Hedging of barrier options

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Author: Natalia Dolgova

Supervisor: Prof. Dr. Paolo Vanini

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Abstract

The hedging approaches for barrier options in the literature are based on assumptions that make these methods difficult to implement. Either one requires a not existing excessive liquidity of hedging instruments, or not acceptable sizes of the hedge positions follow or the hedge errors are not acceptable in the crucial barrier price region. Based on these observations we propose a Vega-matching strategy. We show that this approach leads to a better hedging performance in most cases compared to dynamic hedging and static hedging of Derman, Kani and Ergener (1994) and of Carr and Chou (1997a). Since the quality of any static hedge changes over time, we finally define an implementable optimization approach which allows us to control the hedging performance over time. The optimal hedging strategy significantly improves the non-optimized ones.

1 Introduction

Barrier options are among the most heavily traded exotic derivatives and perhaps the oldest of all exotic options (Dupont (2001)). These instruments are attractive for clients because they have a lower premium than vanilla options. At the same time they are complicated for traders. The payoff discontinuity and the resulting behavior of the Greeks complicates hedging of these options. The definition of an optimal hedging strategy is still an open problem.

The dynamic Delta-hedging in a Black-Scholes framework (Merton (1973)) leads to difficulties in the hedging of the barrier options. The Delta of the barrier options is very sensitive to changes in the price of the underlying. Hence, traders need to rebalance the hedge very often. This leads to large transaction costs and is a challenge for the operations of the large barrier options positions. To eliminate these problems static hedging approaches were invented (Nalholm and Poulsen (2006)). We consider the methods of Derman et al. (1994) and Carr and Chou (1997a). Although static hedging approaches do not require any rebalancing, they also have some shortcomings. The main obstacle is that replication is impossible in practice because it requires an infinite number of vanilla options. Additionally, these approaches are based on Black-Scholes model assumptions. Since volatility is not constant, model risk follows which triggers hedging errors.

Our goal is to find a hedging strategy which hedges the random payoff of the barrier option more effectively than the above static approaches. To achieve this goal, we define a static Vega-matching approach.

We compare in the first part of this thesis the effectiveness of the dynamic, and static DEK and CarrChou hedging strategies for barrier options. We first define requirements which a hedge should satisfy. Concentrating in this thesis on the most difficult to hedge Down&In put option we investigate these methods using simulated Black-Scholes data as well as historical data. We illustrate the main disadvantages of the approaches which make them difficult or even impossible to implement in practice. Taking those disadvantages into consideration, in a second part we propose the static hedging approach which makes the hedged position Vega-neutral. The hedge portfolio consists of only a few vanilla options and Vega hedging protects from volatility risk. The hedge performance of this method dominates the other approaches.

A problem which is inherent in all static hedging strategies is that the hedge quality can worsen over time. To eliminate this problem, we propose an optimization approach which minimizes the risk of using one particular hedging strategy (Akgün (2007)). The optimization is made over all possible realizations of the hedging strategy and requires a finite set of scenarios only to give useful results. The scenarios are generated using the Black-Scholes model. The state variables are the P&Ls of the hedged barrier option position, the control variables can be strikes of the hedge instruments or the size of the hedge positions. The numerical examples show to what extent non-optimized strategy are improved.

The thesis is organized as follows. Section 2 classifies all barrier options. Based on the analysis of the Greeks behavior and payoff functions of different types of the barrier options we choose a Down&In put for further consideration. Section 3 describes and compares the static approaches of Derman et al. (1994), Carr and Chou (1997a) and the dynamic hedging strategy. Section 4 introduces the Vega-matching strategy and shows its advantages over other strategies. Section 5 defines the optimization approach for static hedging strategies and compares optimal versus non-optimal strategies. Section 6 concludes.

2 Classification of barrier options

Barrier options are path-dependent options which become vanilla ones or worthless depending on whether the underlying hits a prespecified barrier level. Barrier options have different characteristics than plain vanilla options, in particular if the underlying asset price is close to the barrier.

We examine barrier options with one underlying and a single barrier which is either above or below the strike price. These barrier options are called vanilla barrier options. There are different variations and extensions of such barrier options; the so called exotic barrier options (Zhang (1998)). A short overview of the exotic barrier options is given in Appendix A. Since we consider vanilla type options only, we use the expression "barrier options" for "vanilla barrier options".

The barrier options are divided into regular and reverse ones, see Table 1.

Table 1: Types of barrier options.

	Regular	**Reverse**
Knock-out	Down&Out Call Up&Out Put	Up&Out Call Down&Out Put
Knock-in	Down&In Call Up&In Put	Up&In Call Down&In Put

"In" means that a barrier option has a positive value once the barrier level is crossed; "Out" means that a barrier option becomes worthless when the barrier is hit. "Down" ("Up") reflects the position of a barrier below (above) with respect to the option's strike. The terms regular and reverse are defined in the sequel.

2.1 Regular Barrier Options

Regular barrier options are those which are out-of-the-money when the underlying price reaches the barrier.

Knock-out regular barrier options

Knock-out barrier options have a vanilla option payoff at maturity if the barrier was never hit from issuance date to maturity.

For an **Up&Out** put, the barrier B is above the strike K and the payoff $f(S(T))$ at maturity T reads:

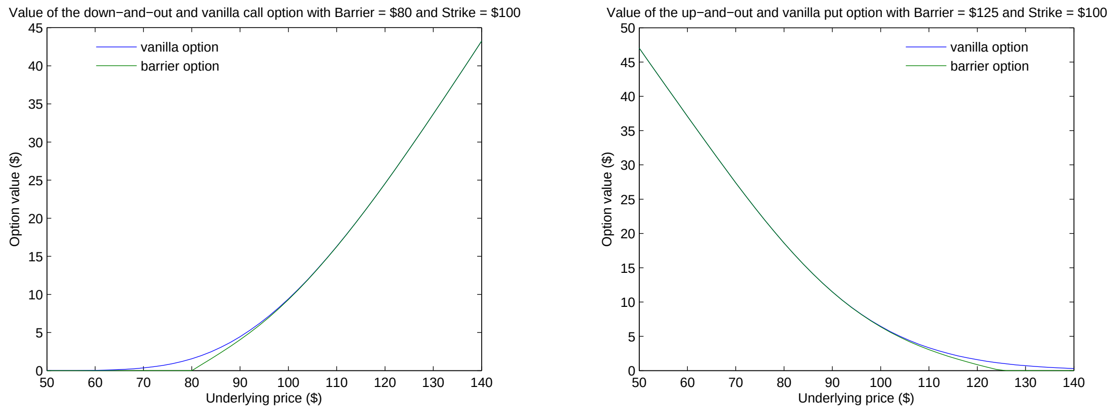
$$f(S(T)) = \begin{cases} 0 & \text{if } S(t^*) \geq B \text{ for } t^* \in [0,T], \\ K - S(T) & \text{if } S(T) < K \ \& \ S(t) < B \text{ for all } t \in [0,T]. \end{cases}$$

For the **Down&Out** call, $B < K$, we have:

$$f(S(T)) = \begin{cases} 0 & \text{if } S(t^*) \leq B \text{ for } t^* \in [0,T], \\ S(T) - K & \text{if } S(T) > K \ \& \ S(t) > B \text{ for all } t \in [0,T]. \end{cases}$$

If the price of the underlying is near to the barrier, Knock-out barrier options are out-of-the-money and have low value. If a barrier is hit these options become worthless. For the buyers of such options, if market rises (falls), the Knock-out regular call (put) is equivalent to a vanilla call (put) but with cheaper costs (Figure 1).

Figure 1: Value of the Knock-out barriers and vanilla options.



Panel left: value of the vanilla call option and the Down&Out Call. Panel right: value of the vanilla put option and the Up&Out Put

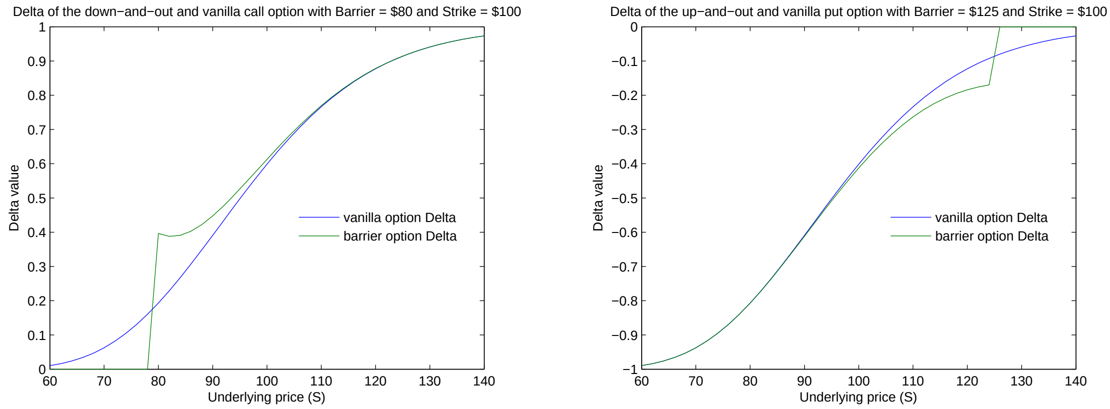
The "Greeks" of the options provide measures of the option price sensitivity to the different factors, see Appendix B for a list of Greeks. We consider in the sequel the Delta (Δ), i.e. a sensitivity of the option price changes to the changes in underlying price and the Vega (ν), i.e a sensitivity of the option price changes to the changes in volatility.

The Delta of a Knock-out barrier option has a kink at the barrier. Figure 2 shows, that an at-the-money vanilla call has Δ of 0.5, whereas the Delta of a barrier option (Δ_B) is larger. Intuitively, if underlying price rises, the price of the vanilla and the barrier option are the same. But if the underlying approaches the barrier the higher probability of the barrier option to become worthless increases its Delta compared to a vanilla option.

Figure 3 below shows the volatility sensitivity of the barrier option. If volatility increases, the price of the barrier option increases in a non-linear way compared to vanilla options. For high volatility, the underlying hits a barrier with higher probability, therefore, the barrier option has a shorter time to stay active and consequently the price of the barrier option is lower than for a corresponding vanilla option.

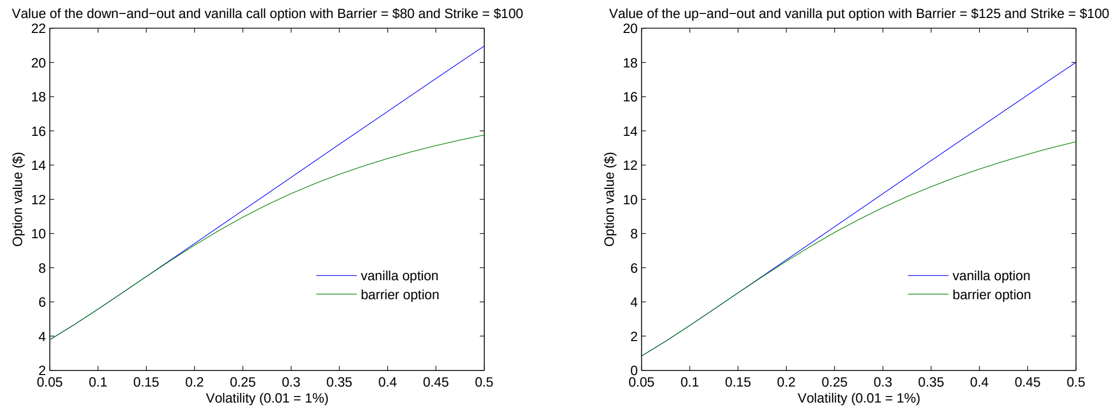
Figure 4 shows the Vega-dependence. If price moves away from the barrier, the Vega of the Knock-out barrier option approaches the Vega of a vanilla option.

Figure 2: Deltas of the Knock-out barriers and vanilla options.



Panel left: Delta of the vanilla call option and the Down&Out call barrier option. Panel right: Delta of the vanilla put option and the Up&Out Put

Figure 3: Value of the Knock-out barriers and vanilla options with respect to the volatility of the underlying.



Panel left: value of the vanilla call option and Down&Out call. Panel right: value of the vanilla put option and the Up&Out Put

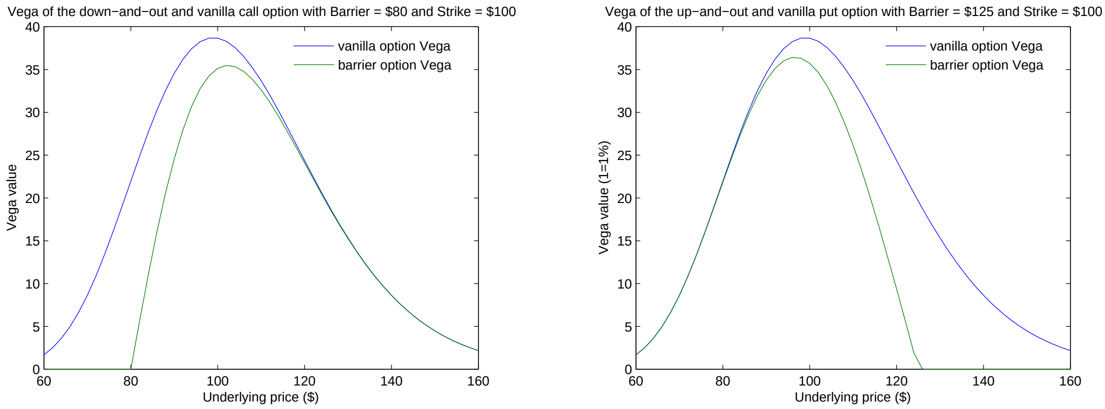
Knock-in regular barrier options

By definition, Knock-in barrier options are out-of-the money vanilla options if a barrier is hit.

For the **Up&In** put, $B > K$, the payoff function is:

$$f(S(T)) = \begin{cases} 0 & \text{if } S(t) < B \text{ for all } t \in [0,T], \\ K - S(T) & \text{if } S(T) < K \ \& \ S(t^*) \geq B \text{ for } t^* \in [0,T]. \end{cases}$$

Figure 4: Vega of the Knock-out barriers and vanilla options.



Panel left: Vega of the vanilla call option and the Down&Out call. Panel right: Vega of the vanilla put option and the Up&Out Put

The **Down&In** call, $B < K$, has the following payoff function:

$$f(S(T)) = \begin{cases} 0 & \text{if } S(t) > B \text{ for all } t \in [0, T], \\ S(T) - K & \text{if } S(T) > K \ \& \ S(t^*) \le B \text{ for } t^* \in [0, T]. \end{cases}$$

We do not analyze these types of the regular barrier options any further since Knock-in options are a combination of a vanilla option and Knock-out options. For example, a portfolio of one Knock-in call and one Knock-out call with the same strike, same barrier and same maturity is equivalent to a vanilla call. Figures of the knock-in regular barrier option payoffs and their Greeks are given in Appendix C.

2.2 Reverse Barrier Options

Reverse barrier options, by definition, knock in or out when they are in-the-money.

Knock-out Reverse barrier options

For the **Down&Out** put, $B < K$, payoff reads:

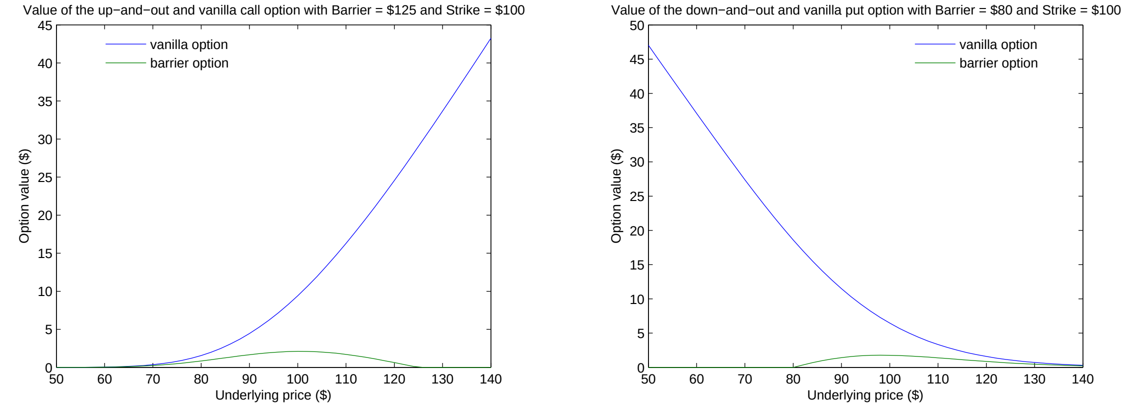
$$f(S(T)) = \begin{cases} 0 & \text{if } S(t^*) \le B \text{ for } t^* \in [0, T], \\ K - S(T) & \text{if } S(T) < K \ \& \ S(t) > B \text{ for all } t \in [0, T]. \end{cases}$$

For the **Up&Out** call, $B > K$, the payoff function is:

$$f(S(T)) = \begin{cases} 0 & \text{if } S(t^*) \ge B \text{ for } t^* \in [0, T], \\ S(T) - K & \text{if } S(T) > K \ \& \ S(t) < B \text{ for all } t \in [0, T]. \end{cases}$$

Reverse Knock-out barrier options have low premium which is restricted by the knock-out feature. Moreover, if the underlying price approaches the barrier, price of the barrier option decreases although an intrinsic value of the option increases. (Figure 5).

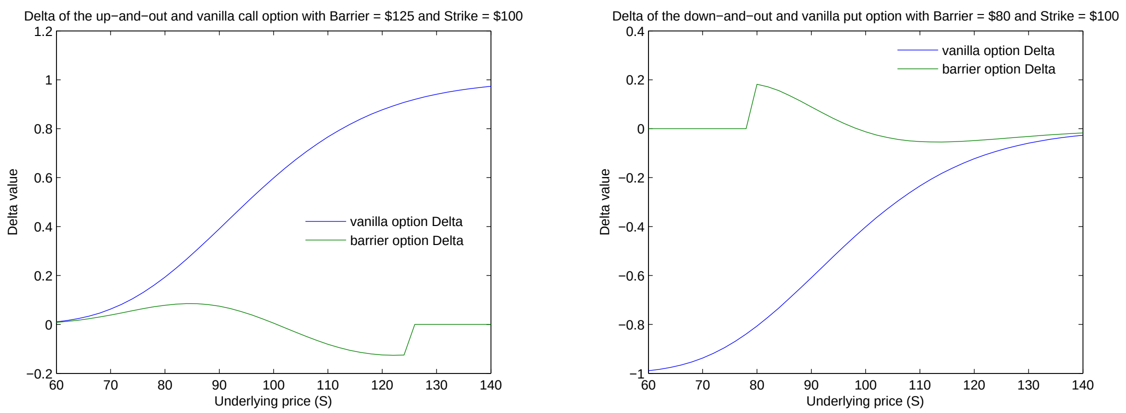
Figure 5: Value of the Knock-out reverse barrier options and the vanilla options.



Panel left: value of the vanilla call option and the Up&Out Call. Panel right: value of the vanilla put option and the Down&Out Put

The Delta moves from a positive to negative value for a call barrier option, see Figure 6. If the underlying price approaches a barrier, the value of the barrier option decreases since the probability is large that the option becomes worthless.

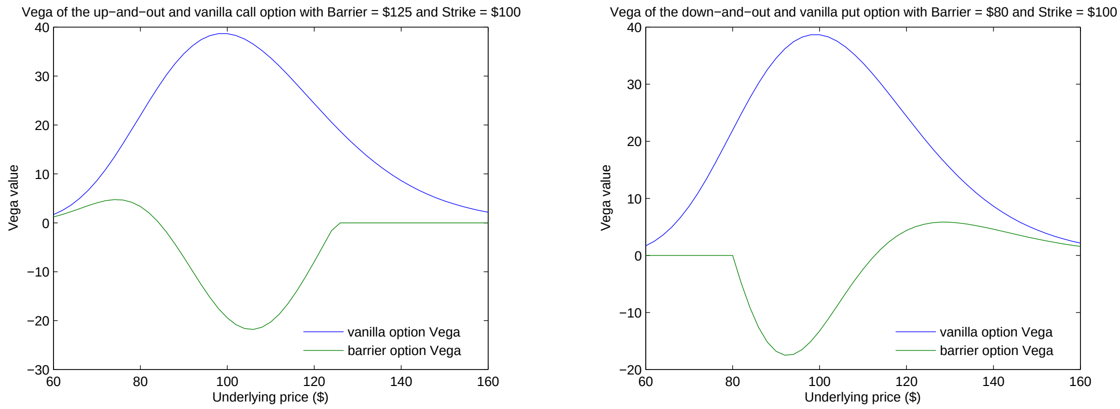
Figure 6: Deltas of the reverse Knock-out barrier options and the vanilla options.



Panel left: Delta of the vanilla call option and the Up&Out call. Panel right: Delta of the vanilla put option and the Down&Out Put

The Vega of the reverse Knock-out barriers changes its sign and becomes negative close to the barrier. Far away from the strike, when the barrier option is out-of-the-money, the option price reacts to changes in the volatility in the same way as for a vanilla option. Near the barrier, if the volatility is large, the probability to hit the barrier rises. Therefore, the price decreases, see Figure 7.

Figure 7: Vega of the Knock-out reverse barrier options and the vanilla options.



Panel left: Vega of the vanilla call option and the Up&Out call. Panel right: Vega of the vanilla put option and the Down&Out Put

Knock-in Reverse barrier options

For the **Down&In** put, $B < K$, the payoff function is:

$$f(S(T)) = \begin{cases} 0 & \text{if } S(t) > B \text{ for all } t \in [0,T], \\ K - S(T) & \text{if } S(T) < K \;\&\; S(t^*) \leq B \text{ for } t^* \in [0,T]. \end{cases}$$

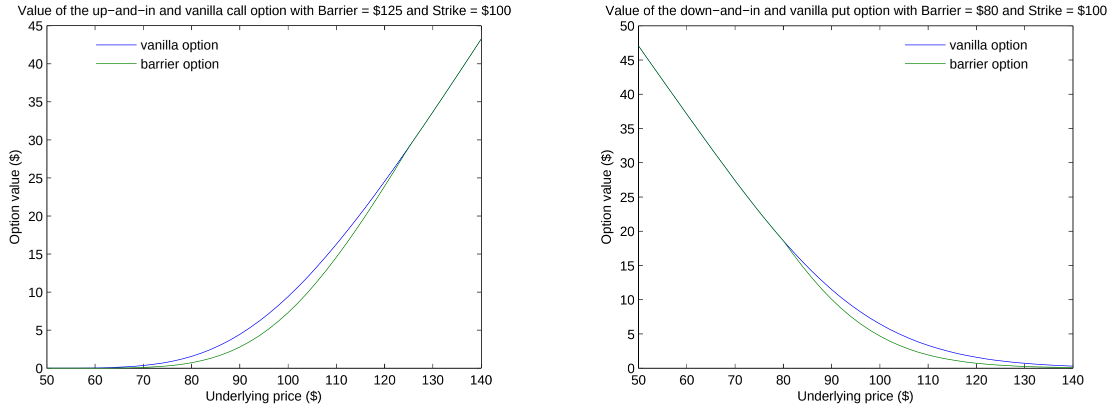
For the **Up&In** call, $B > K$, we have:

$$f(S(T)) = \begin{cases} 0 & \text{if } S(t) < B \text{ for all } t \in [0,T], \\ S(T) - K & \text{if } S(T) > K \;\&\; S(t^*) \geq B \text{ for } t^* \in [0,T]. \end{cases}$$

The closer the underlying price is to the knock-in level, the more expensive is a barrier option due to the higher probability that the barrier is hit. Once the barrier is hit, the value of the option increases linearly similar as for vanilla options, see Figure 8. This kink in the value function is reflected by the discontinuity of the Delta, see Figure 9.

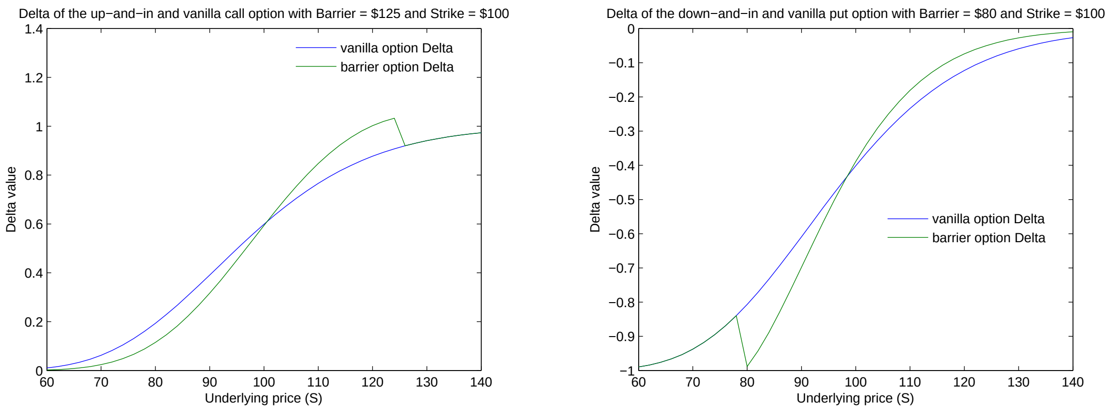
Close to the barrier the Vega is considerably high and distinct from the Vega of a vanilla option due to the high probability that the barrier option becomes worthless. After hitting the barrier, the Vega is the same for a barrier and a vanilla option.

Figure 8: Value of the Knock-in reverse barrier option and the vanilla options.



Panel left: value of the vanilla call option and the Up&In Call. Panel right: value of the vanilla put option and the Down&In Put

Figure 9: Deltas of the reverse Knock-in barrier options and the vanilla options.



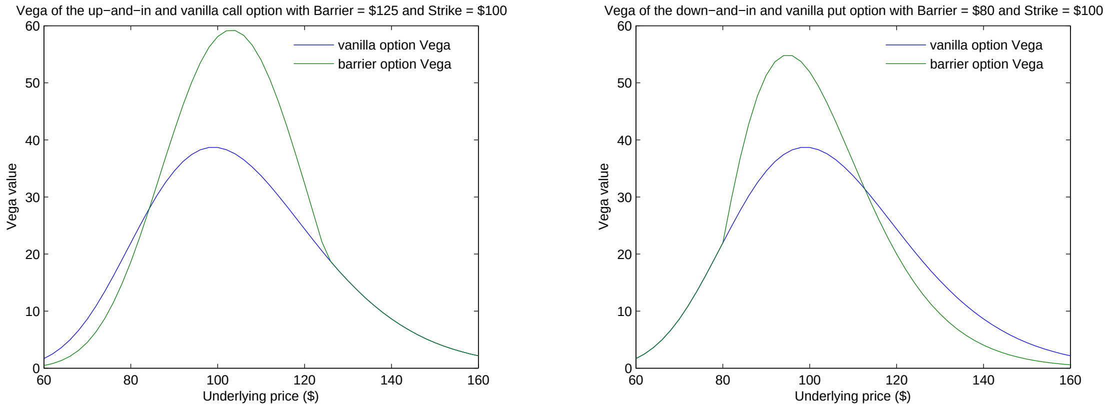
Panel left: Delta of the vanilla call option and the Up&In call. Panel right: Delta of the vanilla put option and the Down&In Put

2.3 Summary

The option price and the Greeks of all discussed barrier options are summarized in Figures 11 and 12. In summary, the shape of various Greeks indicates that hedging barrier options needs a careful analysis. We examine hedging of reverse Knock-in barrier options. We omit to discuss the Knock-out options since:

- the knock-out feature of the barrier option limits the possible payoff and makes the premium of such options insignificant. Hence, these options are more attractive for risk averse

Figure 10: Vega of the Knock-in reverse barrier options and the vanilla options.



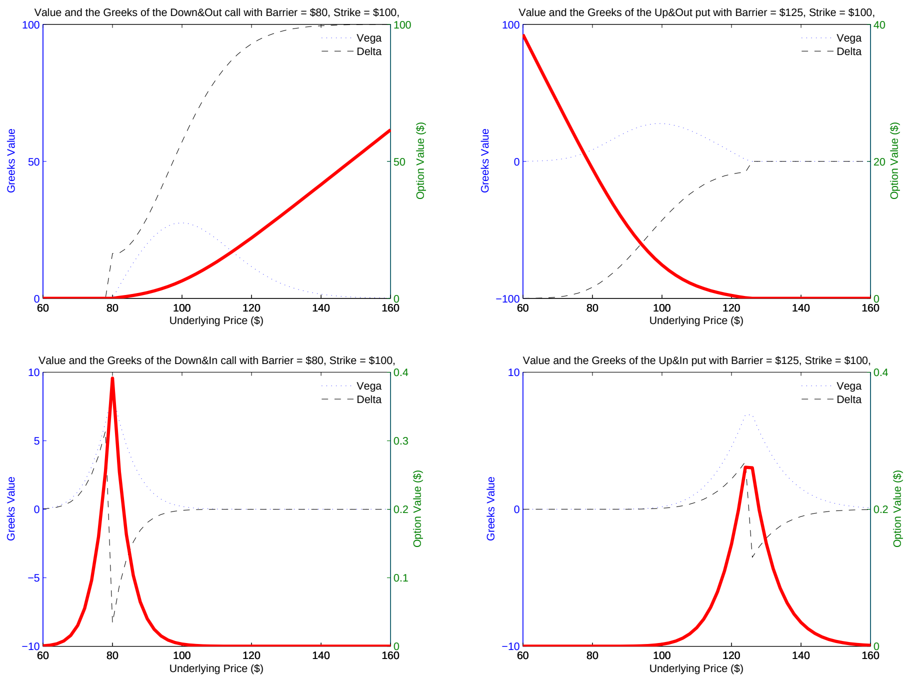
Panel left: Vega of the vanilla call option and the Up&In call. Panel right: Vega of the vanilla put option and the Down&In Put

clients because of much smaller potential losses. Since potential profits are also restricted, Knock-out reverse barrier options are not so popular among the traders.

- the Knock-out barrier options are ideal instruments if expected market movements are small. In this case a knock-in event is less possible and such options give their owners more leverage position. In contrast, when market fluctuations are large, these options are not any longer attractive because a potential payoff is restricted by the knock-out level.

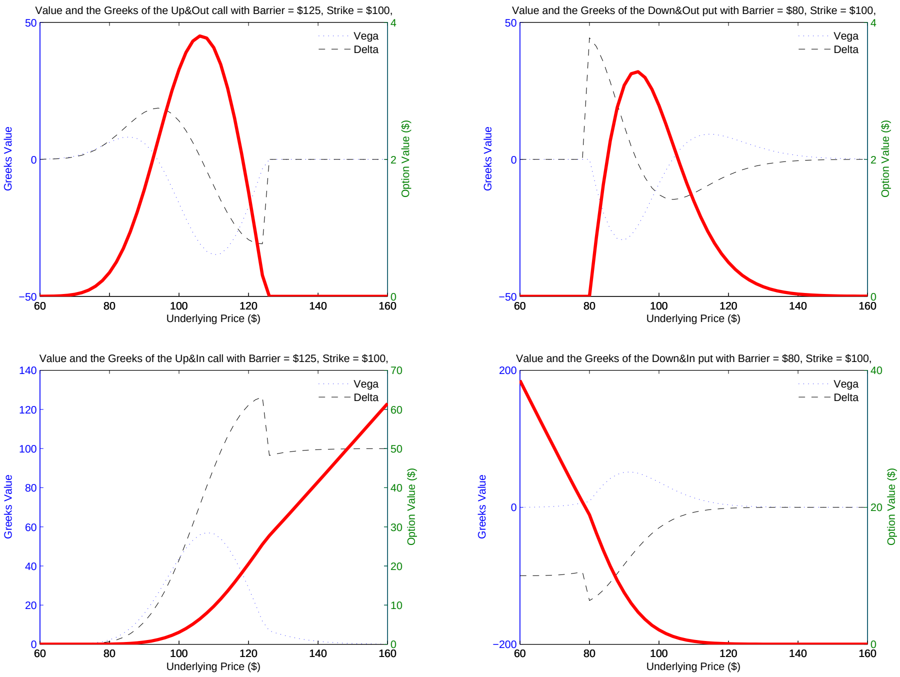
Taking into account the fact that reverse Knock-in barrier options are more popular in the market, the present work focuses on Down&In put option.

Figure 11: Regular barrier options.



To demonstrate the Delta behavior, the Delta here has been multiplied by 100 to shift the decimal.

Figure 12: Reverse barrier options.



To demonstrate the Delta behavior, the Delta here has been multiplied by 100 to shift the decimal.

3 Hedging barrier options

The goal in hedging barrier options is to span a hedge of these products using simpler plain vanilla products. This raises the following questions:

1. Is a perfect hedge possible using simpler products?
2. What kind of simple products are to be used?
3. How many of such products are required? How are they selected?
4. Is such a hedge implementable in the market?

We first formulate requirements which a hedging of barrier options should satisfy. We then present some hedging approaches described in the literature. Finally, we verify whether they meet the hedging requirements. That for, we use historical data.

3.1 Requirements to the hedging of the barrier options

The ideal case of hedging is replication, i.e. the payoffs of the option and the hedge match exactly for all contingencies. Usually replication of the barrier option payoff is not possible due to the market imperfections. Any difference between the payoff of the barrier option and the hedge portfolio is a **hedging error**. Therefore, ideally the hedging error should be zero. There are two hedging methodologies: barrier options are hedged either dynamically by a frequently rebalancing in the underlying or statically, through an initially fixed portfolio of vanilla options.

We define the following requirements which a hedge of a barrier option position should satisfy.

1. **Market liquidity**

Liquidity is by definition the possibility to trade at any time without affecting the asset's price. We require that the instruments of the hedge portfolio are liquid. For a static hedging strategy we require for example that plain vanilla options with all needed strikes and maturities for the hedge exist.

2. **Position size in the hedge portfolio**

Any position in the hedge portfolio should not exceed a predefined size compared to the barrier option size: too large positions are not acceptable to traders. On the other hands, the positions should not be too small in order to be substantial.

3. **Number of hedge positions**

A number of the positions in the hedge portfolio should be minimal given an acceptable hedging error level.

4. **Realization risk**

This requirement holds only for static hedging. Static hedging strategies by definition involve the construction of a hedge portfolio at the beginning, which is not changed over time, together with the liquidation of this portfolio when the barrier is hit. Realization risk means that the hedging error raises over time and in particular at the liquidation time date the characteristics of the underlying such as volatility determine the hedging error value. We require realization risk to be acceptable.

If a constructed hedging strategy does not satisfy these requirements, it either leads to a significant hedging error or the strategy is not implementable in practice.

3.2 Hedging approaches

We always assume a Black-Scholes framework (Black and Scholes (1973)).

3.2.1 Dynamic hedging

We identify dynamic hedging with Delta-hedging (Δ), i.e the first order sensitivity of the option with respect to the underlying price.

The general set-up of the Delta-hedging is as follows: when a hedger sells/buys a barrier option on an underlying stock, he receives/pays the price of the option and sets up a hedge portfolio by buying/selling Δ shares of the stock and putting the rest in the bank account. Over time, the hedger adjusts the hedge portfolio continuously (i.e. infinitely often in any time interval) in order to reinstall Delta-neutrality: the Delta of the joint position barrier option and hedge is zero.

In the Black-Scholes framework the underlying price $S(t)$ satisfies:

$$dS(t) = (r - q)S(t)dt + \sigma S(t)dW(t). \tag{1}$$

q is the dividend yield, σ the volatility of the stock returns per unit of time and W_t a standard Brownian motion. The risk-free asset pays a constant interest rate of $r \geq 0$. The value of the barrier option can be derived in the analytical form (Hull (2003)). The price of a Down&In put $P_{DI}(t)$ with strike K, barrier B and maturity T, if the barrier is not hit, reads:

$$\begin{aligned} P_{DI}(t) &= -S(t)N(-x_1)\exp(-q(T-t)) + K\exp(-r(T-t))N(-x_1 + \sigma\sqrt{T-t}) \\ &\quad + S(t)\exp(-q(T-t))(\frac{B}{S(t)})^{2\lambda}[N(y) - N(y_1)] \\ &\quad - K\exp(-r(T-t))(\frac{B}{S(t)})^{2\lambda-2}[N(y - \sigma\sqrt{T-t}) - N(y_1 - \sigma\sqrt{T-t})], \end{aligned} \tag{2}$$

with $N(\cdot)$ the cumulative normal distribution function and

$$\lambda = \frac{r - q + \sigma^2/2}{\sigma^2}, \tag{3}$$

$$y = \frac{\ln[\frac{B^2}{(S(t)K)}]}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t}, \tag{4}$$

$$x_1 = \frac{\ln(\frac{S(t)}{B})}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t}, \tag{5}$$

$$y_1 = \frac{\ln(\frac{B}{S(t)})}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t}. \tag{6}$$

If the price of the underlying hits the barrier before maturity, the price of the barrier option equals the price of the corresponding vanilla option.

Given the explicit price formula (2) for a Down&In put option, the calculation of the Delta just means to differentiate (2) with respect to $S(t)$, see Appendix D for the result.

The Delta of the barrier option is not a continuous function of the underling S at the barrier. This makes Delta-hedging difficult, see a numerical example in the section 2.4.1.

3.3 Static hedging

Static hedging approaches were put forth by Derman et al. (1994) and Carr and Chou (1997a).

3.3.1 Idea of static hedging

We show that any European security can be statically replicated using a combination of zero coupon bonds, forwards and vanilla European put and call options (Carr and Picron (1999)).

The assumption made in the derivation are:

- The payoff function of an European security $f(S(T))$ is twice differentiable.
- There is no arbitrage and markets are frictionless.

With $I_{(.)}$ an indicator function any payoff can be rewritten as follows:

$$\begin{aligned}
f(S(T)) &= f(S(T))(I_{(S(T)\leq k)} + I_{(S(T)>k)}) + f(k) - f(k)((I_{(S(T)\leq k)} + I_{(S(T)>k)}) \\
&= f(k) - I_{(S(T)\leq k)}[f(k) - f(S(T))] + I_{(S(T)>k)}[f(S(T)) - f(k)] \\
&= f(k) - I_{(S(T)\leq k)}[\int_{S(T)}^{k} f'(u)du] + I_{(S(T)>k)}[\int_{k}^{S(T)} f'(u)du] \\
&= f(k) - I_{(S(T)\leq k)}[\int_{S(T)}^{k} [f'(k) + f'(u) - f'(k)]du] \\
&\quad + I_{(S(T)>k)}[\int_{k}^{S(T)} [f'(k) + f'(u) - f'(k)]du] \\
&= f(k) - I_{(S(T)\leq k)}[\int_{S(T)}^{k} [f'(k) - \int_{u}^{k} f''(\upsilon)d\upsilon]du] \\
&\quad + I_{(S(T)>k)}[\int_{k}^{S(T)} [f'(k) + \int_{k}^{u} f''(\upsilon)d\upsilon]du] \\
&= f(k) - I_{(S(T)\leq k)} \int_{S(T)}^{k} f'(k)du + I_{(S(T)>k)} \int_{k}^{S(T)} f'(k)du \\
&\quad + I_{(S(T)\leq k)} \int_{S(T)}^{k} \int_{u}^{k} f''(\upsilon)d\upsilon du + I_{(S(T)>k)} \int_{k}^{S(T)} \int_{k}^{u} f''(\upsilon)d\upsilon du. \qquad (7)
\end{aligned}$$

Since $f'(k)$ does not depend on u, we have:

$$\begin{aligned}
&-I_{(S(T)\leq k)} \int_{S(T)}^{k} f'(k)du + I_{(S(T)>k)} \int_{k}^{S(T)} f'(k)du \\
&= +I_{(S(T)\leq k)} f'(k)(S(T) - k) + I_{(S(T)>k)} f'(k)(S(T) - k) \\
&= f'(k)(S(T) - k). \qquad (8)
\end{aligned}$$

Changing the order of integration, we get:

$$\begin{aligned}
f(S(T)) &= f(k) + f'(k)(S(T) - k) + I_{(S(T)\leq k)} \int_{S(T)}^{k} \int_{S(T)}^{\upsilon} f''(\upsilon)dud\upsilon \\
&\quad + I_{(S(T)>k)} \int_{k}^{S(T)} \int_{\upsilon}^{S(T)} f''(\upsilon)dud\upsilon. \qquad (9)
\end{aligned}$$

Integrating with respect to u, we have:

$$\begin{aligned}
f(S(T)) &= f(k) + f'(k)(S(T) - k) + I_{(S(T)\leq k)} \int_{S(T)}^{k} f''(\upsilon)(\upsilon - S(T))d\upsilon \\
&\quad + I_{(S(T)>k)} \int_{k}^{S(T)} f''(\upsilon)(S(T) - \upsilon)d\upsilon. \qquad (10)
\end{aligned}$$

Since

$$I_{(S(T)\leq k)}\int_{S(T)}^{k} f''(\upsilon)(\upsilon - S(T))d\upsilon = \int_{0}^{k} f''(\upsilon)(\upsilon - S(T))^{+}d\upsilon \tag{11}$$

and

$$I_{(S(T)\geq k)}\int_{k}^{S(T)} f''(\upsilon)(S(T) - \upsilon)d\upsilon = \int_{k}^{\infty} f''(\upsilon)(S(T) - \upsilon)^{+}d\upsilon. \tag{12}$$

$f(S(T))$ is simplified to:

$$f(S(T)) = f(k) + f'(k)(S(T) - k) + \int_{0}^{k} f''(\upsilon)(\upsilon - S(T))^{+}d\upsilon + \int_{k}^{\infty} f''(\upsilon)(S(T) - \upsilon)^{+}d\upsilon. \tag{13}$$

Thus a European security's payoff can be viewed as the payoff arising from a static position in $f(k)$ zero coupon bond, $f'(k)$ long forwards and an infinite continuum of put and call options. The value of this replicating portfolio $V(t)$ at time t, assuming no arbitrage, is therefore:

$$V(t) = f(k)B(t,T) + f'(k)(S(T) - kB(t,T)) + \int_{0}^{k} f''(\upsilon)P(t,T,\upsilon)d\upsilon + \int_{k}^{\infty} f''(\upsilon)C(t,T,\upsilon)d\upsilon, \tag{14}$$

where

$B(t,T)$ is the price of the zero-coupon bond at time t with maturity T,

$P(t,T,\upsilon)$ is the price of a put at time t, with maturity T and strike υ and

$C(t,T,\upsilon)$ is the price of a call at time t, with maturity T and strike υ.

This idea of replicating of the European security payoff underlies the static hedging approach of Carr and Chou (1997a). The similar idea was developed by Derman et al. (1994). They showed that the payoff of a barrier option can be statically replicated along the barrier and at maturity by portfolio of infinite number of vanilla calls and puts with the same strikes and different maturities.

Since replication requires an infinite number of positions in vanilla instruments, in practice one can only approximately replicate an European security with a non-linear payoff, and hence a hedging error always follows. This approximation is achieved by matching the replicating portfolio's payoff and the payoff of the security, at a finite number of points.

3.3.2 Calendar spread/DEK

The Calender-spread method of Derman et al. (1994) (DEK) hedges the payoff of the barrier options along the barrier and at maturity using a portfolio of vanilla options.

The hedge portfolio for a Down&In put option contains finite number of vanilla puts with strikes which are all equal to the barrier level but with different expiration times. Intuitively, the more options there are in the portfolio, the better is the hedge. DEK-hedge portfolios are constructed as follows. Suppose, a Down&In put is sold. The weights of the options in the hedge portfolio are defined such that the value of this portfolio and the barrier option coincide at the barrier. The hedging option weights are calculated recursively, starting from the option with the longest time to maturity: one assumes that there exists a put vanilla option with time to maturity equal to the barrier option time to maturity. Next one considers a second vanilla option with a shorter time to maturity and calculates its weight in the hedge portfolio given the weight of the first option. The procedure is iterated.

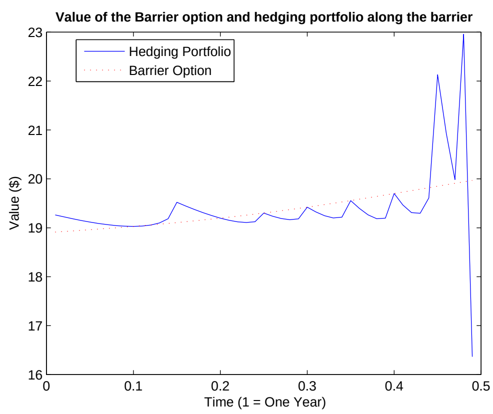
Formally, the value of the barrier option with strike K at the barrier B is matched by a hedge portfolio at N^M matching points $t_j^M \leq T$, $j = 1, ..., N^M$. At time t_j^M we take a position in a put expiring at t_j^H no later than at t_{j+1}^M, such that the value of the hedge portfolio is equal to the value of the barrier option at time t_j^M, if the underlying price equals the barrier. The weights ω_j of the vanilla puts in the hedge portfolio can be found by solving a linear systems of equation recursively. For $j = N^M - 1, ...1$, solve:

$$\omega_j P(B, t_j^M, \sigma^2, B, t_j^H) = -P(B, t_j^M, \sigma^2, K, T) - \sum_{k=j+1}^{N^M-1} \omega_k P(B, t_j^M, \sigma^2, B, t_k^H), \tag{15}$$

where $P(B, t_j^M, \sigma^2, B, t_j^H)$ is the price of a vanilla put with maturity t_j^H, strike B and volatility σ^2 at time t_j^M and $S(t_j^M) = B$.

The Calendar-spread method performs well at the points of time where the hedge portfolio was chosen to match the value of the barrier option. But close to expiry, there is a region of a large absolute mismatch near the barrier (Figure 13). As explained in Derman et al. (1994) the reason for this value gap has to be attributed to the fact that no option with expiry equal to the barrier option and with strike above the barrier is able to hedge the value of the barrier $K - B$. This mismatch can be mitigated by replicating the barrier option during, say, the last month or weeks using daily intervals for the hedge portfolio.

Figure 13: Value of the barrier option and the DEK hedge portfolio.



3.3.3 Strike spread/CarrChou Method

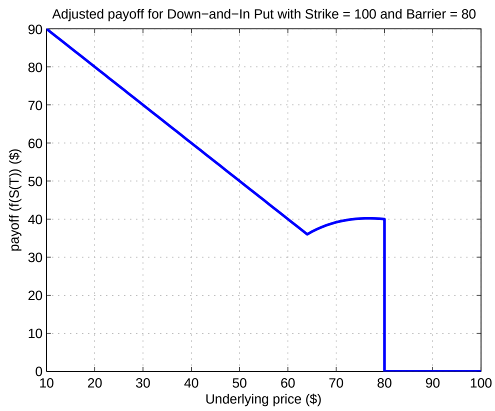
The Strike-spread hedging method of Carr and Chou (1997a) (CarrChou) converts the problem of hedging a barrier option to a problem of hedging a European security with a non-linear payoff function. The payoff function of this European security used to replicate a barrier option is called the *adjusted payoff function.* A static strike-spread hedge portfolio is obtained by hedging the adjusted payoff using a portfolio of finite number of vanilla puts or calls with different strikes.

The adjusted payoff for a Down&In put with a payoff function $g(S(T)) = (K - S(T))^+$ and a barrier B reads:

$$f(S(T)) = \begin{cases} 0, & S(T) \in (B, \infty) \\ g(S(T)) + (\frac{S(T)}{B})^p g(\frac{B^2}{S(T)}), & S(T) \in (0, B), \end{cases} \quad (16)$$

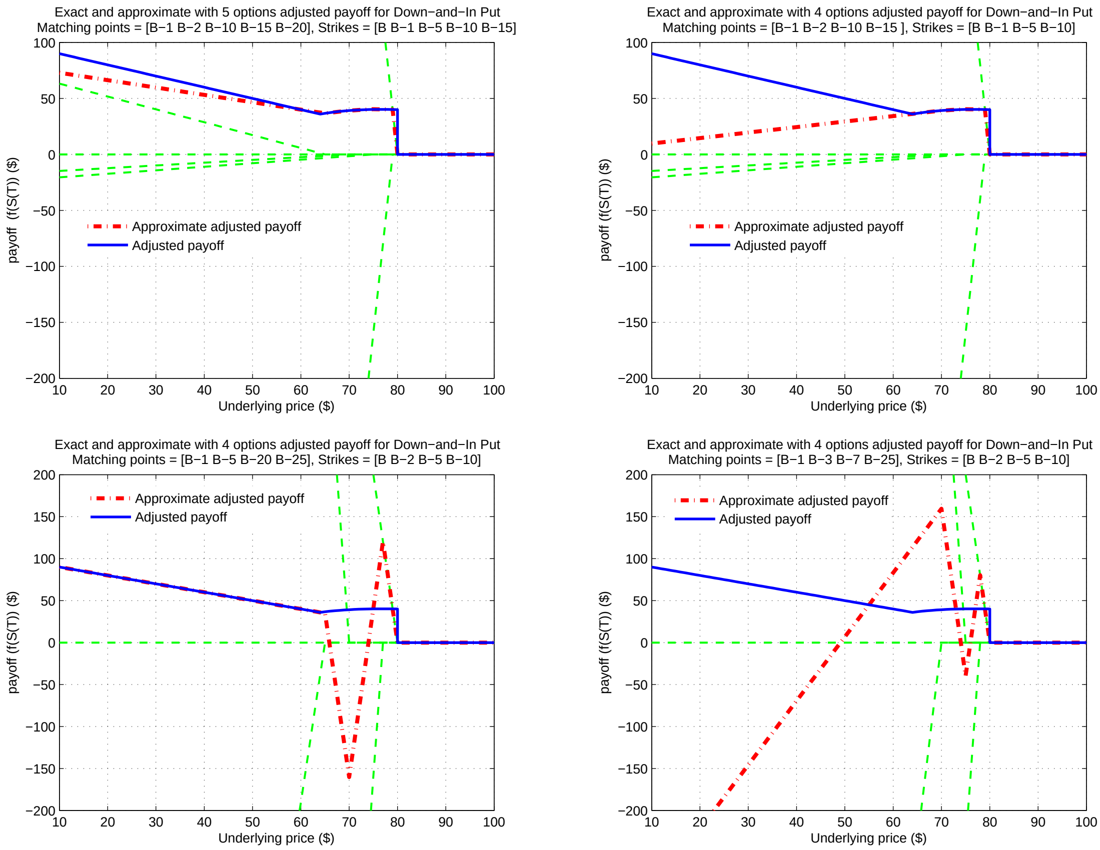
where $p = 1 - \frac{2(r-q)}{\sigma^2}$. We prove in Appendix E the replication (16) for the down-and-in barrier options, see Figure 14 for the adjusted payoff.

Figure 14: The adjusted payoff for the Down&In put.



Contrary to the DEK approach, in the CarrChou approach the approximation to a perfect hedge is due to the discretization of the integrals in (13) over the strikes, i.e hedging the adjusted payoff (16), we use instead of continuum only a finite number of the vanilla options. Since it is important to find a portfolio which matches the adjusted payoff in the non-linear portion of the payoff, as the linear portion is exactly matched by a single position in the non-barrier version of the barrier option, "matching" takes place only in the non-linear region of the adjusted payoff. Figure 15 shows attempts to match adjusted payoffs using different hedge portfolios. Consider, for example the top left panel. The hedge portfolio consists of 5 vanilla put options. Combining these options with some fixed weights in the hedge portfolio, the approximate adjusted payoff follows. The calculation of the weights in the hedge portfolio is discussed below. The upper panel of Figure 15 shows a very good approximation of the non-linear part of the adjusted payoff while in the lower panels strong mismatches arise.

Figure 15: Approximate adjusted payoff.



Green lines reflect the payoff of each single vanilla option in the hedge portfolio.

To create a hedge portfolio of vanilla put options to match the adjusted payoff below the barrier, one needs to define the number (N^M) of the matching points $x_j, j = 1, ..., N^M$. The hedge portfolio by definition takes positions in vanilla puts with strikes $K_j > x_j, j = 1, ...N^M$, and the maturity T equals the barrier option maturity. To determine the positions ω_j in the vanilla options, one solves the following system of equations:

$$\sum_{i=1}^{j} \omega_i (K_i - x_i)^+ = f(x_i), j = 1, ...N^M. \quad (17)$$

One obtains the positions required in the put options such that the adjusted payoff is matched if the final price of the underlying is equal to x_j. This approximation can be made more accurate by increasing the number of points, x_j, at which the hedge portfolio's payoff is matched to the adjusted payoff in the whole non-linear region.

3.4 Hedging performance

We use the hedging error to compare different hedging strategies. The hedging strategies described above are applied to hedge a Down&In Put which is sold at initial time $t = 0$ for a price $P_{DI}(0)$. The price is calculated using the Black-Scholes model. We consider either a dynamic, DEK or CarrChou strategy. If any static hedging is chosen, then we buy at time $t = 0$ a static hedge portfolio constructed either by the DEK or CarrChou method with value $P_H(0)$. In case of the dynamic hedging strategy we calculate the Delta of the barrier option and buy Δ_B amount of the underlying asset.

If the asset price crosses the barrier B, we record a hitting time τ and calculate the liquidation value of the static hedge portfolio $P_H(\tau)$. The price of the barrier option is $P_{DI}(\tau)$ and the hedging performance for the static approaches reads:

$$H(\tau) = \frac{P_{DI}(0) - P_H(0) + exp(-r(0)\tau)(P_H(\tau) - P_{DI}(\tau))}{P_{DI}(0)}. \quad (18)$$

If the barrier was not hit until maturity, the hedge portfolio and the barrier option have zero-value at maturity.

For the dynamic strategy, we adjust the hedge at the next rebalancing time. This accumulates costs C until expiry of the barrier option T or the hitting time τ, where we liquidate the portfolio of the underlying and record a terminal result of the hedging strategy. The hedging error H reads:

$$H(\tau) = \frac{P_{DI}(0) + exp(-r(0)\tau)(P_H(\tau) - C(\tau) - P_{DI}(\tau))}{P_{DI}(0)}. \quad (19)$$

Using this setup we evaluate and compare the three strategies.

3.4.1 Dynamic hedging

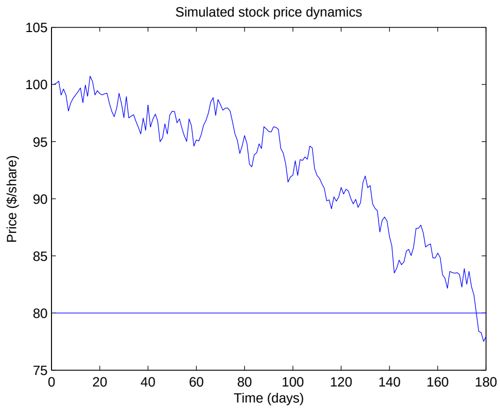
The underlying price dynamics starts at a price of 100. The expected return of the underlying is 3% and its volatility is 20%. The volatility used to simulate the paths is the same as the one used to determine the option value and the Delta. For simplicity, we assume that dividends are zero. We execute the Delta-hedging with daily rebalancing according to the Black-Scholes hedging method for a Down&In put option. All results and data during the hedging exercise are given in the Table 2. The stock price dynamics used for this example is shown in the Figure 16.

At the first day ($t = 0$) we sell the Down&In put option with the Strike 100 and the Barrier 80 for the theoretical price 2.05, see Table 2, Column 9. To hedge this option we sell Δ_B shares, because the $\Delta_B(0) = -0.2980$ is negative. The received amount of money $29.8 = |-0.2980| \times 100$ (Column 3 in the Table) is invested in the bank account until the

Table 2: Delta hedging example.

Day number	Price	Delta of the barrier option	Costs of shares to be short sold	Number of shares to be purchased (+) /short sold (−) additionally	Cumulative costs	Interest rate costs	Cumulative costs with interests	Barrier option price
-1-	**-2-**	**-3-**	**-4-**	**-5-**	**-6-**	**-7-**	**-8-**	**-9-**
#	S	=dV/dS	=1*2	=1(t)*(3(t)-3(t-1))	=5(t-1)+4(t)	=5 * (exp(r(t) * 1/365) - 1)	=5+6	V
0	100.00	-0.2980	-29.80	0.00	-29.80	0.00	-29.80	2.05
1	100.10	-0.2929	-29.32	0.51	-29.28	0.00	-29.29	2.00
2	100.28	-0.2851	-28.59	0.77	-28.51	-0.01	-28.52	1.93
3	99.08	-0.3302	-32.71	-4.46	-32.97	-0.01	-32.98	2.28
	...							
135	87.09	-1.2266	-106.83	-36.66	-109.40	-0.46	-109.86	4.39
136	88.11	-0.9669	-85.19	22.88	-86.52	-0.46	-86.98	3.18
137	88.40	-0.8895	-78.63	6.85	-79.67	-0.47	-80.14	2.82
138	88.04	-0.9622	-84.71	-6.40	-86.07	-0.48	-86.55	3.06
139	86.69	-1.3119	-113.73	-30.32	-116.39	-0.49	-116.88	4.49
140	85.88	-1.5524	-133.32	-20.66	-137.05	-0.50	-137.55	5.54
141	83.50	-2.3283	-194.41	-64.79	-201.83	-0.51	-202.35	10.03
142	83.93	-2.2057	-185.12	10.29	-191.54	-0.53	-192.07	8.96
143	84.62	-1.9757	-167.19	19.46	-172.08	-0.54	-172.63	7.38
144	84.23	-2.1232	-178.84	-12.42	-184.50	-0.56	-185.06	8.05
145	84.46	-2.0475	-172.94	6.40	-178.11	-0.57	-178.68	7.45
146	85.42	-1.6921	-144.53	30.36	-147.75	-0.59	-148.34	5.53
	...							
159	85.24	-1.6019	-136.55	22.11	-138.46	-0.72	-139.18	3.64
160	84.85	-1.7964	-152.43	-16.51	-154.97	-0.73	-155.70	4.09
161	83.33	-2.7902	-232.51	-82.81	-237.78	-0.75	-238.53	7.32
162	83.05	-3.0115	-250.10	-18.38	-256.16	-0.77	-256.93	7.88
163	82.17	-3.6921	-303.39	-55.92	-312.08	-0.79	-312.88	10.57
164	83.64	-2.5700	-214.95	93.85	-218.24	-0.81	-219.05	5.68
165	83.54	-2.6359	-220.22	-5.51	-223.74	-0.83	-224.58	5.62
166	83.48	-2.6761	-223.41	-3.35	-227.10	-0.85	-227.95	5.45
167	83.53	-2.6103	-218.03	5.50	-221.60	-0.87	-222.47	4.99
168	83.37	-2.7432	-228.70	-11.09	-232.69	-0.89	-233.57	5.05
169	82.28	-4.0061	-329.62	-103.91	-336.59	-0.91	-337.51	8.28
170	83.88	-2.0427	-171.35	164.69	-171.90	-0.93	-172.83	3.00
171	82.52	-3.7600	-310.28	-141.72	-313.62	-0.95	-314.57	6.40
172	83.63	-2.0724	-173.32	141.15	-172.47	-0.97	-173.44	2.64
173	82.30	-4.1333	-340.16	-169.61	-342.08	-1.00	-343.08	6.08
174	81.58	-5.6952	-464.60	-127.41	-469.49	-1.04	-470.53	8.87
175	79.95	-1.0000	-79.95	375.38	-94.12	-1.04	-95.16	20.01
	...							
179	77.95	0.0000	0.00	0.00	0.00	0.00	0.00	0.00

Figure 16: Simulated stock price process.



next rebalancing day, i.e. for one day. The cumulative costs are negative, since the Delta of the put option is negative, see Column 8 in Table 2. Hence, the negative cumulative costs are a profit. At time step $t = 1$, the stock price is 100.1 and the Delta is -0.2929. At this point we sell a smaller amount of the underlying stock, $\Delta_B(1) \leq \Delta_B(0)$: we buy back the difference $-0.2929 - (-0.2980) = 0.0051$ at the new price 100.1, see column 5. The cumulative profit decreases (see column 8), because the amount of underlying $\Delta_B(0)$ which was sold at day 0 at a smaller price has to be bought at day 1 at a higher price. This daily procedure is repeated until the barrier is hit or until maturity. In our example, at day 175 the underlying price is $S(175) = 79.95$, i.e. for the first time the barrier of 80 is hit. The Delta is $\Delta(174) = -5.7$. Hence to liquidate the portfolio one has to buy shares for a price of 79.95. This leads to costs of 455.32. The bank account value is 470.53 and the price of the barrier option, which is a vanilla option after hitting the barrier, is 20.01. Discounting all cash-flows at time $t = 175$ back to $t = 0$, we have:

$$H(175) = \frac{2.05 + e^{(-0.03/365*175)}(470.52 - 20.01 - 455)}{2.05} = -131.06\%, \tag{20}$$

Although the method works perfectly in principle and it is easy implementable in practice, the example shows some pitfalls.

First, continuous weight adjustment is impossible because continuous trading is impossible. Hence, in practice weights are adjusted at discrete time intervals. This leads to a hedging error.

Second, the weights adjustment (trade) takes place in the regular intervals, therefore transaction costs are always involved in practice. Since barrier options have a high sensitivity to the underlying price changes (Delta), traders need to trade in the underlying more often. Transaction costs grow proportional to the trade frequency.

3.4.2 Static Calendar-Spread (DEK) hedging example

We sell a Down&In put for the price of 2.05 and construct a DEK-hedge portfolio. The hedge portfolio consists of 8 vanilla put options with strikes at the barrier and maturities equal $0.1479, 0.2466, 0.2959, 0.3452, 0.3945, 0.4438, 0.4784, 0.4932$, see Table 3, where maturities are fractions of one year. We choose the maturities of the hedging options such that close to expiry of the barrier option matching points are more dense. The value of this portfolio is 2.04. Thus, at the beginning of the hedging strategy the cash-flow is equal $2.05 - 2.04 = 0.01$, i.e the hedge has almost the same value as the barrier option. At the hitting time $t = 175$ the barrier option becomes a vanilla put with price 20.01 and the hedge portfolio is liquidated at the price of 17.88. To find the value of the hedge portfolio we check which options in the portfolio are not worthless, i.e. we search those vanilla options with maturities longer then the hitting time. At the day 175, which corresponds to 0.4888 in terms of maturities, only one put vanilla option with maturity 0.4932 is still alive. The price of this option is 0.68. Multiplied with the position in this option in the hedge portfolio, the value of the portfolio is 17.88. Thus, the hedging error is:

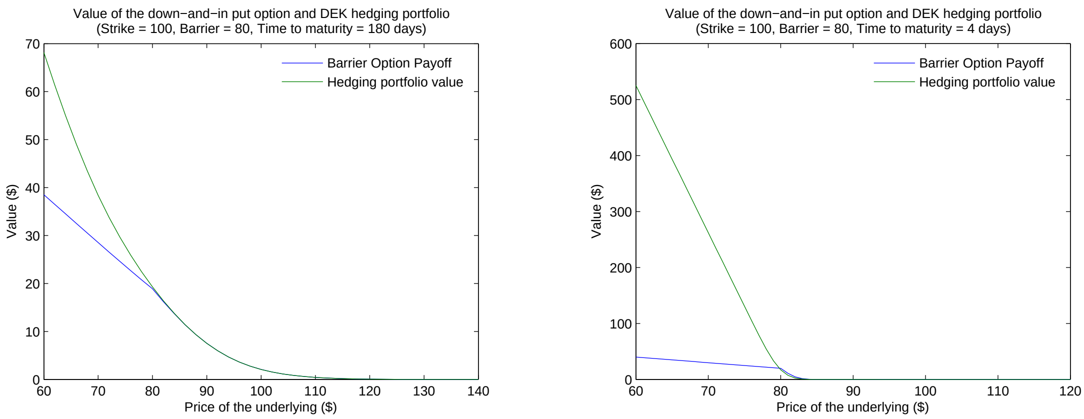
$$H(175) = \frac{2.05 - 2.04 + e^{(-0.03/365*175)}(17.88 - 20.01)}{2.05} = -101.86\%. \tag{21}$$

Table 3: Static Calendar-spread (DEK) hedging strategy example.

position	**strike**	**maturity**	**price**
0	100	0.4932	4.85
-0.6449	80	0.1479	0.00
-0.3891	80	0.2466	0.03
-0.5295	80	0.2959	0.06
-0.7382	80	0.3452	0.09
-0.8481	80	0.3945	0.13
-10.3087	80	0.4438	0.18
-9.0987	80	0.4784	0.22
26.3066	80	0.4932	0.23
value of the portfolio			**2.038597**

Although the hedge portfolio at the beginning was constructed in such way that it closely matched the barrier option payoff, quite a high hedging error occurred at the hitting time. This can be explained as follows. The barrier was hit at day 175 but there is no put option with such a time to maturity in the hedge portfolio. If we suppose that barrier is hit exactly at a time point where there exist a put option, i.e at 0.4784, than the hedge portfolio value at the hitting time is equal to 20.1491. Since the barrier option price is 20.00, a small hedging error of 7.52% follows. Again, to perfectly replicate the barrier option payoff an infinite number of the vanilla options is required. Figure 17 shows the value of the barrier option and DEK hedge portfolio consisting of 8 vanilla options. The left panel in Figure 17 shows the value of the barrier option and the hedge portfolio at the beginning of the hedging strategy. Above the barrier there is a perfect match between the values. The right panel reflects the situation at the hitting time. Below the barrier level there is large mismatch between the barrier option and the hedge portfolio value.

Figure 17: Value of the barrier option and the DEK hedge portfolio.



3.4.3 Static Strike-Spread (CarrChou) hedging example

We consider the same example as for DEK. To construct the portfolio, we choose some matching points below the barrier $(B-1, B-2, B-3) = (79, 78, 77)$ to match the adjusted payoff of the barrier option. We define the strikes of these vanilla put options, which are set above the matching points, as $80, 79, 78$. The value of the portfolio is equal 1.89, see Table 4.

Table 4: Static Strike-spread (CarrChou) hedging strategy example.

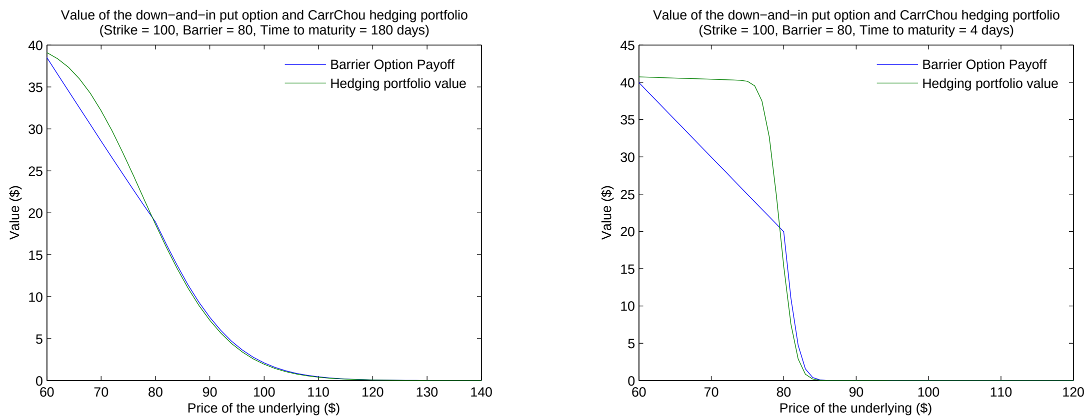
position	**strike**	**maturity**	**price**
0	100	0.4932	4.85388016
-0.03874339	78	0.4932	0.1451598
-40.0369006	79	0.4932	0.18394625
40.1071369	80	0.4932	0.23090185
value of the portfolio			**1.89055**

After the hitting time, the portfolio is reevaluated and sold. At this point of time (day 175) the value of the portfolio is 15.78 and the hedging error is:

$$H(175) = \frac{2.05 - 1.89 + e^{(-0.03/365*175)}(15.78 - 20.01)}{2.05} = -195.7\%. \qquad (22)$$

Figure 18 reflects the most dangerous regions near the barrier level where large losses may occur. If the underlying price is far away from the barrier, almost perfect replication is possible. However, near the barrier there is a mismatch between the value of the barrier option and the hedge portfolio. The closer the hitting time is to maturity of the barrier option, the more pronounced is the mismatch, see Figure 18, right Panel.

Figure 18: Value of the barrier option and the CarrChou hedge portfolio.



Although there is no need to adjust the weights in the hedge portfolio and therefore transaction costs are zero, the static hedging strategies are difficult to implement.

First, the approaches of DEK and CarrChou assume liquidity of all hedging options. Barrier option can only be replicated perfectly using the static replication method with an infinite number of vanilla options with various strike prices and times to maturity. An infinite number of vanilla options with arbitrary strike prices and/or time to maturity do not exist.

Second, the hedge portfolio is not unique and induces large positions in the hedging options. It would require significant costs to construct such a portfolio of the vanilla options in order to hedge a barrier option.

3.5 Testing the different approaches using historical data

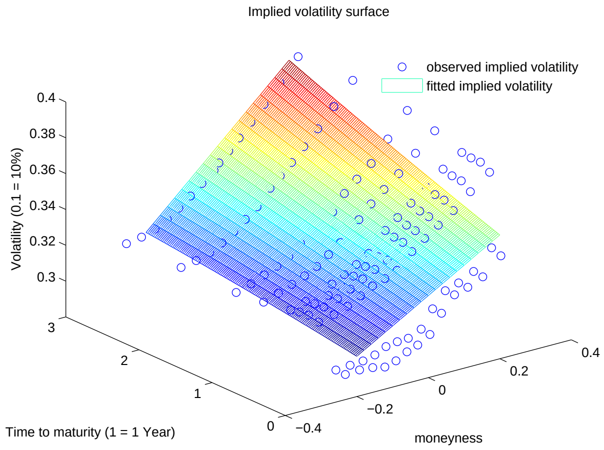
We compare the different hedging strategies for the Down&In put option using market data. We consider share prices of US companies, see Table 5. Data are provided by Wharton Research Data Services, OptionMetrics. We use daily closing prices and zero coupon yield curves. Additionally, since volatility in reality is not constant and differs across strikes and maturity, we use also implied volatility surfaces. To obtain volatilities for needed strikes and maturities, we interpolate at each date an implied volatility surface, see Figure 19. Given the implied volatility surface we try to find a fitting curve function which allows us to get the implied volatility for any maturity and moneyness. For this purpose we chose a non-linear function v with four unknown parameters $x_i, i = 1...4$, i.e.:

$$v = x_1 + x_4 T + x_3 e^{x_2 T} M, \tag{23}$$

where T is time to maturity and M is moneyness.

To find the parameters we solve a non-linear least squares problem, where we minimize the difference between the given implied volatility and the implied volatility derived from equation (23). We use this implied volatility to compute the hedge ratio and option prices.

Figure 19: Implied volatility surface.



$$Moneyness = \frac{\ln(\frac{ForwardPrice}{Strikeprice})}{\sqrt{Timetomaturity}}$$

Table 5: Analyzed companies.

Sector	Name	Ticker	Period
technology	Amazon.com Inc.	AMZN	1996 – 2006
	International Business Machines Corp.	IBM	1996 – 2006
	Microsoft Corp.	MSFT	1996 – 2006
	Marvell Technology Group Ltd.	MRVL	1996 – 2006
	Siemens AG	SI	1996 – 2006
financial	American Express Co.	AXP	1996 – 2006
industrial goods	Boeing Co.	BA	1996 – 2006
basic material	BP plc	BP	1996 – 2006
	Exxon Mobil Corp.	XOM	1996 – 2006
conglomerates	General Electric Co.	GE	1996 – 2006
healthcare	Johnson & Johnson	JNJ	1996 – 2006
	Pfizer Inc.	PFE	1996 – 2006

As in previous examples, the results are based on the following hedging exercise. We hedge the sale of a Down&In Put barrier option maturity $T = 180$ and with a lower barrier B at 80% of the strike. The hedging instruments include the underlying stock and vanilla options on the stocks under the consideration. The hedging errors calculated from historical data are shown in Table 6.

Table 6: Hedging error (%) with historical data.

	Amazon	AXP	BA	BP	GE	IBM	JNJ	MRVL	MSFT	PFE	SI	XOM
Dynamic												
mean	-2.61	-14.05	-32.20	-18.66	-15.40	-17.57	-0.66	-1.52	-2.02	-12.03	38.90	-22.95
std	29.16	59.15	76.31	58.78	51.14	56.54	53.86	22.55	47.95	46.91	20.22	66.19
max	85.27	148.71	146.94	95.69	78.71	238.11	219.92	37.25	218.10	214.20	83.08	336.87
min	-116.81	-832.04	-872.50	-857.39	-401.89	-283.25	-453.86	-77.88	-291.32	-224.69	-20.04	-1080.33
CarrChou												
mean	-84.50	-74.29	-158.99	-30.58	-113.07	-805.68	-107.26	70.70	-69.22	-67.49	568.62	-11.47
std	465.19	546.37	646.21	655.41	538.49	3888.66	1226.03	168.57	351.02	441.91	141.17	544.86
max	1116.72	593.34	740.50	1596.20	936.63	2153.42	1472.41	378.69	365.03	506.78	1647.05	1171.40
min	-1708.79	-2004.28	-4225.84	-4025.61	-2891.82	-38901.97	-11104.22	-792.24	-1384.32	-2498.03	151.72	-3448.73
DEK												
mean	0.27	21.36	22.21	14.43	15.03	32.48	36.87	0.02	8.48	32.51	-14.10	16.99
std	9.12	82.01	56.09	67.12	67.83	146.30	201.70	8.05	40.41	78.58	5.35	93.96
max	67.35	829.21	727.86	646.32	662.66	1443.39	2504.57	77.25	688.59	521.67	0.25	1132.52
min	-38.28	-189.55	-46.27	-179.73	-58.97	-131.06	-49.92	-68.49	-155.13	-402.92	-28.68	-28.58

We conclude:

1. Delta hedging with non-constant volatility causes the Delta-approach to lose its quality as a hedge measure. Changes in implied volatilities impact the option's value such that the Deltas need to be adjusted.

2. The static hedging strategies show a low performance when using implied volatility than for the constant volatility case. Using implied volatility we price the barrier option and the hedging vanilla options in the static portfolio with different volatilities. This leads to a larger mismatch between a hedge portfolio and a barrier option. The Strike-spread approach, in particular, shows the worst results. The weights in the CarrChou hedge portfolio (17) should be calculated under constant volatility, since the adjusted payoff function $f(S(T))$ (16) for the Down&In put option holds only in a Black and Scholes framework.

3. One more feature using real world data is the existence of the jumps in the stock dynamics. The presence of jumps significant influences the Delta- and the Calendar-spread hedging strategies. The Calendar-spread hedging strategy matches a payoff of the barrier option exactly at the barrier level but a price of the underlying at the hitting time can be beyond the barrier. A mismatch between the barrier option value and the value of the hedge portfolio follows. Further more Delta-hedging is inefficient in the presence of jumps. As the Delta of the barrier options can have a very large value, especially near the barrier, a daily rebalancing of the hedge portfolio leads to large costs.

4 Alternative hedging strategy for the barrier option - Vega-matching

This section presents an alternative strategy to statically hedge the random payoff of the barrier option. Contrary to the DEK and CarrChou approaches, this hedging strategy requires a smaller number of hedge instruments.

A major problem which traders face is stochastic volatility. Vega-hedging is applied to protect a position from volatility risk. The strategy described below is constructed in such a way that a static hedge portfolio for a barrier option provides Vega-neutrality. We call this strategy Vega-matching (VM). Residual delta risk can be hedged using additionally Delta-hedging.

We consider a Down&In put with barrier B and strike K. To hedge a short position P of this option we construct a hedge portfolio using 3 vanilla options as follows:

1. We take a long a position $P_1 = -P$ in the vanilla put option with a strike K_1 equal to $K_1 = 0.5 \times (B + K)$;

2. The second long position $P_2 = -2 \times P$ in the hedge portfolio is a position in the vanilla put option with the strike $K_2 = 0.5 \times (B + K_1)$;

3. The last position P_3 in the hedge portfolio is a vanilla put with strike $K_3 = B$ such that the difference between the Vega of the barrier option ν_B and the portfolio Vega equals zero, i.e Vega-neutrality follows. Formally,

$$P_3 = \frac{\nu_B - (\nu_1 + 2 \times \nu_2)}{\nu_3}, \tag{24}$$

where ν_1, ν_2, ν_3 are the Vegas of the vanilla put options in the hedge portfolio. In the Black-Scholes framework the formula for the Vega of the vanilla option with strike K and maturity T is:

$$\nu = SN(-d_1)\sqrt{T}, \tag{25}$$

where

$$d_1 = \frac{\ln(S)/K) + (r - q + \sigma^2/2)(T)}{\sigma\sqrt{T}}, \tag{26}$$

Numerically, the Vega of the Down-and-In put ν_B is approximated by:

$$\nu_B = \frac{P_{DI}(\sigma + \delta) - P_{DI}(\sigma)}{\delta} \tag{27}$$

with δ a small number.

As was shown in the previous section the volatility skew causes a mismatch between a static hedge portfolio (DEK, CarrChou) and a barrier option. To take this drawback into consideration in the VM strategy one uses implied volatilities to calculate option prices.

To show the performance of the VM hedging strategy, we reconsider the example of the previous section. We hedge a Down&In put option with strike price 100 and barrier 80. We sell it at the first day for a price of 2.05. The strikes of the hedging vanilla put options are $90, 85, 80$. To find their positions in the hedge portfolio we first fix positions of the options with strikes $90, 85$. Then we calculate the Vegas of the vanilla options. They are $18.25, 11.53, 5.91$ and ν_B is 36.24, see Table 7. Finally, the position of the vanilla put option with $K_3 = B$ is derived, i.e.:

$$P_3 = \frac{36.24 - (1 \times 18.25 + 2 \times 11.52)}{5.9} = -0.8570. \tag{28}$$

Table 7: Positions in the hedge portfolio.

	Position	**Vega**
Down&InPut	-1	36.24
Vanilla Put with Strike = 90	1	18.25
Vanilla Put with Strike = 85	2	11.53
Vanilla Put with Strike = 80	-0.857	5.91
Total		**0.0**

In summary, the VM hedge portfolio costs 2.50, see Table 8.

Table 8: Hedge portfolio of the VM hedging strategy.

position	**strike**	**maturity**	**price**
1	90	0.4932	1.438
2	85	0.4932	0.6324
-0.8570	80	0.4932	0.2309
value of the portfolio			**2.5049**

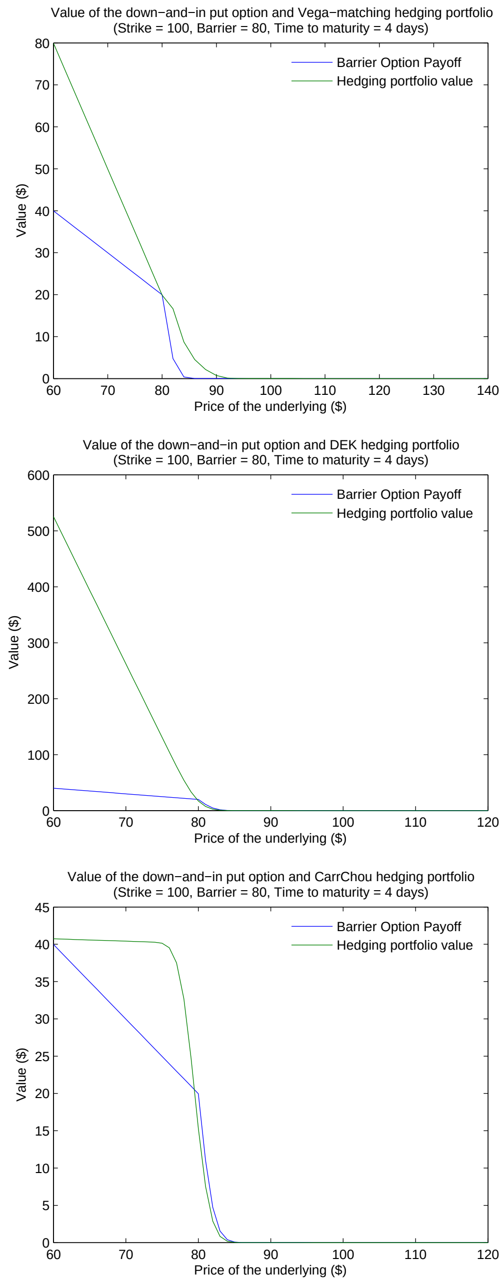
In our example the barrier is hit at the day 175. At this time we liquidate the portfolio receiving 19.48. The barrier option in turn costs 20.01, therefore the hedging error H is:

$$H(175) = \frac{2.05 - 2.50 + e^{(-0.03/365 \times 175)}(19.48 - 20.01)}{2.05} = -47.73\%. \tag{29}$$

Figure 20 shows the value matching between the barrier option and different hedging strategies when the barrier is hit. We note, that the VM hedging strategy represents the smallest mismatch below the barrier level.

The performance results of the VM hedging strategy are given in the Tables 9 and 10. Table 9 shows the hedging error of the VM approach in comparison with the hedging strategies described in Section 2. The results are obtained with simulated data in the Black-Scholes framework. We observe that the VM strategy does not outperform other strategies. The reason is that the VM hedging strategy matches the Vega but not the value of the hedge

Figure 20: Value of the Down&In Put and the hedge portfolios at hitting time.



portfolio and the barrier option. Moreover, when the barrier is not hit and the Down&In put is worthless, the hedge portfolio can have a value, since strikes of some hedging vanilla options are above the barrier level. This induces a hedging error.

We next use the historical data of Section 2.5. The VM hedging strategy performs better than the other strategies because it does not rely on Black and Scholes assumptions, compare

Table 10 and Table 6 in Section 2.5.

Table 9: Hedging error in Black-Scholes world(%).

	Static hedging			**Dynamic hedging**
Hedging Error (%)				
	VM	**DEK**	**CarrChou**	
Mean	5.68	5.00	4.94	3.39
Standard deviation	105.93	44.85	761.25	44.98
Maximum	937.57	1939.63	964.07	594.69
Minimum	-77.90	-918.59	-4630.01	-1069.19

Table 10: Hedging error (%) of the VM strategy with historical data.

	Amazon	AXP	BA	BP	GE	IBM	JNJ	MRVL	MSFT	PFE	SI	XOM
mean	6.94	10.69	9.19	6.04	10.28	19.03	19.07	8.86	8.86	15.42	13.38	11.00
std	13.94	21.18	24.59	24.43	16.18	61.73	30.94	16.55	16.55	37.63	5.10	22.70
max	184.80	383.81	480.90	221.46	252.78	573.91	302.08	146.17	146.17	519.80	34.61	376.90
min	-26.99	-45.20	-47.52	-194.92	-84.65	-103.38	-84.90	-58.08	-58.08	-38.55	-17.05	-145.43

4.1 Summary

There is still no perfect match between the barrier option value and the hedge portfolio using the VM hedging strategy. But the VM strategy has some significant advantages compared to the other strategies:

- the VM hedge portfolio have only 3 vanilla option positions;
- positions in the portfolio are not large with respect to the position of the hedged barrier option;
- the strategy does not assume a Black-Scholes world;
- the strategy is easily implementable.

5 Optimization of static hedging strategies

The performance of the described static hedging strategies strongly depends on the hitting events and namely on the fact, whether the hitting event occurs or not. If a hitting event is realized, the performance depends on the hitting time, the price of the underlying and the volatility, etc. Even if there is an almost perfect match between a barrier option and a hedge portfolio at the beginning, it is possible that a large mismatch occurs over time, in particular, at the hitting time. Therefore, it is important to take into consideration all potential realizations of the hedging strategy. We consider such an approach, see Akgün (2007).

The general idea of the optimization method presented below is the following: we minimize the losses of a particular hedging strategy with respect to certain parameters of this strategy. Such a strategy is called an optimal hedging strategy. In other words, the optimization allows us to find those parameters of the hedging strategy which support the best performance of this strategy over all possible outcomes.

5.1 An optimization approach

We specify the optimization approach. An appropriate objective function is the expected shortfall with confident level α, (ES_α)[1]. All possible outcomes of a hedging strategy can be thought as a set of scenarios Ω. The performance of the hedging strategy H depends on these scenarios, i.e. $H = H(\tau, \omega) := H(S(\tau, \omega))$, where τ is hitting time. H is defined by equation (18). The optimal policy is some parameter X of a chosen hedging strategy h which provides minimal losses of using this strategy. Examples of X are sets of strikes and positions. The set of all possible hedging strategies is DEK, CarrChou, VM, i.e. we choose a parameter set X^* such that for example DEK is optimal.

To find optimal parameters X^* we solve the following problem:

$$\min_X ES_\alpha(H(\tau, \omega)). \tag{30}$$

Subject to:

$$\frac{P_{DI}(0) - P_H(0)}{P_{DI}(0)} \geq C_1 \tag{31}$$

$$E(H^{X^*}(\tau, \omega)) \geq E(H^X(\tau, \omega)) \tag{32}$$

$$X^* \leq C_2 \tag{33}$$

Inequality (31) is the restriction of the initial net cash-flow of the hedging strategy. $P_H(0)$

[1]By definition, ES_α is the expectation (i.e. the mean) of the losses under the condition that a threshold, which is fallen short of in $\alpha \times 100\%$ of all cases, has already been fallen short of. $ES_\alpha(x) := E[x|x \geq VaR_\alpha(x)]$.

is the hedge portfolio value, $P_{DI}(0)$ is value of the barrier option at the beginning of the hedging strategy and C_1 is a constant. The restriction (32) is the performance requirement, i.e. at each hitting time τ the expected performance of the optimized strategy $H^{X^*}(\tau, \omega)$ should not be smaller than that one of the original strategy $H^X(\tau, \omega)$, (33) restricts optimized parameters of the hedging strategy by some constant C_2. The objective of the approach (30) is to minimize the expected shortfall ES_α over all hedging errors.

The selection of the restrictions for optimization problem depends on the hedging strategy, traders preferences, capital requirements etc.

A successful implementation of this approach requires two crucial assumptions:

1. Assumptions of the Black-Scholes model hold.

2. A set of scenarios and probabilities for the stock and the option prices is computed, i.e we consider a finite set $\Omega = \omega_1, ...\omega_W$ of scenarios.

To provide a finite scenarios set we work in a discrete time setup and we assume that there are a limited set N of the hitting times $\tau = \tau_1, \tau_2, ...\tau_N$. To create scenarios we construct a fixed grid of stock prices $S(\omega)$ and hitting times. Suppose that the grid consists of the points $(\tau^i, S(\omega_j))$, for $i = 1, ...N$ and $j = 1, 2, ..., M$. S follows a geometric Brownian motion with constant drift and volatility. Let p^{ij} denote the probability of reaching a point $(\tau^i, S(\omega_j))$. These probabilities are calculated under the Black-Scholes assumptions, see Appendix F. We then calculate the corresponding hedging errors and construct its probability distribution on the grid. We call this distribution a P&L-profile.

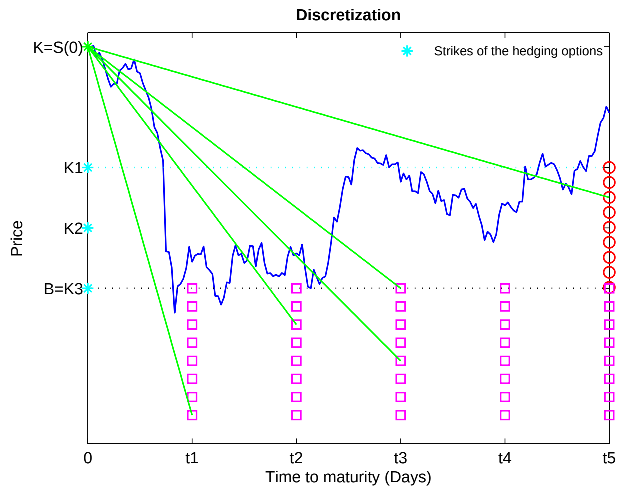
5.2 Optimizing the Vega-matching strategy

We optimize the VM hedging strategy for a Down&In put option. We consider the hedging of a short position in a Down&In put on IBM stock. At the first day we sell the Down&In put with strike 81.85, maturity $T = 180$ days and 80%-barrier $B = 65.48$ for price $P_{DI}(0) = 5.12$. To price options at any time during the hedging strategy we use the volatility smile of the first day. To hedge the position of the barrier option we choose the VM hedging strategy and construct a hedge portfolio of 3 vanilla options with positions P_1, P_2, P_3. The value of this portfolio is $P_{VM} = 4.57$.

Step one. Construction of the P&L profile.

To create a P&L profile for this strategy we construct first the finite state space Ω, see Figure 21. To achieve this goal, we have to consider two cases: the barrier is hit and the barrier is not hit. We first consider the case when the barrier is hit. With $N = 5$, the hitting times τ are $36, 72, 108, 144, 180$. For each τ_i we split the interval $[\lambda B, B]$, where $\lambda \in [0, 1)$, into $M = 8$ subintervals. We assume that every point in these interval is a possible price level after the barrier was hit, i.e. for $\lambda = 0.85$ we have $\{(S(1), \tau_i) = 0.85B, (S(2), \tau_i), ..., (S(M), \tau_i) = B\}$ for each τ_i, $i = 1, ...5$. Thus we get a grid of size 5×8 and calculate the state probabilities p^{ij}, see Figure 22, left panel.

Figure 21: Discretization.



Squares demonstrate discretization points along the barrier and circles reflect discretization at maturity above the barrier level. Hitting times τ on the graphic are denoted by $t1, t2, t3, t4, t5$.

We consider a case where the barrier is not hit. The value of the barrier option is always zero. And we are interested only in such cases where the value of the hedge portfolio differs from zero, i.e. when $S(T) < max(K_H)$. We split the region $[B, max(K_H)]$ into M smaller intervals and calculate p^{Tj}, see Figure 22, left panel. Given these probabilities we calculate the hedging error for each point according to the equations (18). The P&L-profile consists of all these hedging errors.

Step two. Optimization

Given the probability distribution of the hedging error for the VM strategy we choose confidence level $\alpha = 95\%$ and minimize $ES_{95\%}(H(\tau, \omega))$ with respect to the positions in the hedge portfolio, $X = [P_1, P_2, P_3]$, i.e we solve (34-38):

$$\min_X ES_{95\%}(H(\tau, \omega)). \tag{34}$$

s.t.

$$E(H_{X^*}(T, \omega)) \geq E(H_X(T, \omega)) \tag{35}$$

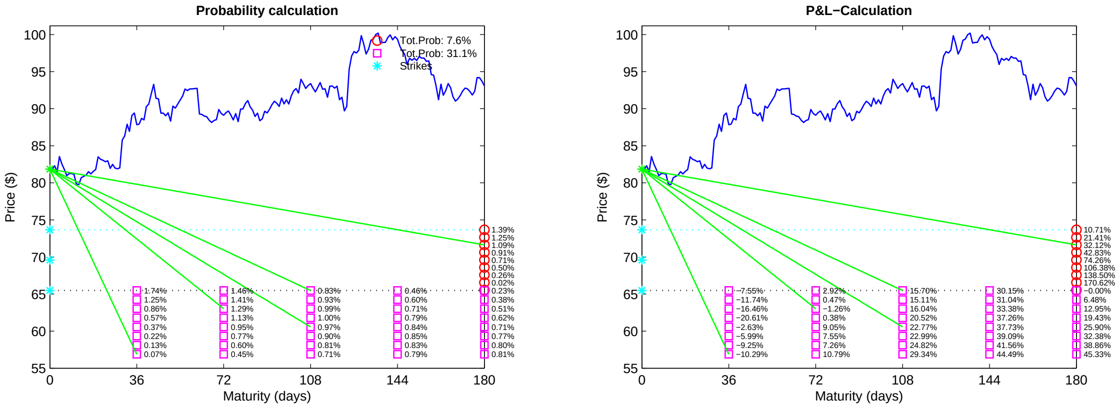
$$E(H_{X^*}(\tau, \omega)) \geq E(H_X(\tau, \omega)) \tag{36}$$

$$\frac{P_{DI}(0) - P_H(0)}{P_{DI}(0)} \geq -10\% \tag{37}$$

$$X^* \leq 4 \tag{38}$$

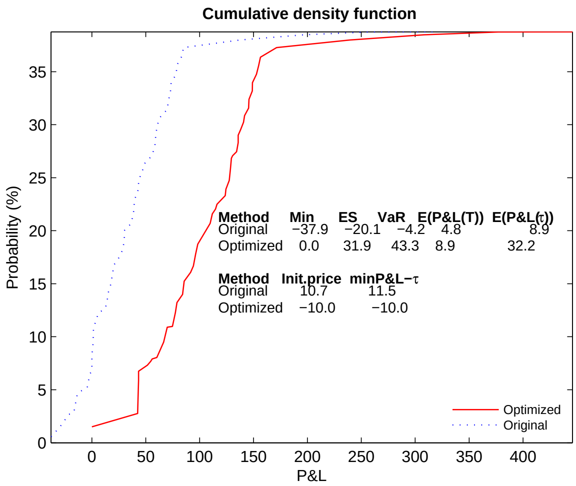
The optimal positions in the hedge portfolio are $[2.2, 1.3, -2.3]$ and the corresponding

Figure 22: Discretization of the state space and probability distribution of the P&L.



expected shortfall is 31.9. This value is much lower than the expected shortfall of the original VM strategy see Figure 23.

Figure 23: Cumulative density function. Case 1.



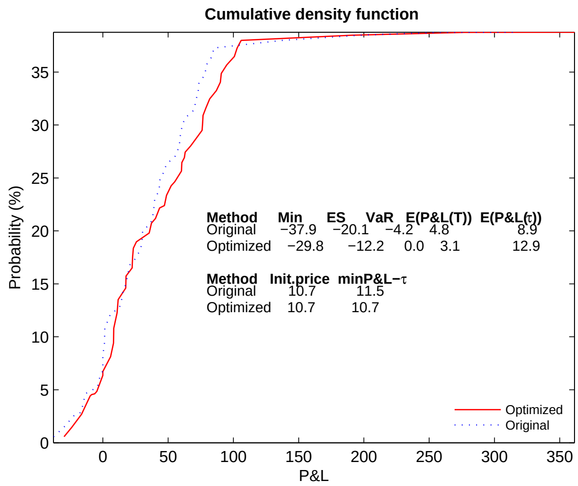
We consider how the optimal hedging strategy depends on conditions of the optimization problem. We do not set here any requirement regarding the hedging error of the strategy but strengthen the initial constraint (37): the price of the hedge portfolio can not be larger than the barrier option price (39).

$$\frac{P_{DI}(0) - P_H(0)}{P_{DI}(0)} \geq 0 \tag{39}$$

$$X^* \leq 4 \tag{40}$$

The optimal positions are now $[0.3, 0.4, -3.0]$. The expected shortfall equals -12.2, see Figure 24. This example shows that manipulating the restrictions significantly changes the performance of the hedging strategy, i.e. the shadow price of the constraints are not small.

Figure 24: Cumulative density function. Case 2.



5.3 Summary

We summarize the findings of the optimization approach.

Pro's

1. Optimized hedging strategies dominate pure hedging strategies.
2. Positions in the hedge portfolio can be restricted in size.
3. Availability of the strikes in the market is taken into account.
4. The discretization of the state space makes the approach easily implementable.

Con's

1. The approach is model-dependent.

6 Conclusion

We analyze the static methods of Derman et al. (1994), Carr and Chou (1997a) and the dynamic hedging for barrier options. We investigate performance of these strategies using Black-Scholes generated data and illustrate the strengths and weaknesses of these approaches. In summary, these strategies fail to satisfy hedging requirements which make them implementable in practice. Tests with historical data reveal that the performance of all three strategies is even worse since volatility is not constant. To improve the hedging effectiveness we propose Vega-matching strategy which does not require unrealistic Black-Scholes assumptions and can be easily constructed. We show that this strategy outperforms all three above mentioned strategies under real market conditions.

Analyzing hedging error as a function of a future hitting event we find that all described static hedging approaches, including the Vega-matching strategy, have one common shortcoming: the hedging error is increasing over time. To cope with this problem we propose an optimization approach. Based on the finite set of scenarios we control the performance of the hedging strategy over all possible outcomes. We test the optimization on Vega-matching hedging approach. The results reveal that the optimization significantly improves the performance of the original hedging strategy. We conclude that the hedging of the barrier option using optimized VM strategy can be easily applied in practice and results in an acceptable hedging error.

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A Exotic barrier option

Table 11: Exotic barrier options

Criteria to classification	Exotic barrier option	Definition
Type of option	Barrier option with rebate	A rebate for the knock-in barrier option will be paid at expiration by the seller of the barrier option to the holder of the option if the option failed to knock in during its lifetime. For the knock-out barrier option a rebate will be paid by the seller of the option, if the option knocks out. There are two possibilities, either rebate will be paid at expiry of the barrier option or at the first hitting time.
	Alternative barrier option	A barrier option on the one asset with a barrier in another one.
	Digital barrier option	Digital option (an option that pays out a fixed amount if the option at maturity is in-the-money) that includes a barrier which, if reached during the life, affects the existence of the option.
	Rainbow barrier option	Either barrier option which based on more than one underlying. For example, for knock-in rainbow with two assets as underlying option will be active even one of the underlier never reaches the barrier during the time to maturity.
	Parisian barrier option	Barrier option, with a payoff which depends on the history of the underlying asset price. These options either become active (knock-in) or inactive (knock-out) when the asset price has spent a predetermined time, without interruption, either above (an "up" option) or below (a "down" option) a barrier.
	Step option	The final payoff of this kind of option looses its value at a rate proportional to the time the underlying spends above the barrier.
	Exploding option	This option represents a certain payoff if and when a certain price is reached between the initiation and maturity. It is said then to "explode". This type of barrier options is equivalent to a reverse knock-out with a rebate equal to the exploding payoff at the time of the termination of the option.
	Asian barrier option	Barrier option written on the moving arithmetic or geometric average.
Type of the barrier	Double-barrier option	An option with two distinct barriers that define the allowable range for the price fluctuation of the underlying asset. In order for the investor to receive a payout, one of two situations must occur; the price must reach the range limits (for a knock-in) or the price must avoid touching either limit (for a knock-out). A double barrier option is a combination of two dependent knock-in or knock-out options. If one of the barriers are reached in a double knock-out option, the option is worthless. If one of the barriers are reached in a double knock-in option, the option comes alive
	Floating barrier option	The barrier of this option may either increase or decrease with time, or follow some other deterministic paths.
	Forward-start barrier option	This option has a barrier, which is active not immediately after signing the contract but at some time in the future.
	Early-ending barrier option	This is the option with the barrier stopping before the expiration of the option.
	Window barrier option	Such option has a barrier which is only active within one or more time periods during the option's life.
Style of execution	European barrier option	An option that can only be exercised at the end of its life.
	American barrier option	This type of the option can be exercised anytime during its life.

B Greeks

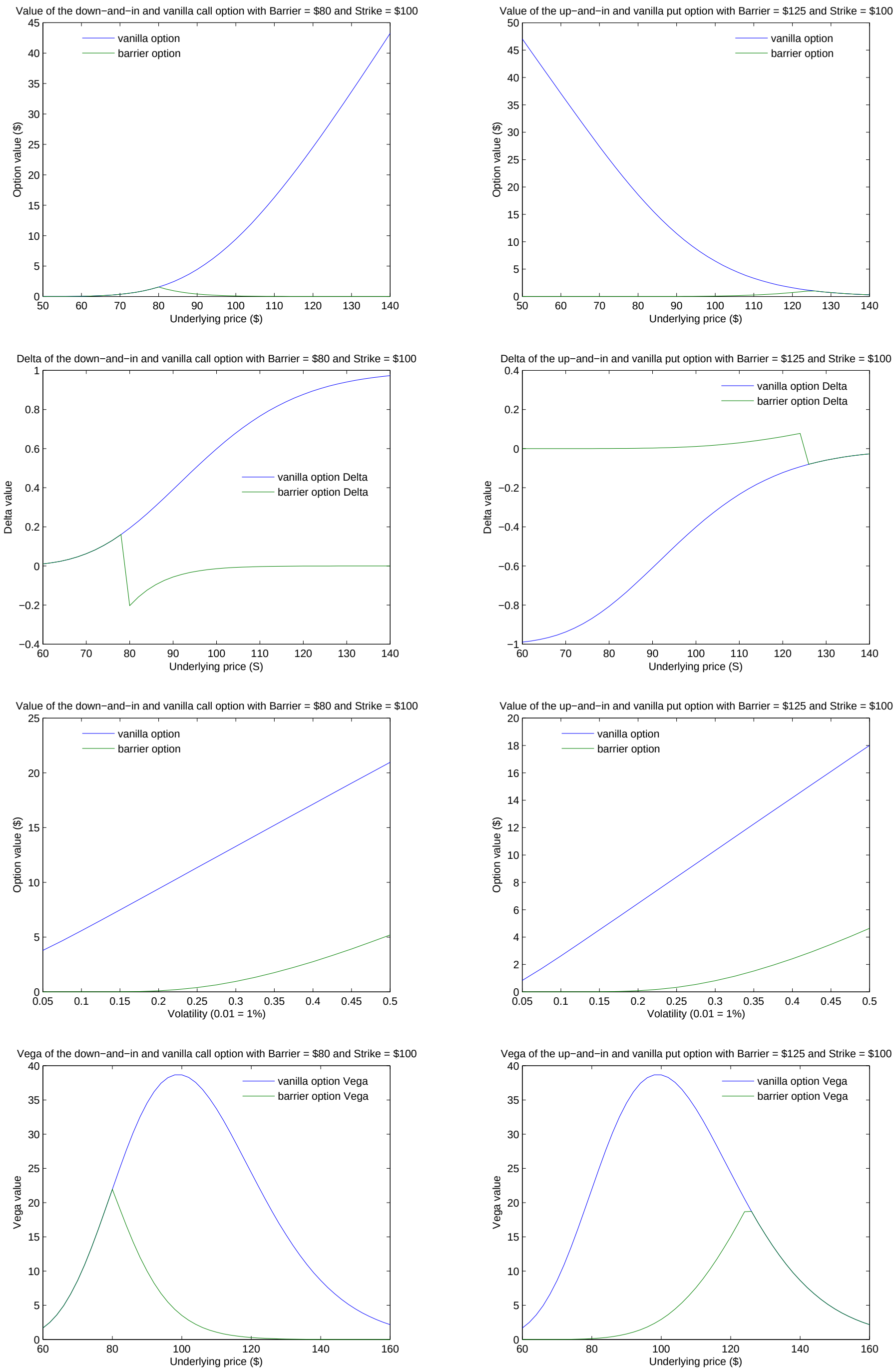
Table 12: Greeks

The Greeks are the quantities representing the market sensitivities of options. Each Greek represents a specific measure of risk in owning an option.(Taleb (1996))

Greeks	Derivation	Definition
Delta	$\frac{\partial \text{Option Value}}{\partial \text{Underlying price}}$	Option price sensitivity to changes in the price of the underlying asset
Gamma	$\frac{\partial^2 \text{Option Value}}{\partial \text{Underlying price}^2}$	Delta sensitivity to changes in the underlying asset price
Vega	$\frac{\partial \text{Option Value}}{\partial Volatility}$	Option price sensitivity to changes in volatility
Theta	$\frac{\partial \text{Option Value}}{\partial \text{Time to maturity}}$	Option price sensitivity to changes in time to maturity
Rho	$\frac{\partial \text{Option Value}}{\partial \text{Risk-free interest rate}}$	Option price sensitivity to changes in the risk-free interest rate
Volga	$\frac{\partial^2 \text{Option Value}}{\partial Volatility^2}$	The second order sensitivity of the option price to implied volatility or Vega sensitivity to changes in the volatility
Vanna	$\frac{\partial^2 OptionValue}{\partial \text{Underlying Price} \partial Volatility}$	The sensitivity of Delta to a unit change in volatility or cross-sensitivity of option value to changes in underlier price and underlier volatility
Delta decay	$\frac{\partial Delta}{\partial \text{Time to maturity}}$	Delta sensitivity to changes in time to maturity

C Knock-in regular barrier options

Figure 25: Knock-in regular barrier options.



D Delta of the Down&In barrier option

For $t < T$:

If $S(t^*) > B$ for any $t^* \in [0,t]$, then:

$$
\begin{aligned}
\Delta(P_{DI}(t)) \quad = \quad & -N(-x_1) + \frac{S(t)\exp(-q(T-t))N(-x_1)}{S(t)\sigma\sqrt{T-t}} \\
& -\frac{K\exp(-r(T-t))N(-x_1+\sigma\sqrt{T-t})}{S(t)\sigma\sqrt{T-t}} \\
& + \exp(-q(T-t))\Big(\frac{B}{S(t)}\Big)^{2\lambda}[N(y)-N(y_1)] \\
& -2\lambda(\frac{B}{S(t)})^{2\lambda}\exp(-q(T-t))[N(y)-N(y_1)] \\
& + \exp\left(-q(T-t)\right)\frac{(\frac{B}{S(t)})^{2\lambda}}{\sigma\sqrt{T-t}}[N(y1)-N(y)] + \\
& K\exp(-r(T-t))\Big(\frac{(\frac{B}{S(t)})^{2\lambda-2}}{S(t)}\Big)(2\lambda-2) \\
& [N(y-\sigma\sqrt{T-t})-N(y_1-\sigma\sqrt{T-t})] \\
& -K\exp(-r(T-t))\frac{(\frac{B}{S(t)})^{2\lambda-2}}{S(t)\sigma\sqrt{T-t}}[N(y_1-\sigma\sqrt{T-t})-N(y-\sigma\sqrt{T-t})], \qquad (41)
\end{aligned}
$$

where

$N(\cdot)$ is the cumulative normal distribution function and

$$\lambda = \frac{r-q+\sigma^2/2}{\sigma^2}, \tag{42}$$

$$y = \frac{\ln(\frac{B^2}{S(t)K})}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t}, \tag{43}$$

$$x_1 = \frac{\ln(\frac{S(t)}{B})}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t}, \tag{44}$$

$$y_1 = \frac{\ln(\frac{B}{S(t)})}{\sigma\sqrt{T-t}} + \lambda\sigma\sqrt{T-t}. \tag{45}$$

If $S(t^*) \leq B$ for any $t^* \in [0,t]$, then:

$$\Delta(P(t)) = -exp(-q(T-t))N(-d_1). \tag{46}$$

For $t = T$:

If $S(t^*) \leq B$ for $t^* \in [0,T]$ & $K > S(T)$, then

$$\Delta(P_{DI}(T)) = -1. \tag{47}$$

If $S(t^*) \leq B$ for $t^* \in [0, T]$ & $K \leq S(T)$, or if $S(t^*) > B$ for all $t^* \in [0, t]$, then $\Delta(P_{DI}(T)) = 0$.

E Equivalent payoff for the down-and-in barrier options in the Black-Scholes model

First we give the lemma without proof which underlies the further discussion in this section, for proof see Carr and Chou (1997b).

Lemma. Under the assumptions of the Black-Scholes model, consider a European security expiring at time T with payoff:

$$f_1(S(T)) = \begin{cases} g(S(T)), & S(T) \in (A, C) \\ 0, & otherwise. \end{cases}$$

Further, for $B \geq 0$, consider another European security with maturity T and the payoff:

$$f_2(S(T)) = \begin{cases} (\frac{S(T)}{B})^p g(\frac{B^2}{S(T)}), & S(T) \in (\frac{B^2}{C}, \frac{B^2}{A}) \\ 0, & otherwise. \end{cases}$$

where $p = 1 - \frac{2(r-q)}{\sigma^2}$ and r, q, and σ are the constant interest rate, dividend yield and volatility, respectively. Then, for any $t \in [0, T]$ the value of these two securities is equal when $S(t) = B$. Note, A can be 0 and C can be ∞.

We use the result of this lemma to show that a barrier option can be replicated with a European security with a specifically chosen payoff function. This argument only holds in the Black-Scholes world, which is the main assumption in the previous lemma. We consider only the case of a down-and-in barrier with payoff function $g(S(T))$ and barrier level B ($< S(0)$).

Consider a long position in two European securities with payoff functions at time T as follows:

$$f_*(S(T)) = \begin{cases} 0, & S(T) \in (B, \infty) \\ g(S(T)), & otherwise, \end{cases}$$

$$f_2(S(T)) = \begin{cases} 0, & S(T) \in (B, \infty) \\ (\frac{S(T)}{B})^p g(\frac{B^2}{S(T)}), & otherwise. \end{cases}$$

We claim that the value of these two securities is the same as the value of a down- and-in barrier option with payoff function $g(S(T))$ and barrier level B ($< S(0)$). In order to prove the claim, we now consider two scenarios for the stock price dynamics:

1. The stock price never hits the barrier level B over the life of these European securities.
2. The stock price does hit the barrier level B at some time over the life of these European securities.

Under scenario 1 the payoffs at maturity of both considered securities are zero. The combined portfolio payoff is also zero and exactly equal to that of the barrier option under this scenario.

Scenario 2. By above lemma at the time when the stock hits the barrier, the value of the European security with payoff $f_2(S(T))$ is exactly the same as the value of the a European security with payoff function:

$$f_1(S(T)) = \begin{cases} g(S(T)), & S(T) \in (B, \infty) \\ 0, & otherwise. \end{cases}$$

Hence, at time when $S(t) = B$ we can sell the payoff $f_2(S(T))$ and buy the $f_1(S(T))$ without any costs. The new portfolio payoff at maturity is:

$$f_1(S(T)) + f_*(S(T)) = \left\{ \begin{array}{ll} g(S(T)) + 0 = g(S(T)), & S(T) \in (B, \infty) \\ 0 + g(S(T)), & otherwise \end{array} \right\} = g(S(T)),$$

which is equal to the payoff of the barrier option under this scenario.

Thus the down-and-in barrier options can be replicated with a European security with payoff function:

$$f(S(T)) = f_*(S(T)) + f_2(S(T)) = \left\{ \begin{array}{ll} 0, & S(T) \in (B, \infty) \\ g(S(T)) + (\frac{S(T)}{B})^p g(\frac{B^2}{S(T)}), & S(T) \in (0, B]. \end{array} \right.$$

Therefore replicating a European security with such a payoff function is the same as replicating the barrier option.

F Calculation of the states probabilities for optimization approach

Reflection theorem.

Consider the Black-Scholes model with constant short term interest rate r and a risky asset whose price is a Geometric Brownian motion,

$$dS(t) = \mu S(t)dt + \sigma S(t)dW^Q(t), \tag{48}$$

under the equivalent martingale measure Q. $\mu = r - d$, where d is dividend yield of the risky asset.

Put

$$p = 1 - \frac{2\mu}{\sigma^2}, \tag{49}$$

and consider a simple claim with a payoff at time T specified by a payoff function g (a 'g-claim' for short). Its arbitrage-free time-t price is

$$\pi^g(t) = e^{-r(T-t)}E_t^Q(g(S(T))) = e^{-r(T-t)}f(S(t),t), \tag{50}$$

where $f(S(t),t) = E_t^Q(g(S(T)))$. Let $H > 0$ be a constant and define a new function g by

$$\hat{g}(x) = \Big(\frac{x}{H}\Big)^p g\Big(\frac{H^2}{x}\Big). \tag{51}$$

This function is called *the reflection of g through H.*

If all assumptions above hold, then the arbitrage-free time-t price of the simple claim with payoff function g is:

$$\pi^{\hat{g}}(t) = e^{-r(T-t)}\Big(\frac{S(t)}{H}\Big)^p f\Big(\frac{H^2}{S(t)},t\Big). \tag{52}$$

The proof can be found in Poulsen (2004).

Geometric Brownian motion and its minimum.

Let us look at the simple claim with payoff function $h = g - \hat{g}$. Note that $h(x) = g(x)$ if $x > B$ and equation (52) tells us that the time-t price of the h-claim is

$$\pi^h(t) = e^{-r(T-t)}(f(S(t),t) - \Big(\frac{S(t)}{B}\Big)^p f\Big(\frac{B^2}{S(t)},t\Big)). \tag{53}$$

If $S(t) = B$, then the h-claim has a price of 0.

Now consider the payoff function $v(x) = 1_{x \geq K}$ and put $g(x) = v(x)1_{x \geq B}$ for some B, where $0 \leq B \leq min(K, S(0))$ (so really, $v = g$). Then

$$f(x,t) = \Phi\Big(\frac{ln(\frac{x}{K}) + (\mu - \sigma^2/2)(T-t))}{\sigma\sqrt{T-t}}\Big), \tag{54}$$

Let π^h denote the price of the associated h-claim option. On the one hand

$$e^{rT}\pi^h(0) = E^Q(1_{S(T)\geq K}1_{m(T)>B}) = Q(S(T) \geq K, m(T) > B), \tag{55}$$

where $m(T) = min_{u \le T} S(u)$ denotes the running minimum of $S(u)$. On the other hand equation (53) tells us that

$$e^{rT}\pi^h(0) = \Phi\Big(\frac{ln(\frac{S(0)}{K}) + (\mu - \sigma^2/2)(T))}{\sigma\sqrt{T}}\Big) - \Big(\frac{S(0)}{B}\Big)^p \Phi\Big(\frac{ln(\frac{B^2}{S(0)K}) + (\mu - \sigma^2/2)(T))}{\sigma\sqrt{T}}\Big). \quad (56)$$

Combining these two equation we receive exactly the probability that barrier B was not hit during the maturity T and $S(T)$ is at some particular level above B. We use this to calculate state probabilities at maturity in the optimization approach.

Define the the first hitting time to the level B as $\tau := inf\{u|S(u) = B\}$ and put $K = B$ in (55), then we have:

$$\begin{aligned} Q(\tau \le x) \quad &= \quad 1 - Q(\tau > x) \\ &= 1 - Q(m(x) > B) \\ &= \Phi\Big(\frac{ln(\frac{B}{S(0)}) - (\mu - \sigma^2/2)T)}{\sigma\sqrt{T}}\Big) \\ &+ \Big(\frac{S(0)}{B}\Big)^p \Phi\Big(\frac{ln(\frac{B}{S(0)}) + (\mu - \sigma^2/2)T)}{\sigma\sqrt{T}}\Big). \end{aligned} \quad (57)$$

This gives us the state probabilities along the barrier in the optimization approach.