];
R=[0:0.0012:0.0012*(length(TMid)+length(TBefore)-1)]; length(TMid)
length(TBefore) length(R) D=zeros(length(R),length(R));

for i=1:length(R)
```

```
    for j=1:length(TBefore)
        D(i,j)=DeltaVBefore(m,To,TBefore(j),alpha,vol,StrikeLow,StrikeUp
                                                        ,Nominal,rate,N,R(i));
    end

    for j=1:length(TMid)
    D(i,length(TBefore)+j)=DeltaVMid(m,2,To,TMid(j),alpha,vol,StrikeLow
                                                  ,StrikeUp,Nominal,rate,N,R(i));
    end
end

T=[TBefore,TMid];
surf(T,R,D)
ylabel('Constant Forward Rate')
xlabel('Valuation Time')
zlabel('Delta')
title('Delta Vs ConstantForward Rate & Valuation Time')
```

A.5 Plotting Δ-Surface

Exactly the same code is used as in previous section except now $VBefore$ is replaced with $DeltaVBefore$ and $VMid$ is replaced by $DeltaVMid$.

A.6 Plotting Lines of Constant Time

```
%Plotting Values and Deltas VS Forward for constant yield curve
%type1 denotes Before (0) or Mid (1)
%type2 denotes Value (1) Delta (2)

function shorty=solsurfseperate(type1,type2)
m=5;
To=0.2;
alpha=0.25;
vol=0.1;
StrikeLow=0.07;
StrikeUp=0.085;
Nominal=100;
rate=0.08;
N=8;

if type1==0 & type2==1
    tvalue=0.12;
```

```
        k=0;
        for j=0:0.0001:0.18
        k=k+1;
        A(k)=VBefore(m,To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N,j);
        end
        x= (1/alpha).*(exp([0:0.0001:0.18].*alpha)-1);
        plot(x,A)
        xlabel('Constant Forward Rate')
        ylabel('VBefore')
        title('VBefore VS Constant Forward Rate')
    elseif type1==0 & type2==2
        tvalue=0.12;
        k=0;
        for j=0:0.0001:0.18
        k=k+1;
        A(k)=DeltaVBefore(m,To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N,j);
        end
        x= (1/alpha).*(exp([0:0.0001:0.18].*alpha)-1);
        plot(x,A)
        xlabel('Constant Forward Rate')
        ylabel('DeltaVBefore')
        title('DeltaVBefore VS Constant Forward Rate')
    elseif type1==1 & type2==1
        tvalue=1.3;
        k=0;
        for j=0:0.0001:0.18
        k=k+1;
        A(k)=VMid(m,2,To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N,j);
        end
        x= (1/alpha).*(exp([0:0.0001:0.18].*alpha)-1);
        plot(x,A)
        xlabel('Constant Forward Rate')
        ylabel('VMid')
        title('VMid VS Constant Forward Rate')
    elseif type1==1 & type2==2
        tvalue=1.3;
        k=0;
        for j=0:0.0001:0.18
        k=k+1;
        A(k)=DeltaVMid(m,2,To,tvalue,alpha,vol,StrikeLow,StrikeUp,Nominal,rate,N,j);
        end
        x= (1/alpha).*(exp([0:0.0001:0.18].*alpha)-1);
        plot(x,A)
        xlabel('Constant Forward Rate')
```

```
    ylabel('DeltaVMid')
    title('DeltaVMid VS Constant Forward Rate')
end
```

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