Introduction of the Variance Gamma Options Pricing Model

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Introduction

- Variance Gamma Option Pricing Model (VGOPM) is proposed by by Madan, Carr and Chang in 1998
- The major benefit of the model over the tradition Black-Scholes Option Pricing Model (BSOPM) is that it includes two more parameters, Skewness & Kurtosis, which provide a better fit to the return distribution in the realistic world.
- This paper will briefly discuss the performance of the VGOPM and provide an empirical test using Hang Seng Index option (The result is coming from my Mphil thesis in 1999)

Objectives

- Determine whether the VG process outperform the normal distribution in modeling the Hang Seng index log-return.
- Compare the pricing quality of symmetric / asymmetric VGOPM and BSOPM
- Test the ability of VGOPM in reducing the pricing biases

Literature Review

- Black-Scholes Option Pricing Model
- Variance Gamma Option pricing model

Black-Scholes option pricing model

- Black-Scholes (1973) derived a close form option pricing model as show below:

$$C = S \cdot N(d_1) - K \cdot N(d_2) \cdot e^{-rT}$$

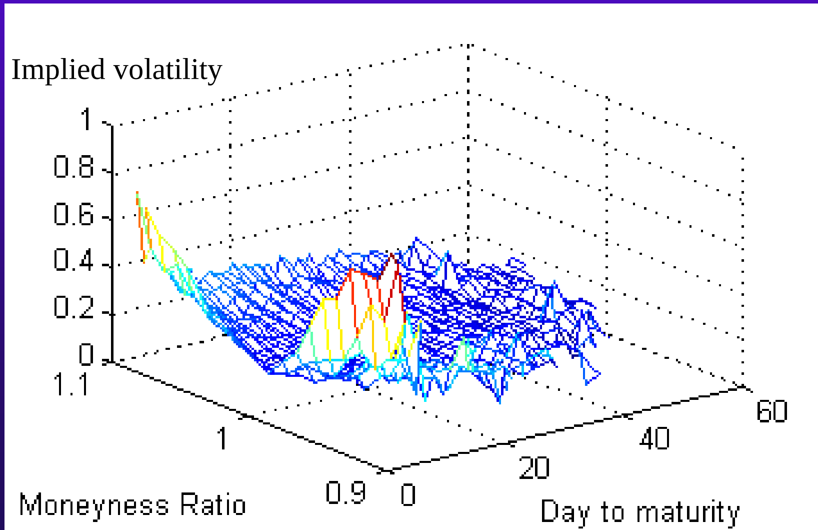
$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left[r + \left(\frac{\sigma^2}{2}\right)\right]T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

$\{C\}$ is the value of a call option, $\{S\}$ is the price of the underlying asset, $\{K\}$ is the strike price, $\{r\}$is free interest rate (HIBOR),$\{T\}$is time to maturity, and $\{\sigma\}$ is the volatility of the spot price. *N( ) is* cumulative density function

Problems in Black-Scholes Option Pricing Model

- Mispricing of BSOPM (Black 1975, MacBeth and Menville (1979), Bhattacharya (1980), Rubinstein (1985) ….etc)
- Volatility smile, Moneyness effect, and Term structure(Cotner and Horrell (1989) , Rubinstein (1985) , Sheikh (1991) …etc)



Variance Gamma option pricing model

- Madan, Carr and Chang (1998) based on Madan and Mile (1991) derived the close form solution of VG Option pricing Model:

$$C = S\Psi\left(d\sqrt{(1-c_1)/v}, (\alpha+\sigma')\sqrt{v/(1-c_1)}, t/v\right)$$
$$- Ke^{-rt}\Psi\left(d\sqrt{(1-c_1)/v}, (\alpha+\sigma')\sqrt{v/(1-c_1)}, t/v\right)$$

$$d = \frac{1}{\sigma'}\left[\ln\left(\frac{S_0}{K}\right) + rt + \frac{t}{v}\ln\left(\frac{1-c_1}{1-c_2}\right)\right]$$

$$c_1 = \frac{v(\alpha+\sigma')^2}{2} \qquad \sigma' = \frac{\sqrt{2}\sigma^2/\sqrt{v}}{\sqrt{2\sigma^2/v+\theta^2}}$$

$$c_2 = \frac{v\alpha^2}{2} \qquad \alpha = \theta/\sigma^2$$

$\Psi()$ is a modified BasselK function. In asymmetric case, there are two new parameters: v is a measure of excess kutosis, α is a measure of risk aversion. In symmetric case: α is equal to zero.

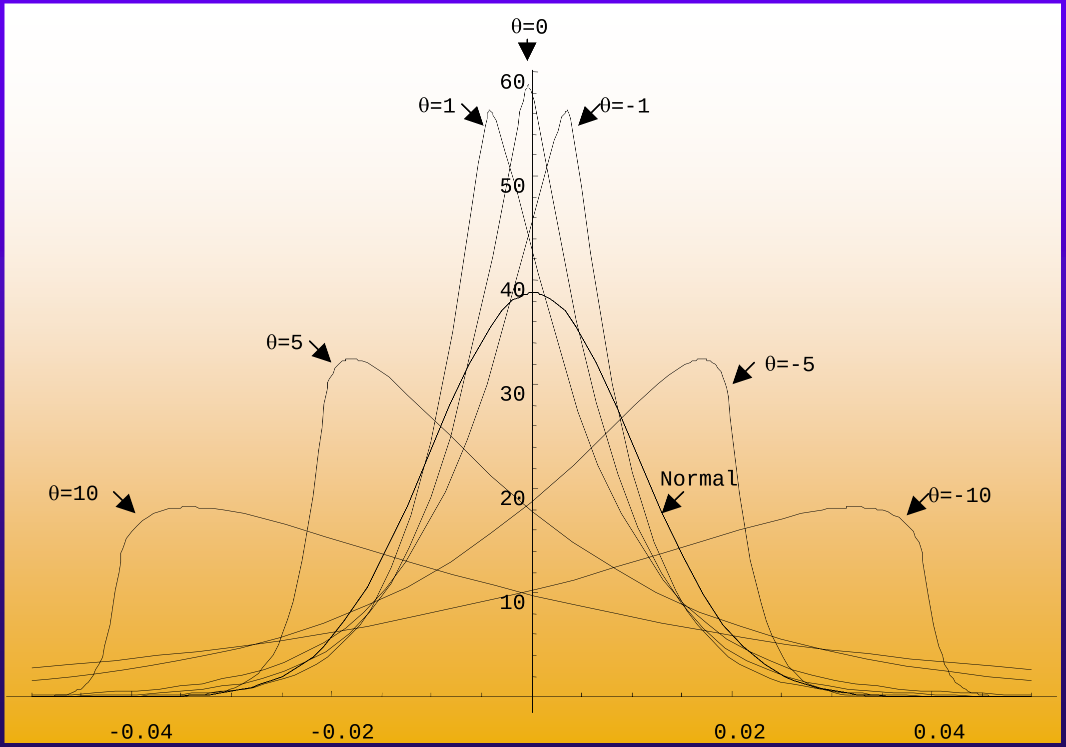
Advantage of using Variance Gamma Option Pricing Model

- The VG process is a pure jump process. It allows infinite price jump in any time interval. Madan and Seneta(1987) shows that this process is more consistent with the observed market data.

- The VGOPM takes Skewness and Kurtosis into consideration, which enhanced the capability of handling long tailed and skewed distribution

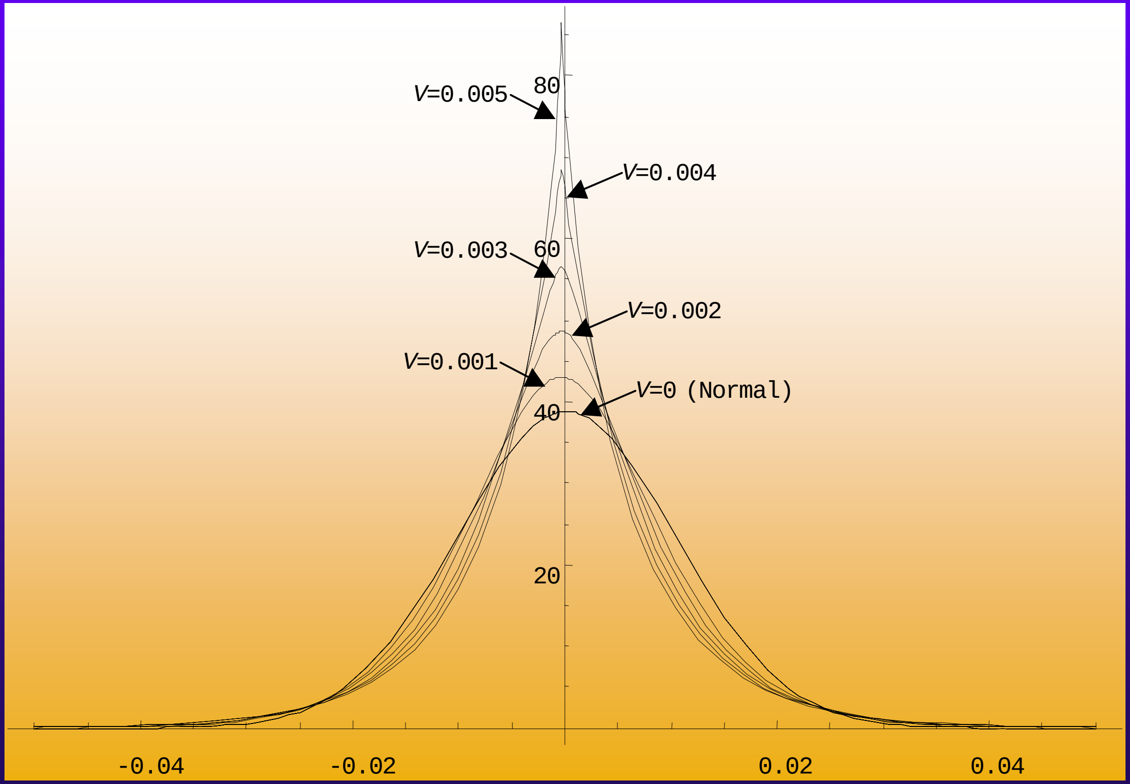
Skewness parameter

- The θ in VG process represent the skewness of the distribution. If $\theta = 0$ the distribution is symmetric, if $\theta > 0$ (< 0), the distribution skew to left (right)



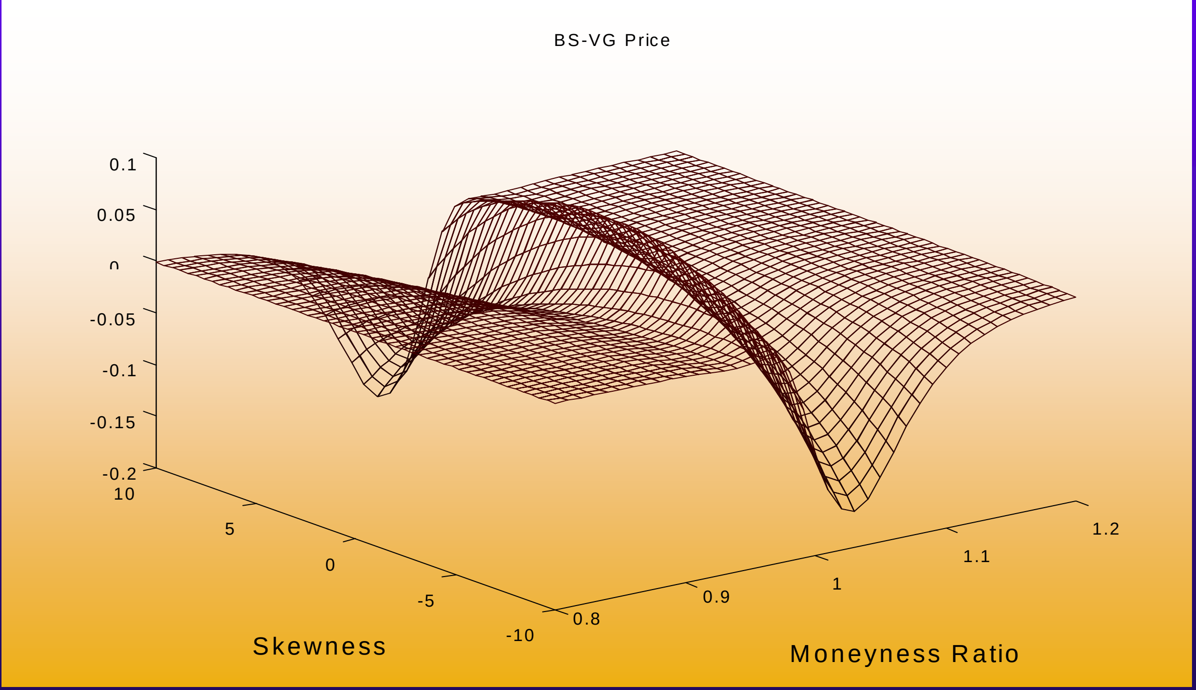
Kurtosis parameter

- The *v* in VG process is the excess kurtosis of the distribution. For normal case $v = 0$. When *v* is greater, the tail is longer.



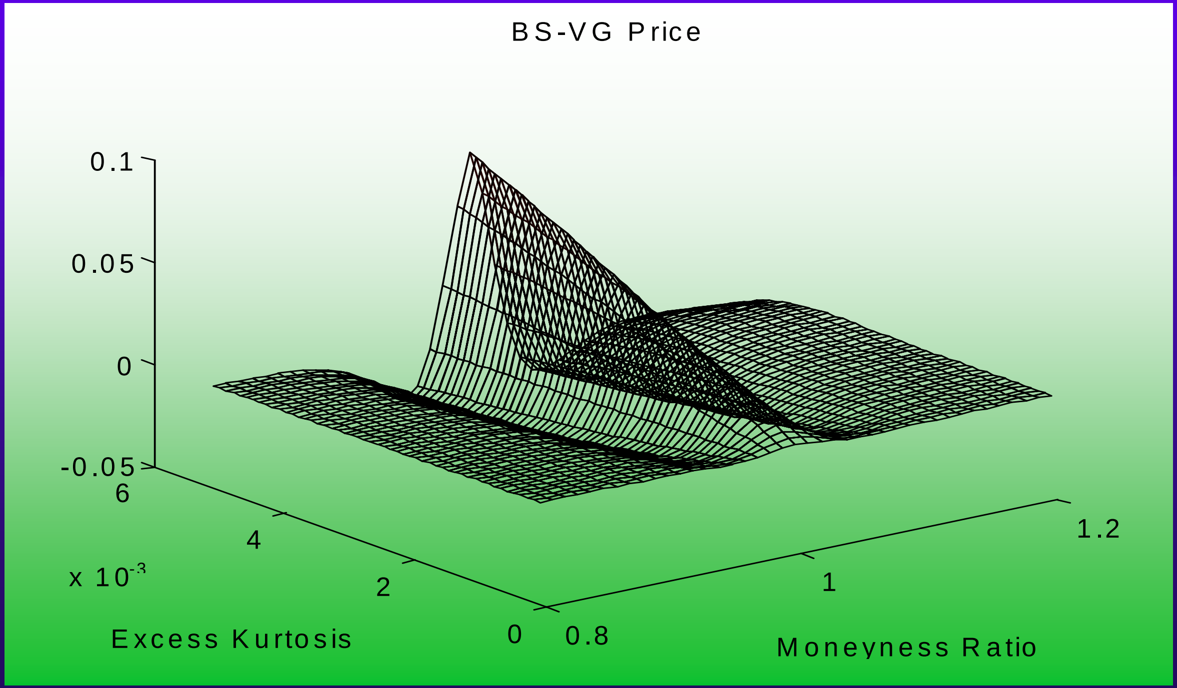
Pricing different between BSOPM and VGOPM

- Skewness effect shows that when the market is highly skewed, the price difference between BSOPM and VGOPM (the W-shape) across the moneyness axis will be skewed and the magnitude will also be increased



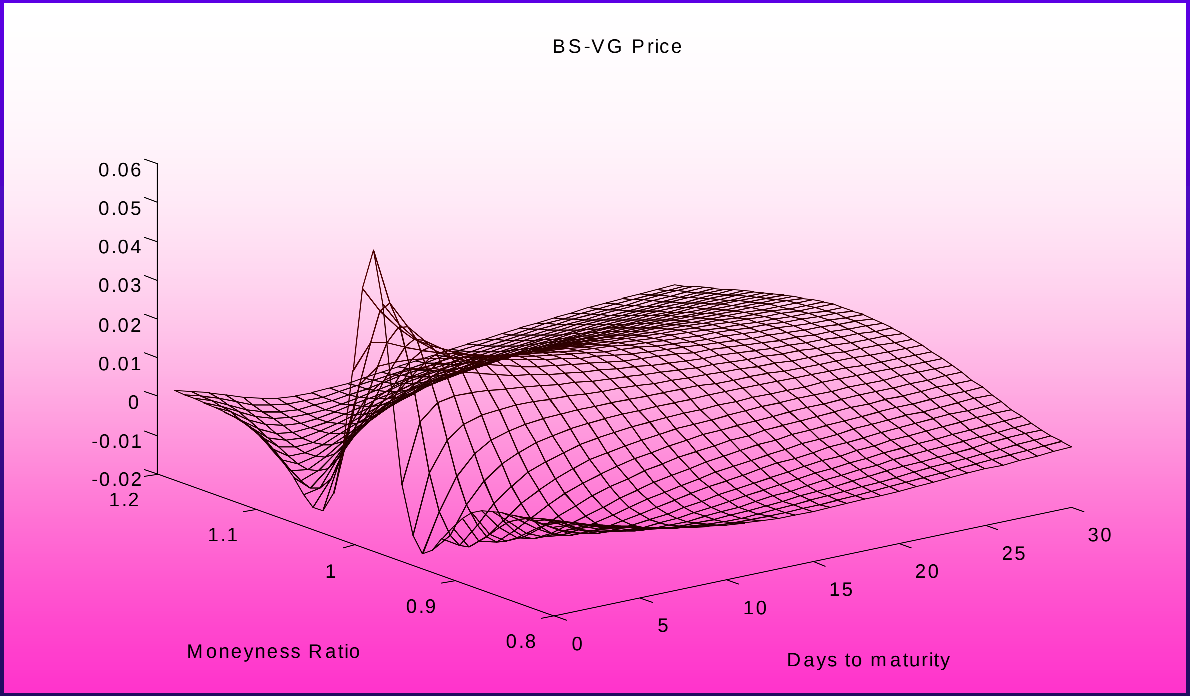
Pricing different between BSOPM and VGOPM

- Kurtosis effect shows that when kurtosis increase (increase the tail), the price difference across the moneyness axis will form a W-shape and the magnitude increase.



Pricing different between BSOPM and VGOPM

- When Skewness and Kurtosis are fixed, vary the time to maturity will also affect the shape of the price difference. When time maturity is shorter, the W-shape will enlarge.



Data

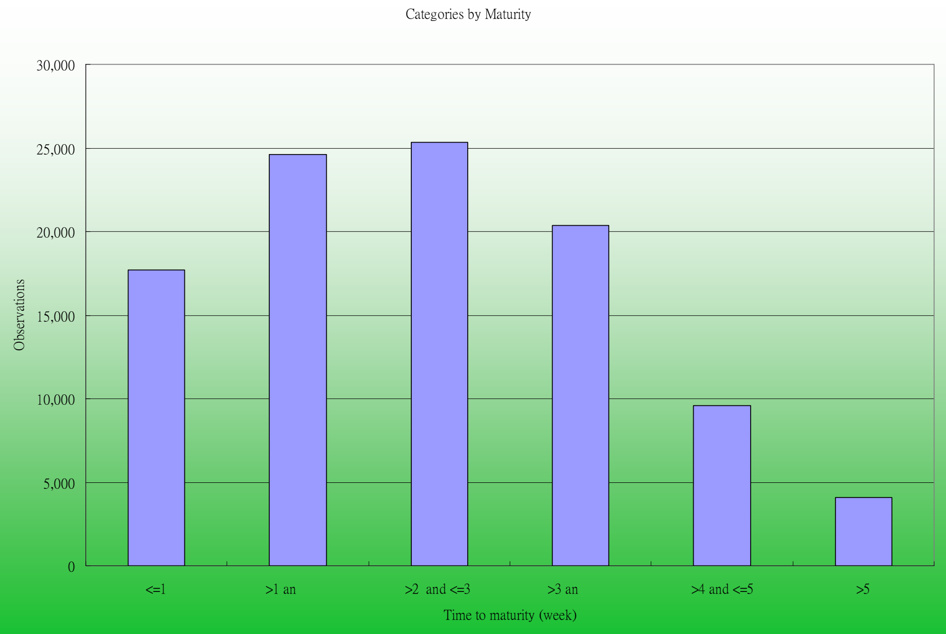
- Data Source (1 Mar 93 ~ 31 May 97)
  - Hang Seng Index minute-by-minute data (Hang Seng Index Service Limited)
  - Hang Seng Index Futures tick data (Hong Kong Futures Exchange)
  - Hang Seng Index Options tick data (Hong Kong Futures Exchange)
  - 1~12 month HIBOR daily data (DataStream)
- Data for analysis
  - 101,645 option data with maturity < 60 days were selected from the original data set
  - Options data were matched with the corresponding index and future data when trading time are within 1-minute
  - The data are also matched with the corresponding months HIBOR

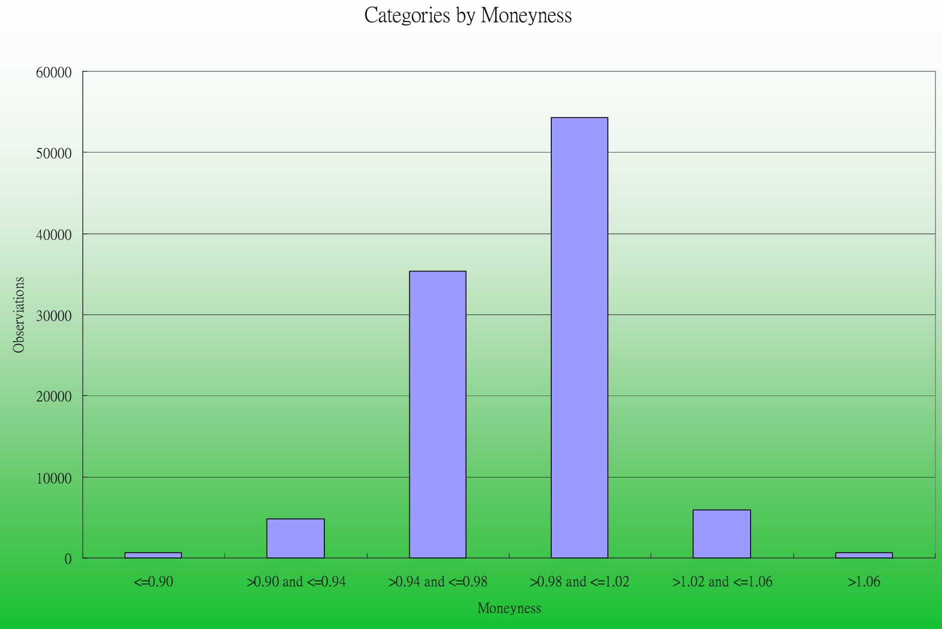
Some points that distinguish this study from others

- Many studies use end-of-the-day option, index, and future prices to perform test. This study use intraday tick data to perform test.
- In previous studies, the data may not synchronous. Here we synchronized all the relevant tick data when trading time within a minute.
- Huge amount of data (covering 3 year, over 110,000 options) is used to perform the empirical test, therefore, the result is more reliable (law of large number)

General picture of the option data

- The following graphs show that the most popular options are at-the-money option and with 2~3 weeks time to maturity.





Methodology

- The historical approach
- The implied approach
- Comparison of pricing performance using various performance measures

The Historical approach

- Parameters are assumed to be constant throughout the day
- The parameters are calculated by using historical daily data of the underlying asset
- 3 time-window have been employed (30,60 and 90 days)
- For BSOPM, the parameters are calculated based on normal distribution assumption
- For SVGOPM and AVGOPM, the parameters are calculated by maximum likelihood method

The Implied approach

- Parameters are allowed to vary intra-daily
- The parameters are estimated by using intraday tick data of the options
- 3 time-window have been employed (10,20 and 30 trades)
- For both BSOPM and VGOPM, the parameters are calculated by inverting the option pricing model

Various performance measures used in the comparison

- Mean error and mean percentage error comparison
- Mean absolute error and mean absolute percentage error
- Mean squared error and mean squared percentage error

Results and Findings

- Basic descriptive statistics of Hang Seng Index return
- Pricing Biases of BSOPM on Hang Seng Index options
- Modeling the HSI returns by a VG Process vs a log-normal process
- Comparing BSOPM and VGOPM using the historical approach
- Comparing BSOPM and VGOPM using the implied approach
- Implied Volatility Smile effect of VGOPM in comparison to that of the BSOPM

Hang Seng Index return

- Descriptive Statistics of the HSI daily return

Period	Mean	Std. Deviation	Skewness	Excess Kurtosis	Number of days
All	.0005837	.012012	-0.358	3.359	745
Jun 94~ Nov 94	-.0009595	.012945	-0.124	1.133	126
Dec 94~ May 95	.0008713	.016769	0.067	0.473	121
Jun 95~ Nov 95	.0003328	.009164	0.516	0.568	127
Dec 95~ May 96	.0011309	.012075	-2.204	13.054	122
Jun 96~ Nov 96	.0013739	.008274	-0.237	0.549	126
Dec 96~ May 97	.0007883	.011249	-0.105	1.059	123

Hang Seng Index return

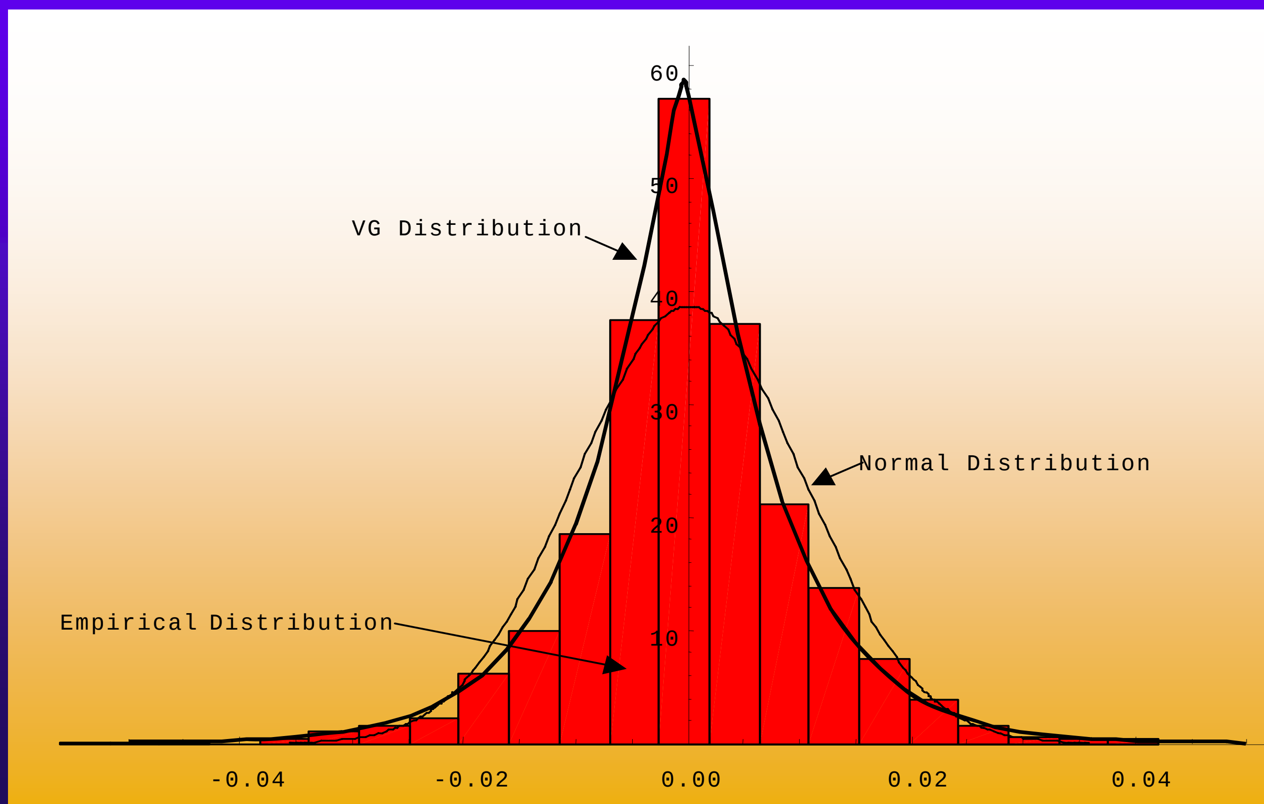
- Maximum likelihood estimations

	Normal Distribution	Symmetric VG Process	Asymmetric VG Process
Mean return (*m*)	0.148005 (0.1127)	0.128793 (0.0921)	0.15669 (0.0920)
Standard De viation (*sd*)	0.190162 (0.0078)	0.188242 (0.0063)	0.188203 (0.0063)
Excess Kurtosis(*v*)	--	0.0028627 (0.0005)	0.00287145 (0.0005)
Skewness(*q*)	--	--	0.0920816 (0.1686)
Log likelihood	2227.24	2275.58	2275.69

The values in brackets represent the standard error of the corresponding estimated value

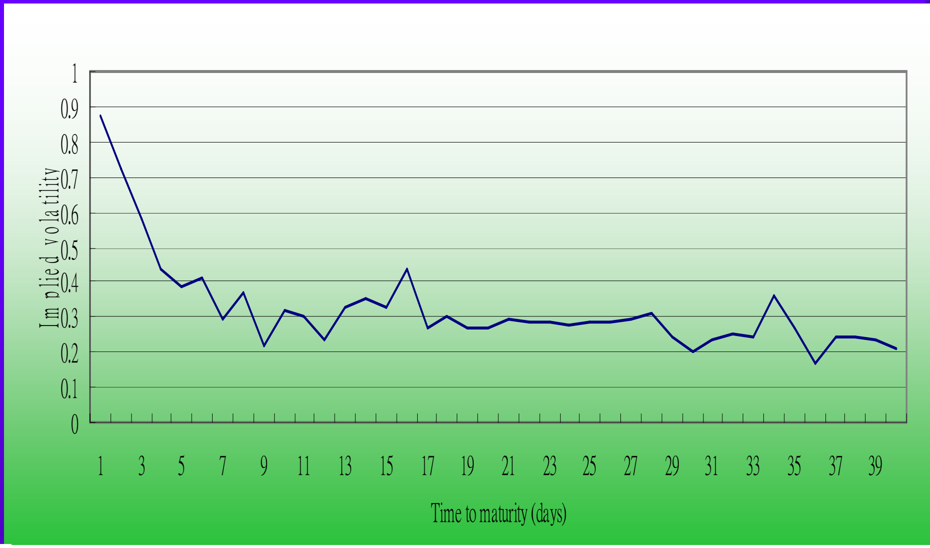
Hang Seng Index return

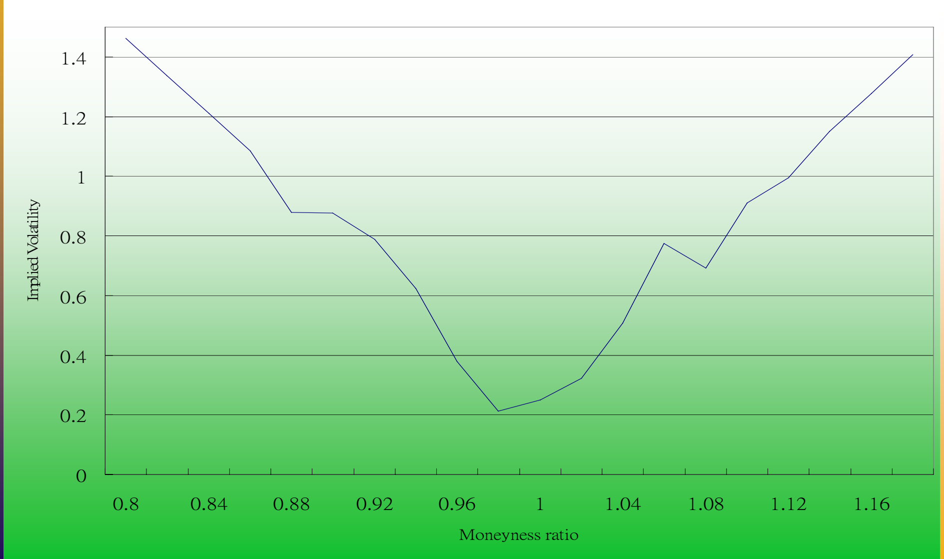
- The following graph shows the histogram of Hang Seng index log-return. The graph demonstrated that the log-return of Hang Seng index is long-tailed, therefore, VG process is more suitable than the normal distribution.

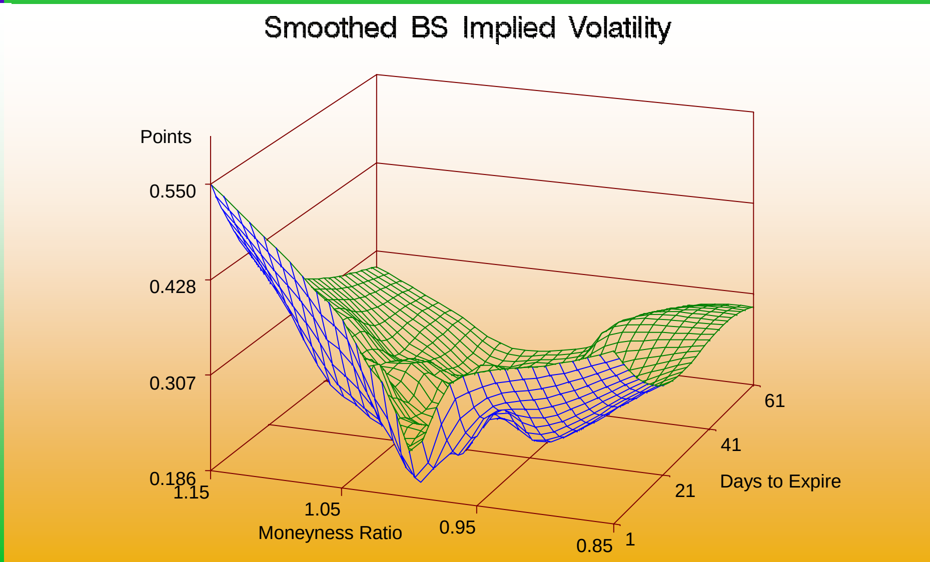


Pricing Biases of BSOPM on Hang Seng Index options

- Consistent with previous studies, pricing biases occurred in BSOPM when applying to Hang Seng Index Options







Comparing BSOPM and VGOPM using the historical approach

- The historical structured parameters of different models (Average value)

	Mean M	Volatility σ	Excess Kurtosis ν	Skewness θ
BSHIS30	0.1622 (0.0375)	0.1835 (0.0614)	--	--
BSHIS60	0.1635 (0.0341)	0.1894 (0.0581)	--	--
BSHIS90	0.1745 (0.0336)	0.1952 (0.0582)	--	--
SVGHIS30	0.1345 (0.0354)	0.1851 (0.0592)	0.0018 (0.0001)	--
SVGHIS60	0.1145 (0.0246)	0.1901 (0.0570)	0.0019 (0.0001)	--
SVGHIS90	0.0979 (0.0211)	0.1955 (0.0582)	0.0020 (0.0001)	--
AVGHIS90	0.1291 (0.0108)	0.1917 (0.0027)	0.0021 (0.0001)	0.2466 (0.1404)

The values in brackets represent the standard error of the corresponding estimated value

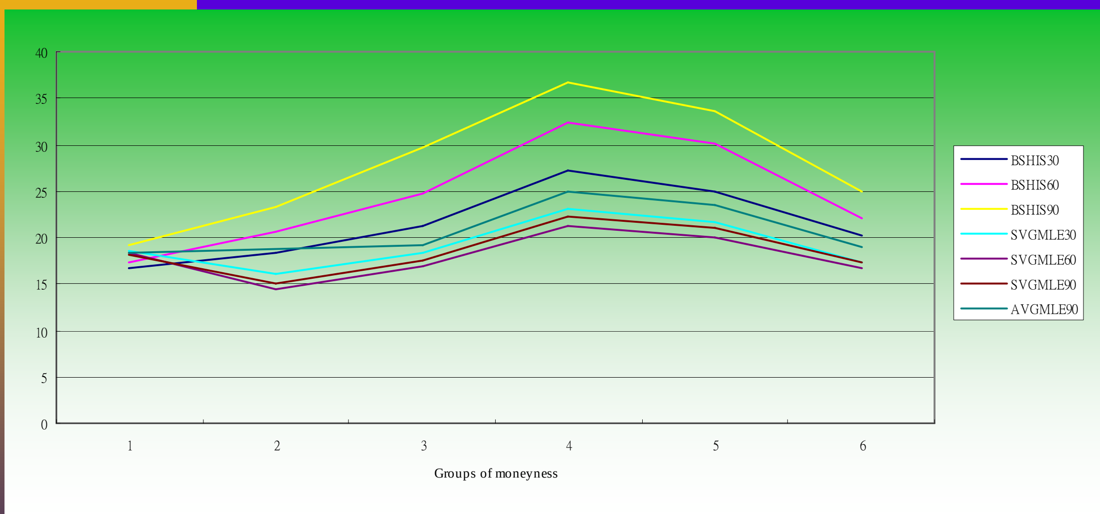
Pricing error via historical approach

- The mean absolute errors and mean absolute percentage errors of SVGOPM and AVGOPM are much better than that of BSOPM.
- When compare the 60 days time-window, the use of SVGOPM reduced the mean absolute error and mean absolute percentage error by around 40%

Model	Mean absolute error	Mean absolute error (%)	observation
BSHIS30	25.5773 (29.18)	32.44% (.5423)	101645
BSHIS60	30.4291 (29.33)	39.11% (.5800)	101645
BSHIS90	34.3852 (30.69)	43.67% (.6032)	101645
SVGMLE30	21.8842 (20.06)	26.81% (.3355)	101645
SVGMLE60	20.1968 (18.82)	24.29% (.2996)	101645
SVGMLE90	21.1391 (20.44)	25.12% (.3056)	101645
ASVGMLE90	23.7182 (27.34)	28.29% (.4110)	98673

Subgroup analysis: breakdown in moneyness - mean absolute error

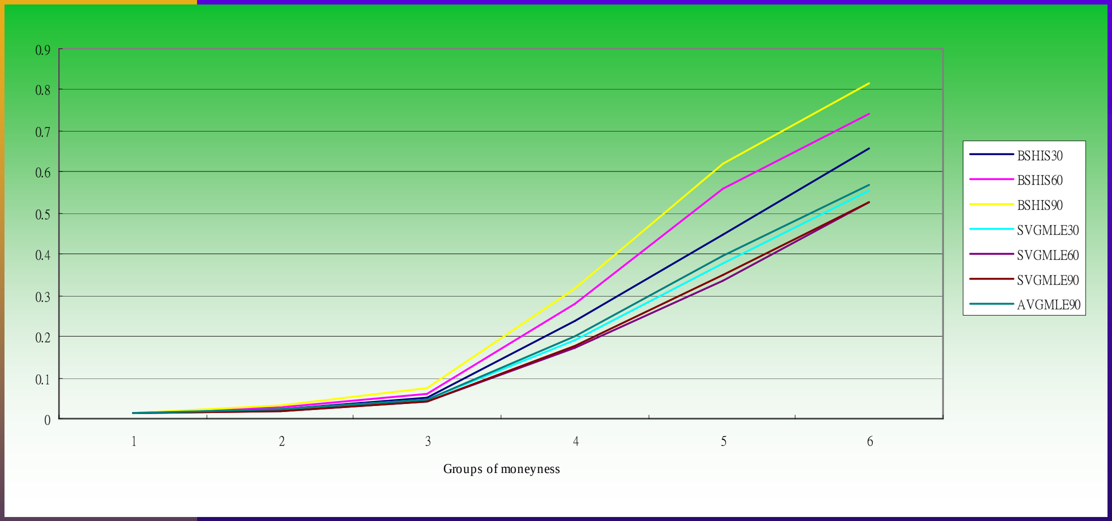
MNYGRP	BSHIS30	BSHIS60	BSHIS90	SVGMLE30	SVGMLE60	SVGMLE90	AVGMLE90
1	16.765762	17.354509	19.235109	18.541621	18.406335	18.161938	18.278644
2	18.407654	20.51644	23.396885	16.1783	14.438938	15.10399	18.797106
3	21.258763	24.663299	29.711161	18.290784	16.90873	17.583864	19.260535
4	27.166915	32.411309	36.705359	23.084723	21.178684	22.193366	24.965966
5	25.003813	30.086759	33.55228	21.585027	19.969845	20.944441	23.550299
6	20.104159	22.071005	24.975176	17.344578	16.776874	17.228272	18.9544



- The error pattern for BSOPM, SVGOPM and AVGOPM are similar.
- The curvature and the mispricing magnitude of the symmetric and asymmetric VGOPM are smaller than that of the BSOPM

Subgroup analysis: breakdown in moneyness - mean absolute error (%)

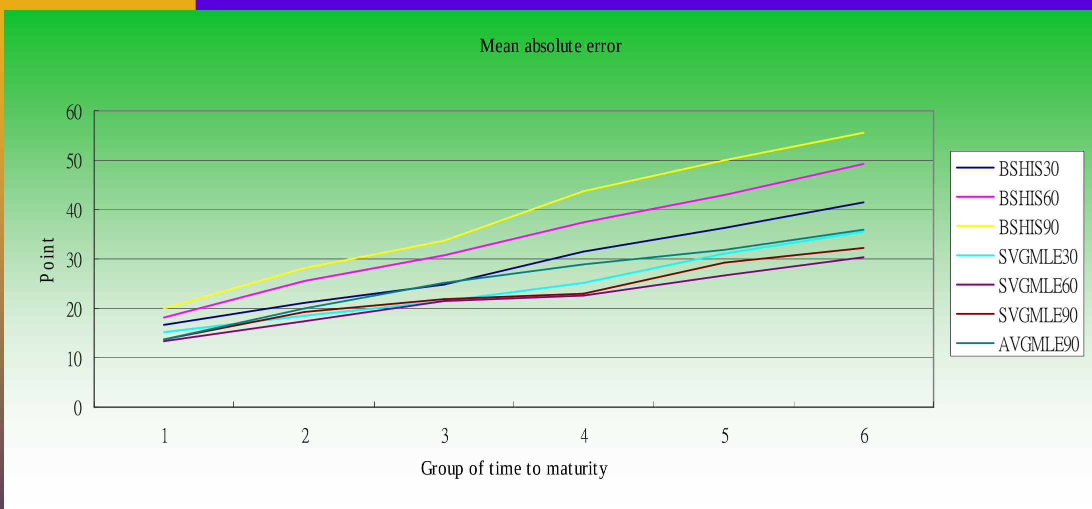
MNYGRP	BSHIS30	BSHIS60	BSHIS90	SVGMLE30	SVGMLE60	SVGMLE90	AVGMLE90
1	1.318%	1.359%	1.527%	1.469%	1.456%	1.443%	1.455%
2	2.500%	2.795%	3.186%	2.122%	1.870%	1.985%	2.425%
3	5.286%	6.143%	7.410%	4.463%	4.090%	4.233%	4.655%
4	23.695%	27.918%	31.662%	19.030%	17.230%	17.895%	20.078%
5	44.697%	55.914%	61.839%	37.600%	33.571%	35.003%	39.848%
6	65.823%	74.246%	81.422%	55.323%	52.717%	52.479%	56.870%



- The error pattern for BSOPM, SVGOPM and AVGOPM are similar.
- The curvature and the mispricing magnitude of the symmetric and asymmetric VGOPM are smaller than that of the BSOPM

Subgroup analysis: breakdown in time to maturity- mean absolute error

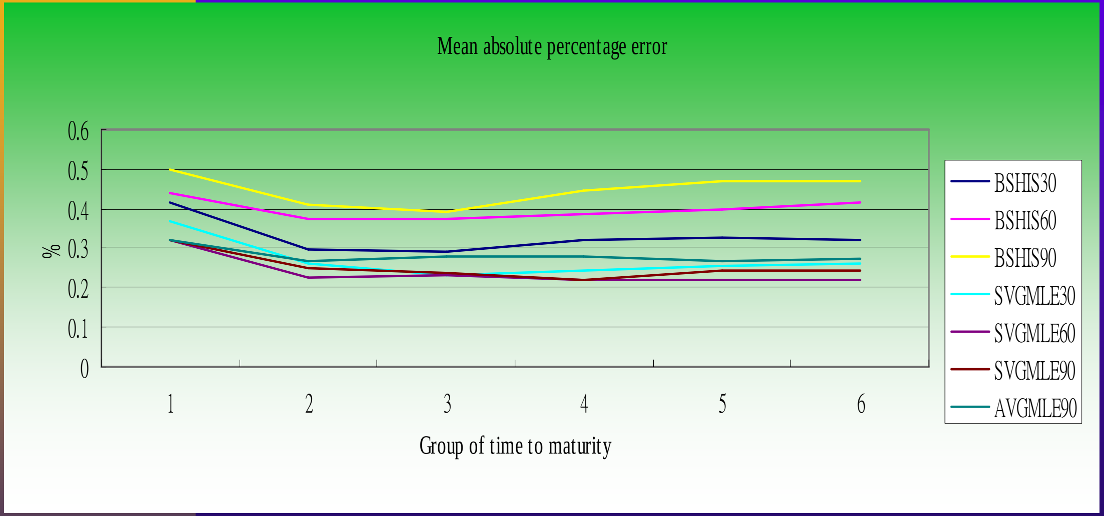
TTMGRP	BSHIS30	BSHIS60	BSHIS90	SVGMLE30	SVGMLE60	SVGMLE90	AVGMLE90
1	16.8514	17.9694	19.9903	15.2099	13.3368	13.7157	13.8142
2	21.1941	25.3809	28.0633	18.5632	17.3965	19.1705	20.0142
3	24.7357	30.6903	33.7033	21.5433	21.5567	21.9592	25.3686
4	31.3259	37.3243	43.7702	25.0724	22.776	22.9596	29.0454
5	36.2252	43.0382	50.0625	30.9803	26.7003	29.1841	31.9471
6	41.4009	49.2884	55.544	35.7228	30.2671	32.1697	35.8058



- All the models show an upward trend when time to maturity increases.
- The slope of the error curves for BSOPM are greater than that for both symmetric and asymmetric VGOPM, therefore, VGOPM should provide a better pricing quality

Subgroup analysis: breakdown in time to maturity- mean absolute error (%)

TTMGRP	BSHIS30	BSHIS60	BSHIS90	SVGMLE30	SVGMLE60	SVGMLE90	AVGMLE90
1	41.49%	44.09%	49.74%	36.65%	31.91%	31.91%	32.12%
2	29.92%	37.61%	41.24%	25.95%	22.71%	24.70%	26.51%
3	28.86%	37.16%	39.25%	23.29%	23.36%	23.78%	28.18%
4	31.99%	38.33%	44.76%	24.52%	22.15%	22.08%	28.01%
5	32.81%	39.62%	46.66%	25.28%	22.23%	24.06%	26.84%
6	32.02%	41.47%	46.88%	26.18%	22.02%	24.31%	27.45%



- The error patent for all models is similar (a horizontal line)
- The magnitude of the errors for VGOPM are smaller than that for BSOPM. The percentage error reduced by more than 1/3

Comparing BSOPM and VGOPM using the implied approach

- The implied volatility (Average value)

	Implied volatility		N
	Mean	Standard deviation	
BSIV10	0.1958	0.05627	93922
BSIV20	0.1948	0.05523	86903
BSIV30	0.1936	0.05451	79905
SVGIV10	0.1966	0.05587	93922
SVGIV20	0.1955	0.05497	86903
SVGIV30	0.1943	0.05428	79905
AVGIV10	0.1961	0.05596	93922
AVGIV20	0.1951	0.05509	86903
AVGIV30	0.1939	0.05445	79905

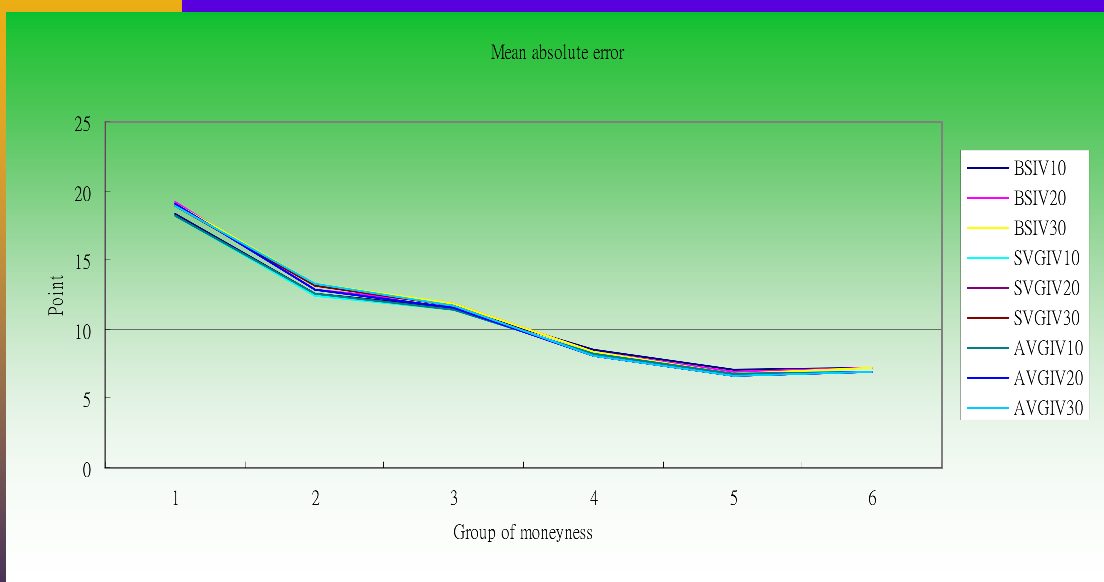
Pricing error via implied approach

- The mean absolute errors and mean absolute percentage errors of SVGOPM and AVGOPM are smaller than that of BSOPM.
- When compare the 30 trades time-window, the use of SVGOPM reduced the mean absolute error and mean absolute percentage error by 3% and 5.6% respectively

Model	Mean absolute error	Mean Absolute error (%)	observation
BSIV10	8.160 (16.27)	10.31% (0.267)	93922
BSIV20	8.030 (16.05)	10.12% (0.274)	86903
BSIV30	7.967 (15.99)	10.01% (0.275)	79905
SVGIV10	7.874 (15.57)	9.70% (0.270)	93922
SVGIV20	7.768 (15.36)	9.55% (0.274)	86903
SVGIV30	7.725 (15.40)	9.45% (0.274)	79905
AVGIV10	7.895 (15.61)	9.72% (0.270)	93922
AVGIV20	7.785 (15.39)	9.56% (0.274)	86903
AVGIV30	7.740 (15.42)	9.46% (0.274)	79905

Subgroup analysis: breakdown in moneyness - mean absolute error

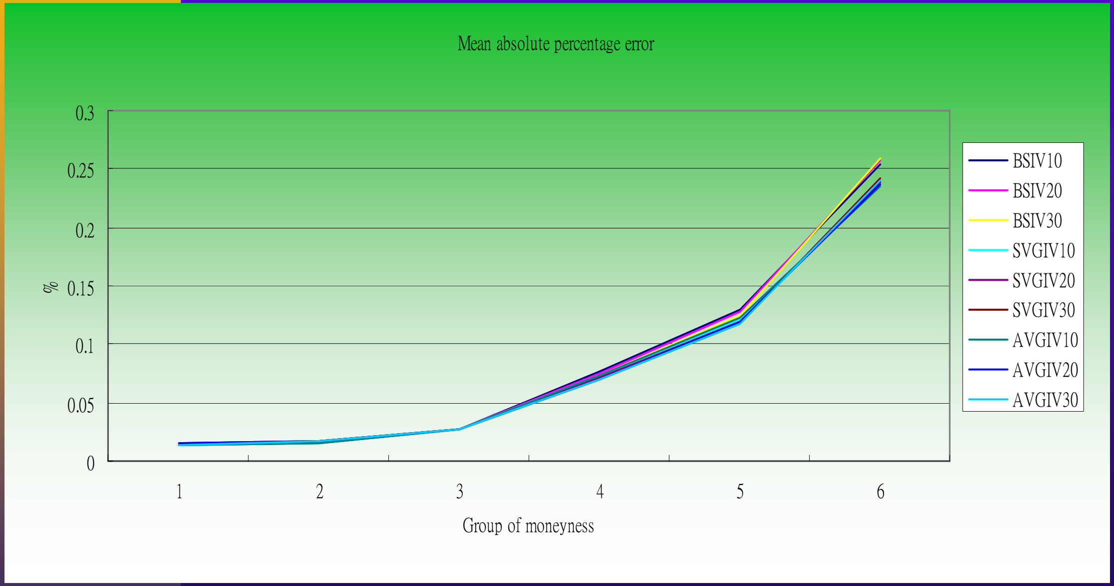
MNYGRP	BSIV10	BSIV20	BSIV30	SVGIV10	SVGIV20	SVGIV30	AVGIV10	AVGIV20	AVGIV30
1	18.3581	19.2279	19.0416	18.1856	19.0556	18.9046	18.1504	19.1298	18.9710
2	12.5228	12.8908	13.2445	12.4764	12.8130	13.1655	12.5097	12.8806	13.2229
3	11.5744	11.8085	11.9187	11.4451	11.6179	11.7340	11.4049	11.5848	11.7092
4	8.5594	8.4020	8.3137	8.2548	8.1281	8.0504	8.2664	8.1320	8.0535
5	7.0734	6.8944	6.8254	6.7942	6.6430	6.6092	6.8450	6.6936	6.6544
6	7.2231	7.2008	7.1883	6.9016	6.9048	6.9136	6.8910	6.8849	6.9055



- The error patent for all model are similar.
- All models performed worse when pricing deep-in-the-money option and then the error decreased gradually as moneyness increases.

Subgroup analysis: breakdown in moneyness - mean absolute error (%)

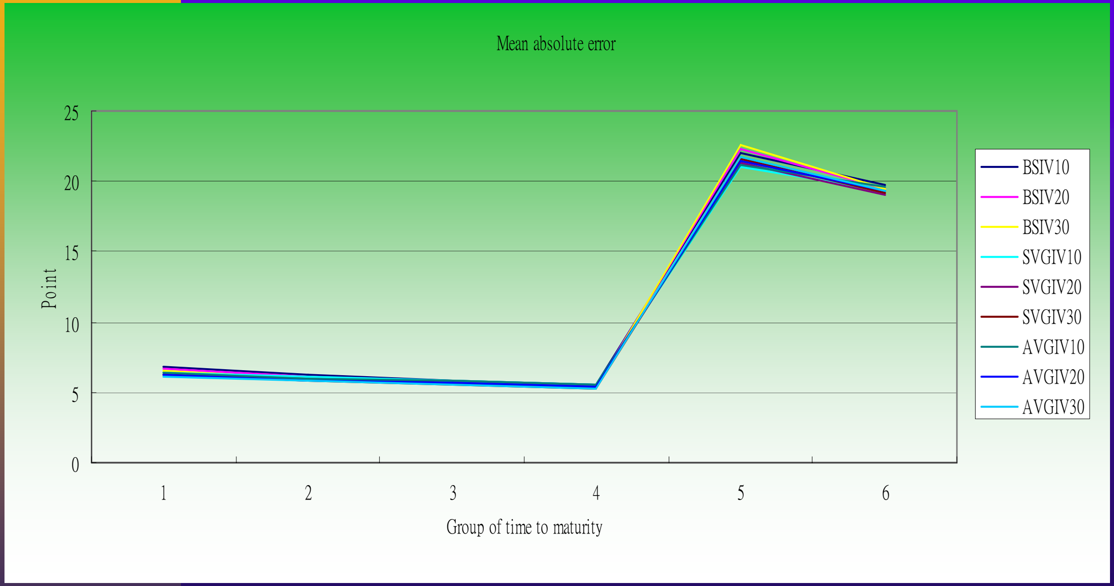
MNYGRP	BSIV10	BSIV20	BSIV30	SVGIV10	SVGIV20	SVGIV30	AVGIV10	AVGIV20	AVGIV30
1	1.410%	1.470%	1.448%	1.395%	1.456%	1.436%	1.391%	1.462%	1.442%
2	1.610%	1.642%	1.662%	1.604%	1.632%	1.652%	1.607%	1.641%	1.661%
3	2.683%	2.722%	2.729%	2.657%	2.685%	2.693%	2.646%	2.676%	2.686%
4	7.619%	7.423%	7.332%	7.254%	7.084%	7.026%	7.247%	7.071%	7.004%
5	12.990%	12.727%	12.520%	12.124%	11.907%	11.716%	12.217%	11.999%	11.808%
6	25.463%	25.673%	25.934%	23.697%	23.943%	24.164%	23.512%	23.757%	23.996%



- The error patent for all model are similar
- The error increases in an accelerating manner form group 1 to group 6
- The error for VGOPM is slightly smaller than that for BSOPM

Subgroup analysis: breakdown in time to maturity - mean absolute error

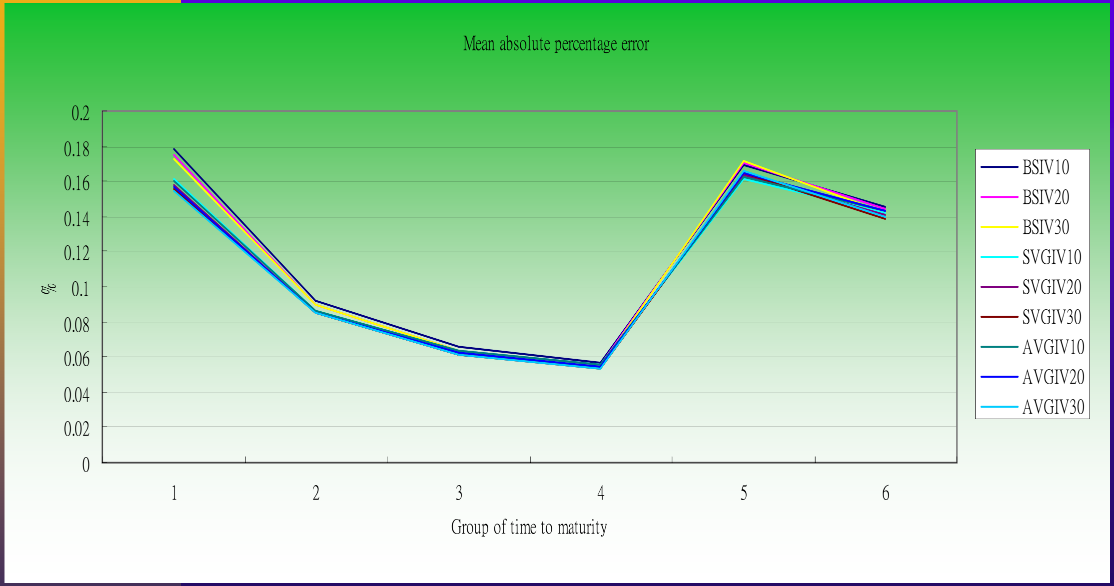
TTMGRP	BSIV10	BSIV20	BSIV30	SVGIV10	SVGIV20	SVGIV30	AVGIV10	AVGIV20	AVGIV30
1	6.8135	6.6601	6.5716	6.3852	6.2631	6.1991	6.3557	6.2365	6.1771
2	6.2476	6.0775	6.0356	6.0567	5.9125	5.8838	6.0292	5.8828	5.8566
3	5.8926	5.7405	5.6535	5.7754	5.6423	5.5613	5.7827	5.6437	5.5615
4	5.5955	5.4339	5.3222	5.4883	5.3441	5.2429	5.5137	5.3534	5.2516
5	22.0444	22.2918	22.5553	21.0304	21.2969	21.6247	21.2085	21.4765	21.7878
6	19.7773	19.4642	19.4997	19.3147	19.0455	19.1184	19.5346	19.2779	19.3447



- The mean absolute error of group 1 to group 4 (spot month) is around 5 to 6 points, but for group 4, it suddenly shoots up to 22 points, then return to 20 points for group 6

Subgroup analysis: breakdown in time to maturity - mean absolute error (%)

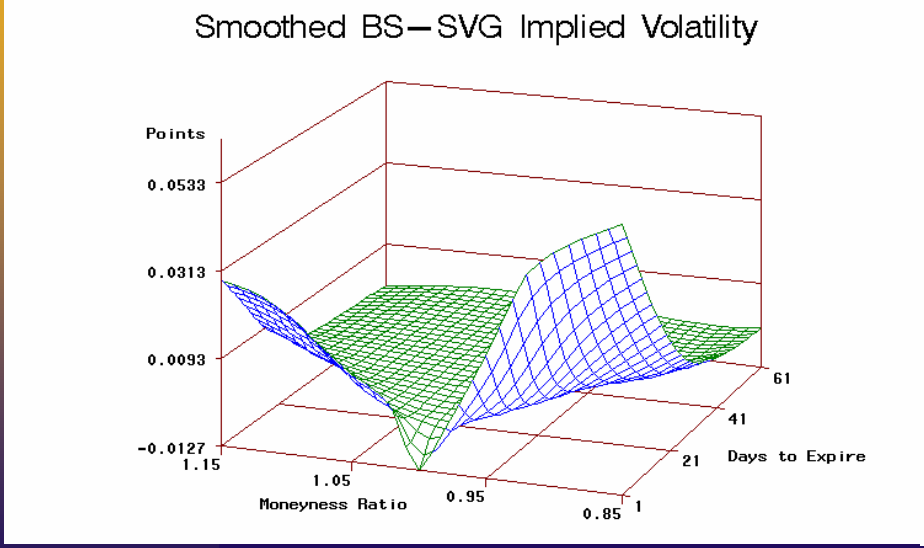
TTMGRP	BSIV10	BSIV20	BSIV30	SVGIV10	SVGIV20	SVGIV30	AVGIV10	AVGIV20	AVGIV30
1	17.873%	17.515%	17.243%	16.105%	15.755%	15.546%	16.060%	15.704%	15.486%
2	9.190%	8.985%	8.928%	8.689%	8.529%	8.496%	8.676%	8.516%	8.478%
3	6.620%	6.406%	6.312%	6.380%	6.192%	6.106%	6.385%	6.195%	6.107%
4	5.687%	5.582%	5.428%	5.543%	5.451%	5.300%	5.573%	5.472%	5.322%
5	16.941%	17.031%	17.167%	16.150%	16.338%	16.493%	16.299%	16.488%	16.636%
6	14.585%	14.484%	14.201%	14.224%	14.126%	13.872%	14.355%	14.270%	14.024%

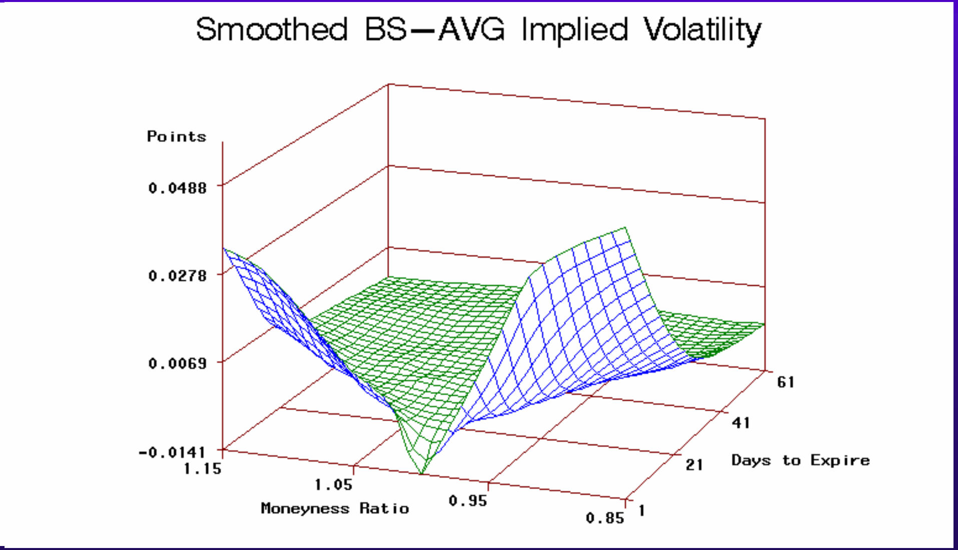


- The error dropped dramatically from 16% in group 1 to 9% in group 2, then dropped slowly to 6% in group 4. Again, there is a great change after group 4 and Black to 16% in group 5.

Volatility Smile effect of BSOPM and VGOPM

- Although the smile effect still occurred in VGOPM, the following graphs shows the reduction of the smile effect of SVGOPM and AVGOPM over the BSOPM.





Conclusions

- The VG process is a better fit for the log return distribution of Hang Seng index options in the study period
- Both the Symmetric and Asymmetric VGOPM reduced the smile effect of BSOPM.

Conclusions

- In the historical approach, the SVGOPM and AVGOPM outperform the BSOPM is quite prominent, especially when the moneyness increase (become more out of the money), and time to maturity increase.
- In general, compared with the BSOPM, the SVGOPM and the AVGOPM reduced the pricing error by 25% to 60%

Conclusions

- Using the implied approach, four out of six performance measures show that VGOPM is better than BSOPM.
- The improvement of pricing quality of the VGOPM in the implied approach is not as good as that in historical approach