

# Pricing equity-linked pure endowments with risky assets that follow Lévy processes

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## Abstract

We investigate the pricing problem for pure endowment contracts whose life contingent payment is linked to the performance of a tradable risky asset or index. The heavy tailed nature of asset return distributions is incorporated into the problem by modeling the price process of the risky asset as a finite variation Lévy process. We price the contract through the principle of equivalent utility. Under the assumption of exponential utility, we determine the optimal investment strategy and show that the indifference price solves a non-linear partial-integro-differential equation (PIDE). We solve the PIDE in the limit of zero risk aversion, and obtain the unique risk-neutral equivalent martingale measure dictated by indifference pricing. In addition, through an explicit-implicit finite difference discretization of the PIDE we numerically explore the effects of the jump activity rate, jump sizes and jump

skewness on the pricing and the hedging of these contracts.

*Key words:* Equity-Linked Pure Endowments, Equity Indexed Annuities,  
Indifference Pricing, Exponential Utility, Lévy Processes, Heavy tailed distribution

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## 1 Introduction

In recent years equity-indexed annuities (EIAs) have increased in popularity and have broken the US \$6 billion per annum mark. Under these contracts, the insured makes an initial deposit (or several deposits) and during the deferral period the interest accrual on the fund is linked to the performance of a stock or index, typically the Standard and Poors (S&P) 500 Index. The popularity of these contracts is due mainly to the guaranteed minimum return that the insured receives; this guarantee allows her to benefit from the upside potential of equity growth without full exposure to the downside risk. Tiong (2000) studied the pricing of the embedded financial option in such contracts with the asset price following a geometric Brownian motion. The author carried out the analysis in a risk-neutral framework through Esscher transforms (Gerber and Shiu, 1994) and developed explicit pricing results within that framework. Although the mortality risk component was ignored in the paper, it was the first published results for pricing EIA products.

EIA products are fairly complicated instruments and untangling the role that mortality risk plays is best studied by first analyzing their basic building blocks: equity-linked pure endowments. The economic market for such products is incomplete due to the presence of mortality risk and standard no-arbitrage arguments do not provide unique prices. Early work on pricing equity-linked contracts using a combination of no-arbitrage arguments and actuarial principles was carried out by Brennan and Schwartz (1976, 1979a,b). Option pricing in incomplete markets has been studied by many authors including: Föllmer and Sondermann (1986); Föllmer and Schweizer (1991); Schweizer (1996) who introduce the so-called risk-minimization hedging schemes (later used by Møller (1998, 2001) for equity-linked insurance products); Carr, Geman, and Madan (2001) who put forward a new methodology which interpolates between no-arbitrage pricing and expected utility maximization; and Bühmann, Delbaen, Embrechts, and Shiryaev (1996) who use Esscher transformations methods to sin-

gle out a particular measure from the collection of equivalent martingale measures under incomplete markets. Young (2003) uses the principle of equivalent utility (Bowers, Gerber, Hickman, Jones, and Nesbitt, 1997) to investigate the pricing problem for equity-linked life insurance policies with fixed premium and death benefit linked to the performance of an underlying index rather than interest accrual. She demonstrated that the single benefit premium for such products satisfies a Black-Scholes-like PDE with an additional non-linearity term due to the presence of mortality risk. Moore and Young (2003) then elaborated on Young's earlier work and developed various analytic bounds for the premiums, in addition to providing a qualitative analysis of the effects of the force of mortality, and the insurer's risk preference on the premium.

In this paper we follow the approach of Young (2003); however, we introduce an additional source of market incompleteness in an attempt to capture the heavy-tailed nature of the distribution of asset returns. Such a generalization is not only theoretically interesting, it is also practically relevant as it is well known that the historical distribution of asset returns have significant heavy-tails. Furthermore, the implied risk-neutral distributions from put and call option prices indicate that investors require a significant premium for taking on the risk of these heavy tails. In this paper we choose to model the asset dynamics via an exponential Lévy process, containing diffusive and jump components, as this class is very large and contains many well known and widely used models; furthermore, they also provide some level of analytical tractability. In section 2 we provide a brief review of finite variation exponential Lévy processes. Lévy processes have been used extensively for pricing financial derivatives without insurance risk exposure; see for example Chan (1999), Benhamou (2000), Gallucio (2001) and Lewis (2001). Jaimungal (2004) studied the pricing problem for various EIA products with jumps in the underlying risky asset. In that work, the asset dynamics is assumed to follow an exponential Lévy process directly under the risk-neutral measure and mortality risk was treated via the actuarial present value principle. Within this framework,

he obtains closed form solutions for the premium and the analogs of the Black-Scholes Greek hedging parameters. This paper extends on that work as well, as we use the principle of equivalent utility as the fundamental pricing principle to incorporate the insurer's risk preference and the asset dynamics is modeled directly in the real world probability measure.

Several authors have used utility methods for option pricing in the presence of transaction costs (Hodges and Neuberger, 1989; Davis, Panas, and Zariphopoulou, 1993; Barles and Soner, 1998); while others have used utility methods for hedging when the underlying asset is not tradable but an asset correlated with the underlying is (Henderson and Hobson, 2002; Musiela and Zariphopoulou, 2003) (for a comprehensive review of indifference pricing and its applications see Carmona (2005)). Young and Zariphopoulou (2002, 2003) used utility methods to price insurance products that contained uncorrelated insurance and financial risks. In this paper, the pure endowment payment at maturity (contingent on survival) is an explicit function of the underlying asset or index and, as such, contains combined insurance and financial risk. In section 3 and 4 we obtain the Hamilton-Jacobi-Bellman (HJB) equation which the value function in the absence and presence of the insurance risk satisfies. We then focus on the case of exponential utility for two reasons: (i) the results are, to an extent, analytically tractable and (ii) the premiums obtained are independent of the insurer's initial wealth. Delbaen, Grandits, Rheinlander, Samperi, Schweizer, Schweizer, and Stricker (2002), generalizing results of Rouge and El Karoui (2000), study indifference pricing with exponential utility in a general semi-martingale context through a dual representation for the pricing problem. They illustrate that the indifference price can be obtained by minimizing a relative entropy minus a correction term, which depends on the option value, and also obtain the optimal investment strategy. We choose to analyze the problem using HJB equations through the so-called primal approach.

In section 5 we find that the indifference price satisfies a non-linear HJB equation due to the

presence of the jump and mortality risk components. We then show that in the limit of zero risk-aversion, the price can be written in terms of a discounted expectation in a particular risk-neutral measure and obtain the explicit hedging strategy that an insurer would follow. Finally, in section 6, we use an implicit-explicit finite-difference scheme to perform numerical experiments on the pricing problem and explore the effects of the force of mortality, the jump sizes, the jump activity rate and the insurer's risk preference on the pricing and hedging of the pure endowment contract. We find that jumps can introduce significant corrections to the diffusive case.

## 2 Lévy Processes for Modeling Asset Prices

Assume that the insurer invests in a riskless Money-market account whose price process is denoted  $\{M_t = e^{\int_0^t r ds}\}_{0 \leq t \leq T}$  with the short rate, or force of interest,  $r$  a strictly positive constant. Also, the insurer is able to invest in a risky asset whose price process is denoted  $\{S_t\}_{0 \leq t \leq T}$ . Let  $(\Omega, \mathbb{F}, \mathbb{P})$  denote a probability space with the natural filtration  $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$  generated by  $S_t$ . In this paper we assume that  $S_t$  is a geometric Lévy process with the canonical decomposition into a drift, a pure diffusion and a pure jump process; that is,

$$S_t = S_0 \exp \{ \mu t + \sigma X_t + J_t \} . \quad (1)$$

In the above,  $\{X_t\}_{0 \leq t \leq T}$  is a  $\mathbb{P}$ -standard Brownian motion and  $\{J_t\}_{0 \leq t \leq T}$  is a pure  $\mathbb{P}$ -Lévy jump process with random jump measure  $\mu(dy, dt)$ . The jump measure  $\mu(dy, dt)$  counts the number of jumps arriving in the time interval  $[t, t + dt)$  of magnitude  $[y, y + dy)$ . The process  $J_t$  can be written

$$J_t = \int_0^t \int_{-\infty}^{\infty} y \mu(dy, ds) . \quad (2)$$

The predictable compensator  $A_t$  of the jump component is defined as the  $\mathcal{F}_t$ -adapted process which makes  $J_t - A_t$  a  $\mathbb{P}$ -martingale. This process is expressed in terms of the Lévy measure (or density)  $\nu(dy)$ , which is independent of  $t$  because of the stationarity of Lévy processes (see Sato, 1999), as follows:

$$A_t = t \int_{-\infty}^{\infty} \nu(dy) . \quad (3)$$

We assume that the jump process has finite variation, that is,

$$\int_{-\infty}^{\infty} |y| \nu(dy) < +\infty . \quad (4)$$

The Lévy-Khintchine representation of the log-stock process is  $(\mu, \sigma, \nu)$ , and the characteristic function of  $Y_t \equiv \ln(S_t)$  is given by

$$\Phi_{Y_t}(z) = \mathbb{E} [\exp\{izY_t\}] = \exp\{t \Psi_{Y_t}(z)\} , \quad (5)$$

where the cumulant  $\Psi_{Y_t}(z)$  is provided by the Lévy-Khintchine theorem resulting in

$$\Psi_{Y_t}(z) = i\mu z - \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{\infty} (e^{izy} - 1) \nu(dy) . \quad (6)$$

The diffusion parameter  $\mu$  is chosen so that the price process  $S_t$  has an observed drift of  $\hat{\mu}$ .

This can be achieved by setting

$$\hat{\mu} = \Psi_{Y_t}(-i) \quad \Rightarrow \quad \mu = \hat{\mu} - \frac{1}{2}\sigma^2 - \int_{-\infty}^{\infty} (e^y - 1) \nu(dy) , \quad (7)$$

where we have further assumed the integrability condition

$$\int_{-\infty}^{\infty} (e^y - 1) \nu(dy) < +\infty \quad (8)$$

on the Lévy density. To ensure the absence of arbitrage in the market we assume that the observed drift  $\hat{\mu}$  is greater than the force of interest  $r$ . The class of models introduced above

is very large and encapsulates many of the well known processes. Here are a few examples:

- **The Black-Scholes Model.** Choosing a vanishing jump measure leads to the classical model of Black and Scholes (1973).
- **A Toy Jump-Diffusion Model.** Consider a compound Poisson process with arrival rate  $\nu > 0$ . Conditional on a jump occurring, it is of size  $\ln|1 + \epsilon| > 0$  with probability  $0 \leq p \leq 1$  and of size  $\ln|1 - \epsilon| < 0$  with probability  $(1 - p)$ , with  $0 < \epsilon \ll 1$ . The parameter  $p$  corresponds to an asymmetry in the frequency of the two kinds of jumps. The Lévy density corresponding to this choice is

$$\nu(dy) = \nu \left( p \delta(y - \ln|1 + \epsilon|) + (1 - p) \delta(y - \ln|1 - \epsilon|) \right) dy , \quad (9)$$

where  $\delta(\cdot)$  represents the Dirac delta function.

- **The Merton Jump-Diffusion Model.** This model, introduced in Merton (1976), consists of a diffusion component plus a compound Poisson process with arrival rate  $\nu > 0$ ; conditional on a jump occurring, its size is normally distributed about  $\eta \in \mathbb{R}$  with variance  $\zeta^2 > 0$ . The corresponding Lévy jump measure is

$$\nu(dy) = \frac{\nu}{\sqrt{2\pi\zeta^2}} \exp \left\{ -\frac{(y - \eta)^2}{2\zeta^2} \right\} dy . \quad (10)$$

- **The Variance-Gamma Process.** The variance-Gamma (VG) process was first introduced by Madan and Seneta (1990). This process corresponds to setting the diffusion coefficient  $\sigma = 0$  in (1) and keeping only the drift term, while the jump component has Lévy density

$$\nu(dy) = \frac{1}{\nu|y|} \exp(b y - a|y|) dy , \quad (11)$$

for  $y \in \mathbb{R}/\{0\}$ , with

$$a = \left[ \frac{2}{\nu\sigma^2} + \left( \frac{\theta}{\sigma^2} \right)^2 \right]^{1/2} , \quad b = \frac{\theta}{\sigma^2} \quad \text{and} \quad \nu > 0 . \quad (12)$$



In the limit  $\nu \rightarrow 0$ , the VG model reduces to the Black-Scholes model with volatility  $\sigma$ .

- **The Normal Inverse Gaussian Process.** The normal inverse Gaussian process has been studied in Rydberg (1997) and Bandorff-Nielsen (1998). It is obtained by setting  $\sigma = 0$  and has Lévy density

$$\nu(dy) = \delta\alpha \frac{K_1(\alpha|y|) e^{\beta y}}{|y|\pi} \quad 0 \leq |\beta| < \alpha, \quad (13)$$

where  $K_1(\cdot)$  is the modified Bessel function of the second kind of order 1.

Up to this point we have focused on describing the dynamics of the logarithm of the risky asset's price process  $\ln(S_t)$ ; however, to determine the indifference price we will need the dynamics of the risky asset's price  $S_t$  itself. This can be obtained by applying Itô's Lemma for jump processes:

$$\begin{aligned} dS_t &= S_0 d(\exp\{\mu t + \sigma X_t + J_t\}) \\ &= \left( \mu + \frac{1}{2}\sigma^2 \right) S_{t-} dt + \sigma S_{t-} dX_t + S_{t-} dJ_t + \int_{-\infty}^{\infty} [S_{t-} e^y - S_{t-} - y S_{t-}] \mu(dy, dt) \\ &= S_{t-} \left\{ \left( \mu + \frac{1}{2}\sigma^2 \right) dt + \sigma dX_t + \int_{-\infty}^{\infty} [e^y - 1] \mu(dy, dt) \right\} \\ &= S_{t-} \left\{ \left( \mu + \frac{1}{2}\sigma^2 \right) dt + \sigma dX_t + d\tilde{J}_t \right\}. \end{aligned} \quad (14)$$

where  $\{\tilde{J}_t\}_{0 \leq t \leq T}$  denotes a new jump process with random jump measure  $\tilde{\mu}(dy, dt)$ , and the  $t_-$  subscript represents the process just prior to any jump at time  $t$ . The process  $\tilde{J}_t$  undergoes a jump of size  $e^y - 1$  whenever  $J_t$  undergoes a jump of size  $y$ . In terms of the random jump measure  $\mu(dy, ds)$  the process  $\tilde{J}_t$  can be defined as

$$\tilde{J}_t \equiv \int_0^t \int_{-\infty}^{\infty} (e^y - 1) \mu(dy, ds) = \int_0^t \int_{-\infty}^{\infty} y \tilde{\mu}(dy, ds). \quad (15)$$

The predictable compensator  $\tilde{A}_t$  of  $\tilde{J}_t$  can be written

$$\tilde{A}_t = t \int_{-\infty}^{\infty} (e^y - 1) \nu(dy) = t \int_{-1}^{\infty} y \tilde{\nu}(dy), \quad (16)$$

where

$$\tilde{\nu}(dy) = \frac{f(\ln(1+y))}{1+y} \mathbb{I}_{\{y>-1\}} dy. \quad (17)$$

In the above the original Lévy measure was assumed to be of the form  $\nu(dy) = f(y) dy$ . Notice that the jumps of  $\tilde{J}_t$  are strictly greater than  $-1$  which prevents the process  $S_t$  from reaching zero.

We assume that the insurer can trade in both the risky and riskless assets. Let  $w > 0$  denote the initial wealth of the insurer at time  $t$ ,  $\{\pi_s\}_{t \leq s \leq T}$  denote the amounts invested in the risky asset at time  $s$ , and  $\{\pi_s^M\}_{t \leq s \leq T}$  denote the amounts invested in the money-market account at time  $s$ . The wealth process of the insurer at time  $s$  is  $W_s = \pi_s + \pi_s^M$  and we restrict the class of admissible trading strategies  $\{(\pi_s, \pi_s^M)\}$  to those that are  $\mathcal{F}_t$ -adapted and self-financing. Due to the self-financing restriction, the wealth dynamics is  $dW_s = (dS_s/S_{s-}) \pi_{s-} + (dM_s/M_{s-}) \pi_{s-}^M$  resulting in

$$\begin{cases} dW_s = \left[ r W_{s-} + \left( \mu + \frac{1}{2} \sigma^2 - r \right) \pi_{s-} \right] ds + \sigma \pi_{s-} dX_s + \pi_{s-} d\tilde{J}_s, & \forall s \geq t, \\ W_t = w. \end{cases} \quad (18)$$

Now that the wealth dynamics is known, we focus on determining the maximum expected utility of terminal wealth first in the absence of and then in the presence of the insurance risk.

### 3 Value Function without the Insurance Risk

The insurer seeks to maximize its expected utility of wealth at the end of the term  $T$  of the pure endowment. Merton (1971, 1969) was the first to study the optimal investment

problems associated with utility maximization and our calculations reduce to these classic results in the appropriate limit. Corcuera, Guerra, Nualart, and Schoutens (2004) analyze the optimal investment problem for Lévy processes by introducing a complete set of so called power-jump assets. We refrain from introducing additional assets and instead focus on the problem where only the riskless and risky assets are tradable. Define the value function of the insurer who does not accept the insurance risk as follows:

$$V(w, t) = \sup_{\{\pi_s\} \in \mathcal{S}} \mathbb{E}[u(W_T) | W_t = w] , \quad (19)$$

in which the function  $u$  is an increasing concave utility function of wealth representing the insurer's risk preference and  $\mathcal{S}$  is the set of square integrable self-financing trading strategies for which  $\int_t^T \pi_s^2 ds < +\infty$ . This further restriction ensures uniqueness of (18) and avoids non-degenerate solutions of the ensuing Hamilton-Jacobi-Bellman (HJB) equation. For details on the rigorous derivation of the HJB equations and technical integrability conditions see for example Fleming and Soner (1993).

To obtain an HJB equation, let  $\{\pi_s\}$  be fixed at  $\pi$  on  $[t, t+h]$ , with  $h \ll 1$ , after which it follows the optimal process  $\{\pi_s^*\}$ . Then, by the dynamic programming principle,

$$V(w, t) \geq \mathbb{E} \left[ V \left( W_{(t+h)^-}^\pi, t+h \right) \middle| W_t = w \right] = V(w, t) + \mathbb{E} \left[ \int_t^{t+h} dV(W_{s-}, s) \middle| W_t = w \right] . \quad (20)$$

Apply Itô's Lemma to  $V$  to obtain

$$\begin{aligned} dV(W_{s-}, s) = & \left[ V_t(W_{s-}, s) + \left( r W_{s-} + \left( \mu + \frac{1}{2} \sigma^2 - r \right) \pi \right) V_w(W_{s-}, s) + \frac{1}{2} \sigma^2 \pi^2 V_{ww}(W_{s-}, s) \right] ds \\ & + \pi \sigma V_w(W_{s-}, s) dX_s + \int_{-\infty}^{\infty} [V(W_{s-} + \pi y, s) - V(W_{s-}, s)] \tilde{\mu}(dy, ds). \end{aligned} \quad (21)$$

where as usual,

$$V_t \equiv \frac{\partial}{\partial t} V, \quad V_w \equiv \frac{\partial}{\partial w} V, \quad \text{and} \quad V_{ww} \equiv \frac{\partial^2}{\partial w^2} V . \quad (22)$$

Plugging equation (21) into (20) and maximizing over  $\pi$  leads to the HJB equation,

$$\left\{ \begin{array}{l} V_t + rw V_w + \max_{\pi} \left[ \left( \mu + \frac{1}{2} \sigma^2 - r \right) \pi V_w + \frac{1}{2} \sigma^2 \pi^2 V_{ww} \right. \\ \left. + \int_{-\infty}^{\infty} [V(w + \pi y, t) - V(w, t)] \tilde{\nu}(dy) \right] = 0, \\ V(w, T) = u(w). \end{array} \right. \quad (23)$$

Suppose that the utility  $u$  is exponential; specifically, write  $u(w) = -\frac{1}{\alpha} e^{-\alpha w}$ , for some  $\alpha > 0$ . The parameter  $\alpha$  is the absolute risk aversion  $r_{\alpha}(w) = -u''(w)/u'(w) = \alpha$  as defined by Pratt (1964). The optimal investment problem with exponential utility for assets whose price process follow semi-martingales was investigated by Delbaen, Grandits, Rheinlander, Samperi, Schweizer, Schweizer, and Stricker (2002). The authors develop a dual representation for the pricing problem in terms of minimizing a relative entropy minus a correction term. We continue with the primal approach using exponential utility and we find the following expression for  $V$ ,

$$V(w, t) = -\frac{1}{\alpha} \exp \left[ -\alpha w e^{r(T-t)} - m_0 (T-t) \right], \quad (24)$$

in which

$$m_0 = \left( \mu + \frac{1}{2} \sigma^2 - r \right) \bar{\pi}_0 - \frac{1}{2} \sigma^2 \bar{\pi}_0^2 - \int_{-\infty}^{\infty} \left[ e^{-\bar{\pi}_0(e^y-1)} - 1 \right] \nu(dy), \quad (25)$$

and  $\bar{\pi}_0$  is implicitly defined by

$$\bar{\pi}_0 - \frac{1}{\sigma^2} \int_{-\infty}^{\infty} e^{-\bar{\pi}_0(e^y-1)} (e^y - 1) \nu(dy) = \frac{\mu + \frac{1}{2} \sigma^2 - r}{\sigma^2}, \quad (26)$$

or by imposing (7), we have

$$\bar{\pi}_0 - \frac{1}{\sigma^2} \int_{-\infty}^{\infty} \left[ e^{-\bar{\pi}_0(e^y-1)} - 1 \right] (e^y - 1) \nu(dy) = \frac{\hat{\mu} - r}{\sigma^2}. \quad (27)$$

Note that because  $\hat{\mu} > r$ , it follows that  $\bar{\pi}_0 > 0$ . The optimal investment in the risky asset is given by

$$\pi_0^* = \frac{\bar{\pi}_0}{\alpha} e^{-r(T-t)}, \quad (28)$$

independent of wealth because the absolute risk aversion is constant - a well known property of exponential utilities. In the limit of vanishing jump measure we obtain the results of Merton (1971), namely

$$\bar{\pi}_0 \rightarrow \frac{\hat{\mu} - r}{\sigma^2}. \quad (29)$$

It is interesting to determine the behavior of  $\bar{\pi}_0$  as the arrival rate of jumps increases, as the size of the jumps increases, and as the skewness becomes more negative. To this end, write the Lévy density as follows:

$$\nu(dy) = \left\{ v_- f_- \left( \frac{y}{s_-} \right) \mathbb{I}(y < 0) + v_+ f_+ \left( \frac{y}{s_+} \right) \mathbb{I}(y > 0) \right\} dy, \quad (30)$$

with  $v_{\pm}$  and  $s_{\pm}$  strictly positive constants, and with  $f_-(y) \geq 0$  on  $(-\infty, 0)$  and  $f_+(y) \geq 0$  on  $(0, \infty)$  such that  $\nu(dy)$  satisfies conditions (4) and (8). Adjustments to  $v_-$  ( $v_+$ ) correspond to adjustments in the arrival rate of negative (positive) jumps, while keeping the jump sizes themselves fixed. Furthermore, the jump sizes can be scaled by changing  $s_{\pm}$ , while keeping the arrival rates constant. With this representation for the Lévy density, the first order condition (26) becomes

$$\begin{aligned} \bar{\pi}_0 - \frac{v_-}{\sigma^2} \int_{-\infty}^0 \left( e^{-\bar{\pi}_0(e^y-1)} - 1 \right) (e^y - 1) f_-(y/s_-) dy \\ - \frac{v_+}{\sigma^2} \int_0^{\infty} \left( e^{-\bar{\pi}_0(e^y-1)} - 1 \right) (e^y - 1) f_+(y/s_+) dy = \frac{\hat{\mu} - r}{\sigma^2}, \end{aligned} \quad (31)$$

where we have imposed the drift matching condition (7) on  $\mu$ . The partial derivatives with

respect to the various parameters are

$$\frac{\partial}{\partial v_-} \bar{\pi}_0 = \frac{\int_{-\infty}^0 \left( e^{-\bar{\pi}_0(e^y-1)} - 1 \right) (e^y - 1) f_-(y/s_-) dy}{\sigma^2 + \int_{-\infty}^{\infty} e^{-\bar{\pi}_0(e^y-1)} (e^y - 1)^2 \nu(dy)} \leq 0, \quad (32)$$

$$\frac{\partial}{\partial v_+} \bar{\pi}_0 = \frac{\int_0^{\infty} \left( e^{-\bar{\pi}_0(e^y-1)} - 1 \right) (e^y - 1) f_+(y/s_+) dy}{\sigma^2 + \int_{-\infty}^{\infty} e^{-\bar{\pi}_0(e^y-1)} (e^y - 1)^2 \nu(dy)} \leq 0, \quad (33)$$

$$\frac{\partial}{\partial s_-} \bar{\pi}_0 = - \frac{\frac{v_-}{s_-^2} \int_{-\infty}^0 \left( e^{-\bar{\pi}_0(e^y-1)} - 1 \right) (e^y - 1) y f'_-(y/s_-) dy}{\sigma^2 + \int_{-\infty}^{\infty} e^{-\bar{\pi}_0(e^y-1)} (e^y - 1)^2 \nu(dy)}, \quad (34)$$

$$\frac{\partial}{\partial s_+} \bar{\pi}_0 = - \frac{\frac{v_+}{s_+^2} \int_0^{\infty} \left( e^{-\bar{\pi}_0(e^y-1)} - 1 \right) (e^y - 1) y f'_+(y/s_+) dy}{\sigma^2 + \int_{-\infty}^{\infty} e^{-\bar{\pi}_0(e^y-1)} (e^y - 1)^2 \nu(dy)}. \quad (35)$$

Equations (32) and (33) imply that as the jump activity increases, the optimal investment in the risky asset decreases, regardless of whether those jumps are positive or negative. This result seems somewhat surprising at first, after all if the jumps are only positive why should one invest less in the risky asset? The resolution to the seeming paradox is that we have calibrated the diffusive drift so that the realized drift of the asset is fixed at  $\hat{\mu}$  (see (7)). Consequently, with only positive jumps, when the jump activity increases, the calibrated diffusive drift decreases and this overpowers the contribution of the upward jump movements and pushes the optimal investment downwards.

To extract the behavior of the optimal investment as the size of the jump increases, we must focus on (34) and (35). If  $f'_-(y) > 0$  on  $(-\infty, 0)$  and  $f'_+(y) < 0$  on  $(0, \infty)$ , then (34) and (35) are both strictly negative. This implies that if the skewness in the jump distribution

is increased the position in the risky asset decreases, regardless of whether the skewness is positive or negative. The effects of the increase in negative skewness is easily understood since increasing negative skewness, increases the downward jump sizes themselves. The increase in positive skewness causes a decrease in the optimal investment because, as before, such an increase directly reduces the diffusive drift component through (7). The above conditions on the Lévy density are satisfied for example by the the VG model (see (11)) and the Normal Inverse Gaussian model (see (13)). In general, the jump measure is not monotonic on the half-lines, and such cases must be treated individually. We find that the toy jump-diffusion model (9) leads to a decrease in the optimal investment as the positive or negative jump sizes increase. Other more general cases will not be considered further at this time.

In Figure 1, the optimal investment ratio  $\bar{\pi}_0$  is obtained using the toy model defined by (9). The parameters are chosen to illustrate the dependence of  $\bar{\pi}_0$  on the jump activity rate, the jump size and the jump skewness. Notice that as the activity rate and jump sizes increase, the optimal investment decreases (as dictated by the above analysis). Furthermore, as the process is made more asymmetric by increasing the upward-jump probability  $p$  the optimal investment increases; however, for all jump sizes, the Merton result (29) provides an upper bound.

#### 4 Value Function with the Insurance Risk

Again, the insurer seeks to maximize its expected utility of wealth at time  $T$ , the end of the term of the pure endowment. If the insured is still alive at time  $T$ , then the insurer will pay to the insured  $g(S_T)$ . In this case, the value function of the insurer is given by

$$U(w, S, t) = \sup_{\{\pi_s\} \in \mathcal{S}} \mathbb{E} [u(W_T - g(S_T)) | W_t = w, S_t = S] . \quad (36)$$

By adapting the argument from the preceding section, we obtain the following HJB equation for  $U$ :

$$\left\{ \begin{array}{l} U_t + rwU_w + \left(\mu + \frac{1}{2}\sigma^2\right) SU_S + \frac{1}{2}\sigma^2 S^2 U_{SS} + \lambda_x(t)(V - U) \\ \quad + \max_{\pi} \left[ \left(\mu + \frac{1}{2}\sigma^2 - r\right) \pi U_w + \frac{1}{2}\sigma^2 U_{ww} + \sigma^2 \pi S U_{wS} \right. \\ \quad \quad \left. + \int_{-\infty}^{\infty} [U(w + \pi y, S + Sy, t) - U(w, S, t)] \tilde{\nu}(dy) \right] = 0, \\ U(w, S, T) = u(w - g(S)), \end{array} \right. \quad (37)$$

in which  $\lambda_x(t)$  is the deterministic hazard rate, or force of mortality, for the buyer of insurance at age  $(x + t)$ . The term  $\lambda_x(t)(V - U)$  arises because when the individual dies (with instantaneous probability  $\lambda_x(t)$ ), then the insurer no longer faces the insurance risk and  $U$  reverts to  $V$ .

As in the previous section, suppose the utility is exponential; then,  $U$  can be written as  $U(w, S, t) = V(w, t)\phi(S, t)$ , in which  $\phi$  solves

$$\begin{aligned} & \phi_t + \left(\mu + \frac{1}{2}\sigma^2\right) S\phi_S + \frac{1}{2}\sigma^2 S^2 \phi_{SS} + \lambda_x(t)(1 - \phi) \\ & + m_0\phi - \max_{\pi} \left[ \left(\left(\mu + \frac{1}{2}\sigma^2 - r\right) \pi \alpha(t) - \frac{1}{2}\sigma^2 \pi^2 \alpha^2(t)\right) \phi + \sigma^2 \pi S \alpha(t) \phi_S \right. \\ & \quad \left. - \int_{-\infty}^{\infty} [e^{-\pi \alpha(t)(e^y - 1)} \phi(S e^y, t) - \phi(S, t)] \nu(dy) \right] = 0, \end{aligned} \quad (38)$$

where  $\alpha(t) = \alpha e^{r(T-t)}$ , and the boundary condition becomes  $\phi(S, T) = e^{\alpha g(S)}$ . It is now possible to obtain the optimal investment strategy in the presence of the insurance risk. However, we defer that calculation to the next section where the insurer's indifference price is obtained.



## 5 Indifference Price of the Equity-Indexed Pure Endowment

Indifference pricing was first introduced by Hodges and Neuberger (1989) for pricing claims in the presence of transaction costs. In the present context, the insurer's indifference price is the price  $P = P(w, S, t)$  for accepting to insure the pure-endowment at time  $t$  which makes the insurer indifferent between accepting the risk with additional wealth  $P$  or not insuring the risk (with no additional wealth). That is,  $P$  solves

$$U(w + P, S, t) = V(w, t). \quad (39)$$

Suppose that utility is exponential, then  $P$  is independent of wealth and is related to  $\phi$  via

$$P(S, t) = \frac{1}{\alpha(t)} \ln \phi(S, t), \quad (40)$$

or equivalently,  $\phi(S, t) = \exp \{ \alpha(t) P(S, t) \}$ , from which it follows that

$$\phi_t = \phi \alpha(t) (-rP + P_t), \quad (41)$$

$$\phi_S = \phi \alpha(t) P_S, \quad (42)$$

$$\phi_{SS} = \phi \alpha(t) (P_{SS} + \alpha(t) P_S^2). \quad (43)$$

By plugging these expressions into equation (38), we obtain

$$\begin{aligned} rP = P_t + \left( \mu + \frac{1}{2} \sigma^2 \right) S P_S + \frac{1}{2} \sigma^2 S^2 (P_{SS} + \alpha(t) P_S^2) + \frac{\lambda_x(t)}{\alpha(t)} (e^{-\alpha(t)P} - 1) \\ + \frac{m_0}{\alpha(t)} - \max_{\pi} \left[ \left( \mu + \frac{1}{2} \sigma^2 - r \right) \pi - \frac{1}{2} \sigma^2 \pi^2 \alpha(t) + \sigma^2 \pi S \alpha(t) P_S \right. \\ \left. - \frac{1}{\alpha(t)} \int_{-\infty}^{\infty} \left\{ e^{\alpha(t)(P(Se^y, t) - P(S, t) - \pi(e^y - 1))} - 1 \right\} \nu(dy) \right], \end{aligned} \quad (44)$$

where the boundary condition is now  $P(S, T) = g(S)$ . Equation (44) is consistent with the results of Moore and Young (2003) in the limit of vanishing jump measure. The optimal

investment  $\pi^*$  solves the first-order necessary condition

$$\pi - \frac{1}{\sigma^2 \alpha(t)} \int_{-\infty}^{\infty} e^{\alpha(t)(P(S e^y, t) - P(S, t) - \pi(e^y - 1))} (e^y - 1) \nu(dy) = \frac{\mu + \frac{1}{2}\sigma^2 - r}{\sigma^2 \alpha(t)} + S P_S, \quad (45)$$

or by imposing the drift matching condition (7), we have

$$\pi - \frac{1}{\sigma^2 \alpha(t)} \int_{-\infty}^{\infty} \left[ e^{\alpha(t)(P(S e^y, t) - P(S, t) - \pi(e^y - 1))} - 1 \right] (e^y - 1) \nu(dy) = \frac{\hat{\mu} - r}{\sigma^2 \alpha(t)} + S P_S. \quad (46)$$

The optimal investment can be reduced to the calculation of the additional investment in the risky asset due to the presence of insurance risk. That is, write the optimal investment as  $\pi^* = \frac{\bar{\pi}_0}{\alpha(t)} + \pi_1$ , then  $\pi_1$  satisfies

$$\pi_1 - \frac{1}{\sigma^2 \alpha(t)} \int_{-\infty}^{\infty} \left[ e^{\alpha(t)(P(S e^y) - P(S) - \pi_1(e^y - 1))} - 1 \right] (e^y - 1) \hat{\nu}(dy) = S P_S, \quad (47)$$

where a new Lévy density  $\hat{\nu}(dy)$  has been introduced. The new Lévy density is related to the original Lévy density via

$$\hat{\nu}(dy) = \exp \{ -\bar{\pi}_0 (e^y - 1) \} \nu(dy). \quad (48)$$

It is interesting that the realized drift of the asset  $\hat{\mu}$  does not appear explicitly in (47); however, the optimal investment without insurance risk  $\bar{\pi}_0$  does depend explicitly on  $\hat{\mu}$  (see (26)) and this implicitly affects  $\pi_1$ . Solving (44) is notoriously difficult and in general one must resort to numerical methods. In section 6 we do just that, but first we obtain the solution in two particular cases: (i) a risk-neutral insurer and (ii) a pure endowment.

## 5.1 Particular Solutions

### 5.1.1 Risk Neutral Insurer

Consider the ratio of the difference between the optimal investment with and without insurance risk and the stock price,

$$\Delta(S, t) \equiv \frac{\pi^* - \pi_0^*}{S} = \frac{\pi_1}{S}. \quad (49)$$

This ratio represents the optimal number of units of the risky asset that account for the insurance risk, and it is the analog of the Black-Scholes Delta hedging parameter  $\Delta(S) = P_S$ . In the limit as  $\alpha \searrow 0$  the insurer becomes risk-neutral, and (47) can be solved explicitly leading to the insurer investing the following additional amount in the risky asset due to the insurance risk:

$$\pi_1 = \frac{S P_S + \frac{1}{\sigma^2} \int_{-\infty}^{\infty} (P(S e^y, t) - P(S, t)) (e^y - 1) \hat{\nu}(dy)}{1 + \frac{1}{\sigma^2} \int_{-\infty}^{\infty} (e^y - 1)^2 \hat{\nu}(dy)}. \quad (50)$$

The insurer's indifference Delta hedging parameter is therefore

$$\Delta(S, t) = \frac{P_S + \frac{1}{\sigma^2} \int_{-\infty}^{\infty} \frac{P(S e^y, t) - P(S, t)}{S} (e^y - 1) \hat{\nu}(dy)}{1 + \frac{1}{\sigma^2} \int_{-\infty}^{\infty} (e^y - 1)^2 \hat{\nu}(dy)}. \quad (51)$$

The term in the denominator accounts for the additional variance in the process due to the presence of jumps, while the additional term in the numerator accounts for change in value of the contract due to jumps. If there are no jump components in the asset dynamics, then the Delta reduces to the Black-Scholes result. Also, if the price function is independent of  $S$ , such as when the endowment is not linked to the asset return, the Delta is exactly zero. Furthermore, if the price function is linear in  $S$ ,  $P(S, t) = a S h(t)$ , then  $\Delta(S) = a h(t)$ , the

same result as the Black-Scholes case.

It is instructive to investigate the deviations from the Black-Scholes result by using a familiar pay-off function. Consider the price function at maturity for a call pay-off struck at  $K$ , so that  $P(S, T) = g(S) = \max(S - K, 0)$ . In this case, the indifference pricing methodology dictates that just prior to maturity the insurer hold the following additional number of units of the risky in excess due to the pure endowment:

$$\Delta(S, T) = \begin{cases} \frac{\int_{\ln(K/S)}^{\infty} (e^y - 1) (e^y - \frac{K}{S}) \hat{\nu}(dy)}{\sigma^2 + \int_{-\infty}^{\infty} (e^y - 1)^2 \hat{\nu}(dy)}, & S < K, \\ \frac{\sigma^2 - \int_{-\infty}^{\ln(K/S)} (e^y - 1) (1 - \frac{K}{S}) \hat{\nu}(dy) + \int_{\ln(K/S)}^{\infty} (e^y - 1)^2 \hat{\nu}(dy)}{\sigma^2 + \int_{-\infty}^{\infty} (e^y - 1)^2 \hat{\nu}(dy)}, & S \geq K. \end{cases} \quad (52)$$

Without jumps, this reduces to the Black-Scholes Delta  $\Delta(S, T) = \mathbb{I}(S \geq K)$ ; however, the presence of upward and downward jumps modifies the strategy. The Delta in the out-of-the-money region ( $S < K$ ) is only zero if upward jumps are absent; otherwise, it is strictly positive. Similarly, the Delta in the in-the-money region ( $S \geq K$ ) reaches unity only if all jumps are absent.

In Figure 2, we have computed the Delta of a call option struck at  $K = 100$ , just prior to maturity, using the toy jump-diffusion model (9) for several model parameters. As the upward jump probability increases, the Delta in both the out-of-the-money and the in-the-money regions increases and is due to the increasing chance that asset will jump into the in-the-money region. Although the curves as a function of activity rate in the bottom panel appear similar to those in the top panel the behavior is quite different. For a fixed jump probability,

the Delta increases with the activity rate in the out-of-the money region, while it decreases in the in-the-money region. This behavior is also reasonable, since if the asset price is in the out-of-the money region, an increase in the activity rate increases the probability that the asset will jump into to the in-the-money region; on the other hand, if the asset price is in the in-the-money region, an increase in the activity rate corresponds to a higher probability that the asset will jump out-of-the money.

Now that the optimal investment is obtained, we find that in the limit of zero risk-aversion  $\alpha \searrow 0$  the HJB equation (44) reduces to,

$$\begin{cases} (r + \lambda_x(t))P = P_t + (r - \eta) S P_S + \frac{1}{2}\sigma^2 S^2 P_{SS} + \int_{-\infty}^{\infty} [P(Se^y, t) - P(S, t)] \hat{\nu}(dy) \\ P(S, T) = g(S) \end{cases} \quad (53)$$

where the drift compensator for jumps  $\eta$  is,

$$\eta = \int_{-\infty}^{\infty} (e^y - 1) \hat{\nu}(dy). \quad (54)$$

The Feynman-Kač theorem for Lévy processes now supplies a solution to the pricing equation (53) in expectation form,

$$P(S, t) = \mathbb{E}^{\mathbb{Q}} \left[ e^{-\int_t^T (r + \lambda_x(s)) ds} g(S_T) \middle| \mathcal{F}_t \right], \quad (55)$$

where the asset price process  $S_t$  can be written in terms of a pure diffusion and pure jump process as follows:

$$S_t = S_0 \exp \left\{ \left( r - \frac{1}{2}\sigma^2 - \eta \right) t + \sigma \hat{X}_t + \hat{J}_t \right\}. \quad (56)$$

Under the  $\mathbb{Q}$ -measure  $\hat{X}_t$  is a standard Brownian process and  $\hat{J}_t$  is a pure Lévy jump process with Lévy density  $\hat{\nu}(dy)$  and corresponding random jump measure  $\hat{\mu}(dy, dt)$ ,

$$\hat{J}_t = \int_0^t \int_{-\infty}^{\infty} y \hat{\mu}(dy, ds) . \quad (57)$$

Equivalently,

$$\frac{dS_t}{S_{t-}} = (r - \eta) dt + \sigma dX_t + \int_{-\infty}^{\infty} (e^y - 1) \hat{\mu}(dy, dt) . \quad (58)$$

Consequently, by comparing (56) with (1) it is possible to identify the change of the model parameters that transform the real world measure  $\mathbb{P}$  to the risk-neutral measure  $\mathbb{Q}$  under the indifference pricing principle. The mapping can be summarized as follows:

$$\{\mu, \sigma, \nu(dy), \lambda_x(s)\} \mapsto \left\{ r - \frac{1}{2}\sigma^2 - \eta, \sigma, \exp\{-\bar{\pi}_0(e^y - 1)\} \nu(dy), \lambda_x(s) \right\} \quad (59)$$

Notice that this mapping preserves the finite activity and/or finite variation properties of the real-world Lévy density. Furthermore, under the  $\mathbb{Q}$ -measure the asset return has relatively more frequent negative jumps and relatively less frequent positive jumps than under the original  $\mathbb{P}$ -measure. Very large negative jumps gain a factor of  $e^{\bar{\pi}_0}$  in their activity rate, while large positive jump components of size  $y$  are heavily suppressed by a factor of  $e^{-\bar{\pi}_0 e^y}$ .

It is well-known that in incomplete markets there exist many equivalent martingale measures Harrison and Pliska (1981); however, imposing additional criteria (such as indifference pricing) may force a particular measure to stand out. Indeed, the Radon-Nikodym derivative process which induces the measure change described above is uniquely given as follows:

$$\left( \frac{d\mathbb{Q}}{d\mathbb{P}} \right)_t = \exp \left\{ -\frac{1}{2} \left( \frac{\mu - r}{\sigma} \right)^2 t - \left( \frac{\mu - r}{\sigma} \right) X_t - t \int_{-\infty}^{\infty} (H(y) - 1) \nu(dy) \right\} \prod_{s \leq t} H(\Delta J_s) , \quad (60)$$

where

$$H(y) = \exp \{ -\bar{\pi}_0 (e^y - 1) \} . \quad (61)$$

By applying (7) we find that  $\mathbb{E}^{\mathbb{Q}}[S_T | \mathcal{F}_t] = e^{r(T-t)} S_t$  verifying that the measure  $\mathbb{Q}$  is a risk-neutral measure.

To illustrate the effects of the measure change induced by indifference pricing consider the toy model described by (9). Under the risk-neutral measure  $\mathbb{Q}$ , the possible jump sizes do not change; however, the activity rate and upward jump probabilities become respectively,

$$v_{\mathbb{Q}} = v \left\{ e^{-\bar{\pi}_0 \epsilon} p + e^{\bar{\pi}_0 \epsilon} (1 - p) \right\} \quad \text{and} \quad p_{\mathbb{Q}} = \frac{e^{-\bar{\pi}_0 \epsilon} p}{e^{-\bar{\pi}_0 \epsilon} p + e^{\bar{\pi}_0 \epsilon} (1 - p)} . \quad (62)$$

Recall that  $\bar{\pi}_0$  implicitly depends on  $v$  and  $p$  through equation (26). In Figure 3 the risk-neutral upward jump probabilities  $p_{\mathbb{Q}}$  are display as a function of the jump size  $\epsilon$  and real-world activity rate  $v$ . The real-world upward jump probability was set fixed at  $p = 50\%$  for all curves. The general trend is that  $p_{\mathbb{Q}}$  decreases as  $v_{\mathbb{P}}$  increases. This implies that the risk neutral distribution of returns becomes more skewed to the downside as the rate of arrival of jumps increases even though in the real world the distribution of jumps is symmetric.

### 5.1.2 Pure Endowment

It is often useful to redefine the pricing problem in terms of the “moneyness” parameter  $z = \ln(S)$  by introducing the function  $\bar{P}(z, t) \equiv \alpha(t) P(e^z, t)$  and the scaled optimal investment ratio  $\bar{\pi} = \alpha(t)\pi$ . In this case the HJB equation (44) becomes

$$\begin{aligned} 0 = & \bar{P}_t + \mu \bar{P}_z + \frac{1}{2} \sigma^2 \left( \bar{P}_{zz} + \bar{P}_z^2 \right) + \lambda_x(t) \left( e^{-\bar{P}} - 1 \right) \\ & + m_0 - \max_{\bar{\pi}} \left[ \left( \mu + \frac{1}{2} \sigma^2 - r \right) \bar{\pi} - \frac{1}{2} \sigma^2 \bar{\pi}^2 + \sigma^2 \bar{\pi} \bar{P}_z \right] \end{aligned}$$

$$- \int_{-\infty}^{\infty} \left\{ e^{\bar{P}(z+y,t) - \bar{P}(z,t) - \bar{\pi}(e^y - 1)} - 1 \right\} \nu(dy) \Big], \quad (63)$$

with boundary condition  $\bar{P}(z, T) = \alpha g(e^z)$ . The first order condition becomes

$$\bar{\pi} - \frac{1}{\sigma^2} \int_{-\infty}^{\infty} \left[ e^{\bar{P}(z+y,t) - \bar{P}(z,t) - \bar{\pi}(e^y - 1)} - 1 \right] (e^y - 1) \nu(dy) = \frac{\hat{\mu} - r}{\sigma^2} + \bar{P}_z, \quad (64)$$

with the optimal investment in the risky asset being

$$\pi^* = \frac{\bar{\pi}}{\alpha} e^{-r(T-t)}. \quad (65)$$

Suppose that the pay-off at maturity is independent of the asset level  $g(S_T) = B$ , that is, a guaranteed payment of  $B$  upon survival to  $T$ . In this case,  $\bar{P}(z, t)$  is independent of  $z$ , (64) reduces to (26) and the HJB equation (63) reduces to,

$$0 = \bar{P}_t(z, t) + \lambda_x(t) \left( e^{-\bar{P}(z,t)} - 1 \right), \quad \bar{P}(z, T) = \alpha B. \quad (66)$$

The above ordinary differential equation can be solved by quadratures resulting in a price at time  $t$  of

$$P(S, t) = \frac{1}{\alpha} e^{-r(T-t)} \ln \left\{ 1 + \left( e^{\alpha B} - 1 \right) {}_{T-t}p_{x+t} \right\}, \quad (67)$$

where the actuarial symbol for survival probability has been introduced:

$${}_t p_x = \exp \left\{ - \int_0^t \lambda_x(s) ds \right\}. \quad (68)$$

As a consequence of the insurer's risk preference, the premium is not a linear function of the endowment size. It is therefore interesting to investigate the behavior in the limit of large and small endowment sizes. When the endowment is small,  $B \ll 1$ ,

$$P(S, t) = e^{-r(T-t)} {}_{T-t}p_{x+t} \left\{ B + \frac{1}{2} {}_{T-t}q_{x+t} B^2 \right\} + o(B^2), \quad (69)$$



in which  ${}_tq_x = 1 - {}_tp_x$ . When the endowment is large,  $B \gg 1$ ,

$$P(S, t) = e^{-r(T-t)} \left\{ B - \frac{T-t p_{x+t}}{\alpha} \right\} + O\left(e^{-\alpha B}\right). \quad (70)$$

In the limit as  $\alpha \rightarrow 0$ , the price reduces to the actuarial present value of a pure endowment of size  $B$ ,

$$P(S, t) = {}_{T-t}p_{x+t} e^{-r(T-t)} B. \quad (71)$$

This result is consistent with the risk-neutral expectation in (55).

## 6 Numerical Experiments

In this section, we discretize the indifference pricing equation (63) by using a combination of implicit and explicit finite differences. Let  $\bar{P}_i^{(n)} = \alpha(T - n\Delta t) P(\exp(-z_{min} + i\Delta z), T - n\Delta t)$ , where the  $(z, t)$  plane has been discretized into blocks of size  $(\Delta z, \Delta t)$ ;  $-z_{min}$  represents the minimum  $x$  value; and there are  $M$  points in the  $z$ -direction and  $N$  points in the  $t$ -direction. The maximization term will be valued explicitly, and let  $m_i^{(n)}$  denote its value at the point  $z = -z_{min} + i\Delta z$ ,  $t = T - n\Delta t$ . We employ an implicit scheme for the first and second derivatives of  $\bar{P}$ , while the non-linear exponential term and the quadratic term  $\bar{P}_z^2$  are evaluated explicitly. The resulting finite-difference scheme is

$$\begin{aligned} & -\frac{1-\theta}{2}(\tilde{\mu} + \tilde{\sigma}^2)\bar{P}_{j+1}^{(n+1)} + \left(1 + (1-\theta)\tilde{\sigma}^2\right) \bar{P}_j^{(n+1)} - \frac{1-\theta}{2}(-\tilde{\mu} + \tilde{\sigma}^2) \bar{P}_{j-1}^{(n+1)} \\ & = +\frac{\theta}{2}(\tilde{\mu} + \tilde{\sigma}^2)\bar{P}_{j+1}^{(n)} + \left(1 - \theta\tilde{\sigma}^2\right) \bar{P}_j^{(n)} + \frac{\theta}{2}(-\tilde{\mu} + \tilde{\sigma}^2) \bar{P}_{j-1}^{(n)} \\ & \quad + \frac{\tilde{\sigma}^2}{8} \left(\bar{P}_{j+1}^{(n)} - \bar{P}_{j-1}^{(n)}\right)^2 + \lambda_j^{(n)} \Delta t \left(e^{-\bar{P}_j^{(n)}} - 1\right) + (m_0 - m_j^{(n)})\Delta t, \end{aligned} \quad (72)$$

where  $0 \leq \theta \leq 1$  is the arbitrary implicit-explicit interpolation parameter, and the scaled drift and variances have been introduced as follows:

$$\tilde{\mu} = \frac{\Delta t}{\Delta z} \mu \quad \text{and} \quad \tilde{\sigma}^2 = \frac{\Delta t}{(\Delta z)^2} \sigma^2 . \quad (73)$$

When  $\theta = 1$ , the fully explicit method is recovered, while  $\theta = 0$  corresponds to the fully implicit method. For PDEs that are linear in spatial derivatives,  $\theta = \frac{1}{2}$  provides results that are second order in both  $\Delta z$  and  $\Delta t$ . Although the HJB equations are not linear in spatial derivatives, we use a value of  $\theta = \frac{1}{2}$ , which was found to give good convergence results.

We perform numerical experiments using a pay-off function which provides the return on the risky asset with a floor protection and a cap on the return,

$$g(S_T) = \begin{cases} S_0 e^{\gamma(T-t)}, & \frac{1}{T-t} \ln\left(\frac{S_T}{S_0}\right) < \gamma , \\ S_T, & \gamma \leq \frac{1}{T-t} \ln\left(\frac{S_T}{S_0}\right) < \kappa , \\ S_0 e^{\kappa(T-t)}, & \frac{1}{T-t} \ln\left(\frac{S_T}{S_0}\right) \geq \kappa . \end{cases} \quad (74)$$

Asymptotically far from  $z = 0$ , the pay-off function is flat; consequently, we impose the following two boundary conditions:

$$\bar{P}_0^{(n)} = \bar{P}_1^{(n)} , \quad \bar{P}_{M-2}^{(n)} = \bar{P}_{M-1}^{(n)} . \quad (75)$$

These conditions corresponding to setting the partial derivative of the premium with respect to the asset level to be zero on the boundaries of the discretized  $(z, t)$ -plane.

Figure 4 shows the price and hedging position in the risky asset for a 10-year pure endowment with risk-aversion parameter  $\alpha = 0.1$ . The toy model with Lévy density (9) was used as the driving jump process. For large spot values, the premium decreases as the jump activity rate increases; while for small spot values, the premium increases as the jump activity rate

increases. Furthermore, the hedging position decreases as the jump activity rate increases. Figure 5 shows the price and hedging position in the risky asset for a 10-year pure endowment for various risk-aversion parameters. As the insurer becomes more risk-averse, the premium increases while the hedging position decreases. Both of these results are consistent with intuition.

## 7 Conclusions

In this paper, we analyzed the problem of pricing equity-linked pure endowments when the underlying risky asset follows a Lévy process. We derived the PDE that the indifference price satisfies under exponential utility and obtain the explicitly solution in the limit in which the investor becomes risk-neutral. The main pricing result is that the price is obtained by computing a discounted expectation in a particular risk-neutral measure. We explicit construct the unique equivalent Martingale measure induced by the indifference pricing principle, in the limit of zero risk aversion. For a general risk-averse investor, we investigated the sensitivity of the optimal investment in the risky asset to jump sizes, jump activity and jump asymmetry and numerical confirm the sensitivity calculations. In general we do not analytically solve the PDE; however, we implement a explicit-implicit finite-difference scheme to carry out numerical experiments, and the qualitative results indicate that jumps can significantly change the dynamic positions in the risky asset that an insurer would hold. These results are all consistent with our intuition on how heavy-tailed return distributions affect pricing and hedging. Future research directions include incorporating stochastic volatility into the asset dynamics and including the effects of stochastic interest rates.

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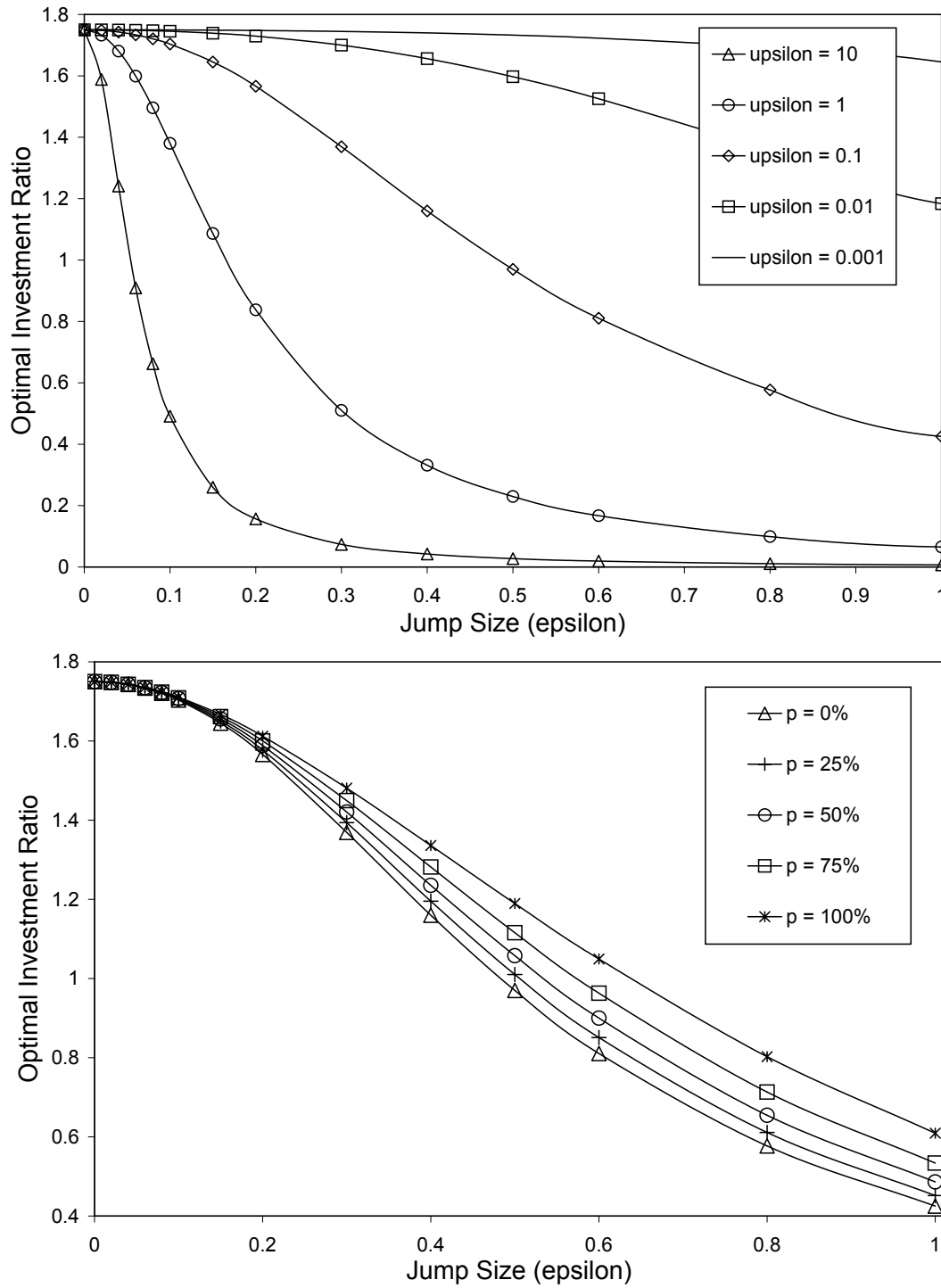


Fig. 1. These diagrams depict the optimal investment ratio  $\bar{\pi}_0$  as a function of the jump size with  $r = 5\%$ ,  $\hat{\mu} = 12\%$ ,  $\sigma = 20\%$  using the toy jump-diffusion model. In the top panel, the jump probability is fixed at  $p = 50\%$  with varying activity rates; while in the bottom panel, the activity rate is fixed at  $v = 0.1$  with varying jump probabilities.



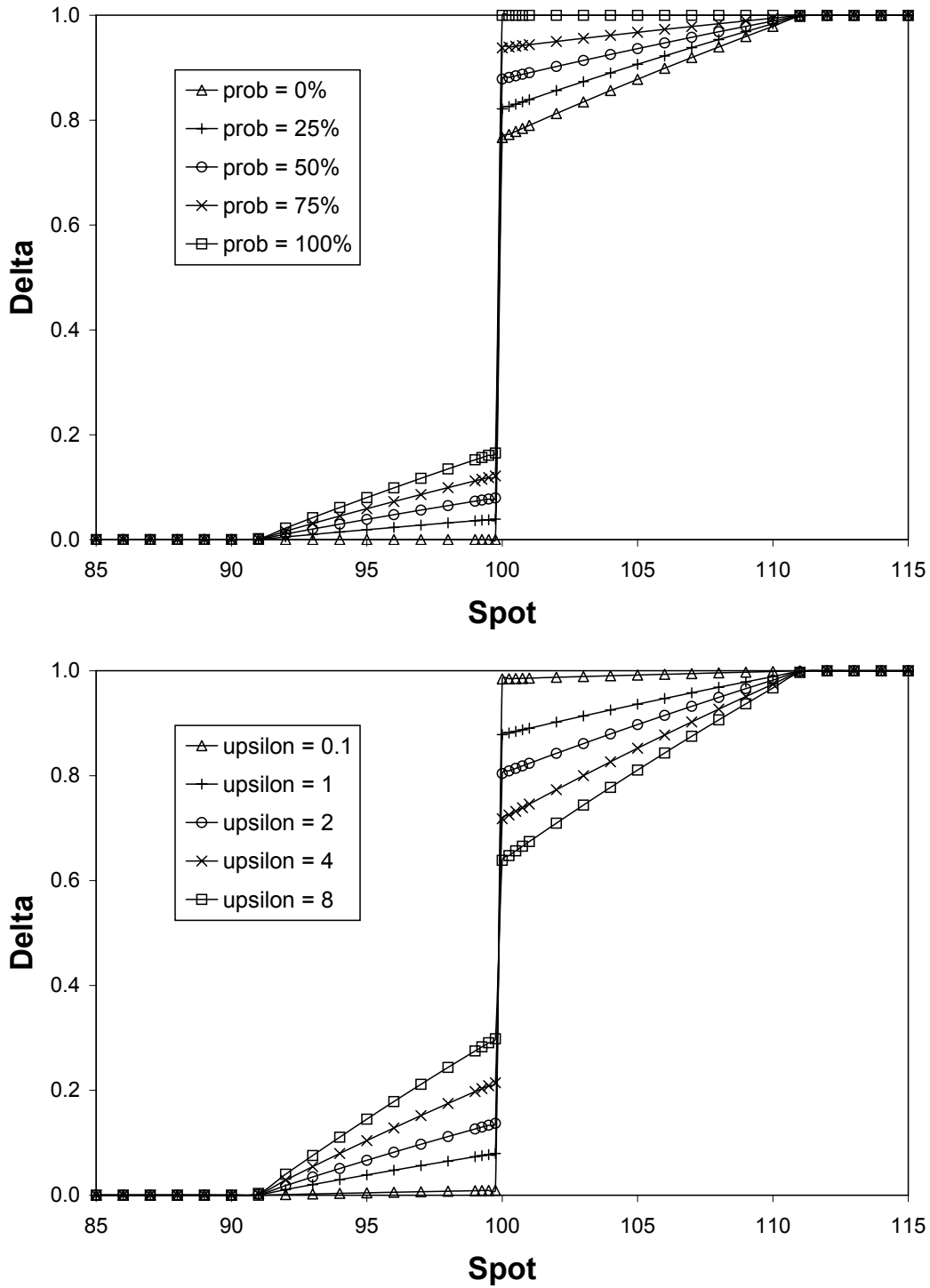


Fig. 2. These diagrams depict the Delta of a call struck at  $K = 100$  prescribed by the indifference pricing principle using the toy jump-diffusion model. In the top panel, the jump activity rate is fixed at  $v = 1$ ; while in the bottom panel, the upward jump probability is fixed at  $p = 50\%$ . In both panels the jump size  $\epsilon = 0.1$ .

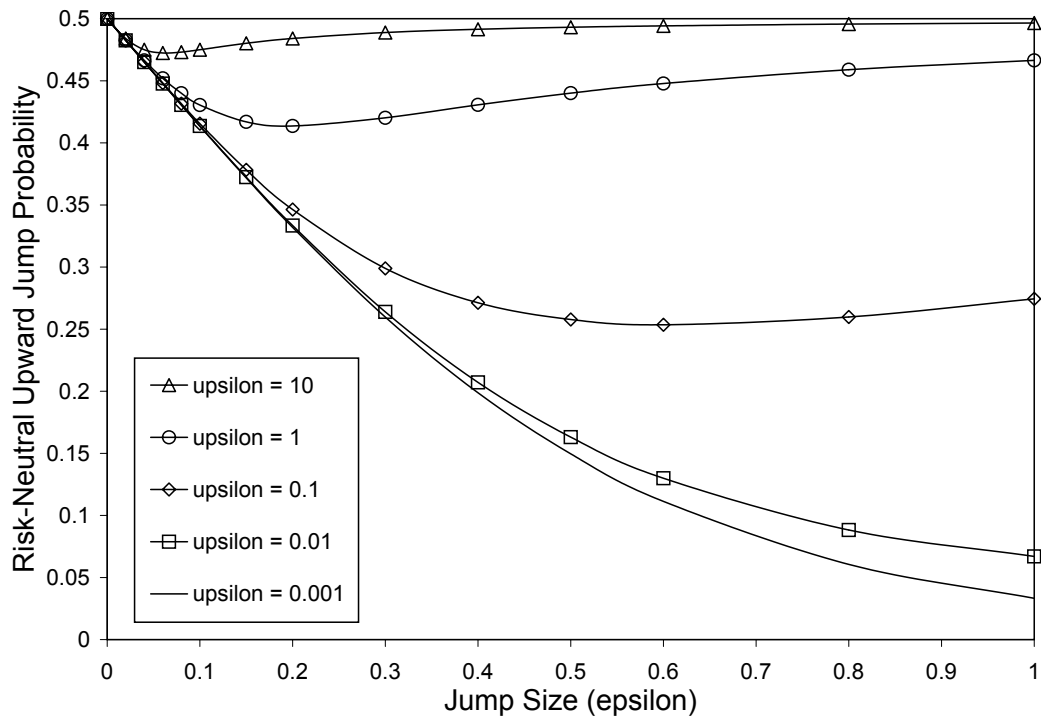


Fig. 3. This diagram depicts the change in the jump probabilities induced by the measure change induced by the indifference pricing principle using the toy jump-diffusion model. The real-world jump probability  $p = 50\%$  and the jump size  $\epsilon = 0.1$ .

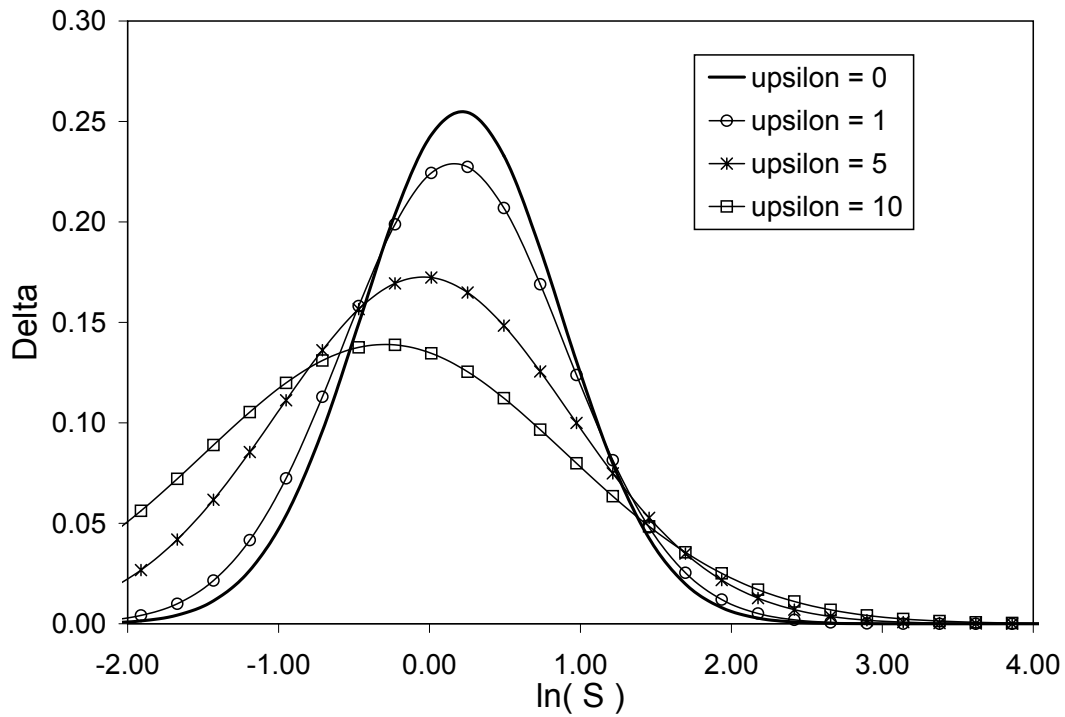
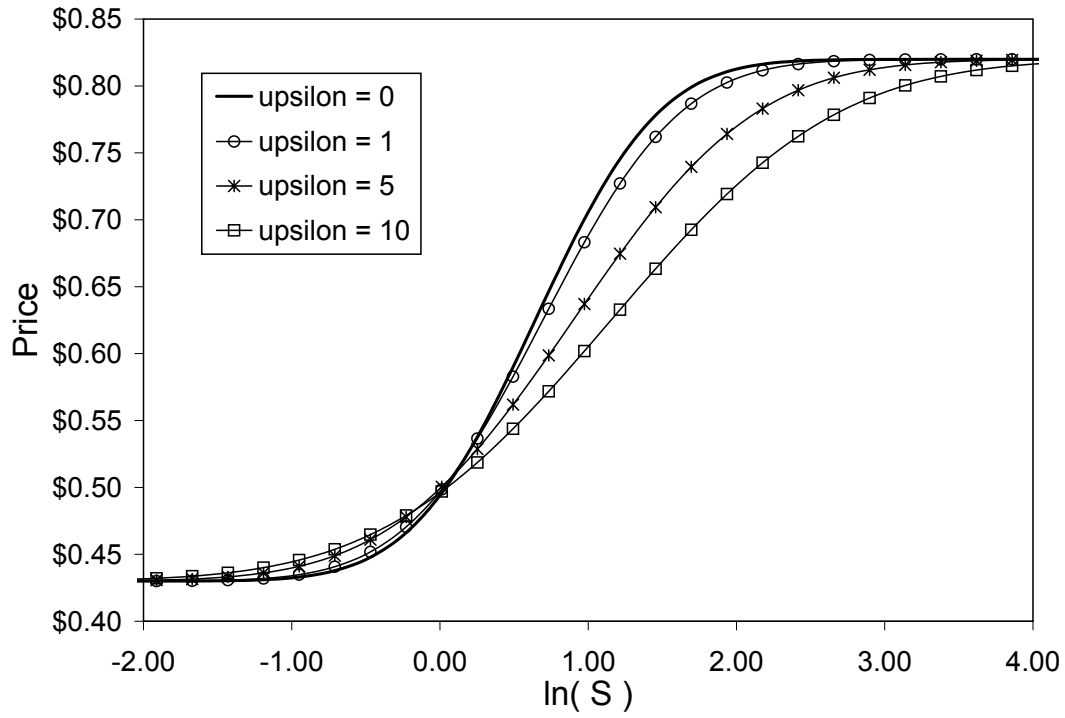


Fig. 4. These diagrams depict the price and hedging position in the risky asset with varying activity rates. The term of the contract is 10 years with  $\alpha = 0.1$ ,  $\lambda_x(t) = 0.1$ ,  $r = 5\%$ ,  $\hat{\mu} = 12\%$ ,  $\sigma = 20\%$ ,  $\epsilon = 0.1$ ,  $p = 50\%$ ,  $\gamma = 6\%$  and  $\kappa = 12\%$ .

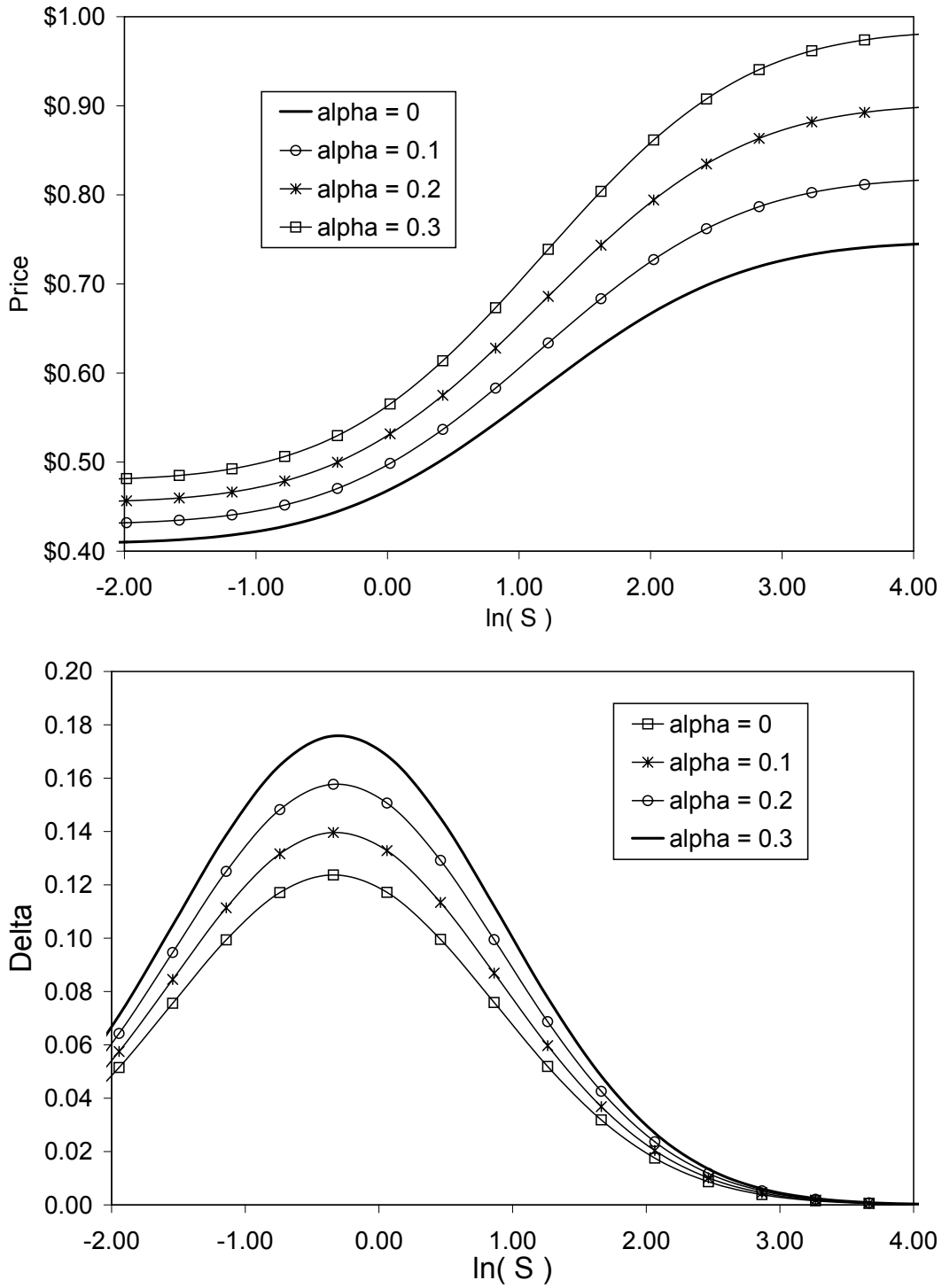


Fig. 5. These diagrams depict the price and hedging position in the risky asset with varying risk aversion parameters. The term of the contract is 10 years with  $\lambda_x(t) = 0.1$ ,  $r = 5\%$ ,  $\hat{\mu} = 12\%$ ,  $\sigma = 20\%$ ,  $\epsilon = 0.1$ ,  $v = 10$ ,  $p = 50\%$ ,  $\gamma = 6\%$  and  $\kappa = 12\%$ .