Exotic equity options

Dr. Graeme West
Financial Modelling Agency
graeme@finmod.co.za

October 7, 2009

Contents

Chapter 1

Review of distributions and statistics

For this course, we will value equity options in the risk neutral world where we have

$$dS = (r - q)S\,dt + \sigma S\,dZ \tag{1.1}$$

where the time is measured in years. One of the most important factors of this formulation is that the risk free rate, the dividend yield, and the volatility are all constant. Whilst the risk free and the dividend yield assumptions are not too problematic (in an equity derivative environment), the volatility assumption is untenable. Volatility is certainly a function of time (this part is quite easy) but is also a function of how the stock price evolves: so $\sigma = \sigma(S, t)$, which is called the local volatility. Models of the volatility skew or smile are thus crucial. The development of the theory has branched into local volatility models and stochastic volatility models, with the latter now predominant theoretically but the former still in heavy use (although theoretically inferior, they are computationally almost instantaneous, whereas stochastic volatility pricing almost always reduces to Monte Carlo). The key evolution is Dupire [1994], Dupire [1997], Derman [1999], Derman and Kani [1998], Heston [1993], Hull and White [1987], Fouque et al. [2000], Hagan et al. [2002]. In all cases, vanilla options and the vanilla skew are used to calibrate the model, which is then used for pricing of exotic options. But, for the rest of this course, we will assume that volatility is constant, or (at the worst) that it has a term structure. Only in some specific instances will we allow volatility to be dependent on the strike or on the evolution of spot, and we don't allow for jumps in the stock price (the stock price is a diffusion).

1.1 Distributional facts

A basic statistical result we shall use repeatedly is that if the random variable Z has probability density function f, and g is a suitably defined function then

$$\mathbb{E}\left[g(Z)\right] = \int f(s)g(s)\,ds \tag{1.2}$$

where the integration is done over the domain of f. This allows us to work out $\mathbb{E}\left[Z\right]$ and $\mathbb{E}\left[Z^2\right]$ for example, by putting $g(s) = s$ and $g(s) = s^2$ respectively. This result is known as 'the Law of the Unconscious Statistician'.

Now, note from statistics that if $X = \ln W \sim \phi(\Psi, \Sigma)$ [1] then the relevant probability density functions are

$$f_X(x) = \frac{1}{\sqrt{2\pi\Sigma}} \exp\left[-\tfrac{1}{2}\frac{(x-\Psi)^2}{\Sigma}\right] \tag{1.3}$$

$$f_W(x) = \frac{1}{\sqrt{2\pi\Sigma}x} \exp\left[-\tfrac{1}{2}\frac{(\ln x-\Psi)^2}{\Sigma}\right] \tag{1.4}$$

Of course the domain for f_X is $\mathbb{R}$ while the domain for f_W is $(0,\infty)$.
In the risk neutral formulation above, by Itô's lemma

$$X := \ln\left(\frac{S(T)}{S(t)}\right) \sim \phi\left(\left(r-q-\frac{\sigma^2}{2}\right)\tau, \sigma^2\tau\right) \tag{1.5}$$

Note now that $S(T) = S(t)e^X$: a very useful representation for European derivatives. Let

$$m_\pm = r - q \pm \frac{\sigma^2}{2} \tag{1.6}$$

So $X \sim \phi(m_-\tau, \sigma^2\tau)$ and so the probability density function for X is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}} \exp\left[-\tfrac{1}{2}\frac{(x-m_-\tau)^2}{\sigma^2\tau}\right] \tag{1.7}$$

and the probability distribution for $S(T)$ is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}x} \exp\left[-\tfrac{1}{2}\frac{(\ln x - \ln S(t) - m_-\tau)^2}{\sigma^2\tau}\right] \tag{1.8}$$

Now if $\ln Y \sim \phi(\Psi, \Sigma)$ then for $k > 0$

$$\begin{aligned}\mathbb{E}[Y^k] &= \frac{1}{\sqrt{2\pi\Sigma}} \int_{-\infty}^{\infty} e^{kx} \exp\left[-\tfrac{1}{2}\frac{(x-\Psi)^2}{\Sigma}\right] dx \\ &= \exp\left(k\Psi + \tfrac{1}{2}k^2\Sigma\right) \end{aligned} \tag{1.9}$$

which will be crucial in Chapter 10. Thus in the above risk neutral setting we have

$$\mathbb{E}_t^{\mathbb{Q}}\left[S(T)^k\right] = S(t)^k \exp\left(\left(k(r-q) + \tfrac{1}{2}(k^2-k)\sigma^2\right)\tau\right) \tag{1.10}$$

The first derivative of the cumulative normal

This is the closed form formula:

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{1.11}$$

The second derivative of the cumulative normal

This is again, given by a closed form formula:

$$N''(x) = -xN'(x) \tag{1.12}$$

[1] By this we mean that the mean is Ψ and the variance is Σ. This could apply to more than one dimension too, in which case Ψ would be the mean vector and Σ the covariance matrix. Furthermore, in general we reserve the symbol σ for the annualised volatility, also known as the volatility measure, and do not use it as the standard deviation of some distribution.

I admit this is a change of notation for us, so there could be some residual errors in the notation in this document. I welcome corrections.

The inverse of the cumulative normal

Given an input y, the Inverse Standard Normal Integral gives the value of x for which $N(x) = y$, where $N(\cdot)$ denotes the Cumulative Standard Normal Integral.

The Moro transform Moro [February 1995] to find this function is the most well known algorithm. Having the ability to generate normally distributed variables from a (quasi) random uniform sample is clearly important in work involving any Monte Carlo experiments, and the Moro transformation is fast and accurate to about 10 decimal places.

For another approach, we can use our existing cumulant function and any version of Newton's method. As pointed out in Acklam [2004], having a double precision function has some rather pleasant spin-offs. Given a function that can compute the normal cumulative distribution function to double precision, the Moro approximation of the inverse normal cumulative distribution function can be refined to full machine precision, by a fairly straightforward application of Newton's method. In fact, higher degree methods such as Newton's second order method (sometimes called the Newton-Bailey method) or a third order method known as Halley's method will be the fastest, and are very amenable here, because the Gaussian function is so easily differentiated over and over - see Acklam [2004] and Acklam [2002].

The Newton-Bailey method would be as follows:

$$
\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n) - y}{f'(x_n) - \frac{(f(x_n)-y)f''(x_n)}{2f'(x_n)}} \\
&= x_n - \frac{f(x_n) - y}{f'(x_n) + \frac{(f(x_n)-y)x_n f'(x_n)}{2f'(x_n)}} \\
&= x_n - \frac{f(x_n) - y}{f'(x_n) + \frac{1}{2}(f(x_n) - y)x_n}
\end{aligned}
$$

Earlier versions of excel had an absurd error in the NORMSINV function: it would return impossible values for inputs within 0.0000003 of 1 or 0 respectively. Given that such values close to 0 or 1 on occasion are provided by uniform random number generators, this approach is to be avoided. Also note that the random number generator rand()/rnd() in excel/vba is absurd as it can (and does) return the value 0 and 1. This will cause either your own inverse function, or NORMSINV, to fail.

1.2 Risk Neutral Probabilities

We can speed up and simplify the calculation of the risk-neutral probabilities in option premium formulae. As usual in option pricing, we have

$$
\begin{aligned}
\tau &= T - t \\
d_{\pm} &= \frac{\ln \frac{f}{K} \pm \frac{1}{2}\sigma^2 \tau}{\sigma\sqrt{\tau}}
\end{aligned}
$$

where f denotes the forward level for spot-type options and the futures level for options involving futures.

Certain special cases apply, where the formula does not make sense in a pure sense, but can be made sense of mathematically by taking limits. This occurs if any of forward/future, strike, term or volatility are zero. The appropriate outcome in these cases (in the sense of a limit) is determined by testing:

- when the strike K is zero, $d_{\pm} = \infty$ which will give $N(d_{\pm}) = 1$ and $N'(d_{\pm}) = 0$,
- when f is zero, $d_{\pm} = -\infty$ which will give $N(d_{\pm}) = 0$ and $N'(d_{\pm}) = 0$,

- when either term or volatility are zero, and f is greater than the strike, $d_{\pm} = \infty$ which will give $N(d_{\pm}) = 1$ and $N'(d_{\pm}) = 0$,
- when either term or volatility are zero, and f is less than the strike, $d_{\pm} = -\infty$ which will give $N(d_{\pm}) = 0$ and $N'(d_{\pm}) = 0$.

1.3 Bivariate cumulative normal

The probability density function of the bivariate normal distribution is

$$\phi_2(X, Y, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[\frac{-(X^2 - 2\rho XY + Y^2)}{2(1-\rho^2)}\right] \tag{1.13}$$

The cumulative bivariate normal distribution is the function

$$N_2\left(x,\ y,\ \rho\right) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{x} \int_{-\infty}^{y} \exp\left[\frac{-(X^2 - 2\rho XY + Y^2)}{2(1-\rho^2)}\right]\ dY\ dX \tag{1.14}$$

Again, approximations are required. The most common algorithm is that of Drezner [1978], which appears in both [Hull, 2002, Appendix 12C] and in [Haug, 1998, Appendix A.2], for example.

We have adapted one of the algorithms from Genz [2004], namely, the modification of the algorithm of Drezner and Wesolowsky [1989]. This algorithm tests against the previous independent implementations, and it can be verified using numerical integration that it is accurate to at least 14 decimal places.

Adaptation was needed because the algorithm calculated the complementary probability that $X \geq x$, $Y \geq y$ given the correlation coefficient. The algorithm has been adapted to return the more usual probability that $X \leq x$, $Y \leq y$.

Limiting cases are important for the bivariate cumulative normal. Note that in the sense of a limit

$$N_2(x, y, 1) = N(\min(x, y)) \tag{1.15}$$

$$N_2(x, y, -1) = \begin{cases} 0 & \text{if} \quad y \leq -x \\ N(x) + N(y) - 1 & \text{if} \quad y > -x \end{cases} \tag{1.16}$$

We have that

$$\begin{aligned} \frac{\partial}{\partial x} N_2\left(x,\ b,\ \rho\right) &= \frac{\partial}{\partial x} \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{x} \int_{-\infty}^{b} \exp\left[\frac{-(X^2 - 2\rho XY + Y^2)}{2(1-\rho^2)}\right]\ dY\ dX \\ &= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{b} \exp\left[\frac{-(x^2 - 2\rho xY + Y^2)}{2(1-\rho^2)}\right]\ dY \\ &= N'(x) N\left(\frac{b - \rho x}{\sqrt{1-\rho^2}}\right) \end{aligned} \tag{1.17}$$

and hence by the Fundamental Theorem of Calculus

$$\int_{-\infty}^{a} N'(x) N\left(\frac{b - \rho x}{\sqrt{1-\rho^2}}\right)\ dx = N_2(a, b, \rho)$$

by manipulating with the constants we get

$$\int_{-\infty}^{a} N\left(K + Lx\right) N'(x)\ dx = N_2\left(a, \frac{K}{\sqrt{L^2+1}}, \frac{-L}{\sqrt{L^2+1}}\right) \tag{1.18}$$

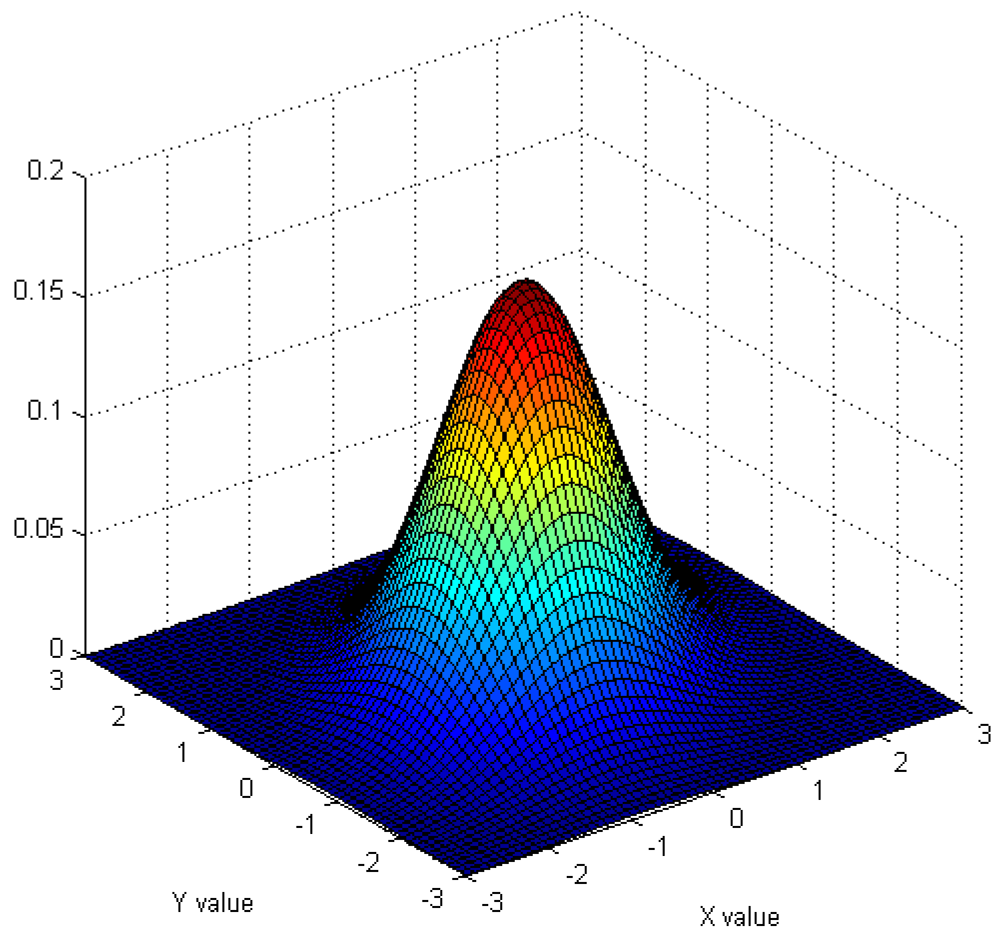

Figure 1.1: The bivariate normal pdf

It follows by completing the square from this that

$$\int_{-\infty}^{a} e^{Ax} N\left(K + Lx\right) N'(x)\, dx = e^{\frac{A^2}{2}} N_2\left(a - A,\ \frac{K + AL}{\sqrt{L^2+1}}, \frac{-L}{\sqrt{L^2+1}}\right) \tag{1.19}$$

1.4 Trivariate cumulative normal

The cumulative trivariate normal distribution is the function

$$N_3(x_1, x_2, x_3, \Sigma) = \frac{1}{(2\pi)^{3/2}\sqrt{|\Sigma|}} \int_{-\infty}^{x_1}\int_{-\infty}^{x_2}\int_{-\infty}^{x_3} \exp\left(-\tfrac{1}{2}\underline{X}'\Sigma^{-1}\underline{X}\right) dX_3\, dX_2\, dX_1 \tag{1.20}$$

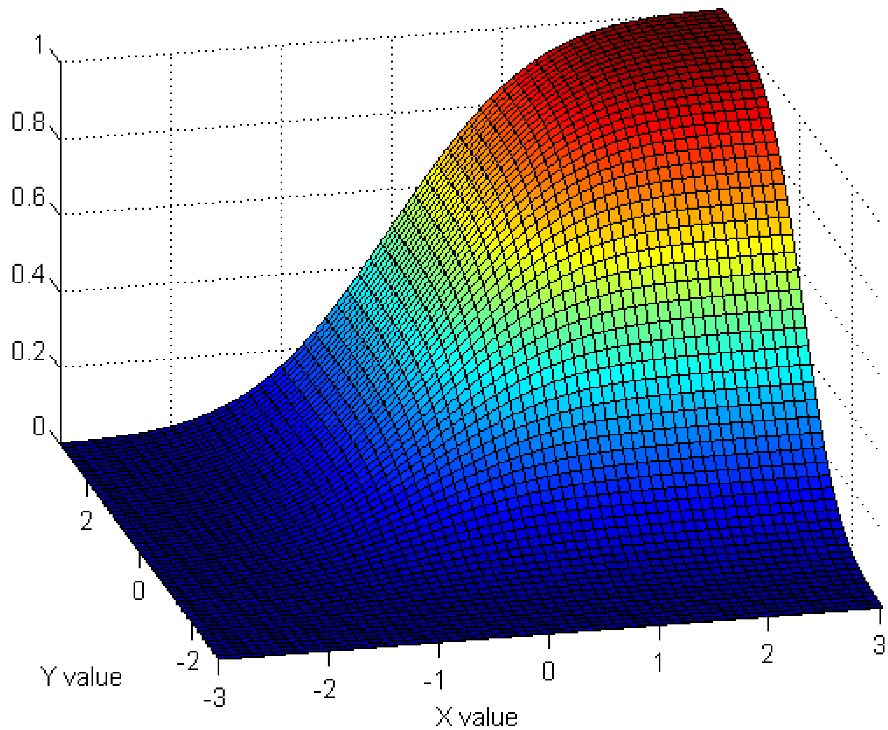

Figure 1.2: The bivariate cumulative normal function, $\rho = 50\%$

where Σ is the correlation matrix between standardised (scaled) variables X_1, X_2, X_3, and $|\cdot|$ denotes determinant. Denote by $N_3(x_1, x_2, x_3, \rho_{21}, \rho_{31}, \rho_{32})$ the function $N_3(x_1, x_2, x_3, \Sigma)$ where $\Sigma = \begin{bmatrix} 1 & \rho_{21} & \rho_{31} \\ \rho_{21} & 1 & \rho_{32} \\ \rho_{31} & \rho_{32} & 1 \end{bmatrix}$.
Again, approximations are required. Code for the trivariate cumulative normal is not generally available. There are a few highly non-transparent publications, for example Schervish [1984], but this code is known to be faulty. We have used the algorithm in Genz [2004]. This has required extensive modifications because the algorithms are implemented in Fortran, using language properties which are not readily translated. The function in Genz [2004] returns the complementary probability, again, we have modified to return the usual probability that $X_i \le x_i$ $(i = 1, 2, 3)$ given a correlation matrix. Again, it is claimed that this algorithm is double precision; high accuracy (of our vb and c++ translations) has been verified by testing against Niederreiter quasi-Monte Carlo integration (using the Matlab algorithm qsimvn.m, also at the website of Genz).
As before, one can show that

$$N_3(x_1, x_2, x_3, \Sigma) = \int_{-\infty}^{x_3} N'(x) N_2 \left(\frac{x_1 - \rho_{13} x}{\sqrt{1 - \rho_{13}^2}}, \frac{x_2 - \rho_{23} x}{\sqrt{1 - \rho_{23}^2}}, \frac{\rho_{12} - \rho_{13}\rho_{23}}{\sqrt{1 - \rho_{13}^2}\sqrt{1 - \rho_{23}^2}} \right) dx \tag{1.21}$$

Many of the issues surrounding developing robust code for these cumulative functions are discussed in West [2005].

1.5 Exercises

1. Write vba code for the Newton-Bailey method of finding the cumnorm inverse function. Use the double precision cumnorm function provided. Use 'newx' below as your first estimate, where 'y' is the input:

```
r = Sqr(-2 * Log(Min(y, 1 - y)))
newx = r - (2.515517 + 0.802853 * r + 0.010328 * r ^ 2) /
           (1 + 1.432788 * r + 0.189269 * r ^ 2 + 0.001308 * r ^ 3)
If y < 0.5 Then newx = -newx
```

2. Show that if S is subject to GBM with drift μ and volatility σ,

$$\mathbb{E}[S(T)^k] = S(t)^k \exp\left(\left(k\mu + \tfrac{1}{2}(k^2 - k)\sigma^2\right)(T - t)\right)$$

3. Formally verify (1.15) and (1.16).

4. Verify (1.17), (1.18) and (1.19).

5. Find the integral $\int_\alpha^\infty \cdots$ in place of (1.18).

6. (exam 2004) Consider the bivariate normal cumulative function $N_2(x, y, \rho)$. Recall this is the probability that $X \leq x$, $Y \leq y$ where X and Y are normally distributed variables which are correlated with correlation coefficient ρ. So

$$N_2(x, y, \rho) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(X, Y, \rho)\, dY\, dX$$

where f is the relevant probability density function. Let $M_2(x, y, \rho)$ be the complementary probability i.e. it is the probability that $X \geq x$, $Y \geq y$. Also $N(\cdot)$ is the usual cumulative normal function. Prove that

$$N_2(x, y, \rho) = M_2(x, y, \rho) + N(x) + N(y) - 1$$

(Think before you dive in headfirst. Very simple, elegant proofs are possible.)

Chapter 2

Structures

Any piecewise linear payoff can be decomposed into some linear combination of (calls, puts,) asset or nothing and cash or nothing options. Although there are theorems that deal with this, their notational complexity conceals the fact that the procedure one needs to invoke is fairly routine. First, we will see some of the ideas in play in typical option payoff profiles.

2.1 Spreads

A spread (vertical) has options at two strikes at the same expiry date on the same stock. The options are either both calls or both puts, with one long and the other short.

- A bull call spread is long the call at the lower strike and short the call at the higher strike.
- A bear call spread is short the call at the lower strike and long the call at the higher strike.
- A bull put spread is long the put at the lower strike and short the put at the higher strike.
- A bear put spread is short the put at the lower strike and long the put at the higher strike.

2.2 Collars

Suppose we have a long position in stock. We might want to avoid massive losses in the event that the stock price falls dramatically by buying an out the money put. Rather than paying for the put, we sell an out the money call to the same counterparty. This structure is called a collar. To emphasise that there is no premium, it is sometimes called a zero-cost collar.
Similarly if we have a short position in the stock we might go long an out the money call and short an out the money put.
Such a zero-cost collar might be called a range forward, a cylinder or a tunnel.

2.3 Straddles and strangles

- A straddle is long 1 call and long 1 put at the same strike price and expiration and on the same stock.

- A strangle is long 1 call at a higher strike and long 1 put at a lower strike in the same expiration and on the same stock.

Such long positions makes money if the stock price moves up or down well past the strike prices of the strangle. Long straddles and strangles have limited risk but unlimited profit potential.
Such short positions makes money if the stock price stays at or about the strike(s). Short straddles and strangles have unlimited risk and limited profit potential.

2.4 Butterflies and condors

- A butterfly is long a call at strike X_1, short two calls at X_2, and long a call at X_3, with $X_3 - X_2 = X_2 - X_1$.
- A condor (wingspread) has options at four strikes, with the same distance between the each wing strike and the lower or higher of the body strikes. Thus, a call is long a call at strike X_1, short one call at X_2, short a call at X_3, and long a call at X_4, with $X_4 - X_3 = X_2 - X_1$ and $X_3 > X_2$.

The identical structure can be manufactured with puts instead of calls!

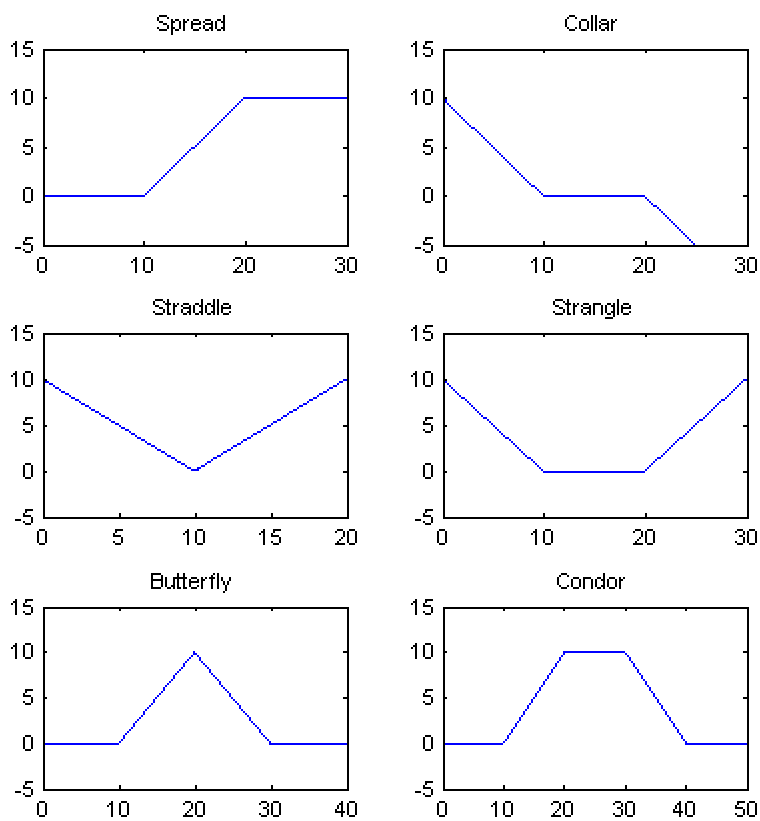

Figure 2.1: The terminal payoff of some of the structures discussed here

Chapter 3

Review of vanilla option pricing

3.1 Deriving the Black-Scholes formula

By the principle of risk-neutral valuation, the value of a European call option is

$$V = e^{-r\tau}\,\mathbb{E}_t^{\mathbb{Q}}\left[\max(Se^X - K, 0)\right] \tag{3.1}$$

where X has the meaning of (1.5). We now calculate:

$$\begin{aligned}
V &= e^{-r\tau}\,\mathbb{E}_t^{\mathbb{Q}}\left[\max(Se^X - K, 0)\right] \\
&= e^{-r\tau}\frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}}\int_{-\infty}^{\infty}\max(Se^x - K, 0)\exp\left[-\tfrac{1}{2}\left(\frac{x - m_-\tau}{\sigma\sqrt{\tau}}\right)^2\right]dx \\
&= e^{-r\tau}\frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}}\int_{\ln\frac{K}{S}}^{\infty}(Se^x - K)\exp\left[-\tfrac{1}{2}\left(\frac{x - m_-\tau}{\sigma\sqrt{\tau}}\right)^2\right]dx \\
&= e^{-r\tau}S\frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}}\int_{\ln\frac{K}{S}}^{\infty}e^x\exp\left[-\tfrac{1}{2}\left(\frac{x - m_-\tau}{\sigma\sqrt{\tau}}\right)^2\right]dx \\
&\quad - e^{-r\tau}K\frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}}\int_{\ln\frac{K}{S}}^{\infty}\exp\left[-\tfrac{1}{2}\left(\frac{x - m_-\tau}{\sigma\sqrt{\tau}}\right)^2\right]dx
\end{aligned}$$

Now, for the first integral, we complete the square:

$$\begin{aligned}
x - \tfrac{1}{2}\left(\frac{x - m_-\tau}{\sigma\sqrt{\tau}}\right)^2 &= x - \tfrac{1}{2}\frac{x^2 - 2m_-\tau x + m_-^2\tau^2}{\sigma^2\tau} \\
&= -\tfrac{1}{2}\frac{x^2 - 2m_-\tau x - 2x\sigma^2\tau + m_-^2\tau^2}{\sigma^2\tau} \\
&= -\tfrac{1}{2}\frac{x^2 - 2m_+\tau x + m_-^2\tau^2}{\sigma^2\tau} \\
&= -\tfrac{1}{2}\frac{(x - m_+\tau)^2 - m_+^2\tau^2 + m_-^2\tau^2}{\sigma^2\tau} \\
&= -\tfrac{1}{2}\left(\frac{x - m_+\tau}{\sigma\sqrt{\tau}}\right)^2 + (r - q)\tau
\end{aligned}$$

so

$$
\begin{aligned}
V &= e^{-q\tau} S \frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}} \int_{\ln\frac{K}{S}}^{\infty} \exp\left[-\tfrac{1}{2}\left(\frac{x - m_+\tau}{\sigma\sqrt{\tau}}\right)^2\right] dx \\
&\quad - e^{-r\tau} K \frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}} \int_{\ln\frac{K}{S}}^{\infty} \exp\left[-\tfrac{1}{2}\left(\frac{x - m_-\tau}{\sigma\sqrt{\tau}}\right)^2\right] dx \\
&= e^{-q\tau} S N\left(\frac{m_+\tau - \ln\frac{K}{S}}{\sigma\sqrt{\tau}}\right) - e^{-r\tau} K N\left(\frac{m_-\tau - \ln\frac{K}{S}}{\sigma\sqrt{\tau}}\right) \\
&= e^{-q\tau} S N(d_+) - e^{-r\tau} K N(d_-)
\end{aligned}
$$

where the meaning of d_+ and d_- will be established now.
The put formula follows by put-call parity, or by mimicking the argument.

3.2 Vanilla pricing methods for equity options

Note that in all cases

$$
\begin{aligned}
V &= \xi\eta[\mathbb{f}N(\eta d_+) - KN(\eta d_-)] && (3.2) \\
d_\pm &= \frac{\ln(\mathbb{f}/K) \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} && (3.3)
\end{aligned}
$$

where

- ξ is $e^{-r\tau}$ for an European Equity Option and for Standard Black, and 1 for SAFEX Black Futures Options and SAFEX Black Forward Options.
- $\eta = 1$ for a call and $\eta = -1$ for a put,
- $\mathbb{f} = f = Se^{(r-q)\tau}$ is the forward value for an European Equity Option and for SAFEX Black Forward Options, and $\mathbb{f} = F$ is the futures value for Standard Black and SAFEX Black Futures Options.

3.3 A more general result

In full generality, we have the following result.

Lemma 3.3.1. Suppose we have a vanilla European call or put on a variable Y, strike K, where the terminal value of Y is lognormally distributed, $\log Y \sim \phi(\Psi, \Sigma)$. Then the option price is given by

$$
\begin{aligned}
V_\eta &= e^{-r\tau}\eta\left[e^{\Psi + \frac{1}{2}\Sigma} N(\eta d_+) - KN(\eta d_-)\right] && (3.4) \\
d_+ &= \frac{\Psi + \Sigma - \log K}{\sqrt{\Sigma}} && (3.5) \\
d_- &= \frac{\Psi - \log K}{\sqrt{\Sigma}} && (3.6)
\end{aligned}
$$

Check that the Black-Scholes formula follows as a special case of this, and be able to prove this result. (It is in the tutorial. Simply follow the scheme already seen for Black-Scholes.)

3.4 Implied volatility

For any of the 4 option types we will on occasion know all of the inputs except the volatility, and know the premium, and require the volatility that, when input, will return the correct premium. Such a volatility is known as the implied volatility. It can be found using the Newton-Rhapson method, although one has to be careful, because an injudicious seed value will cause this method to not converge. In Manaster and Koehler [1982], a seed value of the implied volatility is given which guarantees convergence.

The argument in Manaster and Koehler [1982] is unnecessarily complicated, and can easily be understood as follows: premium as a function of volatility is an increasing function, bounded below by the intrinsic value and above by the price of the underlying. It is initially convex up and subsequently convex down. Thus, choosing the point of inflection as the seed value, guarantees convergence, no matter which way the iteration, which will be monotone and quadratic in speed, will go. By simple calculus, one finds this point of inflection, for any of the four methods, to be

$$\sigma = \sqrt{\frac{2}{\tau}\left|\ln\frac{f}{K}\right|} \tag{3.7}$$

However, note that this method fails outright if the option is at the money forward.

An alternative to use the first estimate of Corrado and Miller [1996], modified to ensure valid computation. This estimate is the root of a quadratic, but a naïve application will run into the problem of having complex roots. Thus, a first estimate which is always valid is:

$$\sigma = \frac{\sqrt{2\pi}}{\xi(f+K)\sqrt{\tau}}\left[V - \frac{\xi\eta(f-K)}{2} + \sqrt{\max\left(0, \left(V - \frac{\xi\eta(f-K)}{2}\right)^2 - \frac{(\xi(f-K))^2}{\pi}\right)}\right] \tag{3.8}$$

The code will then expand this point to an interval in which the root must lie, and then use Brent's algorithm.

3.5 Calculation of forward parameters

Forward quantities are calculated as follows:

$$r(0;T_1,T_2) = \frac{r_2T_2 - r_1T_1}{T_2 - T_1} \tag{3.9}$$

$$q(0;T_1,T_2) = \frac{q_2T_2 - q_1T_1}{T_2 - T_1} \tag{3.10}$$

$$\sigma(0;T_1,T_2) = \sqrt{\frac{\sigma_2^2T_2 - \sigma_1^2T_1}{T_2 - T_1}} \tag{3.11}$$

where time is measured in years. Alternatively, for dividends, we may simply calculate the forward values or the present value of the forward values. For the volatility, this is the at the money volatility. Inclusion of the skew is always tricky and requires additional assumptions.

3.6 Exercises

1. Repeat the derivation of the Black-Scholes formula, this time for puts.

2. Prove Lemma 3.3.1.

3. Verify that, with the usual notation, $f\, N'(d_1) = K\, N'(d_2)$. Torturous, long, solutions are problematic. That does not mean leave out details!

4. (a) Make sure your cumnorm function is working. Approximately, on what domain does it return values which are different from 0 or 1? Why is this not the whole real line?
   (b) Write a d_1 and a d_2 function. Be sure to accommodate the special cases discussed in §1.2.
   (c) Write a SAFEX Black option pricing function (inputs F, K, σ, valuation date, expiry date and style).
   (d) Make sure that the function works for the special cases already discussed. This work should be done by the d_i functions, not by the option pricing functions.
   (e) Draw graphs of the option values for varying spot/future and varying time to expiry.
   (f) Extend to a Black-Scholes option pricing function (inputs S, r, q, K, σ, valuation date, expiry date and style).

5. (exam 2004) A supershare option entitles the holder to a payoff of $\frac{S(T)}{X_L}$ if $X_L \leq S(T) \leq X_H$, and 0 otherwise. The price of a supershare option is given by

$$
\begin{aligned}
V &= \frac{S}{X_L} e^{-q\tau} [N(d_1) - N(d_2)] \\
d_1 &= \frac{\ln \frac{f}{X_L} + \frac{1}{2}\sigma^2 \tau}{\sigma\sqrt{\tau}} \\
d_2 &= \frac{\ln \frac{f}{X_H} + \frac{1}{2}\sigma^2 \tau}{\sigma\sqrt{\tau}}
\end{aligned}
$$

   Create a option pricing calculator in excel, referring to a pricing function written in vba. The time input will be in years i.e. don't use dates. Draw a spot profile of the value of the derivative.

6. (exam 2004) The Standard Black call option pricing formula is

$$
\begin{aligned}
V &= e^{-r\tau}(FN(d_1) - KN(d_2)) \\
\Delta &= e^{-r\tau} N(d_1) \\
\Gamma &= e^{-r\tau} N'(d_1) \frac{1}{F\sigma\sqrt{\tau}} \\
d_{1,2} &= \frac{\ln \frac{F}{K} \pm \frac{1}{2}\sigma^2 \tau}{\sigma\sqrt{\tau}}
\end{aligned}
$$

   (a) Write code to price, and provide Greeks for, a call option using the Standard Black formula. The last input of your list of inputs to the pricing formula will be an optional string parameter. The default will be "p" (for premium). Have "d" (for delta) and "g" (for gamma) other possibilities. Fix the strike, the volatility, the risk free rate, and the term (which will be in years i.e. don't use dates).
   (b) For a range of futures prices, draw graphs (separate sheets for each) of the value, the delta, and the gamma. On each of these above sheets, illustrate the effect of time on each profile by drawing the graphs for 6 months, 1 month and 1 week to expiry.

7. (exam 2003) A chooser option is one that expires after term τ_2. After term $\tau_1 < \tau_2$, however, the holder must decide if the option is a put or a call (European, with identical strikes X). Use put-call parity to find the value (in terms of vanilla options) of this option at the inception of the product. As usual, assume constant term structures of r, q and σ.

8. (a) Write code to price, and provide Greeks for, a European call option using Black-Scholes. The last input of your list of inputs to the pricing formula will be an optional string parameter. The default will be "p" (for premium). Have "d" (for delta) and "g" (for gamma) other possibilities.

   (b) For a range of spot prices, draw graphs (separate sheets for each) of the value, the delta, and the gamma.

   (c) On each of the above sheets, draw several graphs, illustrating the effect of time on each profile i.e. draw the graphs for 1 year to expiry, 6 months, 1 month, 1 week, etc.

Chapter 4

Dividend yields and discrete dividends

Typically European equity options are priced using the Black-Scholes model Black and Scholes [1973] or that model adjusted for dividends by calculating a continuous dividend yield. This has the effect of spreading the dividend payment throughout the life of the option. This is most attractive where the option is on an index (where the index is paying out several dividends, spread out through the period of optionality).

For American equity options with the underlying having no or several dividends, we may argue similarly. Here the approximation of Barone-Adesi and Whaley Barone-Adesi and Whaley [1987] is popular, but we prefer the method of Bjerksund and Stensland Bjerksund and Stensland [1993], Bjerksund and Stensland [2002] as it is computationally far superior, and has been shown to be more accurate in long dated options.

Bjerksund and Stensland [2002] is a more recent improvement over Bjerksund and Stensland [1993].

Another standard approach (for the European case) is to reduce the stock value by the present value of dividends (the escrowed dividend method), or to increase the strike by the future value of dividends. Both are unsatisfactory approaches as they affect the stochastic process on the equity fairly significantly. See Frishling [2002], Bos and Vandermark [2002], Haug et al. [2003].

In the case of only a few dividend payments on the underlying equity, the original approach above - calculating a continuous dividend yield and using that in a closed form formula - is also no longer satisfactory, even for European options. The dividends occur at one or a few discrete times, but we are spreading them out throughout the life of the option by making this assumption, and this has a material effect on the stochastic process for the stock price.

This comment also applies to the classic binomial tree approach for pricing American options developed in Cox et al. [1979]. Use of a binomial tree necessitates that risk free rates are assumed constant, and that there is a constant dividend yield, as described above. This will lead to the same severe problems as before. Note that dividends cannot be made discrete in the tree approach because doing so will make the tree no longer recombine, which is computationally a disaster.

Much theory has been developed to price (European) options under the assumption that the dividends are a known proportion of the stock price on the dividend payment date. See Björk [1998], for example. However, to use this approach alone is academic fiction: companies and brokers think of, predict, and eventually declare the reasonably short dated dividends in a currency unit. Furthermore, companies are very much loathe to reduce the dividend amount year on year, as a significant proportion of stock holders hold the stock purely for the purpose of receiving annuity revenue from the dividends (for example, retirees, who intend living on the dividends, and leaving the stock to their inheritors) and may transfer their holding to another stock if the dividend was decreased significantly (or even, was not keeping up with inflation). Thus, even if the stock price has decreased somewhat, the company will attempt to maintain dividend levels at more or less the same currency level, at least for a while. Thus, the model that dividends are a known proportion of share price is not practicable.

The most meaningful possibility is that the first few dividends are known or predicted in cash, whilst the remaining dividends are predicted as a proportion of stock price. Our preferred approach is as follows: use broker/analyst forecasts in the short and medium term, and then forecast percentage dividends in the long term using the model of [West, 2009, §6.6]. Alternatively, if there are no broker forecasts (for a smaller stock, or simply because we are operating under informational constraints) then all forecasts are percentage dividends based on history.

In the case of an American call with one dividend, the formula of Roll, Geske, Whaley Roll [1977], Geske [1979a], Whaley [1981] is well known (amongst practitioners) to be arbitragable (and not so well known amongst software vendors, who often insist on offering this as the default model). Again, see Frishling [2002], Haug et al. [2003]. Furthermore, their approach does not allow for the pricing of American puts (as is well known, the pricing of American puts is in general more difficult than the pricing of calls).

Thus, for European or American options with a few dividends, one should probably prefer to use a finite difference scheme for pricing. This finite difference scheme easily accommodates the discrete jumps of dividends, and both the cash and proportion formulation. One can use the finite difference approach for any number of dividends if prepared to input them. As the number of dividends increases, the benefits of these approaches are outweighed by the superior speed of using the continuous dividend yield proxy in the Black-Scholes or Bjerksund-Stensland formula.

4.1 Pricing European options by Moment Matching

Let $t = t_0$ with dividends occurring on $t_1, \ \ldots, \ t_n$ and $T = t_{n+1}$. Now if we have a cash dividend D_i on t_i then

$$\begin{aligned} S(t_i) &= S(t_{i-1}) e^{X_i} - D_i \\ \Rightarrow S(t_i)^2 &= S(t_{i-1})^2 e^{2X_i} - 2D_i S(t_{i-1}) e^{X_i} + D_i^2 \end{aligned}$$

and so

$$\begin{aligned} \mathbb{E}[S(t_i)] &= \mathbb{E}[S(t_{i-1})] \mathbb{E}\left[e^{X_i}\right] - D_i \\ \mathbb{E}\left[S(t_i)^2\right] &= \mathbb{E}\left[S(t_{i-1})^2\right] \mathbb{E}\left[e^{2X_i}\right] - 2D_i \mathbb{E}[S(t_{i-1})] \mathbb{E}\left[e^{X_i}\right] + D_i^2. \end{aligned}$$

Here $X_i = \ln\left(\frac{S(t_i^-)}{S(t_{i-1})}\right)$ and we know its distribution as in (1.5).

Otherwise, if we have a simple dividend yield d_i, then

$$\begin{aligned} S(t_i) &= S(t_{i-1}) e^{X_i} (1 - d_i) \\ \Rightarrow S(t_i)^2 &= S(t_{i-1})^2 e^{2X_i} (1 - d_i)^2 \end{aligned}$$

so that

$$\begin{aligned} \mathbb{E}[S(t_i)] &= \mathbb{E}[S(t_{i-1})] \mathbb{E}\left[e^{X_i}\right] (1 - d_i) \\ \mathbb{E}\left[S(t_i)^2\right] &= \mathbb{E}\left[S(t_{i-1})^2\right] \mathbb{E}\left[e^{2X_i}\right] (1 - d_i)^2 . \end{aligned}$$

Here, in both cases, $e^{X_i} = e^{\left(r - \frac{1}{2}\sigma^2\right)\tau_i + \sigma\sqrt{\tau_i}Z}$ and $\tau_i = t_i - t_{i-1}$ and hence

$$\begin{aligned} \mathbb{E}\left[e^{X_i}\right] &= e^{r\tau_i} \\ \mathbb{E}\left[e^{2X_i}\right] &= e^{\left(2r + \sigma^2\right)\tau_i} \text{ from (1.10).} \end{aligned}$$

We then proceed by induction starting with $\mathbb{E}\left[S\left(t_0\right)\right]=S$ and $\mathbb{E}\left[S\left(t_0\right)^2\right]=S^2$ until we reach $\mathbb{E}\left[S\left(t_{n+1}\right)\right]$ and $\mathbb{E}\left[S\left(t_{n+1}\right)^2\right]$.

Now assume that S is lognormally distributed at time T. Clearly this assumption is not mathematically correct, but is is known that the error is not severe unless the dividends are very large. If $\ln S \sim \phi(\Psi, \Sigma)$, where Ψ and Σ are not known a priori, then from (1.9) we have

$$\begin{aligned} \mathbb{E}[S] &= e^{\Psi+\frac{1}{2}\Sigma} && (4.1)\\ \mathbb{E}[S^2] &= e^{2\Psi+2\Sigma} && (4.2) \end{aligned}$$

Hence, given $\mathbb{E}[S]$ and $\mathbb{E}[S^2]$, we can easily solve simultaneously for Ψ and Σ. It follows in our application of these facts that

$$\begin{aligned} \Sigma &= \ln \frac{\mathbb{E}_t^{\mathbb{Q}}\left[S^2\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[S\right]^2} && (4.3)\\ \Psi &= \ln \mathbb{E}_t^{\mathbb{Q}}\left[S\right] - \tfrac{1}{2}\Sigma && (4.4) \end{aligned}$$

Now, $\sqrt{\Sigma}$ is to be thought of as the volatility for the period. In other words, $\ln S \sim \phi(\Psi, \sigma^2(t_n - t))$ where σ is the annualised volatility measure, or $\Sigma = \sigma^2(t_n - t)$. Hence

$$\sigma^2 = \frac{1}{t_n - t}\left[\ln \frac{\mathbb{E}_t^{\mathbb{Q}}\left[S^2\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[S\right]^2}\right] \tag{4.5}$$

where σ denotes the appropriate volatility measure to use in a Black model option valuation. We use Lemma 3.3.1: easier to implement from existing models, one is using Black's model with

- a futures spot of $\mathbb{E}_t^{\mathbb{Q}}\left[S\right]$,
- a strike of K,
- a volatility of σ as in (4.5),
- a risk free rate of r_n,
- a term of $t_n - t$.

4.2 Calculation of the dividend yield

If dividend amounts $D_1,\ D_2,\ \ldots,\ D_n$ are known or predicted (and, as has been discussed, n is sufficiently large that the continuous dividend yield proxy is valid), then the following conversion is necessary. First, we calculate the present value of all the dividends:

$$Q = \sum_{i=1}^{n} D_i e^{-r_i(t_i - t)/365} \tag{4.6}$$

where t_i are the payment dates of the dividends, t is the valuation date, and r_i is the NACC risk free rate for time t_i. The summation is taken over all dividends whose LDR date is after valuation date t and on or before expiry date T. (In other words, the date criterion for inclusion and the discounting date are different.) Then

$$q_d = \frac{-1}{\tau} \ln \frac{S_t - Q}{S_t} \tag{4.7}$$

is the relevant dividend yield. T is the expiry date of the option. See [West, 2009, Chapter 6]. Effectively, the stock price is adjusted from S_t to $S_t - Q = S_t e^{-q\tau}$, where $\tau = \frac{T-t}{365}$.
Alternatively suppose $d_1,\ d_2,\ \ldots,\ d_n$ are simple dividend yields; again these are the dividend yields for dividends whose LDR dates lie in the period $(t, T]$. Then the appropriate adjustment to the stock price is to multiply the price by $\prod_{i=1}^{m}(1-d_i)$. Thus, the percentage dividend yield is

$$q_p = -\frac{1}{\tau}\sum_{i=1}^{m}\ln(1-d_i) \tag{4.8}$$

In this case, we proceed as follows: calculate the dividend yield q_d in (4.7) as if only the cash dividends were going to be paid, and calculate the dividend yield q_p in (4.8) as if only the percentage dividends were going to be paid. Then

$$q = q_d + q_p \tag{4.9}$$

is the appropriate dividend yield which takes into account both the cash and the percentage dividends. Note that this formulation is invalid if the order of the cash and percentage forecasts are mixed, although we think this scenario will be uncommon.

Chapter 5

Binary options and rebates

5.1 European binaries (cash or nothing)

A binary/digital call pays off

$$V(T) = \begin{cases} 1 & \text{if} \quad S(T) > K \\ 0 & \text{if} \quad S(T) < K \end{cases} \tag{5.1}$$

and a binary put pays off

$$V(T) = \begin{cases} 0 & \text{if} \quad S(T) > K \\ 1 & \text{if} \quad S(T) < K \end{cases} \tag{5.2}$$

How do we value these? By the principle of risk neutral valuation, $V(t) = e^{-r\tau}\mathbb{E}_t^{\mathbb{Q}}[V(T)]$, which, for the call, is

$$\begin{aligned} V(t) &= e^{-r\tau}\frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}}\int_{-\infty}^{\infty} \mathbb{1}_{\{Se^x>K\}} \exp\left[-\tfrac{1}{2}\left(\frac{x-m_-\tau}{\sigma\sqrt{\tau}}\right)^2\right] dx \\ &= e^{-r\tau}\frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}}\int_{\ln\frac{K}{S}}^{\infty} \exp\left[-\tfrac{1}{2}\left(\frac{x-m_-\tau}{\sigma\sqrt{\tau}}\right)^2\right] dx \\ &= e^{-r\tau}N(d_-) \end{aligned}$$

Similarly the put is worth

$$V(t) = e^{-r\tau}N(-d_-)$$

In general,

$$V(t) = e^{-r\tau}N(\eta d_-) \tag{5.3}$$

5.2 European asset or nothing

Now the payoff for the call is

$$V(T) = \begin{cases} S(T) & \text{if} \quad S(T) > K \\ 0 & \text{if} \quad S(T) < K \end{cases} \tag{5.4}$$

and for the put is

$$V(T) = \begin{cases} 0 & \text{if} \quad S(T) > K \\ S(T) & \text{if} \quad S(T) < K \end{cases} \tag{5.5}$$

Easily, the value this time is

$$V(t) = S(t)e^{-q\tau}N(\eta d_+) \tag{5.6}$$

Of course, the value of a European vanilla option easily decomposes into a combination of an asset or nothing and a cash or nothing option.

5.3 Rebates

Now, what about variations on this situation? Suppose the life of the option is from 0 to T. We could consider

- A digital type payoff that pays off 1 if $S(t)$ ever reaches B, the payoff occurring at T - these don't occur in reality, but we will use them as building blocks for what follows (European digital option)
- A digital type payoff that pays off 1 if $S(t)$ ever reaches B, the payoff occurring at the first such t (American digital option)
- A digital type payoff that pays off 1 if $S(t)$ never reaches B, the payoff occurring at T (no-hit rebate).

These are important building blocks for barrier options and can be called rebates: the option holder receives a rebate as compensation for the fact that his barrier option has expired worthless. B is called the barrier. In the first two cases, if the barrier is struck by the stock price, it triggers a new event, namely, the cancelation of the position, which is what is called an out barrier option. The option holder immediately receives the rebate as compensation for this cancelation. In the third case, the barrier had to have been struck for the triggering of the option becoming live, which is what is called an in barrier option. If the barrier is never struck, the option holder receives (on termination) the rebate as compensation.
These have been priced in Rubinstein and Reiner [1991].

5.3.1 Some theoretical considerations

Suppose $X(t)$ is Brownian motion. Then for any $\theta > 0$ the stochastic exponential $e^{\theta X(t) - \frac{1}{2}\theta^2 t}$ is a Martingale. This is known as the Doléans-Dade exponential of the martingale $\theta X(t)$.
The optional sampling theorem states that a stopped Martingale is again a Martingale. However, this theorem requires that the stopped process is uniformly integrable. Consider a stopping time τ; we can consider the stopped process $e^{\theta X(t\wedge\tau) - \frac{1}{2}\theta^2 (t\wedge\tau)}$. The exponential function is not uniformly integrable, so we need specific properties of the stopping time to use the optional sampling theorem. IF we can apply the theorem THEN we will be able to conclude that $\mathbb{E}\left[e^{\theta X(t\wedge\tau) - \frac{1}{2}\theta^2 (t\wedge\tau)}\right] = 1$ for all t (in particular $\mathbb{E}\left[e^{\theta X(\tau) - \frac{1}{2}\theta^2 \tau}\right] = 1$).

5.3.2 European digital option

Wystup [2002].[1]

[1]Thanks to Dangerous David Acott, Tom Wasabi McWalter and especially Hardy 'From the Machine' Hully for contributions in this section.

Let $h(t)$, where $0 \le t \le T$, denote the density of the first exit time ie. the first time t for which $S_t = B$. See Figure 5.1. The required valuation is then $V = e^{-rT} \int_0^T h(t)\, dt$.
We first need to find the function $h(t)$, and then perform the integration. We first suppose that $B > S$; the case where $B < S$ is similar but has some subtle differences. We proceed cautiously!

Hitting time for Brownian motion without drift (hit is high)

We first establish the hitting time distribution for Brownian motion without drift.
The first hitting or stopping time is

$$\tau = \inf_{t \ge 0} \{X(t) = b\}$$

We suppose that $b > 0$. We can apply the optional sampling theorem to the stochastic integral because this stopped Brownian motion $X(t \wedge \tau)$ is bounded from above (but not below), and so $e^{\theta X(t\wedge\tau) - \frac{1}{2}\theta^2 (t\wedge\tau)}$ is bounded from above by $e^{\theta b}$, and below by 0.
Thus $\mathbb{E}\left[e^{\theta b - \frac{1}{2}\theta^2 \tau}\right] = 1$ for any $\theta > 0$.
Let p_b be the hitting time distribution. Let $\mathcal{L}$ denote the Laplace transformation. Then

$$\begin{aligned}
\mathcal{L}[p_b](s) &= \int_0^\infty e^{-st} p_b(t)\, dt \\
&= \mathbb{E}\left[e^{-s\tau}\right] \\
&= \mathbb{E}\left[e^{-\sqrt{2s}b + \sqrt{2s}b - s\tau}\right] \\
&= e^{-\sqrt{2s}b}\mathbb{E}\left[e^{\sqrt{2s}b - s\tau}\right] \\
&= e^{-\sqrt{2s}b}
\end{aligned}$$

by putting $\theta = \sqrt{2s}$. Thus

$$\begin{aligned}
p_b(t) &= \mathcal{L}^{-1}[e^{-\sqrt{2s}b}] && (5.7) \\
&= \frac{b}{t^{3/2}\sqrt{2\pi}} \exp\left[-\tfrac{1}{2}\frac{b^2}{t}\right] && (5.8)
\end{aligned}$$

using [Abramowitz and Stegun, 1974, 29.3.82].

Hitting time for Brownian motion with drift (hit is high)

Now let the first hitting time be defined as $\inf_{t\ge 0}\{\alpha t + X(t) = b\}$. Here X is as before, the drift is $\alpha > 0$, and the hit level we seek is $b > 0$.
This time X_b is the Brownian motion which is stopped when $\alpha t + X(t)$ first hits b. Again, X_b is bounded from above by b (we use the fact that $\alpha > 0$).
We have seen the idea already: $\mathbb{E}\left[e^{-s\tau}\right] = \mathbb{E}\left[e^{-\theta b + \theta b - s\tau}\right] = e^{-\theta b}\mathbb{E}\left[e^{\theta b - s\tau}\right] = e^{-\theta b}$, as $\mathbb{E}\left[e^{\theta b - s\tau}\right] = 1$, for some cute choice of θ.
This time,

$$\begin{aligned}
\theta b - s\tau &= \theta[\alpha\tau + X_b(\tau)] - s\tau \\
&= \theta X_b(\tau) - (s - \theta\alpha)\tau
\end{aligned}$$

and so we should choose $s - \theta\alpha = \frac{1}{2}\theta^2$. Solving for $\theta > 0$ yields $\sqrt{2s+\alpha^2} - \alpha$. Again let p_b be the hitting time distribution. Then

$$\begin{aligned}
\mathcal{L}[p_b](s) &= \int_0^\infty e^{-st} p_b(t)\,dt \\
&= \mathbb{E}\left[e^{-s\tau}\right] \\
&= \mathbb{E}\left[e^{(\alpha-\sqrt{2s+\alpha^2})b+(\sqrt{2s+\alpha^2}-\alpha)b-s\tau}\right] \\
&= e^{(\alpha-\sqrt{2s+\alpha^2})b}\mathbb{E}\left[e^{(\sqrt{2s+\alpha^2}-\alpha)b-s\tau}\right] \\
&= e^{(\alpha-\sqrt{2s+\alpha^2})b} \qquad (5.9)
\end{aligned}$$

Thus

$$\begin{aligned}
p_b(t) &= \mathcal{L}^{-1}[e^{(\alpha-\sqrt{2s+\alpha^2})b}] \\
&= e^{\alpha b}\mathcal{L}^{-1}[e^{-\sqrt{2s+\alpha^2}b}] \\
&= e^{\alpha b}\frac{b}{t^{3/2}\sqrt{2\pi}}\exp\left[-\tfrac{1}{2}\frac{b^2}{t}\right]e^{-\frac{1}{2}\alpha^2 t} \\
&= \frac{b}{t^{3/2}\sqrt{2\pi}}\exp\left[-\tfrac{1}{2}\left(\frac{b-\alpha t}{\sqrt{t}}\right)^2\right] \qquad (5.10)
\end{aligned}$$

using the previous Laplace transform result and [Abramowitz and Stegun, 1974, 29.2.12].

Hitting time for the stock price $(B > S)$

We are looking for the first τ satisfying $Se^{m_-\tau+\sigma X(\tau)} = B$, so τ is given by

$$\inf\left\{t \geq 0 : \frac{m_-}{\sigma}t + X(t) = \frac{1}{\sigma}\ln\frac{B}{S}\right\}$$

Making the substitution $\alpha = \frac{m_-}{\sigma}$, $b = \frac{1}{\sigma}\ln\frac{B}{S} > 0$ we have the distribution of this hitting time in (5.10), which we have agreed to name $h(t)$: thus

$$h(t) = \frac{\ln\frac{B}{S}}{\sigma t^{3/2}\sqrt{2\pi}}\exp\left[-\tfrac{1}{2}\left(\frac{\ln\frac{B}{S} - m_- t}{\sigma\sqrt{t}}\right)^2\right] \qquad (5.11)$$

Now, remember if at all possible what we are doing here: we have $V = e^{-rT}\int_0^T h(t)\,dt$. This is

$$\begin{aligned}
V_{\text{up}} &= e^{-rT}\left[\left(\frac{B}{S}\right)^{\frac{2m_-}{\sigma^2}} N(e_+(T)) + N(-e_-(T))\right] \qquad (5.12) \\
e_\pm(t) &= \frac{\pm\ln\frac{S}{B} - m_- t}{\sigma\sqrt{t}} \qquad (5.13)
\end{aligned}$$

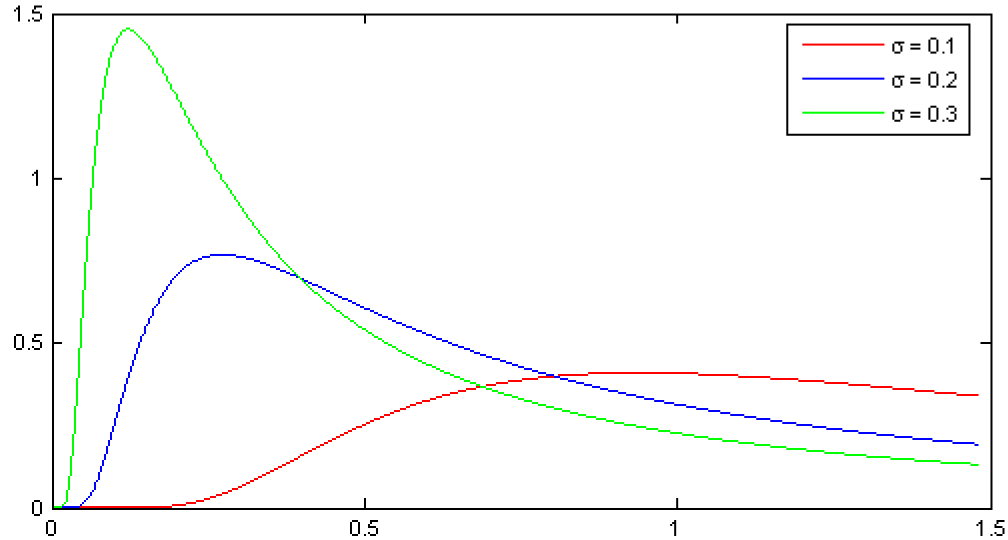

Figure 5.1: First exit time density

To show this, we establish in order the following identities:

$$e_-(t) - e_+(t) = \frac{2}{\sigma\sqrt{t}} \ln \frac{B}{S} \tag{5.14}$$

$$e_-^2 = e_+^2 - 4\frac{m_-}{\sigma^2} \ln \frac{B}{S} \tag{5.15}$$

$$N'(e_-) = N'(e_+) \left(\frac{B}{S}\right)^{\frac{2m_-}{\sigma^2}} \tag{5.16}$$

$$\frac{\partial}{\partial t} e_\pm(t) = \frac{1}{2t} e_\mp(t) \tag{5.17}$$

$$e_\pm(0) = \mp\infty \tag{5.18}$$

$$h(t) = \frac{d}{dt} \left(\frac{B}{S}\right)^{\frac{2m_-}{\sigma^2}} N(e_+(t)) - N(e_-(t)) \tag{5.19}$$

What if $B < S$?

The differences appear already at the first stage! This time X_b is bounded from below (but not above). The Martingale we must consider is $-\sqrt{2s}X_b(t)$, and it then follows that $\mathbb{E}\left[e^{-\sqrt{2s}b - s\tau}\right] = 1$. It then follows that $\mathcal{L}[p(b,t)] = e^{\sqrt{2s}b}$.

One generalises each calculation in turn with the appropriate care. The value of the option is

$$V_{\text{down}} = e^{-rT}\left[\left(\frac{B}{S}\right)^{\frac{2m_-}{\sigma^2}} N(-e_+(T)) + N(e_-(T))\right] \tag{5.20}$$

5.3.3 No hit rebate

Clearly the value of the no-hit rebate is given by $e^{-rT} - V$, because the sum of a hit rebate and a no-hit rebate is cash. This is an example of in-out parity.

5.3.4 American digital options

Wystup [2002]. Now the payoff occurs as soon as the hit occurs, and the value is $\int_0^T e^{-rt} h(t)\, dt$.
By first completing the square, and then following the same basic strategy as before, this value can be shown to be

$$V_\eta = \left(\frac{B}{S}\right)^{\frac{m_- + n_-}{\sigma^2}} N(-\eta e_+(T)) + \left(\frac{B}{S}\right)^{\frac{m_- - n_-}{\sigma^2}} N(\eta e_-(T)) \tag{5.21}$$

$$n_- = \sqrt{m_-^2 + 2\sigma^2 r} \tag{5.22}$$

$$e_\pm(t) = \frac{\pm \ln \frac{S}{B} - n_- t}{\sigma\sqrt{t}} \tag{5.23}$$

5.4 Exercises

1. Verify (5.6).

2. (exam 2003) Suppose we have a power option of term τ. The payoff of this option is $(S(T) - K)^2$ if $S(T) > K$, 0 otherwise. Price such an option, giving all details.

3. Derive the rebate valuation formula (one-hit) where the rebate is paid at the termination of the product. Follow the scheme in the notes i.e. establish the formulae there, in the order they are given. In the last step, do the integral by looking for relevant substitutions from the steps already performed.

Chapter 6

Variance swaps

6.1 Contractual details

A long party in a variance swap will receive realised variance, and pay fixed. Realised variance is defined as

$$\Sigma = \frac{d}{N} \sum_{i=1}^{N} \left(\ln \frac{S_i}{S_{i-1}} \right)^2$$

where $S_0, S_1, \ldots, S_N$ are the stock prices on contractually specified days $t_0, t_1, \ldots, t_N$, and d is the number of contractually specified trade days in the year (so, 252 or 250 or suchlike).

The definition of log returns might or might not be adjusted for dividends.

Very often the payoff to a variance swap will be capped. The default seems to be at a cap level which corresponds to $2.5K$, where K is the strike in volatility terms. As such, these caps are irrelevant (worthless) in the case of index variance swaps. They may have relevance in the pricing of single equity variance swaps.

Note that a position in a capped swap is the same as a position in a swap and a short position in a call.

Thus, we restrict attention to the the case where there is no cap.

6.2 The theoretical pricing model

Here we follow Demeterfi et al. [March 1999].

In a diffusion model, the realized variance for a given evolution of the stock price is the integral

$$\Sigma = \frac{1}{T} \int_0^T \sigma^2(t) \, dt \tag{6.1}$$

This is a good approximation to the contractually defined variance above.

The value of the pay fixed leg of the variance swap with volatility strike K is the expected present value of the payoff in the risk-neutral world

$$V = e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[\Sigma - K^2 \right] \tag{6.2}$$

Here r is the risk-free rate for expiration T. Also, let q be the dividend yield.

Then

$$\frac{dS}{S} = (r-q)dt + \sigma(t)\,dZ$$
$$d\ln S = \left(r - q - \tfrac{1}{2}\sigma^2(t)\right)\,dt + \sigma(t)\,dZ$$

so taking differences and rearranging we get

$$\sigma^2(t)\,dt = 2\left(\frac{dS}{S} - d\ln S\right)$$

Writing this in the integral form we have

$$\int_0^T \sigma^2(t)\,dt = 2\int_0^T \frac{dS}{S} - 2\ln\frac{S(T)}{S(0)}$$

Now $\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{dS}{S}\right] = (r-q)T$ - this is dynamically replicated by trading in futures - so the only problem is the log contract. In fact the log contract can theoretically be replicated using a continuum of option positions.

Let S_* be some fixed point. We decide what this means and what it should be in due course. Firstly, define another exotic payoff f by

$$f(S(T)) := \frac{S(T)-S_*}{S_*} - \ln\frac{S(T)}{S_*} \tag{6.3}$$

Thus

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}[\Sigma] &= \frac{2}{T}\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{dS}{S} - \ln\frac{S(T)}{S(0)}\right] \\
&= 2(r-q) - \frac{2}{T}\mathbb{E}^{\mathbb{Q}}\left[\ln\frac{S(T)}{S(0)}\right] \\
&= 2(r-q) - \frac{2}{T}\mathbb{E}^{\mathbb{Q}}\left[\frac{S(T)-S_*}{S_*} - f(S(T)) + \ln\frac{S_*}{S(0)}\right] \\
&= 2(r-q) - \frac{2}{T}\left[\frac{S(0)e^{(r-q)T}-S_*}{S_*} - \mathbb{E}^{\mathbb{Q}}[f(S(T))] + \ln\frac{S_*}{S(0)}\right]
\end{aligned}$$

But now, we discover the remarkable

$$f(S(T)) = \int_0^{S_*} \frac{1}{K^2}\max(K-S(T),0)\,dK + \int_{S_*}^{\infty}\frac{1}{K^2}\max(S(T)-K,0)\,dK \tag{6.4}$$

and so

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}[\Sigma] \\
&= 2(r-q) - \frac{2}{T}\frac{S(0)e^{(r-q)T}-S_*}{S_*} + \frac{2e^{rT}}{T}\left[\int_0^{S_*}\frac{1}{K^2}p(K)\,dK + \int_{S_*}^{\infty}\frac{1}{K^2}c(K)\,dK\right] - \frac{2}{T}\ln\frac{S_*}{S(0)}
\end{aligned} \tag{6.5}$$

We thus replicate f by trading a continuum of puts with strikes from 0 to S_* and a continuum of calls with strikes from S_* to ∞. The weight of the options in the continuum is proportional to $\frac{1}{K^2}$. So, this clearly motivates us to choose S^* to be a point below which we prefer to use puts, and above which we prefer to use calls for replication. In this we prefer to use out-the-money options at all times, because they are more liquid than in-the-money options. So S_* might be the liquid strike on the skew closest to the forward level of S, for example.

This argument is not only valid in the Black-Scholes world. It is also valid in the local or stochastic (without jumps) volatility world.

6.3 Super-replication

The only difficulty is that to replicate we require a continuum of options. In reality of course this is impossible. Thus we seek a super-replication strategy.

Of course it is the payoff of f that needs to be super-replicated, everything else can be done straightforwardly. On the region $[K_1, K_m]$ we can super-replicate the payoff f with a portfolio of options: exactly those options used to calculate the theoretical variance.

For the region $[K_n, K_m]$ we choose the following calls:

- $\frac{f(K_{n+1})-f(K_n)}{K_{n+1}-K_n}$ many calls struck at K_n, plus
- $\frac{f(K_{j+1})-f(K_j)}{K_{j+1}-K_j} - \frac{f(K_j)-f(K_{j-1})}{K_j-K_{j-1}}$ many calls stuck at K_j for $j = n+1, \ \ldots, \ m-1$.

This super-replicates in the region $[K_n, K_m]$ and sub-replicates in the region $[K_m, \infty)$. We choose in addition

- $f'(K_m) - \frac{f(K_m)-f(K_{m-1})}{K_m-K_{m-1}}$ many calls struck at K_m.

which improves things in the region $[K_m, \infty)$. Note that super-replication in that region is impossible.

For the region $[K_1, K_n]$ we choose the following puts:

- $\frac{f(K_{n-1})-f(K_n)}{K_n-K_{n-1}}$ many puts struck at K_n, plus
- $\frac{f(K_{j-1})-f(K_j)}{K_j-K_{j-1}} - \frac{f(K_j)-f(K_{j+1})}{K_{j+1}-K_j}$ many puts struck at K_j for $j = n-1, \ \ldots, \ 2$.

This super-replicates in the region $[K_1, K_n]$ and sub-replicates in the region $[0, K_1]$. We choose in addition

- $-f'(K_1) - \frac{f(K_1)-f(K_2)}{K_2-K_1}$ many puts struck at K_1.

which improves things in the region $[0, K_1]$. Note that super-replication in that region is impossible.

6.4 Exercises

1. What is the price of an off-the-run variance swap i.e. one that has already started? Write it as a function of history and the price of a just starting variance swap.
2. The notional of the variance swap is usually quoted in 'vega notional' $\mathcal{V}_N$, where the vega notional and the cash (variance) notional $\$_N$ are related by $\$_N = \frac{\mathcal{V}_N}{2K}$. Show that with this definition, if the realised volatility is $K + \Delta$, where Δ is small, then the payoff is approximately $\mathcal{V}_N \Delta$.

Chapter 7

Compound options

A compound option is an option in which the underlying is an option. Thus, there is a first exercise date, and if exercised, a vanilla option is born, for a (necessarily subsequent) exercise date.

The following analysis is based on the work of Geske Geske [1979b] who investigated equity options and hypothesised that an equity itself behaved like an option. Geske's model used the assumptions of the Black-Scholes model: constant volatility, constant interest rates, no dividend yield, no transaction costs etc. Logical adjustments to Geske's model allow the three term structures of volatility, interest rates and dividend yields to be reflected in the premium.

$$
\begin{aligned}
T_2 &\sim \text{maturity of underlying option.} \\
T_1 &\sim \text{maturity of compound option, } T_1 \leq T_2. \\
t &\sim \text{valuation date of the compound option.} \\
\tau_2 &= T_2 - T_1, \text{ in years.} \\
\tau_1 &= T_1 - t, \text{ in years.} \\
\tau &= T_2 - t, \text{ in years.} \\
K_2 &\sim \text{strike of underlying option.} \\
K_1 &\sim \text{strike of compound option.} \\
r &\sim \text{risk free rate} \\
\sigma &\sim \text{volatility} \\
q &\sim \text{expected dividend yield} \\
S_t &\sim \text{spot of underlying stock at time } t.
\end{aligned}
$$

The value of $S(T_1)$ that causes the value of the underlying option to equal K_1 at time τ_1 is a key value - it is a boundary value that will determine whether or not the compound option is exercised or allowed to lapse.

Rather we will perform the analysis in terms of returns. Let y be the draw from the normal distribution, so the stock price at time T_1 is $S(y)$ as below. Let us perform the analysis for a call on a call. The other cases are all similar. Let $\mathrm{BS}(y)$ be the value of the underlying option at time T_1 that corresponds to $S(y)$. Thus

$$S(y) = Se^{m_-\tau_1+\sigma\sqrt{\tau_1}y} \tag{7.1}$$

$$\mathrm{BS}(y) = S(y)e^{-q\tau_2}N(d_+) - K_2e^{-r\tau_2}N(d_-) \tag{7.2}$$

$$\begin{aligned} d_\pm &= \frac{\ln\frac{S(y)}{K_2} + m_\pm\tau_2}{\sigma\sqrt{\tau_2}} \\ &= \frac{\ln\frac{S}{K_2} + m_-\tau_1 + m_\pm\tau_2 + \sigma\sqrt{\tau_1}y}{\sigma\sqrt{\tau_2}} \\ &= \frac{\ln\frac{S}{K_2} + m_-\tau_1 + m_\pm\tau_2}{\sigma\sqrt{\tau_2}} + \sqrt{\frac{\tau_1}{\tau_2}}y \end{aligned} \tag{7.3}$$

Let y_* be such that $\mathrm{BS}(y_*) = K_1$. y_* is found using Newton's method, with

$$\begin{aligned} y_0 &= 0 \\ y_{n+1} &= y_n - \frac{\mathrm{BS}(y_n) - K_1}{\frac{\partial}{\partial y}\mathrm{BS}(y)|y_n} \end{aligned}$$

Then

$$\begin{aligned} V &= e^{-r\tau_1}\int_{y=-\infty}^{\infty}[\mathrm{BS}(y) - K_1]^+N'(y)\,dy \\ &= e^{-r\tau_1}\int_{y=y_*}^{\infty}(\mathrm{BS}(y) - K_1)N'(y)\,dy \\ &= e^{-r\tau_1}\int_{y=y_*}^{\infty}(S(y)e^{-q\tau_2}N(d_+) - K_2e^{-r\tau_2}N(d_-) - K_1)N'(y)\,dy \\ &= e^{-r\tau_1}Se^{m_-\tau_1}e^{-q\tau_2}\int_{y=y_*}^{\infty}e^{\sigma\sqrt{\tau_1}y}N(d_+)N'(y)\,dy \qquad (7.4) \\ &\quad -e^{-r\tau_1}K_2e^{-r\tau_2}\int_{y=y_*}^{\infty}N(d_-)N'(y)\,dy \qquad (7.5) \\ &\quad -e^{-r\tau_1}K_1\int_{y=y_*}^{\infty}N'(y)\,dy \qquad (7.6) \end{aligned}$$

(7.4) follows from a careful application of (1.19), and (7.5) follows from a careful application of (1.18). Of course (7.6) is trivial.

Let $\eta = \pm 1$ if the underlying option is a call/put and $\zeta = \pm 1$ if the compound option is a call/put. Then the value of the compound option is

$$\begin{aligned} V &= \eta\zeta Se^{-q\tau}N_2\left(-\eta\zeta(y_* - \sigma\sqrt{\tau_1}),\ \eta d_+,\ \zeta\sqrt{\frac{\tau_1}{\tau}}\right) \\ &\quad -\eta\zeta K_2e^{-r\tau}N_2\left(-\eta\zeta y_*,\ \eta d_-,\ \zeta\sqrt{\frac{\tau_1}{\tau}}\right) \\ &\quad -\zeta K_1e^{-r\tau_1}N(-\eta\zeta y_*) \end{aligned} \tag{7.7}$$

$$d_\pm = \frac{\ln\frac{S}{K_2} + m_\pm\tau}{\sigma\sqrt{\tau}} \tag{7.8}$$

7.1 Exercises

1. Find $\frac{\partial}{\partial y}\mathrm{BS}(y)$.

2. Fill in the missing details in the notes for getting the final pricing formula for a compound call on call option.

3. (exam 2003) Suppose we wish to price a compound call on call, where all of risk free, dividend yield and volatility have a term structure. (There is however no skew structure to volatility.) Let the compound option expire after time τ_1 (in years) with strike X_1 and let the relevant rates be r_1, q_1 and σ_1. Let the vanilla option expire after time τ_2 (in years) with strike X_2 and let the relevant rates be r_2, q_2 and σ_2.

   (a) What are the forward rates for the period from τ_1 to τ_2 in terms of the ordinary rates for τ_1 and τ_2? Denote them henceforth as r_f, q_f and σ_f,

   (b) Now price such an option, following the following hints which come from the notes:

   - Let $S(y)$ be the value of the underlying and $\mathrm{BS}(y)$ be the value of the underlying option at time τ_1 if the draw from the $\phi(0,1)$ distribution has been y. Thus $S(y) =$ .... and $\mathrm{BS}(y) =$ ....
   - Let y_* be such that $\mathrm{BS}(y_*) = X_1$. y_* is found using Newton's method, with .... (include the differentiation).
   - Then $V =$ ....

   For the final part, you may use without proof the following two facts

$$\begin{aligned}
\int_\alpha^\infty N\left(K+Lx\right)N'(x)\,dx &= N_2\left(-\alpha,\ \frac{K}{\sqrt{L^2+1}},\ \frac{L}{\sqrt{L^2+1}}\right) \\
\int_\alpha^\infty e^{Ax}N\left(K+Lx\right)N'(x)\,dx &= e^{\frac{A^2}{2}}N_2\left(-\alpha+A,\ \frac{K+AL}{\sqrt{L^2+1}},\ \frac{L}{\sqrt{L^2+1}}\right)
\end{aligned}$$

4. (exam 2008) This question concerns pricing a complex chooser option. Such an option is valued today t_0; at time t_1 the owner has to choose between owning

   - A call with a maturity date t_C and strike K_C;
   - A put with a maturity date t_P and strike K_P.

   Volatility, dividend yield, and the risk free rate are all constant.

   (a) I can write the value of $S(t_1)$ as $S(t_0)e^{\cdots z}$ where z is a random sample from a normal distribution with mean 0 and standard deviation 1. Complete the $\cdots$ here.

   (b) Hence write the value AT TIME t_1 of the call as a function $\mathrm{BS}^C(z)$ and the value of the put as a function $\mathrm{BS}^P(z)$, being careful to distinguish between $d_\pm^C$ for the call and $d_\pm^P$ for the put.

   (c) Explain why there will be a value of z, call it z^*, where we will be indifferent to selecting the put or the call. A diagram might be useful. Say how z^* will be found, although you are not required to give any details.

   (d) Write down the value of the option in terms of integrals.

   (e) How will these integrals be evaluated? What will the final answer look like? (You are NOT being asked to perform the calculation - or even to write down any formulae.)

Chapter 8

Basket Options

Suppose the assets have value weights w_i in the basket, where $\sum_{i=1}^n w_i = 1$. Suppose we have a covariance matrix Σ. The basket volatility is then $\sqrt{w'\Sigma w}$. A European option on the basket can then be priced in the usual way with this volatility.

These are the forward weights rather than the valuation date weights. The forward weights will be different to the spot weights because the dividend yields / discrete dividends are different.

All analytic models make the assumption that the basket value is actually the geometric mean of the various underlyings rather than the arithmetic mean. This methodology was first proposed in Gentle [1993]. In fact, the above formulation of the basket volatility placed in the geometric Brownian motion framework is implicitly making this assumption. (This is a non-trivial point - see [Musiela and Rutkowski, 1998, §9.9] for an excellent discussion.) Academic and market research has shown however that the effect of this is not severe, especially considering that this assumption has remarkable computational advantages over a lattice methodology such as that of Rubinstein [1994] - this approach will choke on any basket which has many underlyings - even three underlyings will make the calculation slow. An alternative is to work out the first few (even the first two) moments of the arithmetic basket and then fit an analytically tractible distribution to these moments. We will see this applied in the setting of Asian options. Of course, we could use this geometric basket price as a control variate when doing Monte Carlo.

Here is the solution of Gentle [1993]. Suppose the option is on the basket $\sum_{i=1}^n W_i S_i$, where as usual $W_1,\ W_2,\ \ldots,\ W_n$ are the number of each of the shares in the basket. First note that

$$\begin{aligned}
V(0) &= e^{-r\tau}\mathbb{E}\left[\max\left(\eta\left(\sum_{i=1}^n W_i S_i(T) - K\right), 0\right)\right] \\
&= e^{-r\tau}\mathbb{E}\left[\max\left(\eta\left(\sum_{i=1}^n w_i S_i^*(T) - K^*\right), 0\right)\right]\sum_{i=1}^n W_i f_i \\
\text{where } w_i &= \frac{W_i f_i}{\sum_{j=1}^n W_j f_j} \\
S_i^*(T) &= \frac{S_i(T)}{f_i} \\
K^* &= \frac{K}{\sum_{i=1}^n W_i f_i}
\end{aligned}$$

Now $\sum_{i=1}^n w_i S_i^*(T)$ is a weighted arithmetic average of numbers $S_i^*(T)$ with expectations all equal to 1. We hope for a decent approximation of the arithmetic mean by taking the geometric mean, and controlling by the expected difference. Put

$$
\begin{aligned}
Y &:= \prod_{i=1}^{n} (S_i^*(T))^{w_i} \\
\alpha &:= \exp\left(-\frac{\tau}{2}\sum_{j=1}^{n} w_j \sigma_j^2\right)
\end{aligned}
$$

and so $\ln Y \sim \phi\left(\ln \alpha, \sqrt{w'\Sigma w}\sqrt{\tau}\right)$. Then

$$
\begin{aligned}
V(0) &= e^{-r\tau}\mathbb{E}\left[\max\left(\eta\left(\sum_{i=1}^{n} w_i S_i^*(T) - K^*\right), 0\right)\right]\sum_{i=1}^{n} W_i f_i \\
&= e^{-r\tau}\mathbb{E}\left[\max\left(\eta\left(Y + \sum_{i=1}^{n} w_i S_i^*(T) - Y - K^*\right), 0\right)\right]\sum_{i=1}^{n} W_i f_i \\
&\approx e^{-r\tau}\mathbb{E}\left[\max\left(\eta\left(Y + 1 - \alpha e^{1/2w'\Sigma w\tau} - K^*\right), 0\right)\right]\sum_{i=1}^{n} W_i f_i \\
&= e^{-r\tau}\mathbb{E}\left[\max\left(\eta\left(Y - \overline{K}\right), 0\right)\right]\sum_{i=1}^{n} W_i f_i \\
\text{where } & -\overline{K} = 1 - \alpha e^{1/2w'\Sigma w\tau} - K^*
\end{aligned}
$$

Now, using Lemma 3.3.1 and after some tedious manipulations, we have

$$V(0) \approx \eta e^{-r\tau}[\tilde{f}N(\eta d_+) - \tilde{K}N(\eta d_-)] \tag{8.1}$$

$$\tilde{f} = \sum_{i=1}^{n} W_i f_i \alpha e^{1/2w'\Sigma w T} \tag{8.2}$$

$$\tilde{K} = K + \sum_{i=1}^{n} W_i f_i (\alpha e^{1/2w'\Sigma w T} - 1) \tag{8.3}$$

$$d_\pm = \frac{\ln\frac{\tilde{f}}{\tilde{K}} \pm \frac{1}{2}\sqrt{w'\Sigma w}^2\tau}{\sqrt{w'\Sigma w}\sqrt{\tau}} \tag{8.4}$$

Chapter 9

Rainbow options

A version of this chapter appears in Ouwehand and West [2006].

9.1 Definition of a Rainbow Option

Rainbow Options refer to all options whose payoff depends on more than one underlying risky asset; each asset is referred to as a colour of the rainbow. Examples of these include:

- "Best of assets or cash" option, delivering the maximum of two risky assets and cash at expiry Stulz [1982], Johnson [1987], Rubinstein [1991]
- "Call on max" option, giving the holder the right to purchase the maximum asset at the strike price at expriry Stulz [1982], Johnson [1987]
- "Call on min" option, giving the holder the right to purchase the minimum asset at the strike price at expiry Stulz [1982], Johnson [1987]
- "Put on max" option, giving the holder the right to sell the maximum of the risky assets at the strike price at expiry, Margrabe [1978], Stulz [1982], Johnson [1987]
- "Put on min" option, giving the holder the right to sell the minimum of the risky assets at the strike at expiry Stulz [1982], Johnson [1987]
- "Put 2 and call 1", an exchange option to put a predefined risky asset and call the other risky asset, Margrabe [1978]. Thus, asset 1 is called with the 'strike' being asset 2.

Thus, the payoffs at expiry for rainbow European options are:

Best of assets or cash	$\max(S_1,\ S_2,\ \ldots,\ S_n, K)$
Call on max	$\max(\max(S_1,\ S_2,\ \ldots,\ S_n) - K, 0)$
Call on min	$\max(\min(S_1,\ S_2,\ \ldots,\ S_n) - K, 0)$
Put on max	$\max(K - \max(S_1,\ S_2,\ \ldots,\ S_n), 0)$
Put on min	$\max(K - \min(S_1,\ S_2,\ \ldots,\ S_n), 0)$
Put 2 and Call 1	$\max(S_1 - S_2, 0)$

To be true to history, we deal with the last case first.

9.2 Notation and setting

Define the following variables:

- S_i = Spot price of asset i,
- K = Strike price of the rainbow option,
- σ_i = volatility of asset i,
- q_i = dividend yield of asset i,
- ρ_{ij} = correlation coefficient of return on assets i and j,
- r = the risk-free rate (NACC),
- τ = the term to expiry of the rainbow option.

Our system for the asset dynamics will be

$$\underline{dS/S} = \underline{(r-q)}\,dt + A\,\underline{dW} \tag{9.1}$$

where the Brownian motions are independent. A is a square root of the covariance matrix Σ, that is $AA' = \Sigma$. As such, A is not uniquely determined, but it would be typical to take A to be the Choleski decomposition matrix of Σ (that is, A is lower triangular). Under such a condition, A is uniquely determined.
Let the i^{th} row of A be $\underline{a}_i$. We will say that $\underline{a}_i$ is the volatility vector for asset S_i. Note that if we were to write things where S_i had a single volatility σ_i then $\sigma_i^2 = \sum_{j=1}^n a_{ij}^2$, so $\sigma_i = \|\underline{a}_i\|$, where the norm is the usual Euclidean norm. Also, the correlation between the returns of S_i and S_j is given by $\frac{\underline{a}_i \cdot \underline{a}_j}{\|\underline{a}_i\|\|\underline{a}_j\|}$.

9.3 Margrabe option valuation

Margrabe [1978] began by evaluating the option to exchange one asset for the other at expiry. This is justifiably one of the most famous early option pricing papers. This is conceptually like a call on the asset we are going to receive, but where the strike is itself stochastic, and is in fact the second asset. The payoff at expiry for this European option is:

$$\max(S_1 - S_2, 0),$$

which can be valued as:

$$V_M = S_1 e^{-q_1\tau} N(d_+) - S_2 e^{-q_2\tau} N(d_-), \tag{9.2}$$

where

$$d_\pm = \frac{\ln\frac{f_1}{f_2} \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \tag{9.3}$$

$$f_i = S_i e^{(r-q_i)\tau} \tag{9.4}$$

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \tag{9.5}$$

Margrabe derives this formula by developing and then solving a Black-Scholes type differential equation. But he also gives another argument, which he credits to Stephen Ross, which with the hindsight of modern technology, would be considered to be the most appropriate approach to the problem. Let asset 2 be the numeraire in the market. In other words, asset 2 forms a new currency, and asset one costs $\frac{S_1}{S_2}$ in that currency. The risk free rate in this market is q_2. Thus we have the option to buy asset one for a strike of 1. This has a Black-Scholes price of

$$\begin{aligned} V &= \frac{S_1}{S_2} e^{-q_1\tau} N(d_+) - e^{-q_2\tau} N(d_-) \\ d_\pm &= \frac{\ln \frac{\frac{S_1}{S_2}}{1} + (q_2 - q_1 \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}} \end{aligned}$$

where σ is the volatility of $\frac{S_1}{S_2}$. To get from a price in the new asset 2 currency to a price in the original economy, we multiply by S_2: the 'exchange rate', which gives us (9.2).

So, what is σ? We show that (9.5) is the correct answer to this question in §9.4.

9.4 Change of numeraire

Suppose that X is a European–style derivative with expiry date T. Since Harrison and Pliska [1981] it has been known that if X can be perfectly hedged (i.e. if there is a self–financing portfolio of underlying instruments which perfectly replicates the payoff of the derivative at expiry), then the time–t value of the derivative is given by the following *risk–neutral valuation formula*:

$$X_t = e^{-r(T-t)} \mathbb{E}_t^{\mathbb{Q}} [X_T]$$

where r is the riskless rate, and the symbol $\mathbb{E}_t^{\mathbb{Q}}$ denotes the expectation at time t under a *risk–neutral measure* $\mathbb{Q}$. A measure $\mathbb{Q}$ is said to be risk–neutral if all discounted asset prices $\bar{S}_t = e^{-rt} S_t$ are *martingales* under the measure $\mathbb{Q}$, i.e. if the expected value of each $\bar{S}_t$ at an earlier time u is its current value $\bar{S}_u$:

$$\mathbb{E}_u^{\mathbb{Q}}[\bar{S}_t] = \bar{S}_u \qquad \text{whenever } 0 \leq u \leq t$$

(Here we assume for the moment that S pays no dividends.)

Now let $A_t = e^{rt}$ denote the bank account. Then the above can be rewritten as

$$\frac{X_t}{A_t} = \mathbb{E}_t^{\mathbb{Q}} \left[\frac{X_T}{A_T} \right] \qquad \text{i.e.} \qquad \bar{X}_t = \mathbb{E}_t^{\mathbb{Q}} \left[\bar{X}_T \right]$$

Thus $\bar{X}_t$ is a $\mathbb{Q}$–martingale.

In an important paper, Geman et al. [1995] it was shown that there is "nothing special" about the bank account: given an asset $\hat{A}$, we can "discount" each underlying asset using $\hat{A}$:

$$\hat{S}_t = \frac{S_t}{\hat{A}_t}$$

Thus $\hat{S}$ is the "price" of S measured not in money, but in units of $\hat{A}$. The asset $\hat{A}$ is referred to as a *numéraire*, and might be a portfolio or a derivative — the only restriction is that its value $\hat{A}_t$ is strictly positive during the time period under consideration.

It can be shown (cf. Geman et al. [1995]) that in the absence of arbitrage, and modulo some technical conditions, there is for each numéraire $\hat{A}$ a measure $\hat{\mathbb{Q}}$ with the property that each numéraire–deflated asset price process $\hat{S}_t$ is a $\hat{\mathbb{Q}}$–martingale, i.e.

$$\mathbb{E}_u^{\hat{\mathbb{Q}}}[\hat{S}_t] = \hat{S}_u \qquad \text{whenever } 0 \leq u \leq t$$

(Again, we assume that S pays no dividends.) We call $\hat{\mathbb{Q}}$ the equivalent martingale measure (EMM) associated with the numéraire $\hat{A}$. It then follows easily that if a European–style derivative X can be perfectly hedged, then

$$\hat{X}_t = \mathbb{E}_t^{\hat{\mathbb{Q}}}[\hat{X}_T] \qquad \text{and so} \qquad X_t = \hat{A}_t \mathbb{E}_t^{\hat{\mathbb{Q}}}\left[\frac{X_T}{\hat{A}_T}\right]$$

Indeed, if V_t is the value of a replicating portfolio, then (1) $X_t = V_t$ by the law of one price, and (2) $\hat{V}_t = \frac{V_t}{\hat{A}_t}$ is a $\hat{\mathbb{Q}}$–martingale. Thus

$$\hat{X}_t = \hat{V}_t = \mathbb{E}_t^{\hat{\mathbb{Q}}}\left[\hat{V}_T\right] = \mathbb{E}_t^{\hat{\mathbb{Q}}}\left[\hat{X}_T\right]$$

using the fact that $V_T = X_T$ — by definition of "replicating portfolio".

It follows that if N_1, N_2 are numéraires, with associated EMM's $\mathbb{Q}_1, \mathbb{Q}_2$, then

$$N_1(t)\mathbb{E}_t^{\mathbb{Q}_1}\left[\frac{X_T}{N_1(T)}\right] = N_2(t)\mathbb{E}_t^{\mathbb{Q}_2}\left[\frac{X_T}{N_2(T)}\right]$$

Indeed, both sides of the above equation are equal to the time–t price of the derivative.

To get slightly more technical, the EMM $\hat{\mathbb{Q}}$ associated with numéraire $\hat{A}$ is obtained from the risk–neutral measure $\mathbb{Q}$ via a Girsanov transformation (whose kernel is the volatility vector of the numéraire). In particular, the volatility vectors of all assets are the same under both $\mathbb{Q}$ and $\hat{\mathbb{Q}}$.

A minor modification of the above reasoning is necessary in case the assets pay dividends. Suppose that S is a share with dividend yield q. If we buy one share at time $t = 0$, and if we reinvest the dividends in the share, we will have e^{qt} shares at time t, with value $S(t)e^{qt}$. If $\hat{A}$ is the new numeéraire, with dividend yield $\hat{q}$, then it is the ratio

$$\frac{S(t)e^{qt}}{\hat{A}(t)e^{\hat{q}t}}$$

that is a $\hat{\mathbb{Q}}$–martingale, and not the ratio $\frac{S(t)}{\hat{A}(t)}$.

Suppose now that we have n assets $S_1, S_2, \ldots, S_n$, and that we model the asset dynamics using an n–dimensional standard Brownian motion. If $\underline{a}_i$ is the volatility vector of S_i, then, under the risk–neutral measure $\mathbb{Q}$, the dynamics of S_i are given by

$$\frac{dS_i}{S_i} = (r - q_i)\, dt + \underline{a}_i \cdot dW$$

where q_i is the dividend yield of S_i, and W is an n–dimensional standard $\mathbb{Q}$–Brownian motion. When we work with asset S_j as numéraire, we will be interested in the dynamics of the asset ratio processes

$$S_{i/j}(t) = \frac{S_i(t)}{S_j(t)}$$

under the associated EMM $\mathbb{Q}_j$. Now by Ito's formula the *risk–neutral* dynamics of $S_{i/j}$ are given by

$$\frac{dS_{i/j}}{S_{i/j}} = \left(q_j - q_i + \|\underline{a}_j\|^2 - \underline{a}_i \cdot \underline{a}_j\right)\, dt + (\underline{a}_i - \underline{a}_j) \cdot dW$$

However, when we change to measure $\mathbb{Q}_j$, we know that $Y(t) = S_{i/j}(t)e^{(q_i - q_j)t}$ is a $\mathbb{Q}_j$–martingale. Applying Ito's formula again, we see that the risk–neutral dynamics of Y_t are given by

$$\frac{dY}{Y} = \left(\|\underline{a}_j\|^2 - \underline{a}_i \cdot \underline{a}_j\right)\, dt + (\underline{a}_i - \underline{a}_j) \cdot dW$$

Since $Y(t)$ is a $\mathbb{Q}_j$–martingale, its drift under $\mathbb{Q}_j$ is zero, and its volatility remains unchanged. Thus the $\mathbb{Q}_j$–dynamics of $Y(t)$ are

$$\frac{dY}{Y} = (\underline{a}_i - \underline{a}_j) \cdot dW^j$$

where W^j is a standard n–dimensional $\mathbb{Q}_j$–Brownian motion. Applying Ito's formula once again to $S_{i/j}(t) = Y(t)e^{-(q_i-q_j)t}$, it follows easily that the $\mathbb{Q}_j$–dynamics of $S_{i/j}(t)$ are given by

$$\frac{dS_{i/j}}{S_{i/j}} = (q_j - q_i)\, dt + (\underline{a}_i - \underline{a}_j) \cdot dW^j$$

Returning to §9.3, we have $\sigma^2 = \|\underline{a}_1 - \underline{a}_2\|^2 = \|\underline{a}_1\|^2 + \|\underline{a}_2\|^2 - 2\rho\|\underline{a}_1\|\|\underline{a}_2\| = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$, as required.

9.5 For 2 assets: the results of Stulz

Stulz [1982] derives the value of what are now called two asset rainbow options. First the value of the call on the minimum of the two assets is derived, by evaluating the (rather unpleasant) bivariate integral. Then a min-max parity argument is invoked: having a two asset rainbow maximum call and the corresponding two asset rainbow minimum call is just the same as having two vanilla calls on the two assets.

Finally put-call parity results are derived, enabling evaluation of the put on the minimum and the put on the maximum. For the call on the minimum, the payoff at expiry can be expressed as $\max(\min(S_1, S_2) - K, 0)$. The following is the result of Stulz [1982], improving the notation to be consistent with the generalisations that will follow, adjusting for the typo, and including possible dividend yields.

$$\begin{aligned} V_{\min}(t) &= S_1(t)e^{-q_1\tau} N_2(d_-^{2/1}, d_+^1, -\varrho_{12}) \\ &\quad + S_2(t)e^{-q_2\tau} N_2(d_-^{1/2}, d_+^2, -\varrho_{21}) \\ &\quad - Ke^{-r\tau} N_2(d_-^1, d_-^2, \rho) && (9.6) \\ \sigma_{i/j}^2 &= \sigma_i^2 + \sigma_j^2 - 2\rho_{ij}\sigma_i\sigma_j && (9.7) \\ d_\pm^{i/j} &= \frac{\ln \frac{S_i(t)}{S_j(t)} + \left(q_j - q_i \pm \frac{1}{2}\sigma_{i/j}^2\right)(\tau)}{\sigma_{i/j}\sqrt{\tau}} && (9.8) \\ d_\pm^i &= \frac{\ln \frac{S_i(t)}{K} + \left(r - q_i \pm \frac{1}{2}\sigma_i^2\right)(\tau)}{\sigma_i\sqrt{\tau}} && (9.9) \\ \varrho_{ij} &= \frac{\sigma_i - \rho\sigma_j}{\sigma_{i/j}} && (9.10) \end{aligned}$$

If we long both a call on the max and a call on the min, then we can re-express this as long vanilla calls on each of the underlyings. This we could call 'min-max' parity. So

$$c_{\max} + c_{\min} = c_1 + c_2 \tag{9.11}$$

and we get $c_{\max}$. Note that this trick is not available for $n > 2$. Thus the approach of Stulz will not generalise. The payoff at expiry for a put on the minimum is:

$$\begin{aligned} \max(K - \min(S_1, S_2), 0) &= K - \min(S_1, S_2) + \max(\min(S_1, S_2) - K, 0) \\ &= K - S_1 + \max(S_1 - S_2, 0) + \max(\min(S_1, S_2) - K, 0). \end{aligned}$$

and hence we have

$$p_{\min} = Ke^{-r\tau} - S_1e^{-q_1\tau} + V_M + c_{\min} \tag{9.12}$$

For another approach we have a put-call parity:

$$c_{\min}(K) + Ke^{-r\tau} = p_{\min}(K) + c_{\min}(0) \tag{9.13}$$

where the parentheses denote strike. $c_{\min}(0)$ is the cost of the entitlement to the minimum priced of the two assets at expiry. So, either side of (9.13) gives me the better at expiry of K and the minimum priced asset. Now

$$\lim_{x_1 \to \infty} N_2(x_1,\ x_2,\ \rho) = N(x_2)$$

and so plugging into (9.6) we can value $c_{\min}(0)$. Comparing (9.12) and (9.13) we have a rather roundabout way of deriving the result of Margrabe as a corollary of the results of Stulz.

The payoff at expiry for a put on the maximum is:

$$\begin{aligned}\max(K - \max(S_1, S_2), 0) &= K - \max(S_1, S_2) + \max(\max(S_1, S_2) - K, 0) \\ &= K - \max(S_1 - S_2, 0) + S_2 + \max(\max(S_1, S_2) - K, 0)\end{aligned}$$

and hence we have

$$p_{\max} = Ke^{-r\tau} - V_M + S_2 e^{-q_2\tau} + c_{\max} \tag{9.14}$$

As before, alternatively

$$c_{\max}(K) + Ke^{-r\tau} = p_{\max} + c_{\max}(0) \tag{9.15}$$

and from (9.11) we have

$$c_{\max}(0) = S_1 e^{-q_1\tau} + S_2 e^{-q_2\tau} - c_{\min}(0) \tag{9.16}$$

9.6 The general case

In Johnson [1987] extensions of the results of Stulz [1982] are claimed to any number of underlyings. However, the formulae in the paper are actually quite difficult to interpret without ambiguity: they are presented inductively, and the formula (even for $n = 3$) is difficult to interpret with certainty. Moreover, the formulae are not proved - only intuitions are provided - nor is any numerical work undertaken to provide some comfort in the results. The arguments basically involve intuiting what the delta's of the option in each of the n underlyings should be, and extrapolating from there to the price. So one can say 'bravo' given that it is possible to actually formally derive proofs for these many asset pricing formulae.

What we do is construct general Martingale-style arguments for all cases $n \geq 2$ which are in the style of the proof first found by Margrabe and Ross.

Johnson's results are stated for any number of assets. A rainbow option with n assets will require the n-variate cumulative normal function for application of his formulae. As n increases, so the computational effort and execution time for having such an approximation will increase dramatically. In West [2005] we have vb and c++ code for $n \leq 3$ based upon the Fortran of Genz [2004], so here we apply this code to European rainbow options with three stock underlyings, S_1, S_2 and S_3. Code for $n > 3$ does not seem to be available (in any language), at least in a form that would make the computational time better than direct Monte Carlo valuation of the original option.

Using that code, for the case $n = 3$ we can compare Monte Carlo simulation to the prices in Johnson [1987]; see for example Figure 9.1.

9.6.1 Maximum payoffs

We will first price the derivative that has payoff $\max(S_1,\ S_2,\ \ldots,\ S_n)$, where the S_i satisfy the usual properties. In fact, this is notationally quite cumbersome, and all the ideas are encapsulated in any reasonably small value of n, so we choose $n = 4$ (as we will see later, the fourth asset will be the strike).

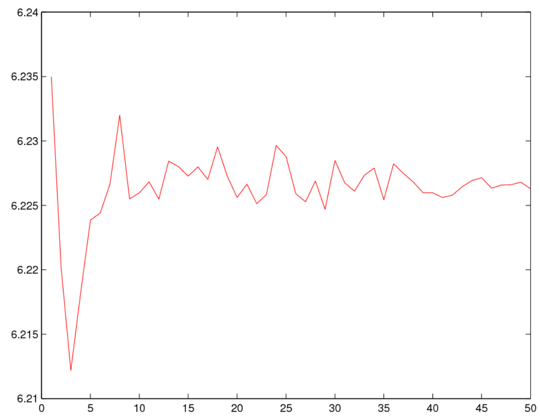

Figure 9.1: Monte Carlo for call on minimum on 3 assets. On the horizontal axis: number of experiments in 1000's, using independent Sobol sequences, on the vertical axis: price. The exact option value using the formula presented here is 6.2273.

Firstly, the value of the derivative is the sum of the value of 4 other derivatives, the i^{th} of which pays $S_i(T)$ if $S_i(T) > S_j(T)$ for $j \neq i$, and 0 otherwise. Let us value the first of these, the others will have similar values just by cycling the coefficients.

We are considering the asset that pays $S_1(T)$ if $S_1(T)$ is the largest price. Now let S_1 be the numeraire asset with associated martingale measure $\mathbb{Q}_1$. We see that the value of the derivative is

$$\begin{aligned} V_1(t) &= S_1(t)e^{-q_1\tau}\mathbb{E}_t^{\mathbb{Q}_1}\left[1; S_{2/1}(T) < 1,\ S_{3/1}(T) < 1,\ S_{4/1}(T) < 1\right] \\ &= S_1(t)e^{-q_1\tau}\mathbb{Q}_1[S_{2/1}(T) < 1,\ S_{3/1}(T) < 1,\ S_{4/1}(T) < 1] \\ &= S_1(t)e^{-q_1\tau}\mathbb{Q}_1[\ln S_{2/1}(T) < 0,\ \ln S_{3/1}(T) < 0,\ \ln S_{4/1}(T) < 0] \end{aligned} \tag{9.17}$$

where $S_{i/j}(T) = \dfrac{S_i(T)}{S_j(T)}$.

Let $\sigma_{i/j} = \|\underline{a}_i - \underline{a}_j\|$. We know that under $\mathbb{Q}_j$ we have $\frac{dS_{i/j}}{S_{i/j}} = (q_j - q_i)dt + (\underline{a}_i - \underline{a}_j)\cdot dW^j$, so $\ln S_{i/j}(T) \sim \phi\left(\ln S_{i/j}(t) + (q_j - q_i - \frac{1}{2}\sigma_{i/j}^2)\tau, \sigma_{i/j}\sqrt{\tau}\right)$.

Hence $\mathbb{Q}_j[S_{i/j}(T) <> 1] = N(\mp d_-^{i/j})$.

Note that $d_\pm^{i/j} = -d_\mp^{j/i}$.

Also, the correlation between $S_{i/k}(T)$ and $S_{j/k}(T)$ is

$$\begin{aligned} \rho_{ij,k} &:= \frac{(\underline{a}_i - \underline{a}_k)\cdot(\underline{a}_j - \underline{a}_k)}{\|\underline{a}_i - \underline{a}_k\|\|\underline{a}_j - \underline{a}_k\|} \\ &= \frac{\underline{a}_i\cdot\underline{a}_j - \underline{a}_i\cdot\underline{a}_k - \underline{a}_k\cdot\underline{a}_j + \sigma_k^2}{\sqrt{(\sigma_i^2 + \sigma_k^2 - 2\underline{a}_i\cdot\underline{a}_k)(\sigma_j^2 + \sigma_k^2 - 2\underline{a}_j\cdot\underline{a}_k)}} \\ &= \frac{\rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k - \rho_{kj}\sigma_k\sigma_j + \sigma_k^2}{\sqrt{(\sigma_i^2 + \sigma_k^2 - 2\rho_{ik}\sigma_i\sigma_k)(\sigma_j^2 + \sigma_k^2 - 2\rho_{jk}\sigma_j\sigma_k)}} \end{aligned} \tag{9.18}$$

Hence $\mathbb{Q}_1[\ln S_{2/1}(T) < 0,\ \ln S_{3/1}(T) < 0,\ \ln S_{4/1}(T) < 0] = N_3(-d_-^{2/1}, -d_-^{3/1}, -d_-^{4/1}, \Omega_1)$ where Ω_1, Ω_2, Ω_3

and Ω_4 are 3×3 matrices; the simplest way to think of them is that they are initially 4×4 matrices, with Ω_k having $\rho_{ij,k}$ in the $(i,j)^{th}$ position, and then the k^{th} row and k^{th} column are removed.
Thus, the value of the derivative that pays off the largest asset is

$$\begin{aligned}
V_{\max}(t) &= S_1(t)e^{-q_1\tau}N_3(-d_-^{2/1}, -d_-^{3/1}, -d_-^{4/1}, \Omega_1) + S_2(t)e^{-q_2\tau}N_3(-d_-^{1/2}, -d_-^{3/2}, -d_-^{4/2}, \Omega_2) \\
&\quad + S_3(t)e^{-q_3\tau}N_3(-d_-^{1/3}, -d_-^{2/3}, -d_-^{4/3}, \Omega_3) + S_4(t)e^{-q_4\tau}N_3(-d_-^{1/4}, -d_-^{2/4}, -d_-^{3/4}, \Omega_4) \\
&= S_1(t)e^{-q_1\tau}N_3(-d_-^{2/1}, -d_-^{3/1}, -d_-^{4/1}, \rho_{23,1}, \rho_{24,1}, \rho_{34,1}) \\
&\quad + S_2(t)e^{-q_2\tau}N_3(-d_-^{1/2}, -d_-^{3/2}, -d_-^{4/2}, \rho_{13,2}, \rho_{14,2}, \rho_{34,2}) \\
&\quad + S_3(t)e^{-q_3\tau}N_3(-d_-^{1/3}, -d_-^{2/3}, -d_-^{4/3}, \rho_{12,3}, \rho_{14,3}, \rho_{24,3}) \\
&\quad + S_4(t)e^{-q_4\tau}N_3(-d_-^{1/4}, -d_-^{2/4}, -d_-^{3/4}, \rho_{12,4}, \rho_{13,4}, \rho_{23,4})
\end{aligned} \tag{9.19}$$

9.6.2 Best and worst of call options

Let us start with the case where the payoff is the best or worst of assets or cash. The payoff at expiry is $\max(S_1,\ S_2,\ S_3,\ K)$. If we consider this to be the best of four assets, where the fourth asset satisfies $S_4(t) = Ke^{-r\tau}$ and has zero volatility, then we recover the value of this option from §9.6.1. This fourth asset not only has no volatility but also is independent of the other three assets.
Thus, $\underline{a}_4 = \underline{0}$, $\rho_{ij,4} = \rho_{ij}$, $\sigma_{i/4} = \sigma_i = \sigma_{4/i}$, $d_\pm^{i/4} = d_\pm^i$, $d_\pm^{4/i} = -d_\mp^i$. Thus

$$\begin{aligned}
V_{\max}(t) &= S_1(t)e^{-q_1\tau}N_3(-d_-^{2/1}, -d_-^{3/1}, d_+^1, \rho_{23,1}, \rho_{24,1}, \rho_{34,1}) \\
&\quad + S_2(t)e^{-q_2\tau}N_3(-d_-^{1/2}, -d_-^{3/2}, d_+^2, \rho_{13,2}, \rho_{14,2}, \rho_{34,2}) \\
&\quad + S_3(t)e^{-q_3\tau}N_3(-d_-^{1/3}, -d_-^{2/3}, d_+^3, \rho_{12,3}, \rho_{14,3}, \rho_{24,3}) \\
&\quad + Ke^{-r\tau}N_3(-d_-^1, -d_-^2, -d_-^3, \rho_{12}, \rho_{13}, \rho_{23})
\end{aligned} \tag{9.20}$$

Now let us consider the rainbow call on the max option.
Recall, this has payoff $\max(\max(S_1, S_2, S_3) - K, 0)$. Note that

$$\begin{aligned}
\max(\max(S_1, S_2, S_3) - K, 0) &= \max(\max(S_1, S_2, S_3), K) - K \\
&= \max(S_1, S_2, S_3, K) - K
\end{aligned}$$

and so

$$\begin{aligned}
V_{\text{cmax}}(t) &= S_1(t)e^{-q_1\tau}N_3(-d_-^{2/1}, -d_-^{3/1}, d_+^1, \rho_{23,1}, \rho_{24,1}, \rho_{34,1}) \\
&\quad + S_2(t)e^{-q_2\tau}N_3(-d_-^{1/2}, -d_-^{3/2}, d_+^2, \rho_{13,2}, \rho_{14,2}, \rho_{34,2}) \\
&\quad + S_3(t)e^{-q_3\tau}N_3(-d_-^{1/3}, -d_-^{2/3}, d_+^3, \rho_{12,3}, \rho_{14,3}, \rho_{24,3}) \\
&\quad - Ke^{-r\tau}[1 - N_3(-d_-^1, -d_-^2, -d_-^3, \rho_{12}, \rho_{13}, \rho_{23})]
\end{aligned} \tag{9.21}$$

Finally, we have the rainbow call on the min option. (Recall, this has payoff $\max(\min(S_1, S_2, S_3) - K, 0)$.) Because of the presence of both a maximum and minimum function, new ideas are needed. As before we first value the derivative whose payoff is $\max(\min(S_1, S_2, S_3), S_4)$.
If S_4 is the worst performing asset, then the payoff is the second worst performing asset. For $1 \le i \le 3$ the value of this payoff can be found by using asset S_i as the numeraire. For example, the value of the derivative that pays S_1, if S_4 is the worst and S_1 the second worst performing asset, is

$$S_1(t)e^{-q_1\tau}N_3(d_-^{2/1}, d_-^{3/1}, -d_-^{4/1}, \rho_{23,1}, -\rho_{24,1}, -\rho_{34,1})$$

If S_4 is not the worst performing asset, then the payoff is S_4. Now the probability that S_4 is the worst performing asset is

$$N_3(d_-^{1/4}, d_-^{2/4}, d_-^{3/4}, \rho_{12,4}, \rho_{13,4}, \rho_{23,4})$$

and so the value of the derivative that pays S_4, if S_4 is not the worst performing asset, is

$$S_4(t)e^{-q_4\tau}[1 - N_3(d_-^{1/4}, d_-^{2/4}, d_-^{3/4}, \rho_{12,4}, \rho_{13,4}, \rho_{23,4})]$$

Thus, the value of the derivative whose payoff is $\max(\min(S_1, S_2, S_3), S_4)$ is

$$\begin{aligned}
V(t) = {} & S_1(t)e^{-q_1\tau}N_3(d_-^{2/1}, d_-^{3/1}, -d_-^{4/1}, \rho_{23,1}, -\rho_{24,1}, -\rho_{34,1}) \\
& + S_2(t)e^{-q_2\tau}N_3(d_-^{1/2}, d_-^{3/2}, -d_-^{4/2}, \rho_{13,2}, -\rho_{14,2}, -\rho_{34,2}) \\
& + S_3(t)e^{-q_3\tau}N_3(d_-^{1/3}, d_-^{2/3}, -d_-^{4/3}, \rho_{12,3}, -\rho_{14,3}, -\rho_{24,3}) \\
& + S_4(t)e^{-q_4\tau}[1 - N_3(d_-^{1/4}, d_-^{2/4}, d_-^{3/4}, \rho_{12,4}, \rho_{13,4}, \rho_{23,4})]
\end{aligned} \tag{9.22}$$

Hence the derivative with payoff $\max(\min(S_1, S_2, S_3), K)$ has value

$$\begin{aligned}
V(t) = {} & S_1(t)e^{-q_1\tau}N_3(d_-^{2/1}, d_-^{3/1}, d_+^{1}, \rho_{23,1}, -\rho_{24,1}, -\rho_{34,1}) \\
& + S_2(t)e^{-q_2\tau}N_3(d_-^{1/2}, d_-^{3/2}, d_+^{2}, \rho_{13,2}, -\rho_{14,2}, -\rho_{34,2}) \\
& + S_3(t)e^{-q_3\tau}N_3(d_-^{1/3}, d_-^{2/3}, d_+^{3}, \rho_{12,3}, -\rho_{14,3}, -\rho_{24,3}) \\
& + Ke^{-r\tau}[1 - N_3(d_-^{1}, d_-^{2}, d_-^{3}, \rho_{12}, \rho_{13}, \rho_{23})]
\end{aligned} \tag{9.23}$$

and the call on the minimum has value

$$\begin{aligned}
V_{\text{cmin}}(t) = {} & S_1(t)e^{-q_1\tau}N_3(d_-^{2/1}, d_-^{3/1}, d_+^{1}, \rho_{23,1}, -\rho_{24,1}, -\rho_{34,1}) \\
& + S_2(t)e^{-q_2\tau}N_3(d_-^{1/2}, d_-^{3/2}, d_+^{2}, \rho_{13,2}, -\rho_{14,2}, -\rho_{34,2}) \\
& + S_3(t)e^{-q_3\tau}N_3(d_-^{1/3}, d_-^{2/3}, d_+^{3}, \rho_{12,3}, -\rho_{14,3}, -\rho_{24,3}) \\
& - Ke^{-r\tau}N_3(d_-^{1}, d_-^{2}, d_-^{3}, \rho_{12}, \rho_{13}, \rho_{23})
\end{aligned} \tag{9.24}$$

9.7 Finding the value of puts

This is easy, because put-call parity takes on a particularly useful role. It is always the case that

$$V_{\text{c}}(K) + Ke^{-r\tau} = V_{\text{p}}(K) + V_{\text{c}}(0) \tag{9.25}$$

where the parentheses denotes strike. V could be an option on the minimum, the maximum, or indeed any ordinal of the basket. If we have a formula for $V_{\text{c}}(K)$, as established in one of the previous sections, then we can evaluate $V_{\text{c}}(0)$ by taking a limit as $K \downarrow 0$, either formally (using facts of the manner $N_2(x, \infty, \rho) = N_1(x)$ and $N_3(x, y, \infty, \Sigma) = N_2(x, y, \rho_{xy})$) or informally (by forcing our code to execute with a value of K which is very close to, but not equal to, 0 - thus avoiding division by 0 problems but implicitly implementing the above-mentioned fact). By rearranging, we have the put value.

9.8 Deltas of rainbow options

By inspecting (9.20) one might expect that

$$\frac{\partial V_{\text{max}}}{\partial S_1} = e^{-q_1\tau}N_3(-d_-^{2/1}, -d_-^{3/1}, d_+^{1}, \rho_{23,1}, \rho_{24,1}, \rho_{34,1})$$

with similar results holding for $\frac{\partial V_{\max}}{\partial S_2}$ and $\frac{\partial V_{\max}}{\partial S_3}$, and indeed for the dual delta $\frac{\partial V_{\max}}{\partial K}$.
Thus turns out to be true in this case, but to claim it as an 'obvious fact' would be erroneous. For a proof one must apply the rather deep homogeneity results of Reiss and Wystup [2001]. The argument of Johnson [1987] is essentially an - almost surely unconscious - anticipatory application of these results: he intuits what $\frac{\partial V}{\partial S_i}$ is and then 'reassembles' V using the result that $V(x_1,\ x_2,\ \ldots,\ x_4) = \sum_{i=1}^{4} x_i \frac{\partial V}{\partial x_i}$ (this is Euler's Homogeneous Function Theorem).
Similar results hold for V_{cmax}, V_{cmin}, V_{pmax} and V_{pmin}.

9.9 Finding the Capital Guarantee on the 'Best of Assets or Cash Option'

We wish to determine the strike K of the 'best of assets or cash' option so that at inception the valuation of the option is equal to K. Denoting the value of such an option as $V(K)$ - implicitly fixing all other variables besides the strike - we wish to solve $V(K) = K$.
To do so using Newton's method is fortunately quite manageable, for the same reasoning that we have already seen. As previously promised we have from (9.20) that

$$\frac{\partial V}{\partial K} = e^{-r\tau} N_3(-d_-^1, -d_-^2, -d_-^3, \rho_{12}, \rho_{13}, \rho_{23})$$

Hence the appropriate Newton method iteration is

$$K_{n+1} = K_n - \frac{V(K_n) - K_n}{\frac{\partial V}{\partial K}|_{K=K_n} - 1} \tag{9.26}$$

and this is iterated to some desired level of accuracy. An alternative would be to iterate $K_{n+1} = V(K_n)$, our differentiation shows that the function V is a contraction, and so this iteration will converge to the fixed point $V(K) = K$ by the contraction mapping theorem.
It is important to note that the process of finding the fair theoretical strike is not just a curiosity. In the first place, it is attractive for the buyer of the option that they will get at least their premium back. (There is a floor on the return of 0%.) Moreover, if K is this fair strike, the trader will strike the option at an K^*, where $K^* > K$, in order to expect fat in the deal.
To see this, we can construct in a complete market a simple arbitrage strategy: imagine that the dealer sells the client for K^* an option struck at K^*, and hedges this with the 'fair' dealer by paying K for an option struck at K.[1] The difference $K^* - K$ is invested in a risk free account for the expiry date. Three cases then arise:

- If $\max(S_1, S_2, S_3) \leq K$ then we owe K^*. The fair trader pays K and we obtain $K^* - K$ from saving, and profit from the time value of $K^* - K$.
- If $K < \max(S_1, S_2, S_3) \leq K^*$, then the fair trader pays S_1 say. We sell this, and obtain the balance to K^* from saving.
- If $K^* < \max(S_1, S_2, S_3)$, then the fair trader pays S_1 say and we deliver this.

[1] The 'fair' dealer is the perfect hedger, whose replicating portfolio ends up with exactly the payoff.

9.10 Pricing rainbow options in reality

The model that has been developed here lies within the classical Black-Scholes framework. As is well known, the assumptions of that framework do not hold in reality; various stylised facts argue against that model. For vanilla options, the model is adjusted by means of the skew - this skew exactly ensures that the price of the option in the market is exactly captured by the model. Models which extract information from that skew and of how that skew will evolve are of paramount importance in modern mathematical finance.

After a moment's thought one will realise what a difficult task we are faced with in applying these skews here. Let us start by being completely naïve: we wish to mark our rainbow option to market by using the skews of the various underlyings. Firstly, what strike do we use for the underlying? How does the strike of the rainbow translate into an appropriate strike for an option on a single underlying? Secondly, suppose we somehow resolved this problem, and for a traded option, wished to know its implied volatility? A familiar problem arises: often the option will have two, sometimes even three different volatilities of one of the assets which recover the price (all other inputs being fixed). To be more mathematical, the map from volatility to price is not injective, so the concept of implied volatility is ill defined. See Figure 9.2.

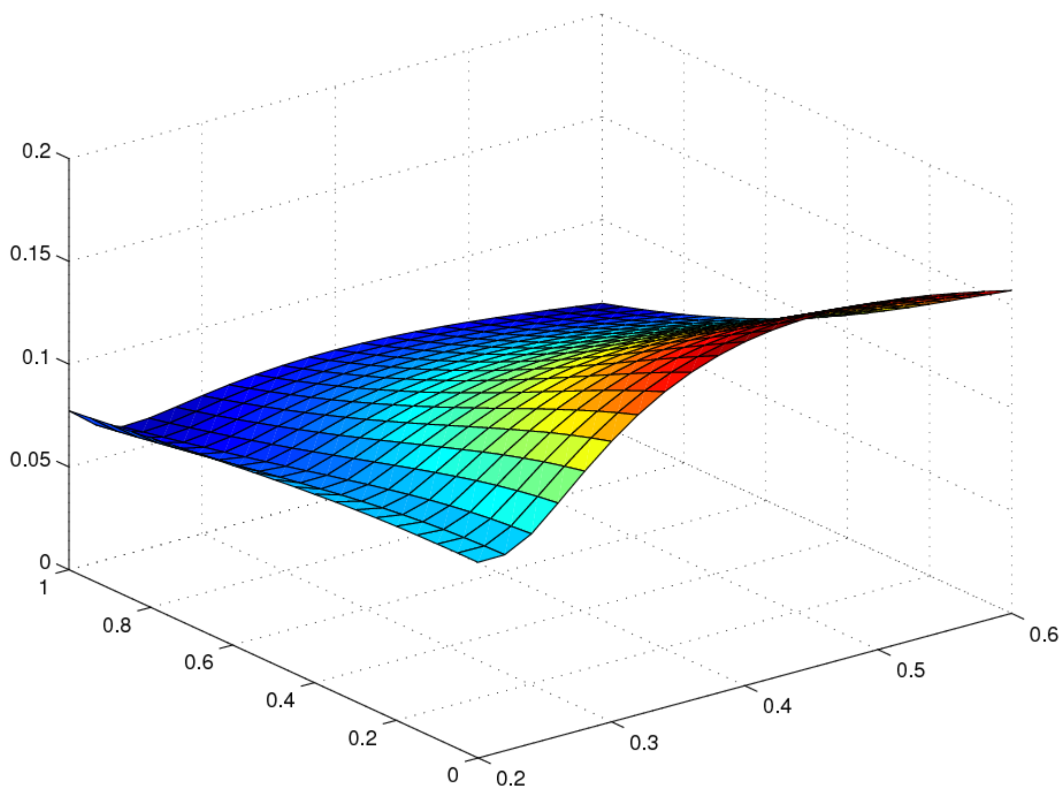

Figure 9.2: The price for a call on the minimum of two assets. $S_1 = 2$, $S_2 = 1$, $K = 1$, $\tau = 1$, $r = 10\%$, $\rho = -70\%$, $20\% \leq \sigma_1 \leq 60\%$, $\sigma_2 \leq 100\%$.

To see the sensitivity to the inputs, suppose to the setup in Figure 9.2 we add a third asset as elaborated in Figure 9.3. Of course the general level of the value of the asset changes, but so does the entire geometry of the price surface.

Another issue is that of the assumed correlation structure: again, correlation is difficult to measure; if there

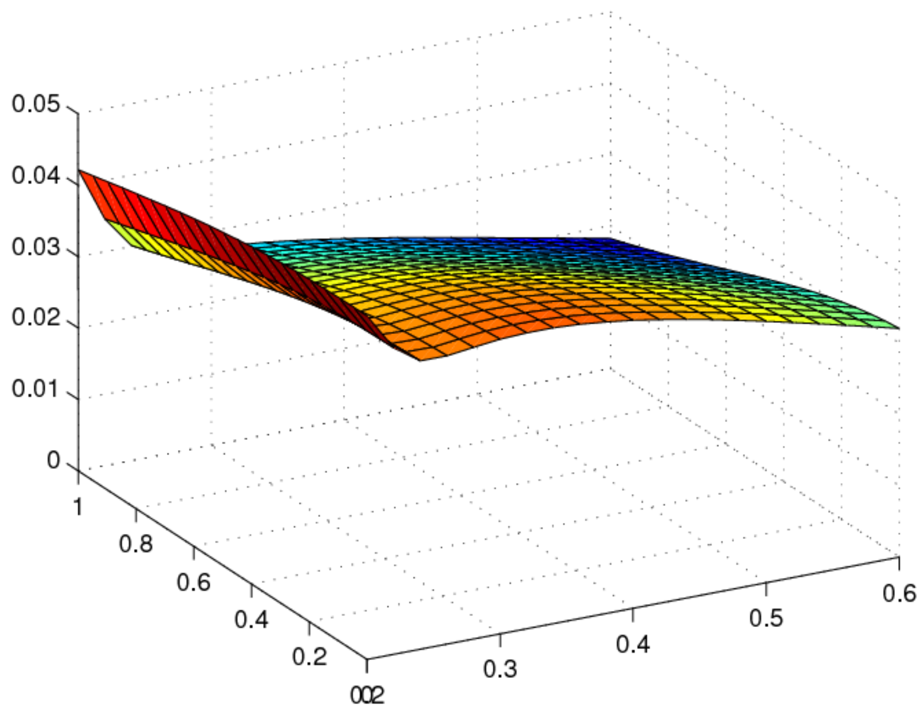

Figure 9.3: The price for a call on the minimum of three assets. As above, in addition $S_3 = 1$, $\sigma_3 = 30\%$ fixed, correlation structure $\rho_{12} = -70\%$, $\rho_{13} = 30\%$, $\rho_{23} = -20\%$.

is implied data, then it will have a strike attached. Finally, the joint normality hypothesis of returns of prices will typically be rejected.

A popular approach is to use skews from the vanilla market to infer the marginal distribution of returns for each of the individual assets and then 'glue them together' by means of a copula function. Given a multivariate distribution of returns, rainbow options can then be priced by Monte Carlo methods.

9.11 Exercises

1. Verify the decomposition for the two asset 'put on the minimum' rainbow in terms of the known pricing formulae by using so-called 'truth tables'. Hence write out in full the pricing formula for this product.

2. Use the Johnson approach for 2 colour call on min rainbow. Now check that the result of Stulz is recovered.

3. As in the notes use the results to Stulz to derive the result of Margrabe.

4. What is put-call parity in the setting of rainbow options? Hence, explain how, if we have a formula for call options, we can get the corresponding put price 'for free'. Illustrate with the 'put on the max' case.

5. Find the requisite derivatives to do the Newton method iteration for the 'best of assets or cash' pricing

formula for the case where there are two assets.

6. You will need to use the function bivarcumnorm from FMAlite.xla.

   (a) In excel create named ranges for q_1, q_2, σ_1, σ_2, X, ρ, r and τ; but NOT for S_1 and S_2.

   (b) Write vba code to price a call on the minimum of two assets. The inputs are all of the above (including S_1 and S_2 obviously). Make sure the code works, pointing your function to all of the named ranges and to two other cells with the stock prices.

   (c) Create a table of prices, with a range of values of S_1 in the rows and a range of values of S_2 in the columns.

   (d) Draw a 3D graph of the prices.

7. (exam 2004) Recall that a call on the minimum of two assets has payoff

$$V(T) = \max(\min(W_1 S_1(T), W_2 S_2(T)) - X, 0)$$

The price of this option is

$$\begin{aligned} V(t) &= W_1 S_1 e^{-q_1\tau} N_2\left(\eta_1 + \sigma_1\sqrt{\tau},\ \frac{\ln\frac{W_2S_2}{W_1S_1} + (q_1 - q_2 - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}},\ \frac{\rho\sigma_2 - \sigma_1}{\sigma}\right) \\ &+ W_2 S_2 e^{-q_2\tau} N_2\left(\eta_2 + \sigma_2\sqrt{\tau},\ \frac{\ln\frac{W_1S_1}{W_2S_2} + (q_2 - q_1 - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}},\ \frac{\rho\sigma_1 - \sigma_2}{\sigma}\right) \\ &- X e^{-r\tau} N_2\left(\eta_1,\ \eta_2,\ \rho\right) \end{aligned}$$

where familiar symbols have their usual meaning, and

$$\begin{aligned} \eta_1 &= \frac{\ln\frac{W_1S_1}{X} + (r - q_1 - \frac{1}{2}\sigma_1^2)\tau}{\sigma_1\sqrt{\tau}} \\ \eta_2 &= \frac{\ln\frac{W_2S_2}{X} + (r - q_2 - \frac{1}{2}\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}} \\ \sigma^2 &= \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 \end{aligned}$$

   (a) Find the price for a derivative which pays the minimum of $W_1S_1(T)$ and $W_2S_2(T)$ at termination.

   (b) By stating and proving put-call parity for minimum rainbow options, now find the price of a put on the minimum of two assets.

   (c) Suppose I have a (cash settled) option which is a call on the minimum of the simple returns of two assets. By writing down the payoff of this option, manipulate into the form given above i.e. find the weights and strike in the above option pricing formula.

Chapter 10

Asian options

(European) Asian or Average Options are options for which the payoff depends on an average of the price of the underlying over some (contractually) specified time interval, or at some discrete times. In reality, only the latter makes sense, and the discrete times could be a quasi-interval in the sense that they are closing prices over a period, such as a month.
There are two fundamental categories of (European) Asian Options. Letting A be a weighted average of historic prices of the underlying:

(a) an average price/rate option has payoff $\max(\eta(A-K),0)$; K is a contractually specified strike).

(b) an average strike options has payoff $\max(\eta(S(T)-A),0)$

For standard Asian Options, all the weights in the weighted average are equal. We will only consider this case.

10.1 Geometric average Asian formula

10.1.1 Geometric Average Price Options

The Geometric Average is defined as

$$A = \sqrt[n]{\prod_{i=1}^{n} S_{t_i}}$$

where the observation dates are $t_1,\ t_2,\ \ldots,\ t_n$.
There is also the notion of a continuous average over an interval of observation $[t^*,T]$, in which case the average is given by

$$\exp\left(\frac{1}{T-t^*}\int_{t^*}^{T}\log S_t dt\right)$$

where $[t^*,T]$ is the interval of observation. However, this is just academic nonsense. Such an average does not exist.

Lemma 10.1.1. If U_i, $i=1,\ 2,\ \ldots,\ q$ is a finite set of independent normal random variables, $U_i \sim \phi(m_i, s_i^2)$, then $\sum_{i=1}^{q} a_i U_i$ is a normal random variable, $\sum_{i=1}^{q} a_i U_i \sim \phi\left(\sum_{i=1}^{q} a_i m_i, \sum_{i=1}^{q} a_i^2 s_i^2\right)$.

Suppose none of the observations have yet been made. Put $S = S_0$, $t = t_0$. By the properties of standard Brownian Motion, $\left\{\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right\}_{i=1,\ 2,\ \ldots,\ n}$ is a set of independent random variables, with

$$\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) \quad \sim \quad \phi\left(m_-(0;t_{i-1},t_i)(t_i - t_{i-1}), \sigma^2(0;t_{i-1},t_i)(t_i - t_{i-1})\right) \qquad i = 1,\ 2,\ \ldots,\ n$$

where

$$m_-(0;t_{i-1},t_i) = r(0;t_{i-1},t_i) - q(0;t_{i-1},t_i) - \tfrac{1}{2}\sigma^2(0;t_{i-1},t_i)$$

Therefore, using the lemma,

$$\begin{aligned}\log\left(\frac{A}{S}\right) &= \frac{1}{n}\sum_{i=1}^{n}\ln S_i - \ln S \\ &=^{1} \frac{1}{n}\sum_{i=1}^{n}(n-i+1)\ln\frac{S_i}{S_{i-1}} \\ &\sim \phi\left(\frac{1}{n}\sum_{i=1}^{n}(n-i+1)m_-(0;t_{i-1},t_i)(t_i - t_{i-1}), \frac{1}{n^2}\sum_{i=1}^{n}(n-i+1)^2\sigma^2(0;t_{i-1},t_i)(t_i - t_{i-1})\right)\end{aligned}$$

and so $\log A :\sim \phi\left(\Psi, \Sigma\right)$ where

$$\begin{aligned}\Psi &= \ln S + \frac{1}{n}\sum_{i=1}^{n}(n-i+1)m_-(0;t_{i-1},t_i)(t_i - t_{i-1}) \\ \Sigma &= \frac{1}{n^2}\sum_{i=1}^{n}(n-i+1)^2\sigma^2(0;t_{i-1},t_i)(t_i - t_{i-1})\end{aligned}$$

Thus from Lemma 3.3.1

$$V_\eta \quad = \quad e^{-r\tau}\eta\left[e^{\Psi+\frac{1}{2}\Sigma}N(\eta d_+) - KN(\eta d_-)\right] \qquad (10.1)$$

$$d_+ \quad = \quad \frac{\Psi + \Sigma - \log K}{\sqrt{\Sigma}} \qquad (10.2)$$

$$d_- \quad = \quad \frac{\Psi - \log K}{\sqrt{\Sigma}} \qquad (10.3)$$

Suppose some observations at time indices $1,\ 2,\ \ldots,\ p$ have already been made. Let the first rate σ_{p+1} actually be denoted $\sigma(0;t_{p+1},t_p)$ and likewise for the other variables i.e. they have a forward notation even though they are spot variables. Then

$$\begin{aligned}\log\left(\frac{A}{S}\right) &= \frac{1}{n}\sum_{i=1}^{n}\ln S_i - \ln S \\ &= \frac{n-p}{n}\left[\frac{1}{n-p}\sum_{i=p+1}^{n}\ln S_i - \ln S\right] + \frac{1}{n}\sum_{i=1}^{p}\ln S_i - \frac{p}{n}\ln S\end{aligned}$$

[1]This is the crucial observation, seen as true by a diagram, and proved by induction.

and so $\log A :\sim \phi(\Psi, \Sigma)$ where

$$\Psi = \frac{1}{n} \sum_{i=p+1}^{n} (n-i+1) m_{-}(0; t_{i-1}, t_i)(t_i - t_{i-1}) + \frac{1}{n} \sum_{i=1}^{p} \ln S_i + \frac{n-p}{n} \ln S$$

$$\Sigma = \frac{1}{n^2} \sum_{i=p+1}^{n} (n-i+1)^2 \sigma^2(0; t_{i-1}, t_i)(t_i - t_{i-1})$$

and we finish as before.

10.1.2 Geometric Average Strike Options

Standard (European) Geometric Average Strike Options are in principle priced similarly, although it is quite tricky, as we have to determine the joint distribution of $S(T)$ and A.

10.2 Arithmetic average Asian formula

The payoff of an average price call is $\max\{A-K, 0\}$ and that of an average price put is $\max\{K-A, 0\}$, where K is the strike price and A is the observed average. A is calculated at a predetermined discrete set of dates, or daily over a certain interval (which may, practically be seen as equivalent to continuous averaging). Of course, the notion of continuous averaging is purely of academic interest.
We develop the following pricing models:

- A simple model for discrete averaging; averaging has not begun. We use the two moment model of Turnbull and Wakeman [1991].
- The simple model for discrete averaging; averaging has begun. Generalisation is quite easy.
- The supposedly high precision model for discrete averaging; we use the four moment model of Turnbull and Wakeman [1991].

10.2.1 Pricing by Moment Matching (TW2)

We follow almost the same process as in §4.1, so let's first calculate the first two moments of the average. Define the following variables:

t	$\sim$	valuation date,
n	$\sim$	number of asset price observations used in the averaging,
S_i	$\sim$	asset price at time $t_i, \{t_1 \leq t_2 \leq \cdots \leq t_n = T\}$
f_i	$\sim$	forward price if $t_i > t$; or the observed price S_i if $t_i \leq t$
σ_i	$\sim$	implied volatility for time $t_i > t$; 0 if $t_i \leq t$
r_i	$\sim$	risk-free rate of interest for time $t_i > t$; 0 if $t_i \leq t$
q_i	$\sim$	expected dividend yield for time $t_i > t$; 0 if $t_i \leq t$

Note that

$$A = \frac{1}{n}\sum_{i=1}^{n} S_i \tag{10.4}$$

$$A^2 = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} S_i S_j \tag{10.5}$$

and so

$$\mathbb{E}_t^{\mathbb{Q}}[A] = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{n}\sum_{i=1}^{n} S_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_t^{\mathbb{Q}}[S_i] = \frac{1}{n}\sum_{i=1}^{n} f_i \tag{10.6}$$

$$\mathbb{E}_t^{\mathbb{Q}}\left[A^2\right] = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}_t^{\mathbb{Q}}[S_i S_j] = \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{E}_t^{\mathbb{Q}}\left[S_i^2\right] + \frac{2}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{j-1}\mathbb{E}_t^{\mathbb{Q}}[S_i S_j]$$

Consider $\mathbb{E}_t^{\mathbb{Q}}[S_i S_j]$, by the tower property for expectations, with $i < j$ we have (in this case $t_i < t_j$):

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}[S_i S_j] &= \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{E}_{t_i}^{Q}[S_i S_j]\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[S_i\, \mathbb{E}_{t_i}^{\mathbb{Q}}[S_j]\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[S_i^2\, e^{(r(i,j)-q(i,j))(t_j-t_i)}\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[S_i^2\right] e^{(r(i,j)-q(i,j))(t_j-t_i)} \\
&= f_i f_j\, e^{\sigma_i^2(t_i-t)}
\end{aligned}$$

where $\cdot(i,j)$ refers to the forward rate from t_i to t_j. Hence:

$$\mathbb{E}_t^{\mathbb{Q}}\left[A^2\right] = \frac{1}{n^2}\sum_{i=1}^{n} f_i^2\, e^{\sigma_i^2(t_i-t)} + \frac{2}{n^2}\sum_{j=2}^{n} f_j \sum_{i=1}^{j-1} f_i\, e^{\sigma_i^2(t_i-t)} \tag{10.7}$$

Now, as in §4.1, we assume that A is lognormally distributed at time t. To be explicit, we assume $\ln A \sim \phi(\Psi, \Sigma)$, and so

$$\Sigma = \ln\frac{\mathbb{E}_t^{\mathbb{Q}}\left[A^2\right]}{\mathbb{E}_t^{\mathbb{Q}}[A]^2} \tag{10.8}$$

$$\Psi = \ln \mathbb{E}_t^{\mathbb{Q}}[A] - \tfrac{1}{2}\Sigma \tag{10.9}$$

and now use Lemma 3.3.1.

Equivalently, and easier to implement from existing models, one is using Black's model with

- a futures spot of $\mathbb{E}_t^{\mathbb{Q}}[A]$,
- a strike of K,
- a volatility of

$$\sigma = \sqrt{\frac{1}{t_n - t}\left[\ln\frac{\mathbb{E}_t^{\mathbb{Q}}[A^2]}{\mathbb{E}_t^{\mathbb{Q}}[A]^2}\right]}$$

- a risk free rate of r_n,
- a term of $t_n - t$.

It seems that it would be exceptionally challenging to work this model to use simple and cash dividends as in §4.1, rather than the dividend yields we have here.

10.2.2 Pricing Asian Forwards

The payoff of a long position in an Asian forward is $A - K$ where A is the average and K is the strike. The value is

$$\begin{aligned} V &= e^{-r_n(t_n-t)}\mathbb{E}_t^{\mathbb{Q}}[A-K] \\ &= e^{-r_n(t_n-t)}\left[\mathbb{E}_t^{\mathbb{Q}}[A]-K\right] \end{aligned} \quad (10.10)$$

10.2.3 The 'full precision' model of Turnbull and Wakeman (TW4)

The first four cumulants of any distribution X are given by Weisstein [1999]

$$\kappa_1(X) = \mathbb{E}_t^{\mathbb{Q}}[X] = \mu \quad (10.11)$$

$$\kappa_2(X) = \mathbb{E}_t^{\mathbb{Q}}\left[(X-\mu)^2\right] \quad (10.12)$$

$$\kappa_3(X) = \mathbb{E}_t^{\mathbb{Q}}\left[(X-\mu)^3\right] \quad (10.13)$$

$$\kappa_4(X) = \mathbb{E}_t^{\mathbb{Q}}\left[(X-\mu)^4\right] - 3\kappa_2(X)^2 \quad (10.14)$$

Suppose A is the average, for which we desire the distribution. Let F be another approximating distribution. Then the Edgeworth series expansion is Turnbull and Wakeman [1991], Giamourides and Tamvakis [2002]

$$a(y) = f(y) - c_1\frac{df}{dy} + \frac{c_2}{2}\frac{d^2f}{dy^2} - \frac{c_3}{6}\frac{d^3f}{dy^3} + \frac{c_4}{24}\frac{d^4f}{dy^4} + \epsilon(y) \quad (10.15)$$

$$c_i = \kappa_i(A) - \kappa_i(F) \quad i = 1,2,3 \quad (10.16)$$

$$c_4 = \kappa_4(A) - \kappa_4(F) + 3c_2^2 \quad (10.17)$$

where a and f are the probability density functions for the distributions of A and F.
We take the approximating distribution to be a lognormal distribution whose first two moments match those of the average distribution. See [Giamourides and Tamvakis, 2002, pg 35]. (This will imply that $c_1 = c_2 = 0$.) Thus, $\ln F \sim \phi(\Psi, \Sigma)$, where Ψ and Σ have their previous meaning.
The prices of average call options are

$$\begin{aligned} c &= e^{-r\tau}\int_K^\infty (y-K)a(y)\,dy \\ &\approx e^{-r\tau}\int_K^\infty (y-K)\left(f(y) - \frac{c_3}{6}\frac{d^3f}{dy^3} + \frac{c_4}{24}\frac{d^4f}{dy^4}\right)\,dy \\ &= {}^{2}e^{-r\tau}\left[\int_K^\infty (y-K)f(y)\,dy - \frac{c_3}{6}\left.\frac{df}{dy}\right|_K + \frac{c_4}{24}\left.\frac{d^2f}{dy^2}\right|_K\right] \quad (10.18) \\ &= V_2 + e^{-r\tau}\left[-\frac{c_3}{6}\,f'(y)|_K + \frac{c_4}{24}\,f''(y)|_K\right] \quad (10.19) \end{aligned}$$

where V_2 is the value found using the TW2 approach. Likewise, or by put-call parity, the formula for average put options is the same.
Note that

$$\begin{aligned} c_3 &= \kappa_3(A) - \kappa_3(F) \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[(A-\mu)^3\right] - \mathbb{E}_t^{\mathbb{Q}}\left[(F-\mu)^3\right] \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[A^3\right] - \mathbb{E}_t^{\mathbb{Q}}\left[F^3\right] \end{aligned} \quad (10.20)$$

[2] Using integration by parts.

and

$$\begin{aligned} c_4 &= \kappa_4(A) - \kappa_4(F) \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[(A-\mu)^4\right] - 3\kappa_2(A)^2 - \mathbb{E}_t^{\mathbb{Q}}\left[(F-\mu)^4\right] + 3\kappa_2(F)^2 \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[(A-\mu)^4\right] - \mathbb{E}_t^{\mathbb{Q}}\left[(F-\mu)^4\right] \\ &= \mathbb{E}_t^{\mathbb{Q}}\left[A^4\right] - \mathbb{E}_t^{\mathbb{Q}}\left[F^4\right] - 4\mu c_3 \end{aligned} \tag{10.21}$$

since there is the equalities between A and F of all first and second order terms.
We calculate $f'(y)$ and $f''(y)$. For this, by straightforward calculus,

$$f(y) = \frac{1}{\sqrt{2\pi\Sigma}y}\exp\left[-\frac{1}{2}\frac{(\ln y - \Psi)^2}{\Sigma}\right] \tag{10.22}$$

$$f'(y) = -f(y)\frac{1}{y}\left[1 + \frac{\ln y - \Psi}{\Sigma}\right] \tag{10.23}$$

$$f''(y) = -f'(y)\frac{1}{y}\left[1 + \frac{\ln y - \Psi}{\Sigma}\right] + f(y)\frac{1}{y^2}\left[1 + \frac{\ln y - \Psi}{\Sigma}\right] - f(y)\frac{1}{y^2\Sigma} \tag{10.24}$$

The final task is to calculate $\mathbb{E}_t^{\mathbb{Q}}\left[A^3\right]$, $\mathbb{E}_t^{\mathbb{Q}}\left[A^4\right]$, $\mathbb{E}_t^{\mathbb{Q}}\left[F^3\right]$, and $\mathbb{E}_t^{\mathbb{Q}}\left[F^4\right]$. The latter two tasks are straightforward because

$$\mathbb{E}_t^{\mathbb{Q}}\left[F^m\right] = \exp\left(\Psi m + \tfrac{1}{2}\Sigma m^2\right) \quad m \in \mathbb{N} \tag{10.25}$$

from (1.9).
Let $\mathbb{E}_t^{\mathbb{Q}}\left[S(t_i)\right] = \mathbb{E}_t^{\mathbb{Q}}\left[S(t_{i-1})\right]\mathbb{E}_t^{\mathbb{Q}}\left[R_i\right]$. In other words, R_i is the price return over $t_i - t_{i-1}$. Note that

$$\ln R_i \sim \phi\left(m_-(0;t_{i-1},t_i)(t_i - t_{i-1}), \sigma^2(0;t_{i-1},t_i)(t_i - t_{i-1})\right) \tag{10.26}$$

and so by (1.9) we have

$$\mathbb{E}_t^{\mathbb{Q}}\left[R_i^m\right] = \exp\left(m\left(r(0;t_{i-1},t_i) - q(0;t_{i-1},t_i)\right)(t_i - t_{i-1}) + \tfrac{1}{2}(m^2-m)\sigma^2(0;t_{i-1},t_i)(t_i - t_{i-1})\right) \tag{10.27}$$

for $m \in \mathbb{N}$ and with exactly the same reasoning

$$\mathbb{E}_t^{\mathbb{Q}}\left[S(t_1)^m\right] = S(t)^m \exp\left(m\left(r_1 - q_1\right)(t_1 - t) + \tfrac{1}{2}(m^2-m)\sigma_1^2(t_1 - t)\right) \tag{10.28}$$

Let us define with backwards induction

$$L_n = 1 \tag{10.29}$$

$$L_i = 1 + R_{i+1}L_{i+1} \tag{10.30}$$

Then (by induction)

$$L_{i+1} = 1 + R_{i+2} + R_{i+2}R_{i+3} + \cdots + \prod_{j=i+2}^{n} R_j \tag{10.31}$$

In particular,

$$L_1 = 1 + R_2 + R_2R_3 + \cdots + \prod_{j=2}^{n} R_j$$

which shows that we have $A = \frac{1}{n}S(t_1)L_1$. Now, the events $S(t_1)$ and L_1 are serial, and hence independent, and so

$$\mathbb{E}_t^{\mathbb{Q}}\left[A^m\right] = \frac{1}{n^m}\mathbb{E}_t^{\mathbb{Q}}\left[S(t_1)^m\right]\mathbb{E}_t^{\mathbb{Q}}\left[L_1^m\right] \tag{10.32}$$

But also there is independence in the expression for L_i (L_{i+1} and R_{i+1} are serial, and hence independent) so

$$\mathbb{E}_t^{\mathbb{Q}}\left[L_i^m\right] = \sum_{j=0}^{m}\binom{m}{j}\mathbb{E}_t^{\mathbb{Q}}\left[R_{i+1}^j\right]\mathbb{E}_t^{\mathbb{Q}}\left[L_{i+1}^j\right] \tag{10.33}$$

Thus we have reduced the task to determining $\mathbb{E}_t^{\mathbb{Q}}\left[L_1^m\right]$.

This we achieve by reverse induction. Note that $L_n = 1$ so $\mathbb{E}_t^{\mathbb{Q}}\left[L_n^m\right] = 1$ for $m = 0, 1, \ldots, 4$. We then use (10.33) to induct backwards in i, finding $\mathbb{E}_t^{\mathbb{Q}}\left[L_i^m\right]$ for $m = 0, 1, \ldots, 4$, until we reach $i = 1$.

10.2.4 The case where some observations have already been made

Suppose $t_p \leq t < t_{p+1}$, so we have observed asset prices at $\{t_1, t_2, \cdots, t_p\}$. Let $\overline{S} = \frac{1}{p}\sum_{i=1}^{p} S_i$. Let A_f be the (unknown) average of the observations still to be made i.e. A_f applies to $\{t_{p+1}, t_{p+2}, \ldots, t_N\}$. Since $A = \frac{p\overline{S}+(n-p)A_f}{n}$, we have that

$$A - K = \frac{p\overline{S} + (n-p)A_f}{n} - K \tag{10.34}$$

$$= \frac{n-p}{n}(A_f - K^*) \tag{10.35}$$

$$K^* = \frac{n}{n-p}K - \frac{p}{n-p}\overline{S} \tag{10.36}$$

which shows (as in Hull [2002]) that the option can now be seen as equivalent to $\frac{n-p}{n}$ newly issued vanilla options with a strike of K^*.

This observation holds for either the 2 moment or 4 moment model, indeed, for any model whatsoever.

If $p\overline{S} \geq nK$ ie. $K^* \leq 0$ then the option is guaranteed to be exercised if a call, and is guaranteed to be worthless if a put. The call can be valued as a type of forward, as in §10.2.2

$$V = e^{-r_n(t_n - t)}\left(\mathbb{E}_t^{\mathbb{Q}}\left[A\right] - K\right) \tag{10.37}$$

$$= e^{-r_n(t_n - t)}\frac{n-p}{n}\left(\mathbb{E}_t^{\mathbb{Q}}\left[A_f\right] - K^*\right) \tag{10.38}$$

10.2.5 The choice of model

The TW2 model performs satisfactorily under all conditions. The worst differences occur with exceptionally high volatilities - with volatilities of 70% the error is of the order of 1%. Under other conditions error is typically of the order of 0.1% to 0.5%.

The question arises as to whether one should, for additional accuracy, implement the TW4 model. To this question the answer is an uncategorical no. For some input parameters this model is slightly more accurate. But for many inputs, the model is grossly less accurate. In fact, for call options with a typical input set of the nature that we have considered here, and volatilities of the order of 50% or greater, the inaccuracy is gross, and exponential with increases in volatility.

The reason for this is that the function $f(y) - \frac{c_3}{6}\frac{d^3 f}{dy^3} + \frac{c_4}{24}\frac{d^4 f}{dy^4}$, which is the approximation for $a(y)$, is not a true pdf. Although it integrates to 1, it is not everywhere positive. In fact, it will have oscillatory behaviour, as is common for 'Taylor-type' partial sums.

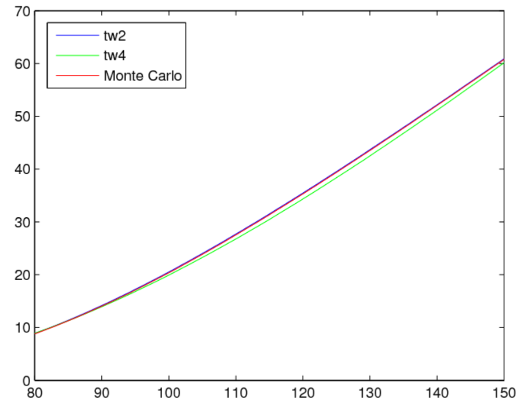

Figure 10.1: Price comparison under varying spot scenarios

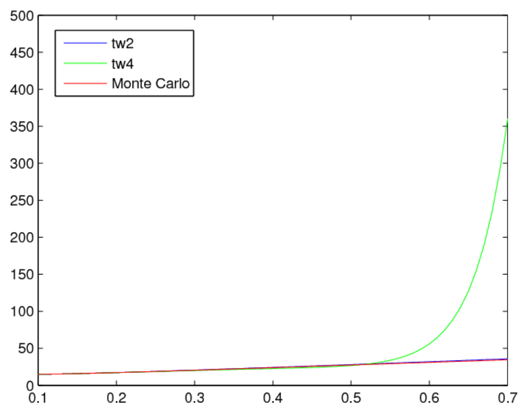

Figure 10.2: Price comparison under varying volatility scenarios: the TW4 model is invalid for high volatility regimes

10.3 Exercises

1. Write a VBA function to price a just started Asian call option using the 2-moment method of Turnbull and Wakemann. After preliminary calculations the function can call the Standard Black function previously built in Chapter 3.

   The function should be passed the spot, strike, style etc. and then also a vertical range of observation dates, a vertical range of risk free rates, a vertical range of volatilities, and a vertical range of dividend yields. The number of observations is variable, and is determined by the function as the length of the vertical ranges (which you may assume the user is bright enough to make equal). You may assume the expiry date is the last observation date.

2. Complete the calculation for finding $\mathbb{E}^{\mathbb{Q}}[S_i S_j]$.

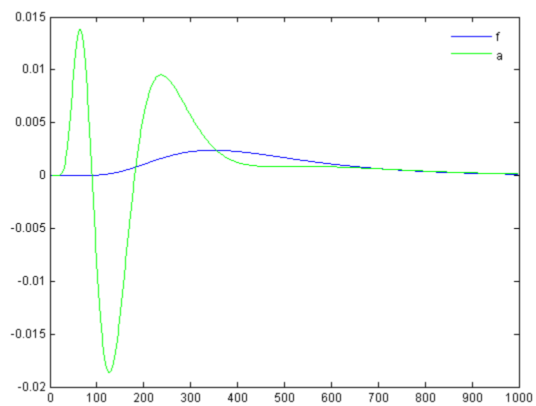

Figure 10.3: The f pdf and the approximation for a.

Chapter 11

Barrier options

A barrier option is an otherwise vanilla call or put option with a strike of K but with an extra parameter B, the barrier: the option only comes into existence (is knocked in) or is terminated (is knocked out) if the spot price crosses the barrier during the life of the option. Because there is a positive probability (in either case) of worthlessness, these options are cheaper than the corresponding vanilla option, and hence possibly more attractive to the speculator.

Even though very much path-dependent, closed form Black-Scholes type formulae for all the possible types of vanilla barrier option were developed in Rubinstein and Reiner [1991] for a stock following geometric Brownian motion and with the barrier continuously monitored. There are eight types: the barrier could be above or below the initial value of S (up or down); the barrier could cause the birth or death of the vanilla option (in or out) and the option could be a call or a put.

11.1 Closed form formulas: continual monitoring of the barrier

We develop the machinery necessary to price vanilla barrier options under the usual Black-Scholes assumptions. The final step - the calculation of the option pricing formula using risk-neutral expectations - is quite fearsome, so we only do it for one case.

The reflection principle

Suppose X is an arithmetic Brownian motion, define the running maximum and minimum by

$$M_t(X) = \max_{s \leq t} X(s) \tag{11.1}$$

$$m_t(X) = \min_{s \leq t} X(s) \tag{11.2}$$

Note that $M_t(X) \geq X(t)$ and $m_t(X) \leq X(t)$.

Suppose we have $K < b$. For every path that ends below K but previously reached b, there is another path that goes above $2b - K$: we simply reflect the path in a mirror at the level b. (Remember, it is arithmetic Brownian motion, and there is no drift.) This is the reflection principle (easy to believe but hard to prove):

$$\mathbb{P}\left[X(t) < K, M_t(X) > b\right] = \mathbb{P}\left[X(t) > 2b - K\right] = 1 - N\left(\frac{2b - K}{\sqrt{t}}\right) \tag{11.3}$$

An aside: hitting times

Recall that

$$T_b(X) = \inf_{t \geq 0} \{X(t) = b\}$$

is the hitting or stopping time: the first time that the process X reached the level b. Suppose $b > 0$. If the process never reaches b, then the hitting time is ∞ (no problem here as $\inf \emptyset = \infty$). Note that $M_t(X) > b \Leftrightarrow T_b(X) < t$. We examine the probability of hitting by time t:

$$\begin{aligned}
\mathbb{P}\left[T_b(X) \leq t\right] &= \mathbb{P}\left[T_b(X) \leq t,\ X(t) < b\right] + \mathbb{P}\left[T_b(X) \leq t,\ X(t) > b\right] \\
&= 2\mathbb{P}\left[T_b(X) \leq t,\ X(t) > b\right] \\
&= 2\mathbb{P}\left[X(t) > b\right] \\
&= 2N\left(\frac{-b}{\sqrt{t}}\right).
\end{aligned}$$

This shows that we eventually hit a.s., although (perhaps paradoxically) one can show that the expected hitting time is infinite. By differentiation, the pdf of $T_b(X)$ is given by $p(b,t) = \frac{b}{t^{3/2}} N'\left(\frac{b}{\sqrt{t}}\right)$. This recovers (5.7).

The joint distribution of the Brownian motion and its running maximum

Let the joint distribution be $f(x,m)$. $f(x,m) = 0$ for $x > m$. Now

$$\int_M^\infty \int_{-\infty}^X f(x,m)\,dx\,dm = \mathbb{P}\left[X(t) < X, M_t(X) > M\right] = \frac{1}{\sqrt{2\pi t}} \int_{2M-X}^\infty e^{-\frac{x^2}{2t}}\,dx$$

from (11.3).
First differentiate w.r.t. M [1]

$$-\int_{-\infty}^X f(x,M)\,dx = \frac{-2}{\sqrt{2\pi t}} \exp\left(-\frac{(2M-X)^2}{2t}\right)$$

Now differentiate w.r.t. X:

$$\begin{aligned}
-f(X,M) &= \frac{-2}{\sqrt{2\pi t}} \cdot \exp\left(-\frac{(2M-X)^2}{2t}\right) \cdot \frac{-2(2M-X)}{2t} \cdot -1 \\
f(X,M) &= \frac{2(2M-X)}{\sqrt{2\pi} t^{3/2}} \exp\left(-\frac{(2M-X)^2}{2t}\right) \qquad (11.4)
\end{aligned}$$

for $X < M$ and $M > 0$, and 0 otherwise. This is [Shreve, 2004, Theorem 3.7.3].

The joint distribution where the Brownian motion has drift

Let $X(t)$ be as before and let $Y(t) = \alpha t + X(t)$, a Brownian motion with drift. We want to know what is the joint density of Y and its maximum. As is well known, if we set $Z(t) = \exp\left(\frac{1}{2}\alpha^2 t - \alpha Y(t)\right)$ and define a new

[1] Here we are using the Leibnitz rule, for differentiation of a definite integral with respect to a parameter Abramowitz and Stegun [1974]:

$$\frac{d}{d\xi} \int_{\psi(\xi)}^{\phi(\xi)} f(\psi,\xi)\,d\psi = f(\phi(\xi),\xi) \frac{d\phi(\xi)}{d\xi} - f(\psi(\xi),\xi) \frac{d\psi(\xi)}{d\xi} + \int_{\psi(\xi)}^{\phi(\xi)} \frac{d}{d\xi} f(\psi,\xi)\,d\psi$$

measure with Radon-Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}} = Z(t)$ then Y is a driftless Brownian motion under $\mathbb{Q}$. Then

$$\begin{aligned}\mathbb{P}\left[Y(t) < Y, M_t(Y) < M\right] &= \mathbb{P}\left[\mathbb{1}_{\{Y(t)<Y, M_t(Y)<M\}}\right] \\ &= \mathbb{Q}\left[\frac{1}{Z(t)}\mathbb{1}_{\{Y(t)<Y, M_t(Y)<M\}}\right] \\ &= \int_{-\infty}^{Y}\int_{0}^{M} \exp\left(-\tfrac{1}{2}\alpha^2 t + \alpha y\right) f(y,m) dm\, dy\end{aligned}$$

Thus, the required pdf is

$$g(Y,M) = \exp\left(-\tfrac{1}{2}\alpha^2 t + \alpha Y\right) \frac{2(2M-Y)}{\sqrt{2\pi} t^{3/2}} \exp\left(-\frac{(2M-Y)^2}{2t}\right)$$

for $Y < M$ and $M > 0$, and 0 otherwise. This is [Shreve, 2004, Theorem 7.2.1].

Aside II: hitting times of Brownian motion with drift

Now we have

$$\begin{aligned}\mathbb{P}\left[M_t(Y) > M\right] &= \mathbb{P}\left[\mathbb{1}_{M_t(Y)>M\}}\right] \\ &= \mathbb{Q}\left[\frac{1}{Z(t)}\mathbb{1}_{\{M_t(Y)>M\}}\right] \\ &= \int_{M}^{\infty}\int_{-\infty}^{m} \exp\left(-\tfrac{1}{2}\alpha^2 t + \alpha y\right) f(y,m) dy\, dm \\ &= 1 - N\left(\frac{M-\alpha t}{\sqrt{t}}\right) + e^{2\alpha M} N\left(\frac{-M-\alpha t}{\sqrt{t}}\right)\end{aligned}$$

the last line being an exercise strictly for masochists, see [Shreve, 2004, Corollary 7.2.2] for example.
Note that differentiation w.r.t. t gives us the density for the first hitting time. Thus the pdf of $T_M(Y)$ is given by $\frac{M}{t^{3/2}} N'\left(\frac{M-\alpha t}{\sqrt{t}}\right)$. This recovers (5.10).

Pricing a barrier option

We consider one special case: pricing an up and out call, with barrier B, strike K, with $S < K < B$. We have $\ln \frac{S(t)}{S} = \left(r - \frac{1}{2}\sigma^2\right)t + \sigma X(t)$ where $X(t)$ is driftless Brownian motion. Put $\alpha = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$ and $Y(t) = \alpha t + X(t)$, so $S(t) = S e^{\sigma Y(t)}$.
Note that $S(t) = B \Leftrightarrow Y(t) = \frac{1}{\sigma}\ln\frac{B}{S} := b$, and $S(t) = K \Leftrightarrow Y(t) = \frac{1}{\sigma}\ln\frac{K}{S} := k$. Thus, the value of the option is

$$\begin{aligned}V &= e^{-rT}\mathbb{E}\left[(S(T)-K)\mathbb{1}_{\{M_T(Y)<b,\, k<Y(T)\}}\right] \\ &= e^{-rT}\int_k^b\int_y^b \left(Se^{\sigma y} - K\right)\exp\left(-\tfrac{1}{2}\alpha^2 T + \alpha y\right)\frac{2(2m-y)}{\sqrt{2\pi}T^{3/2}}\exp\left(-\frac{(2m-y)^2}{2T}\right) dm\, dy \\ &= e^{-rT}\int_k^b \left(Se^{\sigma y} - K\right)\exp\left(-\tfrac{1}{2}\alpha^2 T + \alpha y\right)\frac{1}{\sqrt{2\pi}T^{3/2}}\int_y^b 2(2m-y)\exp\left(-\frac{(2m-y)^2}{2T}\right) dm\, dy \\ &= e^{-rT}\int_k^b \left(Se^{\sigma y} - K\right)\exp\left(-\tfrac{1}{2}\alpha^2 T + \alpha y\right)\frac{1}{\sqrt{2\pi}T^{3/2}}\left[-T\exp\left(-\frac{(2m-y)^2}{2T}\right)\right]_{m=y}^{m=b} dy \\ &= e^{-rT-\frac{1}{2}\alpha^2 T}\frac{1}{\sqrt{2\pi T}}\int_k^b \left(Se^{\sigma y} - K\right)e^{\alpha y}\left[\exp\left(-\frac{y^2}{2T}\right) - \exp\left(-\frac{(2b-y)^2}{2T}\right)\right] dy\end{aligned}$$

The rest is straightforward if irritating. There are 4 definite integrals, each integrating the exponent of a quadratic with leading term $-\frac{1}{2T}$. We complete the square and evaluate each to get

$$\begin{aligned} V = S &\left[N\left(\delta_+\left(\frac{S}{K}\right)\right) - N\left(\delta_+\left(\frac{S}{B}\right)\right)\right] \\ &- e^{-rT}K\left[N\left(\delta_-\left(\frac{S}{K}\right)\right) - N\left(\delta_-\left(\frac{S}{B}\right)\right)\right] \\ &- B\left(\frac{S}{B}\right)^{-2r/\sigma^2}\left[N\left(\delta_+\left(\frac{B^2}{KS}\right)\right) - N\left(\delta_+\left(\frac{B}{S}\right)\right)\right] \\ &+ e^{-rT}K\left(\frac{S}{B}\right)^{-2r/\sigma^2+1}\left[N\left(\delta_-\left(\frac{B^2}{KS}\right)\right) - N\left(\delta_-\left(\frac{B}{S}\right)\right)\right] \end{aligned}$$

where

$$\delta_\pm(s) = \frac{\ln s + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

11.2 Discrete approaches: discrete monitoring of the barrier

Discrete tree type models for barrier options have been in existence for some time, but have in general been quite slow, because of the need to induct backwards through the tree, much like a tree model for an American option. (Some arguments can be invoked to reduce this problem.) Moreover, the payoff can be arbitrarily exotic.

However, the issue of when and how often we can hit the barrier is relevant. The price of a barrier option found using a lattice is sensitively dependent on the location of the barrier within the lattice. Given a lattice (a value of N), the price of a particular style of barrier option with a particular strike, will be the same for the barrier being anywhere between two layers of nodes in the spatial dimension. Hull [2002] defines the *inner barrier* as the layer formed by nodes on the inside of the true barrier and the *outer barrier* as the layer formed by nodes just outside the true barrier. A binomial tree assumes the outer barrier to be the true barrier because the barrier conditions are first used by these nodes. This causes option specification error where the option is priced at a different barrier to the one specified by the contract.

Thus, approaches have been developed to ensure that the barrier coincides withe the outer barrier (or is just to the right side of it). Boyle and Lau [1994] suggest using the standard binomial model of Cox et al. [1979] but with the number of time steps equal to

$$N(i) = \frac{i^2\sigma^2(T-t)}{\left(\ln\frac{S}{B}\right)^2} \tag{11.5}$$

for some choice of i. As Figure 11.1 shows, if the value of N is not chosen to be one of these $N(i)$, it is better to be under rather than over.

11.3 Adjustments for actual frequency of observation

In the real world, the question of whether or not a barrier option has gone in (and is now active) or gone out (and is now worthless) is determined by discrete, daily, closes. This is for legal and technical reasons: it is technically difficult to determine if in continuous time a barrier has been breached, and legally there will be a question of whether or not this has actually happened, because of the problem of having to record tick index data.

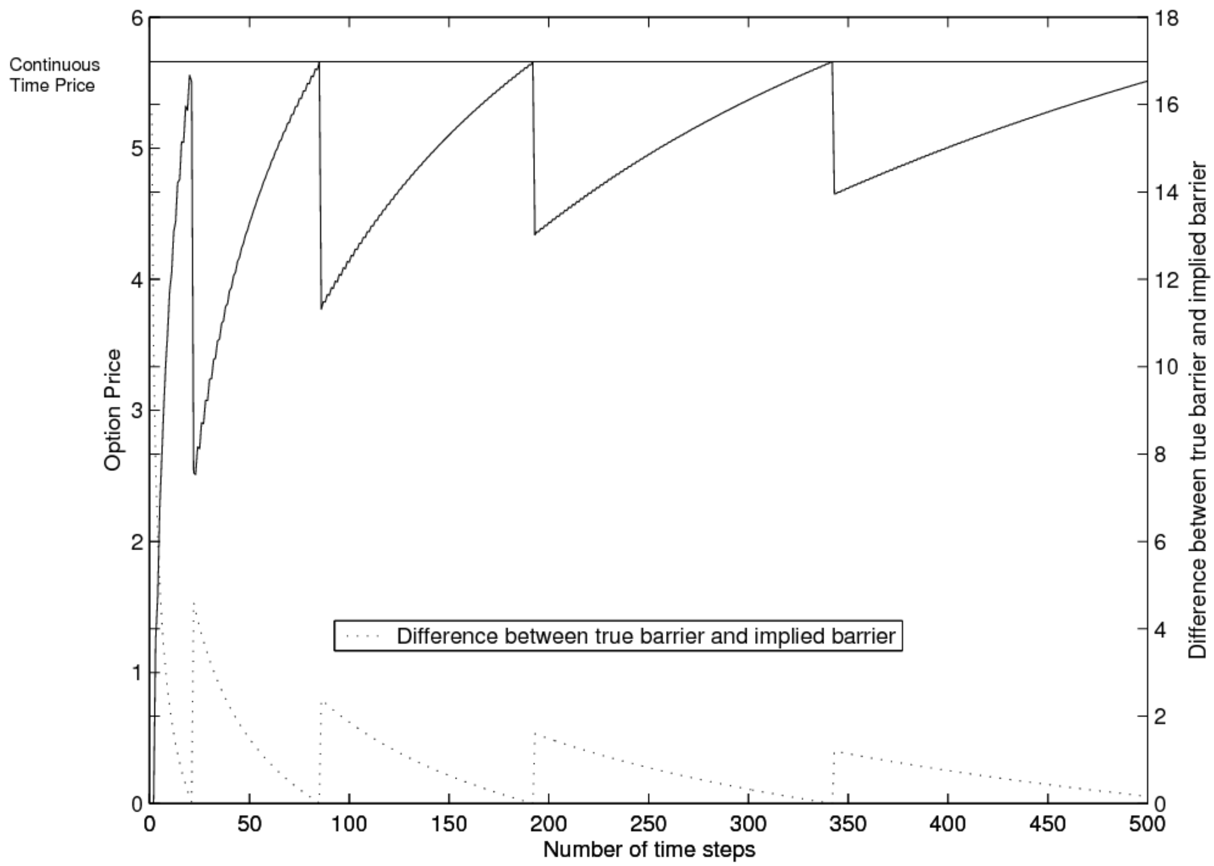

Figure 11.1: Tree model misspecification versus the 'true' price

A first attempt would be to use a tree approach. A naïve first guess is to have a tree with as many steps as the number of business days to expiry. However, this allows the misspecification seen earlier. One can develop models which deal with both of these issues. For example, one builds a tree with a large number of time steps, but where that number is divisible by the number of business days to expiry, and checks the barrier only at the end of each business day. The total number of time steps is chosen with regard to the Boyle and Lau [1994] result. However, computational time obviously becomes an issue.

Note that if an option is subject to discrete observations with N observation points before expiry remaining, then in principle the option price can be calculated exactly via N-variate cumulative normals. For $N \leq 3$ fast, double precision algorithms are available West [2005], for $N > 3$ they are not, and any method of evaluating the cumulative probabilities (numerical or quasi-random integration techniques) will be no quicker - in fact, materially slower - than direct Monte Carlo evaluation of the original product.

The two most feasible approaches are now considered.

11.3.1 Adjusting continuous time formulae for the frequency of observation

In Broadie et al. [1997] the formulae of Rubinstein and Reiner [1991] are considered and a correction for the discreteness is made. They price these options by applying a continuity correction to the barrier. This correction shifts the barrier away from spot by a multiplicative factor of $\exp(\pm\beta\sigma\sqrt{\tau})$, where τ is the time between monitoring (so typically one day) and $\beta = 0.5826\ldots$ is determined from the Riemann zeta function. More formally, let the option which has N remaining equally distributed observations to be made have (unknown) value V_N. Thus, the valuation date is t, the expiry date is T, and there are observations at t_i for $i = 1,\ 2,\ \ldots,\ N$ where $t = t_0 < t_1 < \cdots < t_N = T$, and $t_i - t_{i-1} = \tau$ is constant. Then

$$V_N(B) = V_\infty(Be^{\pm\beta\sigma\sqrt{\tau}}) + o\left(\frac{1}{\sqrt{m}}\right) \tag{11.6}$$

where $\pm$ corresponds to an up/down option.

This approach is highly attractive, not least because it is of course very fast, however, the solution is not as universally accurate as is often thought: it is quite inaccurate under many combinations of pricing inputs - in particular if we are close to expiry, or if the spot is close to the barrier.

This approach was generalised to other discretely monitored path dependent options in Broadie et al. [1999].

11.3.2 Interpolation approaches

The attractive idea of Levy and Mantion [1997] is to find a price for the N-observations remaining option via interpolation, between the known closed form formula (infinitely many observations remaining), and the computed prices for a low number of observations remaining.

The known closed form price is denoted V_∞. The Levy and Mantion [1997] *ansatz* is

$$V_N = V_\infty + aN^{-1/2} + bN^{-1} \tag{11.7}$$

a type of second-order Taylor series expansion. The requirement in each case then is to determine a and b. We have

$$V_1 = V_\infty + a + b \tag{11.8}$$

$$V_2 = V_\infty + a\sqrt{\tfrac{1}{2}} + b\tfrac{1}{2} \tag{11.9}$$

and so a and b are found using Cramer's rule. V_1 and V_2 are found as functions of univariates and bivariates respectively. Nowadays one would extend the ansatz to use V_3 as well.

As an example, let us take this approach (second order) for the up and out call as before.

For V_1 we have payoff if $S(T) \in (K, B)$. This can be evaluated mechanically, or we note that this value is the value of a call struck at K, less the value of a call struck at B, less a cash-or-nothing call struck at B.

For V_2: now is time t_0, observations occur at t_1 and $t_2 = T$, the expiry. If $S(t_1) < B$ and $S(t_2) < B$ then the payoff is $\max(S(t_2) - K, 0)$, 0 otherwise. Currently, the option is in i.e. $S(t_0) < B$.

Let X_1, X_2 be the random $\phi(0,1)$ normal variables that are the 'stock return draws' for the two periods. In other words,

$$\begin{aligned} m_\pm &= r - q \pm \tfrac{1}{2}\sigma^2 \\ S(t_i) &= S(t_{i-1})\exp(m_-\tau + \sigma\sqrt{\tau}X_i) \quad (i = 1,\ 2) \end{aligned}$$

where $\tau = t_1 - t_0 = t_2 - t_1$, measured in years. For there to be a payoff we require

$$\begin{aligned} X_1 &\in \left(-\infty, \frac{\ln\frac{B}{S(t_0)} - m_-\tau}{\sigma\sqrt{\tau}}\right) \\ X_1 + X_2 &\in \left(\frac{\ln\frac{K}{S(t_0)} - 2m_-\tau}{\sigma\sqrt{\tau}}, \frac{\ln\frac{B}{S(t_0)} - 2m_-\tau}{\sigma\sqrt{\tau}}\right) \end{aligned}$$

The crucial point is that because the returns are serial, they are independent, and so (X_1, X_2) are distributed bivariate normal with zero correlation. Thus, the value is

$$
\begin{aligned}
V_2 &= e^{-2r\,\tau} \int_{-\infty}^{x_1^-} \int_{{}^K x_2^- - X_1}^{{}^B x_2^- - X_1} (S(t_0) \exp(2m_- \tau + \sigma\sqrt{\tau}(X_1 + X_2)) - K) N'(X_1) N'(X_2)\, dX_2\, dX_1 \\
x_1^\pm &= \frac{\ln \frac{B}{S(t_0)} - m_\pm \tau}{\sigma\sqrt{\tau}} \\
{}^Y x_2^\pm &= \frac{\ln \frac{Y}{S(t_0)} - 2m_\pm \tau}{\sigma\sqrt{\tau}}
\end{aligned}
$$

which can be routinely evaluated explicitly using the bivariate cumulative normal function, to get

$$
\begin{aligned}
V_2 = S(t_0) e^{-2q\tau} &\left[N_2\left(x_1^+, \frac{{}^B x_2^+}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) - N_2\left(x_1^+, \frac{{}^K x_2^+}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right] \\
- e^{-2r\tau} K &\left[N_2\left(x_1^-, \frac{{}^B x_2^-}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) - N_2\left(x_1^-, \frac{{}^K x_2^-}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right]
\end{aligned}
$$

Chapter 12

Forward starting options

A forward starting option is an option purchased today which will start at some date in the future with the strike being a function of some growth factor. This growth factor we denote with α; typically $\alpha \approx 1$ or $\alpha \approx e^{r(0;T_1,T_2)(T_2-T_1)}$ where T_1 is the forward start date, T_2 is the termination date. Here we will assume that there is no skew or forward skew in volatility.

12.1 Simplest cases

12.1.1 Constant spot

The simplest variation is the 'constant spot' one: here $V(T_2) = \max\left(\eta\left(\frac{S(T_2)}{S(T_1)}M - \alpha M\right), 0\right)$ where M is the constant. Clearly

$$\begin{aligned} V &= e^{-r_2(T_2-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\max\left(\eta\left(\frac{S(T_2)}{S(T_1)}M - \alpha M\right), 0\right)\right] \\ &= e^{-r_2(T_2-t)}\,\mathrm{SBf}(S = M, K = \alpha M, r(0;T_1,T_2), q(0;T_1,T_2), \sigma(0;T_1,T_2), T_1, T_2, \eta) \\ &= e^{-r_1(T_1-t)}\,\mathrm{BS}(S = M, K = \alpha M, r(0;T_1,T_2), q(0;T_1,T_2), \sigma(0;T_1,T_2), T_1, T_2, \eta) \end{aligned}$$

where $\mathrm{SBf}(S, K, r, q, \sigma, t, T, \eta)$ is the SAFEX-Black forward price of an option with spot S, strike K, risk free rate r, dividend yield q, volatility σ, valuation date t, expiry date T, and style η (+1 for a call and -1 for a put).

$\mathrm{BS}(\cdots)$ is the similarly defined Black-Scholes formula. Here we have used the fact that $\mathrm{SBf}(S, K, r, q, \sigma, t, T, \eta) = e^{r(T-t)}\mathrm{BS}(S, K, r, q, \sigma, t, T, \eta)$.

This option is called constant spot because at T_1 the option is on a spot asset worth M. M is fixed at inception; it might be $S(0)$, for example.

12.1.2 Standard forward starting options

The next, slightly more complicated version has payoff $V(T_2) = \max(\eta(S(T_2) - \alpha S(T_1)), 0)$. Superficially it is like the previous case with $M = S(T_1)$, but this is not a constant. The value of the option is, from risk neutral valuation, clearly

$$\begin{aligned} V &= e^{-r_2(T_2-t)}\mathbb{E}_t^{\mathbb{Q}}\left[V(T_2)\right] \\ &= e^{-r_2(T_2-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\max(\eta(S(T_2) - \alpha S(T_1)), 0)\right] \end{aligned} \tag{12.1}$$

We make a key observation which allows us to proceed to deal with the 'stochastic M'. The returns in the period $[t, T_1]$ and $[T_1, T_2]$ are serial, hence independent. Thus

$$\begin{aligned}
& V \\
= \; & e^{-r_2(T_2-t)}\mathbb{E}_t^{\mathbb{Q}}\left[S(T_1)\max\left(\eta\left(\frac{S(T_2)}{S(T_1)}-\alpha\right),0\right)\right] \\
= \; & e^{-r_2(T_2-t)}\mathbb{E}_t^{\mathbb{Q}}\left[S(T_1)\right]\mathbb{E}_t^{\mathbb{Q}}\left[\max\left(\eta\left(\frac{S(T_2)}{S(T_1)}-\alpha\right),0\right)\right] \\
= \; & e^{-r_2(T_2-t)}e^{(r_1-q_1)(T_1-t)}S(t)\mathbb{E}_t^{\mathbb{Q}}\left[\max\left(\eta\left(\frac{S(T_2)}{S(T_1)}-\alpha\right),0\right)\right] \\
= \; & e^{-r_2(T_2-t)}e^{(r_1-q_1)(T_1-t)}\,\mathrm{SBf}(S(t), K=\alpha S(t), r(0;T_1,T_2), q(0;T_1,T_2), \sigma(0;T_1,T_2), T_1, T_2, \eta) \\
= \; & S(t)e^{-q_1(T_1-t)}\,\mathrm{BS}(\mathrm{Spot}=1, K=\alpha, r(0;T_1,T_2), q(0;T_1,T_2), \sigma(0;T_1,T_2), T_1, T_2, \eta)
\end{aligned}$$

12.2 Additive (ordinary) cliquets

Very often a series of these options are traded; the i^{th} element of the series is active in the interval $[T_{i-1}, T_i]$ where $t = T_0 < T_1 < \cdots < T_n$. Each interval is called a tranche. The entire set of options is called a cliquet[1]
Very often a merchant bank will sell the following product to an asset manager: a cliquet of puts struck at $\alpha = 1$ (the floor), and buy from them a cliquet of calls struck at $\omega = 1.20$ (the cap), say. This might be calculated to have net nil premium. Thus, the asset manager has bought a series of forward starting collars. This ensures IN EACH TRANCHE that the asset manager's portfolio is protected against a reduction in the nominal value of their portfolio. In return, they give away the potential performance above a certain level.
The valuation of the cliquet(s) is simply a matter of carefully performing sums with the appropriate α's/ω's, forward rates, $\pm$'s, η's, etc.

12.3 Multiplicative cliquets

Cliquet structures are very common. However, as is usually the case, there are common variations, particularly in South Africa, one of which is clearly superior to the above product: the MULTIPLICATIVE variation, rather than the ADDITIVE variation. The idea is to protect the portfolio against a fall throughout its life, not just at each stage of its life. Suppose for illustration that an unprotected portfolio falls by 10% in each tranche of a five tranche product. A cliquet of protective puts as above would give the option holder a payoff of 10% in the first year, 9% in the second year, 8.1% in the third year, and so on. These cash payoffs are reinvested elsewhere. In the alternative, the payoff is reinvested into the basket, and the protection level is restored for the next tranche. Conversely, if the portfolio outperforms the cap, the asset manager liquidates a portion of the basket down to the cap level and the protection level is continued at the cap level.
If α is the floor level and ω is the cap level then the portfolio is guaranteed to terminate with value in $[\alpha^n, \omega^n]$. Furthermore, the burden of raising or reinvesting cash is removed from the asset manager.
Thus, the vanilla product is a cliquet of options against the spot. The above variation is a cliquet of options against the basket, which may be discretely adjusted up or down at each reset date. Thus, although the spot is subject to geometric Brownian motion, the basket is only subject to GBM in the interval (T_{i-1}, T_i), at each reset, it may jump discontinuously.

[1] A French word for pawl. Lest that translation does not help, a pawl is (according to the Oxford English dictionary) 'a pivoted bar or lever whose free end engages with the teeth of a cogwheel or ratchet, allowing it to move or turn in one direction only'.

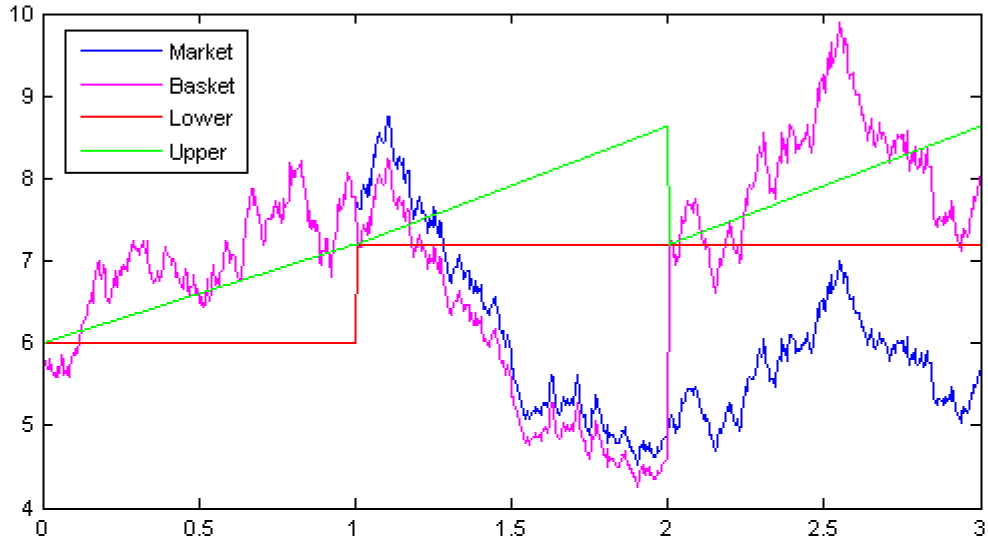

Figure 12.1: The evolution of the market, the basket, and the protection 'triangles'

We will discuss pricing not only at initiation but during the life of the product.
Define (with $\sim$) and notice (with $=$) the following terms:

$$
\begin{aligned}
\alpha &\sim \text{lower strike} \\
\omega &\sim \text{upper strike} \\
B_0^+ &\sim 1 \\
B_i^- &\sim \text{hypothetical/actual basket value immediately before the } i^{th} \text{ resetting} \\
B_i^+ &\sim \text{hypothetical/actual basket value immediately after the } i^{th} \text{ resetting} \\
S_0 &\sim \text{initial spot} \\
S_i &\sim \text{spot at the termination of the } i^{th} \text{ tranche} \\
S &\sim \text{current spot} \\
a &\sim \text{current tranche} \\
\frac{S_i}{S_{i-1}} &= \text{performance of the stock in the } i^{th} \text{ tranche} \\
P_i &\sim \max\left(\alpha, \min\left(\frac{S_i}{S_{i-1}}, \omega\right)\right) \\
B_i^- &= B_{i-1}^+ \frac{S_i}{S_{i-1}} \\
B_i^+ &= B_{i-1}^+ P_i
\end{aligned}
$$

These quantities are built inductively, either for known past data, or in the sense of expectations. We only need to consider the case of expectations. In this, it is key to note that by the independence of serial market events, the expectation of serial market events is the product of their expectations. Thus

$$\mathbb{E}_t^{\mathbb{Q}}\left[B_i^-\right] = \mathbb{E}_t^{\mathbb{Q}}\left[B_{i-1}^+\right]\frac{\mathbb{E}_t^{\mathbb{Q}}\left[S_i\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[S_{i-1}\right]} \tag{12.2}$$

$$\mathbb{E}_t^{\mathbb{Q}}\left[B_i^+\right] = \mathbb{E}_t^{\mathbb{Q}}\left[B_{i-1}^+\right]\mathbb{E}_t^{\mathbb{Q}}\left[P_i\right] \tag{12.3}$$

Firstly the expected values of the S_i are determined from the current spot S and risk free rates r_i and dividend yields q_i. Thus, it all boils down to determining $\mathbb{E}_t^{\mathbb{Q}}[P_i]$. For this,

$$P_i = \max\left(\alpha, \min\left(\frac{S_i}{S_{i-1}}, \omega\right)\right) = \alpha + \max\left(\frac{S_i}{S_{i-1}} - \alpha, 0\right) - \max\left(\frac{S_i}{S_{i-1}} - \omega, 0\right) \tag{12.4}$$

Hence

$$\begin{aligned}
&\mathbb{E}_t^{\mathbb{Q}}\left[P_i\right] \\
&= \alpha + \mathbb{E}_t^{\mathbb{Q}}\left[\max\left(\frac{S_i}{S_{i-1}} - \alpha, 0\right)\right] - \mathbb{E}_t^{\mathbb{Q}}\left[\max\left(\frac{S_i}{S_{i-1}} - \omega, 0\right)\right] \\
&= \alpha + \mathrm{SBf}\left(1, \alpha, r(0; t_{i-1}, t_i), q(0; t_{i-1}, t_i), \sigma(0; t_{i-1}, t_i), t_{i-1}, t_i, 1\right) \\
&- \mathrm{SBf}\left(1, \omega, r(0; t_{i-1}, t_i), q(0; t_{i-1}, t_i), \sigma(0; t_{i-1}, t_i), t_{i-1}, t_i, 1\right)
\end{aligned} \tag{12.5}$$

where $\mathrm{SBf}(S, K, r, q, \sigma, t, T, \eta)$ is the SAFEX Black forward option price with spot S, strike K, risk free rate r, dividend yield q, volatility σ, valuation date t, expiry date T, and style η (+1 for a call and -1 for a put).
The above calculation is for a tranche that is truly in the future. For the active tranche, we would have

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[P_a\right] = \alpha &+ \mathrm{SBf}\left(\frac{S}{S_{a-1}}, \alpha, r(t_a), q(t_a), \sigma(t_a), t, t_a, 1\right) \\
&- \mathrm{SBf}\left(\frac{S}{S_{a-1}}, \omega, r(t_a), q(t_a), \sigma(t_a), t, t_a, 1\right)
\end{aligned} \tag{12.6}$$

To find the price, we require the sum of the present values of the expected payoffs at each tranche expiry. This is given by

$$\begin{aligned}
V(t) &= \sum_{i=a}^{n} e^{-r_i(t_i - t)}\mathbb{E}_t^{\mathbb{Q}}\left[B_i^+ - B_i^-\right] \\
&= \sum_{i=a}^{n} e^{-r_i(t_i - t)}\left[\mathbb{E}_t^{\mathbb{Q}}\left[B_i^+\right] - \mathbb{E}_t^{\mathbb{Q}}\left[B_i^-\right]\right]
\end{aligned} \tag{12.7}$$

The summation is quite intuitive: at time t_i the basket holder gives away the basket B_i^- and receives the basket B_i^+ in its place.
Another variation is that the 'basket corrections' do not take place at the end of each tranche, rather, a single correction take place at the end of the life of the product (the DEFERRED case, as opposed to the canonical NON-DEFERRED case). In this case

$$\begin{aligned}
V(t) &= e^{-r_n(t_n - t)}\mathbb{E}_t^{\mathbb{Q}}\left[B_n^+ - \frac{S(t_n)}{S(t_0)}\right] \\
&= e^{-r_n(t_n - t)}\mathbb{E}_t^{\mathbb{Q}}\left[B_n^+\right] - e^{-q_n(t_n - t)}
\end{aligned} \tag{12.8}$$

In these options, the nature of implied volatility is crucial. Even to have working models of the skew itself can be a difficult exercise, see, for example, Hagan et al. [2002]. Not only is there a skew in volatility, but because there are forward starting options, there are forward starting skews. To obtain forward starting skews is non-trivial.

A simple and commonly chosen option is as follows: assume that the forward skew for the period from T_1 to T_2 is equal to the skew for term $T_2 - T_1$ (that is, for time $t + T_2 - T_1$). Variations are possible: for example, in the case of the SABR skew model, it might be assumed that the unobserved parameters of the skew are constant, rather that the actual skew is constant. The forward atm volatility will be used.

In general, it can be shown that it is impossible to find a static hedge of the forward smile. With no way to lock in this forward smile, the product is model-dependent, i.e, two models that fit the smile would not necessarily give the same value for the product. So, it is important to check any given model's forward smile behaviour. See Quessette [2002].

12.4 Exercises

1. (exam 2004) In the course of pricing forward starting cliquet options, we developed pricing models that depended entirely on the risk neutral expected 'collared performance', which in the i^{th} tranche is denoted P_i. Suppose an asset manager is prepared to participate in the performance of their basket of stocks in the following way: suppose $\beta > \alpha > \gamma$ are constants. If $x = \frac{S(t_i)}{S(t_{i-1})}$, then the collared performance will be $P_i = \begin{cases} x + \alpha - \gamma & \text{if} \quad x < \gamma \\ \alpha & \text{if} \quad \gamma \le x < \alpha \\ x & \text{if} \quad \alpha \le x < \beta \\ \beta & \text{if} \quad \beta \le x \end{cases}$

   Draw the graph of P_i, and then write P_i as a combination of factors which look like option payoffs. Hence determine $\mathbb{E}_t^{\mathbb{Q}}[P_i]$.

2. (exam 2005) Suppose an employer has a pension fund for its employees. The assets of the pension fund are invested in a diverse basket of assets with a volatility of σ and dividend yield of q. Volatility, dividend yield, and the risk free rate are all constant.

   The basket currently has value L. All future income from the assets will be immediately reinvested in the basket as it is received.

   The pension fund will now be closed, that is, there will be no more members added, and no more member contributions received. The fund will now be allowed to 'die out': assume that withdrawals from the pension fund are only made at the end of each year, and only because of death, and that it is estimated (via actuarial considerations) that at the start of the i^{th} year there will be a proportion of p_i members still in the fund.

   The employer has guaranteed the pension fund returns of α per annum: that is, if the assets do not increase by α% in any year then the employer will inject cash into the scheme to that level. They have undertaken to do this until the fund closes completely (until the last member dies).

   Find the value of the obligation that the employer has.

3. (exam 2008) This question concerns a variation on the deferred forward starter product we have seen. Suppose now is time t_0, with annual anniversaries $t_1, t_2, \ldots, t_n$. All term structures are flat. For avoidance of doubt, all the time periods $t_i - t_{i-1}$ are of equal length.

   At the end of n years, a derivative will pay $\prod_{i=1}^{n} p_i - \frac{S(t_n)}{S(t_0)}$, where $p_i = \max\left(1, \beta \frac{S(t_i)}{S(t_{i-1})}\right)$. Find the value of β that makes the product have 0 price.

Bibliography

Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables*. Dover, 1974. 24, 25, 59

Peter John Acklam. A small paper on Halley's method, 2002. URL `http://home.online.no/~pjacklam`. 6

Peter John Acklam. An algorithm for computing the inverse normal cumulative distribution function, 2004. URL `http://home.online.no/~pjacklam`. 6

G. Barone-Adesi and R. Whaley. Efficient analytic approximation of American option values. *Journal of Finance*, 42(2):301–320, 1987. 18

P. Bjerksund and G. Stensland. Closed form approximation of American options. *Scandinavian Journal of Management*, 9:87–99, 1993. 18

P. Bjerksund and G. Stensland. Closed form valuation of American options, 2002. URL `http://nhh.no/for/dp/2002/0902.pdf`. 18

Tomas Björk. *Arbitrage Theory in Continuous Time*. Oxford University Press, 1998. 18

Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 73(May-June):637–659, 1973. 18

Michael Bos and Stephen Vandermark. Finessing fixed dividends. *Risk*, 15(9):157–158, 2002. 18

P Boyle and S H Lau. Bumping up against the barrier with the binomial method. *Journal of Derivatives*, 1 (4), 1994. 61, 62

M. Broadie, P. Glasserman, and S. Kou. Connecting discrete and continuous path-dependent options. *Finance and Stochastics*, 3:55–82, 1999. 63

Mark Broadie, Paul Glasserman, and S.G. Kou. A continuity correction for discrete barrier options. *Journal of Mathematical Finance*, 7(4):325–349, 1997. 62

Charles J. Corrado and Thomas W. Miller. A note on a simple, accurate formula to compute implied standard deviations. *Journal of Banking and Finance*, 20(3):595–603, 1996. 15

John Cox, Stephen Ross, and Mark Rubinstein. Option pricing: a simplified approach. *Journal of Financial Economics*, 7:229–263, 1979. 18, 61

Kresimir Demeterfi, Emanuel Derman, Michael Kamal, and Joseph Zou. More than you ever wanted to know (but less than can be said) about volatility swaps, March 1999. Goldman Sachs Quantitative Strategies Research Notes. 28

E. Derman. Regimes of volatility, 1999. Goldman Sachs Quantitative Strategies Research Notes. 4

Emanuel Derman and Iraj Kani. Stochastic implied trees: Arbitrage pricing with stochastic term and strike structure of volatility. *International Journal of Theory and Applications in Finance*, 1:61–110, 1998. 4

Z Drezner. Computation of the bivariate normal integral. *Mathematics of Computation*, 32:277–279, 1978. 7

Z. Drezner and G.O. Wesolowsky. On the computation of the bivariate normal integral. *Journal of Statist. Comput. Simul.*, 35:101–107, 1989. 7

Bruno Dupire. Pricing with a smile. *Risk*, 7(1), January 1994. 4

Bruno Dupire. Pricing and hedging with smiles. In M.A.H. Dempster and S.R. Pliska, editors, *Mathematics of Derivative Securities*, pages 103–111. Cambridge University Press, Cambridge, 1997. 4

Jean-Pierre Fouque, George Papanicolaou, and K. Ronnie Sircar. *Derivatives in Financial Markets with Stochastic Volatility*. Cambridge University Press, 2000. 4

Volf Frishling. A discrete question. *Risk*, January(1), 2002. 18, 19

H. Geman, N. El Karoui, and J. Rochet. Changes of numeraire, changes of probability measure and option pricing. *Journal of Applied Probability*, 32, 1995. 38

D. Gentle. Basket weaving. *Risk*, 6(6):51–52, 1993. 34

Alan Genz. Numerical computation of rectangular bivariate and trivariate normal and t probabilities. *Statistics and Computing*, 14:151–160, 2004. URL `http://www.sci.wsu.edu/math/faculty/genz/homepage`. 7, 9, 41

R. Geske. A note on an analytic formula for unprotected American call options on stocks with known dividends. *Journal of Financial Economics*, 7(A):375–380, 1979a. 19

R. Geske. The valuation of compound options. *Journal of Financial Economics*, 7:63–81, 1979b. 31

Daniel Giamourides and Michael Tamvakis. Asumptotic distribution expansions in option pricing. *Journal of Derivatives*, summer:33–44, 2002. 53

Patrick S. Hagan, Deep Kumar, Andrew S. Lesniewski, and Diana E. Woodward. Managing smile risk. *WILMOTT Magazine*, September:84–108, 2002. URL `http://www.wilmott.com/pdfs/021118_smile.pdf`. 4, 68

J. M. Harrison and S. R. Pliska. Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes and their Application*, 11:215–260, 1981. 38

Espen G. Haug, Jørgen Haug, and Alan Lewis. Back to basics: a new approach to the discrete dividend problem. *WILMOTT Magazine*, September:37–47, 2003. 18, 19

Espen Gaarder Haug. *The complete guide to option pricing formulas*. McGraw - Hill, 1998. 7

Steven L. Heston. A closed form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2), 1993. 4

John Hull. *Options, Futures, and Other Derivatives*. Prentice Hall, fifth edition, 2002. 7, 55, 61

John Hull and Alan White. The pricing of options on assets with stochastic volatilities. *Journal of Finance*, 42:281–300, 1987. 4

Herb Johnson. Options on the maximum or the minimum of several assets. *Journal of Financial and Quantitative Analysis*, 22:277–283, 1987. 36, 41, 45

Edmond Levy and Francois Mantion. Discrete by nature. *Risk*, 10(1):74–75, 1997. 63

S. Manaster and G. Koehler. The calculation of implied variances from the Black-Scholes model. *Journal of Finance*, 37(1):227–230, 1982. 15

William Margrabe. The value of an option to exchange one asset for another. *The Journal of Finance*, 23: 177–186, 1978. 36, 37

B. Moro. The full Monte. *Risk*, 8(2), February 1995. 6

Marek Musiela and Marek Rutkowski. *Martingale Methods in Financial Modelling*, volume 36 of *Applications of Mathematics: Stochastic Modelling and Applied Probability*. Springer, 1998. 34

Peter Ouwehand and Graeme West. Pricing rainbow options. *WILMOTT Magazine*, May:74–80, 2006. 36

Richard Quessette. New products, new risks. *Risk*, 15(3):97–100, 2002. 69

O. Reiss and U. Wystup. Computing option price sensitivities using homogeneity and other tricks. *Journal of Derivatives*, 9(2):41 – 53, 2001. 45

R. Roll. An analytic valuation formula for unprotected American call options on stocks with known dividends. *Journal of Financial Economics*, 5:251–258, 1977. 19

Mark Rubinstein. Somewhere over the rainbow. *Risk*, 4:63–66, November 1991. 36

Mark Rubinstein. Implied binomial trees. *Journal of Finance*, 49(3):771–818, 1994. 34

Mark Rubinstein and Eric Reiner. Breaking down the barriers. *Risk*, 4(8):28–35, 1991. 23, 58, 62

M. Schervish. Multivariate normal probabilities with error bound. *Applied Statistics*, 33:81–87, 1984. 9

Steven E. Shreve. *Stochastic Calculus for Finance II: continuous time models*. Springer, 2004. 59, 60

René M Stulz. Options on the minimum or the maximum of two risky assets. *Journal of Financial Economics*, XXXIII(1):161–185, 1982. 36, 40, 41

Stuart M. Turnbull and Lee Macdonald Wakeman. A quick algorithm for pricing European average options. *Journal of Financial and Quantitative Analysis*, 26:377–389, 1991. 51, 53

Eric W. Weisstein. Cumulant. *Eric Weisstein's World of Mathematics*, 1999. http://mathworld.wolfram.com/Cumulant.html. 53

Graeme West. Better approximations to cumulative normal functions. *WILMOTT Magazine*, May:70–76, 2005. URL http://www.wilmott.com/pdfs/090721_west.pdf. See the website for vb and c++ implementation of the functions. 9, 41, 62

Graeme West. The Mathematics of South African Financial Markets and Instruments: Lecture notes, 2009. URL http://www.finmod.co.za/safm.pdf. 19, 21

R.E. Whaley. On the valuation of American call options on stocks with known dividends. *Journal of Financial Economics*, 9:207–211, 1981. 19

Uwe Wystup. Rebates, 2002. URL http://www.mathfinance.de.