

**ON THE PRICING OF BERMUDAN
SWAPTIONS WITH AN APPLICATION TO
LIMITED OBSERVED MARKET DATA**

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Abstract

The focus of this thesis is on the risk neutral valuation of Bermudan swaptions and its application to pricing situations where observed market data used for calibration is limited. By exploring the properties of the solution to the optimal stopping problem that specifies the price process of these instruments, a general valuation method suited for practical computations is suggested. The valuation method is based on restricting the evolution of the short rate process to that of a recombining binomial tree and is able to produce fast price estimates of Bermudan swaptions based on limited input data when specifying the dynamics of the short rate process to the Ho-Lee model.

Acknowledgements

This master thesis has been conducted in collaboration with Algorithmica Research AB and I would like to take the opportunity to thank all members of the staff who have helped and supported me on the way to completing the work, especially my supervisor Peter Alaton who has provided me with much advice on various aspects of the interest rate market.

I would furthermore like to thank my supervisor at the Royal Institute of Technology, Professor Boualem Djehiche, for all his encouragement and advice at all stages of the project.

Last but not least, I want to acknowledge all encouragement and support given to me by my family and my girlfriend, Aurora, during this process.

Contents

1	Introduction	7
2	Preliminaries	9
2.1	Zero coupon bonds	9
2.2	The bank account	9
2.3	LIBOR spot rates	10
2.4	Effective zero rates	11
3	Definition of instruments	12
3.1	The interest rate swap	12
3.1.1	The payer swap	12
3.2	The Bermudan swaption	14
3.2.1	The Bermudan payer swaption	15
4	Pricing Bermudan swaptions	17
4.1	Stopping times	18
4.1.1	Stopping times and Bermudan swaptions	18
4.2	Derivation of an optimal stopping time	19
5	Using the pricing envelope in practice	22
5.1	Introducing a new pricing envelope	22
5.1.1	Interpretation of the envelope	24

5.1.2	Extension of the envelope	25
5.2	Pricing with a binomial tree	26
5.2.1	Approximating the pricing procedure	28
5.2.2	The approximation error	29
6	Pricing according to the Ho-Lee short rate model	30
6.1	The Ho-Lee short rate model	30
6.2	Pricing the Bermudan swaption	32
6.3	Calibrating the model using limited data	33
6.4	Disadvantages of the model	34
7	Numerical results	36
7.1	Convergence of the pricing procedure	37
7.2	Convergence of the calibration procedure	38
7.3	Sensitivity measures	39
7.3.1	Price sensitivity w.r.t. the short rate volatility	39
7.3.2	Price sensitivity w.r.t. the initial term structure	40
8	Summary and conclusions	44
A	Pricing zero coupon bonds using the Ho-Lee binomial tree	46

Chapter 1

Introduction

In recent years a lot of attention has been drawn to the problem of accurately and effectively pricing Bermudan swaptions. A number of pricing procedures, such as those described in [1], [3], [8], [9] and [10], have resulted from this research. All these methods have a common property in that they require fairly advanced market data for calibration and furthermore employ computationally intense procedures when calibrating. For a financial institution, the complexity of these procedures is generally manageable, but for non-financial corporates having positions in Bermudan swaptions, it may quickly become a hard problem. For these parties, gathering and processing the extensive collection of market data (such as prices for caps, floors and interest rate swaps) needed to make the pricing procedures work as well as performing and fine tuning the calculations, usually results in a time consuming, or at worst, an impossible task. As a consequence of this, they are thus more or less left without a means to value and assess the risk of their position.

Assuming that these parties receive valuations from the counter party of the Bermudan swaption on regular intervals and that they are able to access widely quoted interest rate curves, there is however as will be revealed, a way for them to retrieve an estimate of the sought information.

The purpose of this thesis is twofold. Firstly, the optimal stopping problem of pricing the Bermudan swaption through risk neutral valuation will be studied and solved as the optimum of a deterministic dynamic programming problem. Secondly, the properties of the derived solution will be explored in order to find a means to convert the theoretical results into a pricing procedure suitable for practical purposes. A specialization of this method, adjusted to the previously described calibration circumstances, will then finally be proposed using the Ho-Lee binomial tree representation of the short rate. Although, this method generally is considered to generate relatively crude results, it will provide a quickly calculated estimate of the price of the Bermudan swaption to the parties lacking the possibility of using more complex models.

The remainder of this thesis is organized as follows. Chapter 2 gives a brief introduction to the concepts used throughout the rest of this thesis when deriving results regarding the pricing of Bermudan swaptions. Chapter 3 provides a definition of the interest rate swap as well as a definition of the Bermudan swaption and its price process. In Chapters 4 and 5 we derive and discuss the theoretical solution to the pricing problem and study how these results may be used in practical pricing situations. In Chapter 6 we then study a special case of the previously derived pricing procedure having a calibration procedure adjusted to the aforementioned limitations in observed market data. This pricing method is obtained by modelling the evolution of the short rate according to the Ho-Lee model in discrete time. Chapter 7 presents some numerical results on the behaviour of the “Ho-Lee pricing method” as well as its calibration procedure. Chapter 8 finally summarizes and discusses the material covered in the thesis and suggests topics that remain to be studied.

Chapter 2

Preliminaries

The purpose of this chapter is to give a brief introduction to the concepts as well as the notation that will be used throughout the rest of this thesis. The interested reader is referred to [2] for a thorough treatment of the covered material.

2.1 Zero coupon bonds

The market we will be working with in subsequent chapters is a complete market free of arbitrage characterized by having zero coupon bonds as its only asset. On a complete and filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0})$ these instruments are defined as follows.

Definition 1. *A zero coupon bond with maturity T is a contract which at time T pays one unit of currency to its holder. The price at time $0 \leq t \leq T$ of a zero coupon bond maturing at time T is denoted by $p(t, T)$, where $(p(t, T) \in \mathcal{F}_t)_{0 \leq t \leq T}$ and $p(t, t) = 1$ for all $t \geq 0$.*

2.2 The bank account

Given the previously introduced bond market, we now construct a derived instrument that will be frequently used in chapters to come. This instrument represents an investment free of risk continuously growing according to an interest rate – the short rate – and is commonly referred to as the bank account.

Definition 2. *The bank account process, $B = (B_t \in \mathcal{F}_t)_{t \geq 0}$, is defined according to the dynamics*

$$\begin{aligned} dB_t &= r_t B_t dt \\ B_0 &= 1 \end{aligned}$$

or equivalently

$$B_t = e^{\int_0^t r_s ds},$$

where the short rate process, $r = (r_t \in \mathcal{F}_t)_{t \geq 0}$, at the time t is given by

$$r_t = - \left. \frac{\partial \log p(t, T)}{\partial T} \right|_{T=t}.$$

The bank account process will be used as a pricing numeraire when performing risk neutral valuation in subsequent chapters. The unique measure under which this numeraire deflates price processes into martingales is denoted by Q . The existence of this martingale measure follows from the assumption that the bond market is free of arbitrage and complete.

Having introduced the bank account process and the measure Q , the price process of a zero coupon bond may be calculated through risk neutral valuation according to the expression presented below. This valuation will be of key importance and used frequently in chapters to come.

Theorem 1. *The price process at the time t of a zero coupon bond having maturity $T \geq t \geq 0$ is given by the expression*

$$p(t, T) = \mathbf{E}^Q \left[\frac{B_t}{B_T} \cdot 1 \mid \mathcal{F}_t \right] = \mathbf{E}^Q \left[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t \right].$$

2.3 LIBOR spot rates

LIBOR spot rates are frequently used when constructing interest rate derivatives characterized by payments calculated according to floating rates. These rates will be used when introducing interest rate swaps and Bermudan swaptions in the next chapter.

Definition 3. *The LIBOR spot rate for the time interval $[t, T]$ where $t \geq 0$ is defined as*

$$L(t, T) = \frac{1}{(T-t)} \left(\frac{p(t, t) - p(t, T)}{p(t, T)} \right) = \frac{1}{(T-t)} \left(\frac{1}{p(t, T)} - 1 \right)$$

or equivalently

$$p(t, T) = \frac{1}{1 + (T-t)L(t, T)}.$$

LIBOR spot rates may in a sense be interpreted as “discrete versions” of the short rate process, r . By comparing the above definition with that of the short rate process, it is clear that $L(t, T)$ is related to r_t through first order approximations of the derivative and the logarithm.

As a final remark it is worth pointing out that the LIBOR spot rate is actually a special case of an interest rate commonly known as the LIBOR forward rate. Since this rate will not be explicitly used in coming chapters, its definition is omitted in order to avoid any confusion. Interested readers are referred to [3].

2.4 Effective zero rates

Effective zero rates may be regarded as internal rates of interest of zero coupon bonds, i.e. rates describing the bonds’ increase in value as time passes, and will be used when performing sensitivity analysis in the chapter covering the behavior of the “Ho-Lee pricing method”.

Definition 4. *The effective zero rate, $y(t, T)$, for the time interval $[t, T]$ where $t \geq 0$ is defined as*

$$y(t, T) = \left(\left[\frac{1}{p(t, T)} \right]^{\frac{1}{T-t}} - 1 \right)$$

or equivalently

$$p(t, T) = \frac{1}{(1 + y(t, T))^{T-t}}.$$

Chapter 3

Definition of instruments

In this chapter we define the transactions and instruments studied in this thesis and present expressions for their corresponding price processes. Although the time structures used in the presentation are somewhat limited in order to maintain manageable expressions, all derived results may easily be adjusted to a more general setup.

3.1 The interest rate swap

An interest rate swap, henceforth abbreviated as an IRS, is a contractual agreement between two parties under which each party agrees to make periodic interest payments to the other for an agreed period of time. The agreement in its most common form states that a series of payments calculated by applying a fixed rate of interest to a notional amount are exchanged for another series of payments calculated at the same notional amount using a floating rate of interest. The fixed rate payment stream is traditionally called the fixed leg, while the floating payments are referred to as the floating leg.

Depending on whether a participant of an IRS agreement receives/pays the fixed leg he is said to hold a receiver/payer swap. That is, the difference between the two setups lies in the sign of the net flow of payments. As a consequence of this, once the price of one type of IRS is known, the price of the other is instantly known by reverting the sign of the first price. Since the difference in prices indeed is minor, we only provide the derivation of the price of the payer swap.

3.1.1 The payer swap

Consider a fixed set of equidistantly spaced dates T_0, T_1, \dots, T_N such that $\delta = T_i - T_{i-1}$ for all $i = 1, 2, \dots, N$. The payer swap is defined according to the following agreement:

- All cash flows will be paid and received at the dates T_1, T_2, \dots, T_N .
- At the beginning of each period, $[T_{i-1}, T_i]$, the LIBOR spot rate, $L(T_{i-1}, T_i)$, is set and the amount $\kappa \delta L(T_{i-1}, T_i)$ is received by the holder at the date T_i where $i = 1, \dots, N$. The amount κ is called the notional amount.
- For the same period, $[T_{i-1}, T_i]$, the holder pays the amount $\kappa \delta R$, at the date T_i where $i = 1, \dots, N$ and R is constant. The rate at which these payments are made, R , is called the swap rate.

Using the above specifics of the contract together with the definition of the LIBOR spot rate, the net cash flow of the payer swap at the date T_i can be expressed as

$$\kappa \delta \left(L(T_{i-1}, T_i) - R \right) = \kappa \left(\frac{1}{p(T_{i-1}, T_i)} - (1 + \delta R) \right). \quad (3.1)$$

We now wish to determine the value of this net income, denoted $N_i(t)$, at a time $t \leq T_0$, i.e. before the first settlement date. Risk neutral valuation gives

$$\begin{aligned} N_i(t) &= \mathbf{E}^Q \left[e^{-\int_t^{T_i} r_s ds} \kappa \left(\frac{1}{p(T_{i-1}, T_i)} - (1 + \delta R) \right) \middle| \mathcal{F}_t \right] \\ &= \kappa \left(\mathbf{E}^Q \left[e^{-\int_t^{T_i} r_s ds} \frac{1}{p(T_{i-1}, T_i)} \middle| \mathcal{F}_t \right] - (1 + \delta R) p(t, T_i) \right) \end{aligned} \quad (3.2)$$

where the cash flow $\kappa(1 + \delta R)$ obviously may be interpreted as the face value of a zero-coupon bond maturing at the date of the net transaction. In order to price the first term in the net income, we need to perform some further calculations. By conditioning the expectation on a larger σ -algebra, $\mathcal{F}_{T_{i-1}} \supseteq \mathcal{F}_t$, and extracting the information stored therein we get

$$\begin{aligned} \mathbf{E}^Q \left[e^{-\int_t^{T_i} r_s ds} \frac{1}{p(T_{i-1}, T_i)} \middle| \mathcal{F}_t \right] &= \\ \mathbf{E}^Q \left[\mathbf{E}^Q \left[e^{-\int_t^{T_{i-1}} r_s ds} e^{-\int_{T_{i-1}}^{T_i} r_s ds} \frac{1}{p(T_{i-1}, T_i)} \middle| \mathcal{F}_{T_{i-1}} \right] \middle| \mathcal{F}_t \right] &= \\ \mathbf{E}^Q \left[e^{-\int_t^{T_{i-1}} r_s ds} \frac{1}{p(T_{i-1}, T_i)} \mathbf{E}^Q \left[e^{-\int_{T_{i-1}}^{T_i} r_s ds} \middle| \mathcal{F}_{T_{i-1}} \right] \middle| \mathcal{F}_t \right] &= \\ \mathbf{E}^Q \left[e^{-\int_t^{T_{i-1}} r_s ds} \middle| \mathcal{F}_t \right]. \end{aligned}$$

From the above calculation, it becomes clear that the first term in the expression for the net cash flow, (3.2), is valued as $\kappa p(t, T_{i-1})$ at the date $t \leq T_0$. Using this fact together with earlier calculations, we finally end up at the net income valuation

$$N_i(t) = \kappa \left(p(t, T_{i-1}) - (1 + \delta R) p(t, T_i) \right). \quad (3.3)$$

Determining the total value at the time $t \leq T_0$ of the payer swap, $\mathbf{PS}(t; \kappa, \delta, R)$, now amounts to summing up all the values of the net incomes and thus we get

$$\mathbf{PS}(t; \kappa, \delta, R) = \sum_{i=1}^N N_i(t) = \kappa \sum_{i=1}^N \left(p(t, T_{i-1}) - (1 + \delta R)p(t, T_i) \right)$$

which simplifies to

$$\mathbf{PS}(t; \kappa, \delta, R) = \kappa \left(p(t, T_0) - p(t, T_N) - \delta R \sum_{i=1}^N p(t, T_i) \right). \quad (3.4)$$

A few remarks are in order. First of all, the valuation formula above is valid only at times prior to the first settlement date, i.e. $t \leq T_0$. At later instants, the value of the IRS is determined by summing up the discounted values of all yet unsettled net cash flows. If for instance, $T_0 \leq T_{k-1} < t \leq T_k \leq T_N$, the value of the payer swap would equal

$$\mathbf{PS}(t; \kappa, \delta, R) = \kappa \left(p(t, T_k) - p(t, T_N) - \delta R \sum_{i=k+1}^N p(t, T_i) \right).$$

Secondly, the swap rate, R , is not to be mistaken for the par swap rate. The par swap rate, $R_p(t)$, is the swap rate at which the price of the IRS equals zero and is calculated using (3.4) as

$$R_p(t) = \frac{p(t, T_0) - p(t, T_N)}{\delta \sum_{i=1}^N p(t, T_i)}. \quad (3.5)$$

Although, IRS contracts in practice are formulated to have an initial price of zero, swap rates do not necessarily have to be calculated at par, i.e. where $R = R_p$.

3.2 The Bermudan swaption

A swaption is an option on an IRS, i.e. an instrument providing its holder the right, but not the obligation, to enter an IRS at a pre-specified date in the future. As the name suggests, an option with the Bermudan characteristic, may be considered as being situated somewhere in the gap between the European option – having a single exercise date – and the American option – having a continuous range of exercise dates – in terms of its set of possible exercise dates. That is to say, a Bermudan swaption is a swaption having multiple discretely positioned exercise dates.

The following section provides a closer definition of the Bermudan swaption in terms of its typical setup and risk neutral valuation. The focus will be on

swaptions having a payer swap as its underlying instrument. These types of swaptions are henceforth referred to as payer swaptions.

3.2.1 The Bermudan payer swaption

Before getting into details about the exercise features of the Bermudan swaption we need to consider the underlying payer swap of the option. Using the setup and results described in the previous section we similarly define this swap as:

- The underlying payer swap has N equidistantly spaced dates, T_1, T_2, \dots, T_N where cash flows occur. The cash flows are identical to those described in the previous section with the exception that the fixed leg follows the swap rate R_s . This swap rate is called the swaption's strike rate.
- The value process of the swap at the time $t \leq T_{N-1}$ is $\mathbf{PS}(t; \kappa, \delta, R_s)$.

There are several possible ways of constructing a Bermudan swaption when it comes to deciding the setup of exercise dates. The arrangement that is used in this thesis is basically the same as described in [3]:

- The swaption may be exercised at one of $M + 1$ number of fixed exercise dates. These dates are denoted T_i^e and all coincide with settlement dates of the underlying IRS, i.e. $T_i^e \in \{T_0, T_1, \dots, T_{N-1}\}$, $i = 0, 1, \dots, M$.
- The first exercise date of the swaption equals¹ the first settlement date of the underlying IRS, i.e. $T_0^e = T_0$.

Before continuing with the risk neutral valuation of the Bermudan swaption we need to determine the option's payoff function. Unlike for instance options on stock, swaptions, are not characterized by strike prices. Instead, swaptions use strike rates specifying the swap rate, $R = R_s$, of the underlying IRS of the option. To make matters more specific, consider for instance the case when the swaption is exercised at the date T_i^e for some $i = 0, 1, \dots, M$. The payoff function generated as a consequence of the exercise is then given by

$$\max\left\{\mathbf{PS}(T_i^e; \kappa, \delta, R_s), 0\right\}.$$

As is clear from the payoff function, one could claim that swaptions use strike prices equalling zero. Remembering that market practice is to construct swaps using par swap rates in order to achieve an initial value of zero, a possible interpretation of this choice of "strike price" is that the holder chooses to exercise

¹This might seem like a limitation to generality at first, but assuming that $T_0^e > T_0$ would in fact be pointless since the holder of the swaption wouldn't be able to access any cash flows settled earlier than T_0^e , which in the context of pricing the swaption makes all settlement dates earlier than T_0^e redundant.

the option only when the strike rate is sufficiently low compared to the par swap rate (the swap rate used for trading an identical swap on the market). In that case, the value of the payer swap would be positive implying that the owner of the swaption would make profit entering the IRS agreement.

Having defined the underlying IRS and the exercise feature of the Bermudan swaption it is possible to calculate its price, $\mathbf{BPS}(T_0^e; \kappa, \delta, R_s)$, at the time T_0^e using risk neutral valuation as

$$\mathbf{BPS}(T_0^e; \kappa, \delta, R_s) = \sup_{\tau \in \mathcal{S}} \mathbf{E}^Q \left[e^{-\int_{T_0^e}^{\tau} r_s ds} \max\{\mathbf{PS}(\tau; \kappa, \delta, R_s), 0\} \middle| \mathcal{F}_{T_0^e} \right] \quad (3.6)$$

where the supremum is taken over the set, \mathcal{S} , of all \mathcal{E} -stopping times with values in the set $\{T_i^e, i = 0, 1, \dots, M\}$. The filtration \mathcal{E} is defined as

$$\mathcal{E} = (\mathcal{F}_{T_i^e})_{i=0}^M. \quad (3.7)$$

The concept of stopping times will be defined and studied more closely in the next chapter which has its focus on solving, i.e. finding the supremum of, the above pricing problem. Until then, the reader may think of these variables as exercise strategies adapted to the stochastic movements of the interest rate market, i.e. strategies deciding at which of the times $T_0^e, T_1^e, \dots, T_M^e$ the option should be exercised.

As a final remark it is important to point out that although studying the pricing of Bermudan swaptions at the instant T_0^e might seem like a large limitation to generality, retrieving the price at any time prior to the first exercise date is in fact quite easily accomplished. The price at any instant $t \leq T_0^e$ is simply calculated as the expectation of the discounted swaption valuation at T_0^e (cf. calculating the price of European options) and the reason we choose to focus on the valuation at the first exercise date is only motivated by the fact that it eases up the notation and keeps focus on the essentials.

Chapter 4

Pricing Bermudan swaptions

In the previous chapter, the price at T_0^e of a Bermudan payer swaption was stated according to the risk neutral valuation expression

$$\mathbf{BPS}(T_0^e; \kappa, \delta, R_s) = \sup_{\tau \in \mathcal{S}} \mathbf{E}^Q \left[e^{-\int_{T_0^e}^{\tau} r_s ds} \max \left\{ \mathbf{PS}(\tau; \kappa, \delta, R_s), 0 \right\} \middle| \mathcal{F}_{T_0^e} \right], \quad (4.1)$$

where the supremum is taken over all \mathcal{E} -stopping times. Though this expression sure enough uniquely determines the price of the option, it cannot be evaluated using straightforward calculations since the price is determined as a solution to a stochastic optimization problem. This particular optimization problem is the focus of this chapter where we introduce the concepts of stopping times and Snell envelopes and solve the above stated valuation problem.

In order to ease up the notation when solving (4.1) the rest of this chapter is focused on the treatment of the problem

$$V = \sup_{\tau \in \mathcal{S}_0} \mathbf{E}^Q [f_\tau | \mathcal{F}_0] \quad (4.2)$$

where $f = (f_i \in \mathcal{F}_i)_{i=0}^M$ is a stochastic process in discrete time and \mathcal{S}_0 belongs to the set $(\mathcal{S}_i)_{i=0}^M$ where \mathcal{S}_i denotes the set of all $\underline{\mathcal{F}}^i$ -stopping times. The filtration $\underline{\mathcal{F}}^i$ is a member of the set $(\underline{\mathcal{F}}^i)_{i=0}^M$ specified by

$$\underline{\mathcal{F}}^i = (\mathcal{F}_k)_{k=i}^M, \quad i = 0, 1, \dots, M.$$

The process f represents the discounted payoff process of the Bermudan payer swaption and is assumed to be limited by the constraint

$$\mathbf{E}^Q [|f_i|] < \infty, \quad i = 0, 1, \dots, M.$$

As a final remark its worth emphasizing that solving (4.1) by working with (4.2) not only simplifies the notation but also generalizes the results somewhat. Even though the process f in this thesis is defined in the context of pricing Bermudan swaptions, the solution to the problem (4.2) is of course valid for any discrete time process limited by the above stated integrability constraint. In other words, by finding the optimal stopping time and the solution to this problem, we have in fact found the solution to the risk neutral valuation problem of any option having a finite set of pointwise positioned exercise dates.

4.1 Stopping times

The solution to the pricing problem (4.2) is found by optimizing on a set of stopping times. In order to find the optimum we thus need to get a proper understanding of these variables.

Definition 5. *A random variable, τ , taking values in $\{i, i + 1, \dots, M\}$ is called a \mathcal{F}^i -stopping time or a stopping time for the filtration \mathcal{F}^i if*

$$\{\tau \leq k\} \in \mathcal{F}_k, \quad k = i, i + 1, \dots, M$$

Stopping times are frequently used in the context of decision making under uncertainty, where they serve as variables determining when particular measures are to be taken, e.g. when to leave a gambling table or when to exercise an option having multiple exercise dates. By definition, a stopping time can only determine whether a certain measure is to be taken at a time provided that the information generated up to that time is available. It can not with certainty assume values corresponding to times yet to come based on what has happened up to present time.

4.1.1 Stopping times and Bermudan swaptions

Knowing the definition of stopping times, a natural question to pose is what motivates the approach of pricing Bermudan swaptions (or any other derivate having an early exercise feature for that matter) using these variables. A key observation needed in order to answer that question, is that some way of representing a strategy determining when to exercise the option, needs to be established. As more money is assumed to be associated with higher utility than less money for all actors in a derivatives market according to arbitrage theory, this strategy should furthermore strive to maximize the payoff received when exercising the swaption.

Bermudan swaptions are instruments traded on an interest rate market associated with uncertainty. As an exercise strategy should take the stochastic movements of this market into consideration a reasonable model of this strategy would be through the use of random variables. Furthermore, the strategy

should have the property that an exercise decision should be made based on all information generated by the market up to that point in time. There would, for instance, be no point in deciding to exercise the swaption in a year from now, as that decision should needless to say be made then, not now. Taking these model demands into account together with the previous discussion on choosing an exercise strategy maximizing the received payoff, the reason why Bermudan swaptions are priced as optimal stopping problems is motivated. The properties of an exercise strategy coincide with those of a stopping time and the task of obtaining a price for the Bermudan swaption becomes equivalent with finding a strategy that maximizes the expectation of the discounted payoff of the option.

4.2 Derivation of an optimal stopping time

We now turn to the problem of finding an optimal stopping for the pricing problem (4.2) and begin by introducing the stochastic process $v = (v_i \in \mathcal{F}_i)_{i=0}^M$ given by

$$\begin{aligned} v_M &= f_M \\ v_i &= \max\left\{f_i, \mathbf{E}^Q[v_{i+1}|\mathcal{F}_i]\right\}, \quad i = 0, 1, \dots, M-1. \end{aligned} \quad (4.3)$$

Using this process we furthermore introduce the $\underline{\mathcal{F}}^i$ -stopping times $\theta_i \in \mathcal{S}_i$ for $i = 0, 1, \dots, M$ defined by

$$\theta_i = \min\left\{k = i, i+1, \dots, M : f_k = v_k\right\}, \quad i = 0, 1, \dots, M. \quad (4.4)$$

The process, v , is commonly referred to as a Snell envelope for the stochastic process f and plays an important role in the problem of pricing Bermudan swaptions as will be revealed shortly. Before reaching this result however, we need to make an important observation.

Lemma 1. *For $i = 0, 1, \dots, M$, the process v satisfies the relation*

$$v_i = \mathbf{E}^Q[f_{\theta_i}|\mathcal{F}_i] \geq \mathbf{E}^Q[f_\tau|\mathcal{F}_i], \quad \tau \in \mathcal{S}_i.$$

Proof. The proof is given by an inductive argument. We begin by noting that the relation by definition is satisfied with equality for $i = M$ and furthermore make the assumption that the relation is valid for the times $i = M, M-1, \dots, k$. Showing that this assumption implies that the relation is valid for $k-1$ then concludes the proof.

Letting $\tau \in \mathcal{S}_{k-1}$ be an arbitrary $\underline{\mathcal{F}}^{k-1}$ -stopping time and introducing $\bar{\tau} \in \mathcal{S}_k$ such that $\bar{\tau} = \max\{\tau, k\}$ we make the following observation for some event $A \in \mathcal{F}_{k-1}$

$$\begin{aligned}
\mathbf{E}^Q[\mathbf{1}_A f_\tau] &= \mathbf{E}^Q[\mathbf{1}_{A \cap \{\tau=k-1\}} f_\tau] + \mathbf{E}^Q[\mathbf{1}_{A \cap \{\tau \geq k\}} f_\tau] \\
&= \mathbf{E}^Q[\mathbf{1}_{A \cap \{\tau=k-1\}} f_{k-1}] + \mathbf{E}^Q[\mathbf{E}^Q[\mathbf{1}_{A \cap \{\tau \geq k\}} f_\tau | \mathcal{F}_{k-1}]] \\
&= \mathbf{E}^Q[\mathbf{1}_{A \cap \{\tau=k-1\}} f_{k-1}] + \mathbf{E}^Q[\mathbf{E}^Q[\mathbf{1}_{A \cap \{\tau \geq k\}} f_{\bar{\tau}} | \mathcal{F}_{k-1}]].
\end{aligned}$$

Since $\{\tau = k - 1\} \in \mathcal{F}_{k-1}$, conditioning on a smaller σ -algebra implies that

$$\begin{aligned}
\mathbf{E}^Q[\mathbf{1}_A f_\tau] &= \mathbf{E}^Q[\mathbf{1}_{A \cap \{\tau=k-1\}} f_{k-1}] + \mathbf{E}^Q[\mathbf{1}_{A \cap \{\tau \geq k\}} \mathbf{E}^Q[f_{\bar{\tau}} | \mathcal{F}_{k-1}]] \\
&= \mathbf{E}^Q[\mathbf{1}_{A \cap \{\tau=k-1\}} f_{k-1}] + \mathbf{E}^Q[\mathbf{1}_{\{A \cap \tau \geq k\}} \mathbf{E}^Q[\mathbf{E}^Q[f_{\bar{\tau}} | \mathcal{F}_k] | \mathcal{F}_{k-1}]] \\
&\leq \mathbf{E}^Q[\mathbf{1}_{A \cap \{\tau=k-1\}} f_{k-1}] + \mathbf{E}^Q[\mathbf{1}_{A \cap \{\tau \geq k\}} \mathbf{E}^Q[v_k | \mathcal{F}_{k-1}]] \\
&\leq \mathbf{E}^Q[\mathbf{1}_A v_{k-1}]
\end{aligned}$$

where the second last inequality follows from the induction hypothesis and the last inequality is an immediate consequence of the definition of the Snell envelope. As the above inequality is valid for all events $A \in \mathcal{F}_{k-1}$, conditioning the above expectations on \mathcal{F}_{k-1} necessarily results in

$$v_{k-1} \geq \mathbf{E}^Q[f_\tau | \mathcal{F}_{k-1}]$$

which proves the inequality relation stated in the lemma. It remains to verify that this inequality is satisfied with equality for the \mathcal{F}^{k-1} -stopping time θ_{k-1} . Returning to the chain of inequalities above, we conclude that

$$\mathbf{E}^Q[\mathbf{1}_A f_{\theta_{k-1}}] = \mathbf{E}^Q[\mathbf{1}_{A \cap \{\theta_{k-1}=k-1\}} f_{k-1}] + \mathbf{E}^Q[\mathbf{1}_{A \cap \{\theta_{k-1} \geq k\}} \mathbf{E}^Q[v_k | \mathcal{F}_{k-1}]].$$

as a consequence of the induction hypothesis. Furthermore the equality relation

$$\mathbf{E}^Q[\mathbf{1}_{A \cap \{\theta_{k-1}=k-1\}} f_{k-1}] + \mathbf{E}^Q[\mathbf{1}_{A \cap \{\theta_{k-1} \geq k\}} \mathbf{E}^Q[v_k | \mathcal{F}_{k-1}]] = \mathbf{E}^Q[\mathbf{1}_A v_{k-1}]$$

must necessarily hold since $v_{k-1} = f_{k-1}$ for the event $\{\theta_{k-1} = k-1\}$ and $v_{k-1} = \mathbf{E}^Q[v_k | \mathcal{F}_{k-1}]$ for the complementary event $\{\theta_{k-1} \geq k\}$ where $v_{k-1} \neq f_{k-1}$. As a final consequence, conditioning the expectation on \mathcal{F}_{k-1} and using the fact that the above relation holds for all events $A \in \mathcal{F}_{k-1}$ results in

$$v_{k-1} = \mathbf{E}^Q [f_{\theta_{k-1}} | \mathcal{F}_{k-1}]$$

which completes the proof. □

Using this lemma we can now construct an optimal stopping time for the problem (4.2) and thereby obtain an explicit expression for the price process of the Bermudan swaption in terms of the Snell envelope, v .

Theorem 2. *The price process, V , satisfies the relation*

$$V = \mathbf{E}^Q [f_{\theta_0} | \mathcal{F}_0] = v_0$$

where v_0 is determined by the recursion

$$\begin{aligned} v_M &= f_M \\ v_i &= \max \left\{ f_i, \mathbf{E}^Q [v_{i+1} | \mathcal{F}_i] \right\}, \quad i = 0, 1, \dots, M-1. \end{aligned}$$

and the optimal $\underline{\mathcal{F}}^0$ -stopping time θ_0 is given by

$$\theta_0 = \min \left\{ k = 0, 1, \dots, M : f_k = v_k \right\}.$$

Proof. The proof follows from Lemma 1 since

$$v_0 = \mathbf{E}^Q [f_{\theta_0} | \mathcal{F}_0] \geq \mathbf{E}^Q [f_\tau | \mathcal{F}_0], \quad \tau \in \mathcal{S}_0.$$

together with the definition of V implies that

$$V = \sup_{\tau \in \mathcal{S}_0} \mathbf{E}^Q [f_\tau | \mathcal{F}_0] = \mathbf{E}^Q [f_{\theta_0} | \mathcal{F}_0] = v_0. □$$

The theorem above provides an optimal stopping time for the problem (4.2) and states the solution of the problem as a Snell envelope for the process f . In the next chapter we discuss how this solution can be used in practice to price Bermudan swaptions.

Chapter 5

Using the pricing envelope in practice

According to the theorem presented in the previous chapter, a Bermudan swaption could be priced through a “back stepping method”, i.e. a method where the price $V = v_0$ is calculated by iteratively constructing the process v in descending order of time. A problem with this procedure is that it typically results in complex and time consuming computations, since it relies on calculating chains of expected values represented by multidimensional integrals. Calculating these integrals using numerical methods is known to be a notoriously cumbersome task (see [6] for further discussion).

In order to avoid this problem a simplified pricing method based on using discrete approximations of the continuous processes v and f is commonly used in practice. The discrete approximations are obtained by restricting the evolution of the stochastic processes to tree structures, which only allow each process to evolve to one of a finite number of states at a given point in time. When applying these approximations, the above discussed continuous integrals are transformed to ordinary sums which considerably lowers the complexity of the calculations.

The focus of this chapter is to study how one of these tree methods can be used to price Bermudan swaptions by restricting the evolution of the short rate process to that of a recombining binomial tree.

5.1 Introducing a new pricing envelope

Using the results presented in the previous chapter together with an adjustment of the set of admissible exercise dates, the initial price of a Bermudan swaption is given as

$$\mathbf{BPS}(T_0^e; \kappa, \delta, R_s) = v_0$$

where v is redefined as

$$\begin{aligned} v_i &= \max\left\{f_i, \mathbf{E}^Q[v_{i+1} | \mathcal{F}_{T_i^e}]\right\}, \quad i = 0, 1, \dots, M-1 \\ v_M &= f_M \end{aligned}$$

and the process f is identified as

$$f_i = e^{-\int_{T_0^e}^{T_i^e} r_s ds} \max\left\{\mathbf{PS}(T_i^e; \kappa, \delta, R_s), 0\right\}, \quad i = 0, 1, \dots, M.$$

We now introduce the envelope $u = (u_i \in \mathcal{F}_{T_i^e})_{i=0}^M$ given by

$$\begin{aligned} u_i &= \max\left\{F_i, \mathbf{E}^Q\left[e^{-\int_{T_i^e}^{T_{i+1}^e} r_s ds} u_{i+1} | \mathcal{F}_{T_i^e}\right]\right\}, \quad i = 0, 1, \dots, M-1 \\ u_M &= F_M \end{aligned} \quad (5.1)$$

where the process $F = (F_i \in \mathcal{F}_{T_i^e})_{i=0}^M$ is defined by

$$F_i = \max\left\{\mathbf{PS}(T_i^e; \kappa, \delta, R_s), 0\right\}, \quad i = 0, 1, \dots, M. \quad (5.2)$$

In order to be able to introduce the binomial tree pricing method as it is typically used in practice (see [3]), we will abandon the pricing of Bermudan swaptions through the use of the process v in favour of the process u . The following theorem verifies the legality of this alternative pricing method.

Theorem 3. *The initial price of the Bermudan swaption is given by the processes u and F defined by (5.1) and (5.2) as*

$$\mathbf{BPS}(T_0^e; \kappa, \delta, R_s) = u_0.$$

Proof. The proof is constructed by an induction argument verifying the relation

$$v_i = u_i e^{-\int_{T_0^e}^{T_i^e} r_s ds}, \quad i = 0, 1, \dots, M$$

which implies that the new pricing method is valid since $v_0 = u_0$. We begin by concluding that the relation by definition of f and F is true at the time T_M^e and furthermore make the assumption that the relation holds for

$T_M^e, T_{M-1}^e, \dots, T_{k+1}^e$ for some $k = 0, 1, \dots, M-2$. This assumption however implies that the relation is valid at T_k^e since

$$\begin{aligned}
v_k &= \max\left\{f_k, \mathbf{E}^Q[v_{k+1} | \mathcal{F}_{T_k^e}]\right\} \\
&= \max\left\{e^{-\int_{T_0^e}^{T_k^e} r_s ds} F_k, \mathbf{E}^Q\left[u_{k+1} e^{-\int_{T_0^e}^{T_{k+1}^e} r_s ds} | \mathcal{F}_{T_k^e}\right]\right\} \\
&= \max\left\{e^{-\int_{T_0^e}^{T_k^e} r_s ds} F_k, \mathbf{E}^Q\left[u_{k+1} e^{-\int_{T_0^e}^{T_k^e} r_s ds - \int_{T_k^e}^{T_{k+1}^e} r_s ds} | \mathcal{F}_{T_k^e}\right]\right\} \\
&= e^{-\int_{T_0^e}^{T_k^e} r_s ds} \max\left\{F_k, \mathbf{E}^Q\left[u_{k+1} e^{-\int_{T_k^e}^{T_{k+1}^e} r_s ds} | \mathcal{F}_{T_k^e}\right]\right\} \\
&= u_k e^{-\int_{T_0^e}^{T_k^e} r_s ds}
\end{aligned}$$

which proves the induction hypothesis and thereby the theorem. □

5.1.1 Interpretation of the envelope

As was previously stated, the price of a Bermudan swaption may be retrieved using the envelope u through the relation $\mathbf{BPS}(T_0^e; \kappa, \delta, R_s) = u_0$ where

$$\begin{aligned}
u_0 &= \max\left\{F_0, \mathbf{E}^Q\left[e^{-\int_{T_0^e}^{T_1^e} r_s ds} u_1 | \mathcal{F}_{T_0^e}\right]\right\} \\
u_1 &= \max\left\{F_1, \mathbf{E}^Q\left[e^{-\int_{T_1^e}^{T_2^e} r_s ds} u_2 | \mathcal{F}_{T_1^e}\right]\right\} \\
&\vdots \\
u_{M-1} &= \max\left\{F_{M-1}, \mathbf{E}^Q\left[e^{-\int_{T_{M-1}^e}^{T_M^e} r_s ds} u_M | \mathcal{F}_{T_{M-1}^e}\right]\right\} \\
u_M &= F_M
\end{aligned}$$

According to this recursion, we may interpret the calculation of the initial price of the swaption, u_0 , as finding the maximum value of either exercising the option immediately and thereby receiving the payoff F_0 , or postponing the exercise decision until later. The value of choosing not to exercise the option at T_0^e is according to the above relations calculated as the discounted initial value of a Bermudan swaption having its first exercise date at T_1^e . The initial value of this option, u_1 , is correspondingly calculated using the same procedure as for u_0 , i.e. by comparing which of the alternatives “exercising immediately” or “postponing the exercise decision” that is associated with the highest expected value.

By repeating this pricing procedure until reaching the last exercise date, T_M^e , the price of the Bermudan swaption is obtained through a dynamic programming relation. The value of u_M , representing the initial (initial meaning the time T_M^e) price of a swaption having its first and only exercise date at T_M^e , must necessarily

satisfy the equality $u_M = F_M$, i.e. the initial price of the option equals the value of its immediate payoff. Since u_M can be calculated explicitly, the sequence $u_{M-1}, u_{M-2}, \dots, u_0$ may be calculated as well in an iterative fashion.

Pricing Bermudan swaptions using the process u could be argued to be more intuitive than the pricing procedure based on the envelope v . Sure enough, both methods are identical in the sense that they render the same valuation by comparing which of the alternatives “immediate exercise” or “postponed exercise” that is more profitable. What separates them, however, is the way these alternatives are quantified. When using the original pricing method based on the processes v and f , the comparison is always done by discounting all values to the time T_0^e . The new method, on the other hand, always performs the discounting to the time of the comparison, which might be considered as more appealing from an intuitive point of view.

5.1.2 Extension of the envelope

It will prove useful to know the value of the process u at all settlement dates between the first and the last exercise date¹ of the Bermudan swaption. For this reason, the envelope is extended to be defined on these dates in this section. As a first step in doing so, we begin by defining the index K (cf. figure 5.1 below) as

$$K = \{l = 1, 2, \dots, N : T_l = T_M^e\} \quad (5.3)$$

and redefine the process u as $u = (u_i \in \mathcal{F}_{T_i})_{i=0}^K$. In order to calculate the value of this envelope at a certain time, we furthermore need to establish a means to identify if this time corresponds to an exercise date or not, as u_i necessarily must be valued differently than (5.1) at non-exercise dates. This is achieved by introducing the indicators E_i defined by

$$E_i = \mathbf{1}_{\{T_i \in \{T_0^e, T_1^e, \dots, T_M^e\}\}}, \quad i = 0, 1, \dots, K. \quad (5.4)$$

When $E_i = 1$, the date T_i corresponds to an exercise date and u_i is calculated in the same manner as before the extension, i.e. through equation (5.1). In the opposite case when $E_i = 0$, the variable u_i however represents the price of a Bermudan swaption having its first exercise date at a later instant than T_i , and should be calculated as the value of (necessarily) keeping the swaption until at least the time T_{i+1} . This value is obtained by excluding the alternative “immediate exercise” in expression (5.1), i.e.

$$u_i = \mathbf{E}^Q \left[e^{-\int_{T_i}^{T_{i+1}} r_s ds} u_{i+1} \middle| \mathcal{F}_{T_i} \right], \quad E_i = 0$$

¹Recall from the definition in the previous chapter that the exercise dates are assumed to coincide with settlement dates of the underlying IRS.

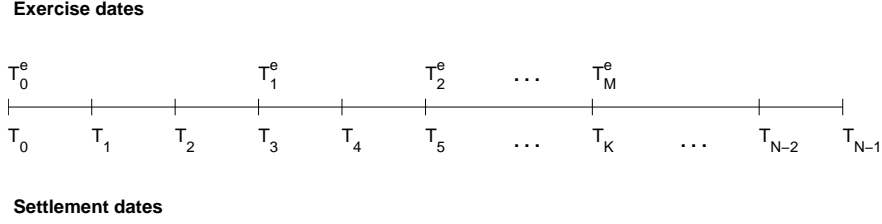


Figure 5.1: Example of a set of exercise dates of the Bermudan swaption together with its underlying IRS's set of settlement dates. Note that the settlement date T_K is defined as the last exercise date of the swaption, i.e. $T_K = T_M^e$.

Using this result, the extended process u is completely specified as

$$u_i = \begin{cases} \max\left\{F_i, \mathbf{E}^Q\left[e^{-\int_{T_i}^{T_{i+1}} r_s ds} u_{i+1} \mid \mathcal{F}_{T_i}\right]\right\}, & E_i = 1 \\ \mathbf{E}^Q\left[e^{-\int_{T_i}^{T_{i+1}} r_s ds} u_{i+1} \mid \mathcal{F}_{T_i}\right], & E_i = 0 \end{cases} \quad (5.5)$$

$$u_K = F_K$$

where the index i is defined on the set $i = 0, 1, \dots, K$ and the process F is extended to $F = (F_i \in \mathcal{F}_{T_i})_{i=0}^K$ and defined as

$$F_i = \max\left\{\mathbf{PS}(T_i; \kappa, \delta, R_s), 0\right\}, \quad i = 0, 1, \dots, K. \quad (5.6)$$

As before, the price of the Bermudan swaption is given by the value of the envelope at the time $T_0^e = T_0$, i.e. as

$$\mathbf{BPS}(T_0^e; \kappa, \delta, R_s) = u_0. \quad (5.7)$$

5.2 Pricing with a binomial tree

The approximate pricing method that is the focus of this chapter is based on restricting the evolution of the short rate process to that of a recombining binomial tree. In the following section we provide a closer examination on how this discretization affects the extended pricing procedure (5.5)-(5.6).

In order to make matters more specific, the short rate process, r , is replaced with the finite state process $\hat{r} = (\hat{r}_i \in \mathcal{F}_{T_i})_{i=0}^{N-1}$. Furthermore the notation $\hat{r}_{i,j}$ is introduced, having the interpretation that the process on the time interval $[T_i, T_{i+1})$ is in the state j , i.e. the value of \hat{r}_i is determined by the index j . Unlike all stochastic processes defined up to this point, \hat{r} is defined on the set of dates $(T_i)_{i=0}^{N-1}$, where the index $N-1$ denotes the ordinal number of the last

settlement date of the swaption's underlying IRS (cf. Chapter 3). The reason for defining the process up to this point in time is that limiting the existence of \hat{r} to a smaller partition of the time axis, causes the value process of the underlying IRS (and thereby the value process of the Bermudan swaption) to be undefined².

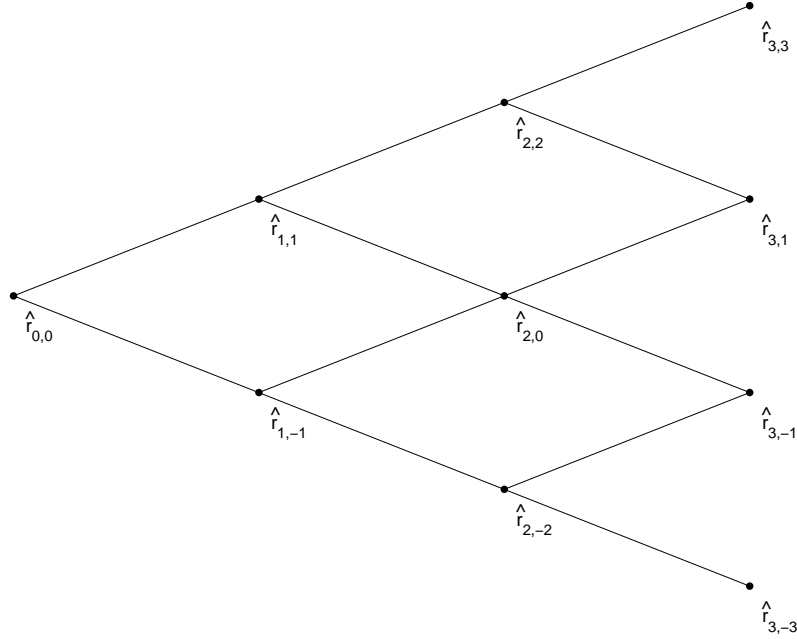


Figure 5.2: Schematic example of a recombining binomial tree evolution of the short rate process \hat{r} . The probability of transcending to a state associated with higher ordinal number equals p_u , while the probability of transition to a state with lower ordinal number is given by $1 - p_u$.

The idea behind using a recombining binomial tree to approximate the short rate process, r , is to limit the number of possible transitions at any given instant of time. This is achieved by only letting the process evolve to one of two states, $j + 1$ and $j - 1$, between the times T_i and T_{i+1} given that it is in state j at T_i (cf. figure 5.2). The probability³ of a transition to the state $j + 1$ is denoted p_u , and the probability of a transition to the $j - 1$ is given by $1 - p_u$. As the short rate is assumed to be known through observation at the time of the pricing, i.e. at $T_0 = T_0^e$, and thus only has one state, the construction of the recombining binomial tree bounds the indices i and j as

$$i = 0, 1, \dots, N - 1 \quad (5.8)$$

$$j = -i, -i + 2, \dots, i - 2, i. \quad (5.9)$$

²This follows from the fact that the value process of the IRS regardless of time contains a term corresponding to the value of a zero coupon bond with maturity T_N .

³This probability is given under the risk neutral measure Q .

5.2.1 Approximating the pricing procedure

We now turn to the question of how the processes u and F are approximated when restricting the evolution of the short rate process to that of a recombining binomial tree. As F by definition is given by

$$F_i = \max\left\{\mathbf{PS}(T_i; \kappa, \delta, R_s), 0\right\}, \quad i = 0, 1, \dots, K$$

the short rate discretization must necessarily cause an approximation, since the value process $\mathbf{PS}(T_i^e; \kappa, \delta, R_s)$ according to (3.4) is constructed as a sum of short rate dependent zero coupon bonds with varying maturities. In order to further investigate the properties of this approximation, we thus need to find out how the price process of a zero coupon bond is approximated as a consequence of introducing the binomial tree. As this task typically generates a somewhat messy notation unless the process \hat{r} is further specified, we postpone the details of this calculation to the next chapter. Until then, we settle with introducing the process $\hat{F} = (\hat{F}_i \in \mathcal{F}_{T_i})_{i=0}^K$ as the approximation of F leaving all details aside. As in the case with the approximated short rate process, $\hat{F}_{i,j}$ denotes that the process \hat{F} is in the state j on the interval $[T_i, T_{i+1})$, where j is bounded by (5.9).

It remains to investigate what effect the binomial tree has on the envelope u defined as

$$\begin{aligned} u_i &= \begin{cases} \max\left\{F_i, \mathbf{E}^Q\left[e^{-\int_{T_i}^{T_{i+1}} r_s ds} u_{i+1} \mid \mathcal{F}_{T_i}\right]\right\}, & E_i = 1 \\ \mathbf{E}^Q\left[e^{-\int_{T_i}^{T_{i+1}} r_s ds} u_{i+1} \mid \mathcal{F}_{T_i}\right], & E_i = 0 \end{cases} \\ u_K &= F_K \end{aligned}$$

where $i = 0, 1, \dots, K$. As was pointed out earlier, the reason for introducing an approximate pricing method based on a binomial tree is motivated by the fact that the process u is partly constructed from an expectation of a continuous random variable. In order to deduce the simplifications induced by this tree, we introduce the process $\hat{u} = (\hat{u}_i \in \mathcal{F}_{T_i})_{i=0}^K$ approximating u , and furthermore let $\hat{u}_{i,j}$ denote that the value of \hat{u}_i is given by the state index j on the interval $[T_i, T_{i+1})$.

By observing the above expression for u , it stands clear that we need to decide on how the stochastic integral over the short rate process (the discount factor) is valued before approximating the aforementioned expected value. This however, turns out to be a relatively easy task since the approximated short rate process \hat{r} by definition is constant over the period $[T_i, T_{i+1})$ at any given state j , meaning that the integral reasonably should be discretized as

$$e^{(T_{i+1}-T_i)\hat{r}_i} \in \mathcal{F}_{T_i}.$$

Using this approximation, the process \hat{u} should then suitably discretize u at all instants of time and states as

$$\begin{aligned}\hat{u}_{i,j} &= \begin{cases} \max\left\{\hat{F}_{i,j}, e^{(T_{i+1}-T_i)\hat{r}_{i,j}} (p_u\hat{u}_{i+1,j+1} + (1-p_u)\hat{u}_{i+1,j-1})\right\}, & E_i = 1 \\ e^{(T_{i+1}-T_i)\hat{r}_{i,j}} (p_u\hat{u}_{i+1,j+1} + (1-p_u)\hat{u}_{i+1,j-1}), & E_i = 0 \end{cases} \\ \hat{u}_{K,j} &= \hat{F}_{K,j}\end{aligned}\tag{5.10}$$

where the indices i and j are bounded by equations (5.8)-(5.9). Besides relying upon the previously stated discretization of the discount factor, this approximation uses the fact that restricting the short rate evolution to that of a recombining binomial tree implies that the expectation could be discretely approximated by a weighted sum. This sum consists of the terms $\hat{u}_{i+1,j+1}$ and $\hat{u}_{i+1,j-1}$ representing the two possible states the process \hat{u} may transcend to at the time T_{i+1} .

5.2.2 The approximation error

Pricing Bermudan swaptions using a recombining binomial tree (or any other tree structure for that matter) naturally generates a valuation error as a consequence of limiting the evolution of all involved processes to a finite set of states. The magnitude of this error does however not only depend on the number of states each process is restricted to. A large contribution of the error is in fact due to the fineness of the mesh on which the approximated short rate process is defined, since this process is assumed to be constant on every interval on the mesh. Even though, we have defined the process \hat{r} to exist only on the settlement dates of the Bermudan swaption's underlying IRS throughout this chapter, this particular partition is in no way the only alternative available when constructing \hat{r} . As a matter of fact, any mesh having equidistant⁴ nodes will do as long as it reaches as far as to the settlement date of the last transaction of the IRS, i.e. the time T_{N-1} . The choice of introducing the approximate pricing method based on a mesh consisting solely of settlement dates, is only motivated by the fact that it eases up the notation and keeps the focus on the essential steps of the method. Though this partition will be used in the next chapter as well for the same reasons, the reader is advised to employ a finer mesh when using the method in practical situations.

As a final comment, it's worth pointing out that choosing to refine the mesh on which the process \hat{r} is constructed, not only limits the pricing error due to the discretization of the time axis. By increasing the number of nodes, the number of states represented in the binomial tree multiplies. This causes a reduction in the error induced by the restriction set on the evolution of the process as well.

⁴In order to produce a binomial tree with the property of being recombining, all points on which the approximated short rate process is defined, need to be equidistantly spaced.

Chapter 6

Pricing according to the Ho-Lee short rate model

The purpose of this chapter is to make the previously introduced approximate pricing method explicit by specifying a model of the short rate process. The model in question was suggested in the seminal article [7] and has since then become widely celebrated as it due to its parameterization, was the first model of the short rate guaranteeing absence of arbitrage.

Besides introducing the Ho-Lee model and presenting its valuation of Bermudan swaptions, this chapter furthermore presents a way of calibrating the parameters of the model given limited observed market data as discussed in the introduction of this thesis. Despite the fact that the Ho-Lee model was introduced some 20 years ago and lack many desirable features of today's short rate models (see [3] for a survey), it still is a competitive alternative when dealing with situations such as these due to the simplicity of its construction.

6.1 The Ho-Lee short rate model

When reading about the Ho-Lee model in recent literature on financial mathematics, the short rate process is usually presented having continuous time dynamics. In the original setup presented by the authors, the dynamics were different however, as the short rate process was defined in a discrete setting and having an evolution given by a recombining binomial tree. As the previously proposed approximate pricing method is constructed through the use of a binomial tree, the model used in this chapter will be constructed similarly to the original Ho-Lee model.

Using the same notation and partition of the time axis as in the previous chapter, the short rate process in discrete time, $\hat{r} = (\hat{r}_i \in \mathcal{F}_{T_i})_{i=0}^{N-1}$, is defined according to the Q -dynamics

$$\hat{r}_i = \theta_i + \sqrt{\delta\sigma^2} \sum_{k=1}^i B_k^Q, \quad i = 0, 1, \dots, N-1 \quad (6.1)$$

where δ (cf. Chapter 3) represents the equidistant time period between two settlement dates of the underlying IRS of the Bermudan swaption and $(B_k^Q)_{k=1}^{N-1}$ is a stochastic sequence consisting of IID symmetric Bernoulli random variables under the measure Q , i.e.

$$\begin{aligned} Q(B_k^Q = +1) &= 1/2, \\ Q(B_k^Q = -1) &= 1/2, \end{aligned} \quad k = 1, 2, \dots, N-1. \quad (6.2)$$

In the above definition, the volatility, σ , of short rate process is defined as a strictly positive deterministic constant, i.e. $\sigma > 0$, while the drift of the process, $(\theta_i)_{i=0}^{N-1}$, is defined as a deterministic sequence on \mathbb{R} .

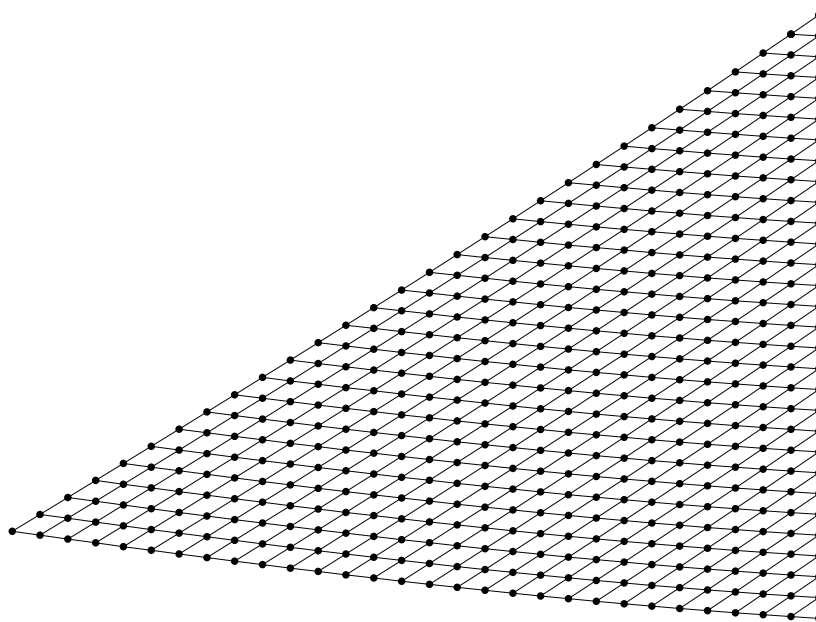


Figure 6.1: Example of the Ho-Lee short rate evolution in a binomial tree.

Using the definition (6.1), the construction (cf. figure 6.1 for a typical example) of the process \hat{r} implies that $\hat{r}_{i,j}$ is given by

$$\hat{r}_{i,j} = \theta_i + j\sqrt{\delta\sigma^2}, \quad \begin{aligned} i &= 0, 1, \dots, N-1 \\ j &= -i, -i+2, \dots, i-2, i \end{aligned} \quad (6.3)$$

where $\hat{r}_{i,j}$ as earlier denotes the value of the short rate process at time T_i when its state is given by the index j . We may furthermore observe that the Ho-Lee

model through its use of symmetric Bernoulli distributed variables, specifies the transition probability from state j to state $j + 1$ (and $j - 1$ for that matter) as $p_u = 1/2$ regardless of the time of the transition.

6.2 Pricing the Bermudan swaption

Given the short rate model (6.1) we are now able to explicitly specify the pricing procedure of Bermudan swaptions according to the method introduced in the previous chapter. Using the same notation as earlier together with the above definitions of $\hat{r}_{i,j}$ and p_u the approximate envelope u is calculated as

$$\begin{aligned} \hat{u}_{i,j} &= \begin{cases} \max\left\{\hat{F}_{i,j}, \frac{1}{2} e^{\delta\theta_i+j\sigma\delta^{3/2}} (\hat{u}_{i+1,j+1} + \hat{u}_{i+1,j-1})\right\}, & E_i = 1 \\ \frac{1}{2} e^{\delta\theta_i+j\sigma\delta^{3/2}} (\hat{u}_{i+1,j+1} + \hat{u}_{i+1,j-1}), & E_i = 0 \end{cases} \\ \hat{u}_{K,j} &= \hat{F}_{K,j} \end{aligned} \quad (6.4)$$

where the indices i, j and K are defined as

$$\begin{aligned} i &= 0, 1, \dots, K \\ j &= -i, -i + 2, \dots, i - 2, i \\ K &= \left\{l = 1, 2, \dots, N : T_l = T_M^e\right\}. \end{aligned} \quad (6.5)$$

Since the process \hat{F} up to this point has not been defined, the above pricing expression is yet to be complete. As was discussed previously, specifying this process requires the calculation of the price of zero coupon bonds, which up until now has been postponed. Having parameterized the short rate model, this can however be with more ease and we begin by introducing the process $\hat{p}^m = (\hat{p}_i^m \in \mathcal{F}_{T_i})_{i=0}^N$ denoting the approximated value process at time T_i of a zero coupon bond having maturity T_m , where

$$i \leq m, \quad i, m = 0, 1, \dots, N. \quad (6.6)$$

Using the same convention as for the process \hat{r} , the variable $\hat{p}_{i,j}^m$ denotes that the price process at time T_i of the zero coupon bond has a value corresponding to the state j . As a consequence of having specified the process \hat{r} , the value of $\hat{p}_{i,j}^m$ is calculated as (the interested reader is referred to the derivation in the appendix)

$$\hat{p}_{i,j}^m = e^{-\delta \sum_{k=i}^{m-1} \theta_k - \delta^{3/2} \sigma j(m-i)} \prod_{k=i+1}^{m-1} \cosh\left(\delta^{3/2} \sigma(m-k)\right) \quad (6.7)$$

which using the expression for the price process of the underlying IRS of the Bermudan swaption as described by equation (3.4) determines \hat{F} as

$$\hat{F}_{i,j} = \max \left\{ \kappa \left(\hat{p}_{i,j}^i - \hat{p}_{i,j}^N - \delta R_s \sum_{k=i+1}^N \hat{p}_{i,j}^k \right), 0 \right\}$$

Simplifying the above expression somewhat finally results in

$$\hat{F}_{i,j} = \max \left\{ \kappa \left(1 - \hat{p}_{i,j}^N - \delta R_s \sum_{k=i+1}^N \hat{p}_{i,j}^k \right), 0 \right\} \quad (6.8)$$

which completely specifies the process \hat{u} and thereby the price of the Bermudan swaption as

$$\mathbf{BPS}(T_0^e; \kappa, \delta, R_s) = \hat{u}_{0,0} \quad (6.9)$$

with \hat{u} and \hat{F} constructed using relations (6.4)-(6.8).

6.3 Calibrating the model using limited data

As was stated in the introduction to this chapter, the Ho-Lee short rate model may be calibrated using relatively simple and easily accessible market data. In this section we aim to motivate this assertion by assign values to all parameters of the model using only:

- The initial term structure, i.e. the curve $p^* = \{p^*(T_0, T) : T \geq T_0\}$ observed on the market.
- A counter party's valuation at $T_0^e = T_0$ of the Bermudan swaption, i.e. the price \mathbf{BPS}^* .

We begin by determining what role the initial term structure plays in the calibration procedure. Using relation (6.7), the initial price of a zero coupon bond having maturity T_m is given by

$$\hat{p}_{0,0}^m = e^{-\delta \sum_{k=0}^{m-1} \theta_k} \prod_{k=1}^{m-1} \cosh(\delta^{3/2} \sigma(m-k))$$

where we have made use of the fact that the only possible state of the short rate process \hat{r} at T_0 is given by $j = 0$. By forming the quotient of this price at two consecutive maturity dates, T_k and T_{k+1} , we furthermore observe that

$$\frac{\hat{P}_{0,0}^{k+1}}{\hat{P}_{0,0}^k} = e^{-\delta\theta_k} \cosh(k\sigma\delta^{3/2}).$$

Rearranging the terms in this relation and inserting prices stored in the observed initial term structure, results in that the sequence of drift parameters, $(\theta_i)_{i=0}^{N-1}$, reasonably should be estimated as

$$\theta_i^{est} = \frac{1}{\delta} \left[\log \left(\frac{p^*(T_0, T_i)}{p^*(T_0, T_{i+1})} \right) + \log \left(\cosh(i\sigma^{est}\delta^{3/2}) \right) \right], \quad (6.10)$$

where $(\theta_i^{est})_{i=0}^{N-1}$ denotes the estimation of the sequence $(\theta_i)_{i=0}^{N-1}$ and σ^{est} denotes the estimation of the volatility of the short rate process. In order to be able to use this expression we obviously need a way of calculating σ^{est} . This task is however quite easily dealt with using the given valuation of the Bermudan swaption as this price implicitly determines the volatility as the solution to the non-linear equation

$$\mathbf{BPS} \left(T_0^e; \kappa, \delta, R_s, \sigma^{est}, (\theta_i^{est}(\sigma^{est}))_{i=0}^{N-1} \right) = \mathbf{BPS}^* \quad (6.11)$$

which expresses the drift of the short rate process using relation (6.10) thus leaving σ^{est} as the only unknown parameter. This equation is quite easily solved using a numerical procedure, for instance the secant method (see the description in [6]). As σ typically lies somewhere in the range of $0.5 \cdot 10^{-2}$ to $1 \cdot 10^{-2}$ in most interest rate markets, qualified starting values for a numerical solver could be extracted from this interval.

Solving equations (6.10)-(6.11) completely determines all parameters of the short rate model based on the limited input data. As the drift of the short rate process is calculated so as to accurately reproduce the initial term structure, the short rate model obtained using the above presented calibration procedure, is furthermore guaranteed to be absent of arbitrage possibilities¹.

6.4 Disadvantages of the model

Unlike many of the short rate models introduced in the last decade, the Ho-Lee model lacks a property that is commonly referred to as “mean reversion”. This property guarantees that the short rate process constantly evolves towards a time varying equilibrium and significantly reduces the probability of the event that the process becomes negative or grows unreasonably large. As the Ho-Lee

¹This statement is on a finer note actually only partly true as only the prices that are actually used in the calibration procedure are correctly represented by the short rate model. All other prices will only be reproduced approximately. In order to reduce the approximation error the short rate process needs to be defined on a finer partition of the time axis (cf. the discussion in the previous chapter).

model lacks mean reversion it thus, in comparison to more recent short rate models, has a relatively high probability of exploding or turning negative at some point in time.

Another disadvantage of the Ho-Lee model follows from the fact that it assumes that the volatility of the short rate is constant in time as well as deterministic. Several studies (see [3] and references therein) have shown that volatility parameterizations such as these are over-simplistic and unable to for instance handle notorious problems such as “volatility smiles”.

Chapter 7

Numerical results

In the following chapter we study the behaviour of the pricing method proposed in the previous chapter in terms of its convergence, calibration to market data and sensitivity to changes in parameter values. Two Bermudan payer swaptions serve as the basis for our conclusions, the first one having an underlying IRS with its last transaction in 5 years (referred to as the 5 year contract) and the second one having an underlying IRS with its last transaction in 10 years (referred to as the 10 year contract). As was pointed out in chapter 3, the presented definitions of IRS agreements and Bermudan swaptions were not given in their most general (or for that matter common) form. In order to construct the two swaptions studied in this chapter according to typical market practice, we thus deviate from our earlier framework somewhat. More specifically, we allow the underlying IRS contracts of both options to make floating rate payments four times a year and fixed rate payments once a year. Furthermore, both swaptions may only be exercised once a year, namely a couple of days before the fixed leg settles. Defining the two Bermudan swaptions according to these criterions, obviously calls for an extension of the results derived up to this point. This extension may however be performed with relative ease since the fundamental properties of the pricing procedure still apply. All that needs to be reformulated is the set of dates on which the pricing envelope and short rate process are defined.

All pricing and calibration is performed on the first exercise date of the two options. The strike rates of the options are calculated as the par swap rates at the same date ($R_s = 3.475\%$ for the 5 year swaption and $R_s = 4.120\%$ for the 10 year swaption). Finally, the curve containing the initial term structure used for obtaining the short rate drift process, $(\theta_i^{est})_{i=0}^{N-1}$, as described in the previous chapter was calculated implicitly using market prices of Swedish IRS contracts and a bootstrapping technique (see [3] for an introduction).

7.1 Convergence of the pricing procedure

Setting the short rate volatility to $\sigma = 0.0075^1$ and varying the equidistant time step, δ , used when discretizing the short rate process, the convergence of the pricing procedure is given by figure 7.1.

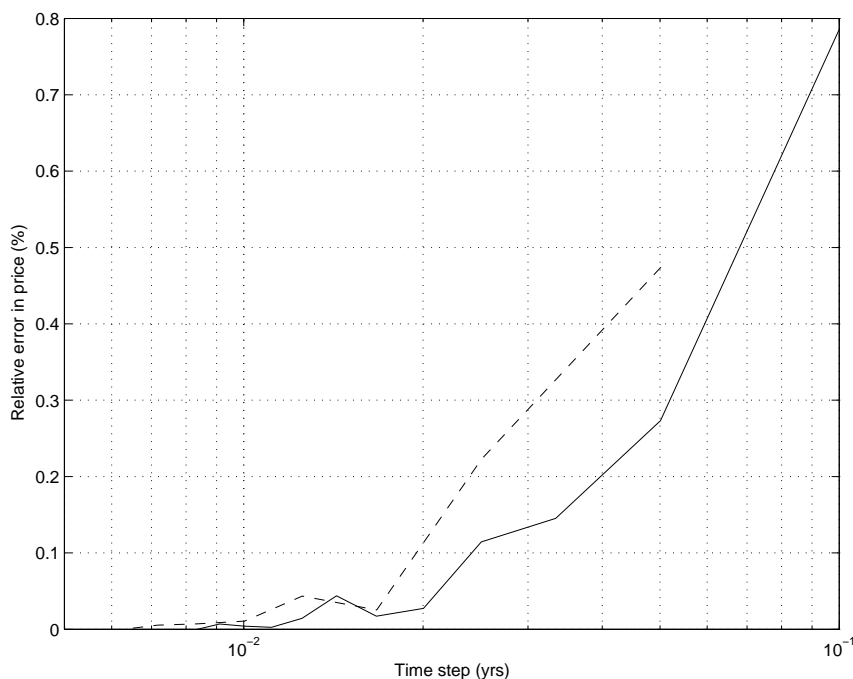


Figure 7.1: Convergence of the pricing procedure measured in relative error when approximating the short rate process with a varying time discretization. The solid curve corresponds to the pricing of the 10 year contract (valued at 5.781% of the notional amount, κ) and the dashed curve to the pricing of the 5 year contract (valued at 2.424% of the notional amount, κ).

As can be observed in the figure, the relative error in price for both contracts decreases steadily when shortening the time step. The error falls almost immediately below the level 0.5% and when reducing the time step to the magnitude of a couple of days (more specifically $\delta = 0.01$) both contracts are valued with a relative error at approximately 0.01%. Comparing these results with the time consumption diagram below, figure 7.2, the pricing of Bermudan swaption contracts through a Ho-Lee binomial tree, could be claimed to produce results with satisfactory accuracy and speed in the context of approximate valuation. In order to retrieve results with an accuracy in the range of 0.5%, the pricing procedure requires less than a second of computation time for both contracts and obtaining results with a relative error less than 0.1% takes approximately 20 seconds for the 10 year contract and 1 second for the 5 year contract.

¹According to market experience, the short rate volatility typically lies somewhere in the interval 0.5% – 1.0%, making the average value 0.75% a reasonable choice.

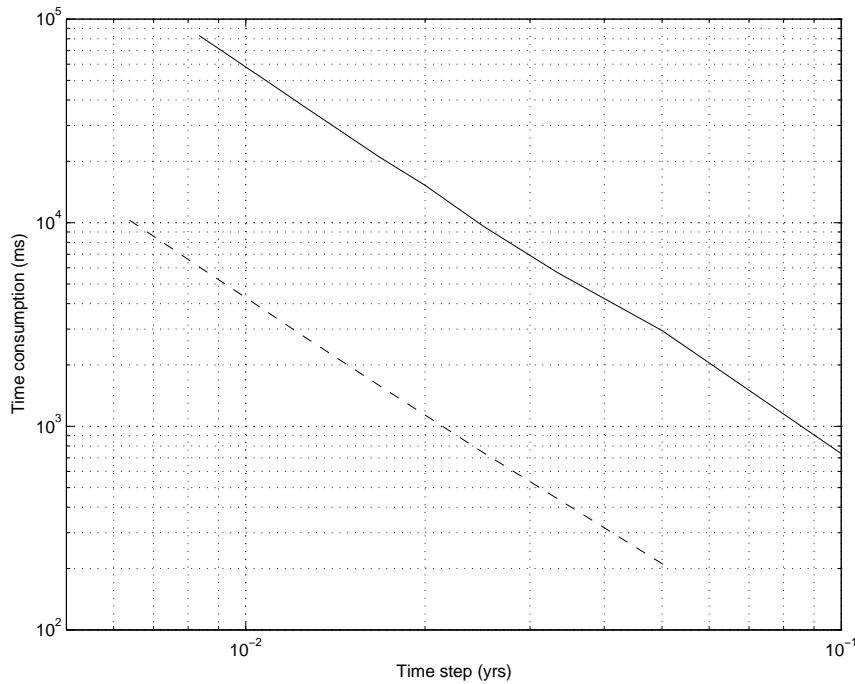


Figure 7.2: Time consumption of the pricing procedure measured in ms when approximating the short rate process with a varying time discretization. The solid curve corresponds to the pricing of the 10 year contract and the dashed curve to the pricing of the 5 year contract.

By studying figure 7.2 we can furthermore conclude that the pricing procedure has quadratic complexity in the sense that halving the time step, δ , results in about a four times larger time consumption of the pricing procedure.

7.2 Convergence of the calibration procedure

By using accurate price data calculated in the same setting as in the previous section, i.e. under the assumption that $\sigma = 0.0075$, the convergence of the calibration procedure was measured by calculating the error in implied volatility when varying the short rate process discretization. The results are presented in figure 7.3 for both contracts.

As can be observed in the figure, the implied volatilities calculated by the calibration procedure converge to the true value when decreasing the time step used when discretizing the short rate process. The convergence of the relative error is however fairly slow as figure 7.4 suggests. For instance, at time steps corresponding to an error in the magnitude of 0.5%, the calibrating procedure requires about 30 seconds of computation time when calibrating to the price of the 10 year swaption and approximately 10 seconds of computation time when

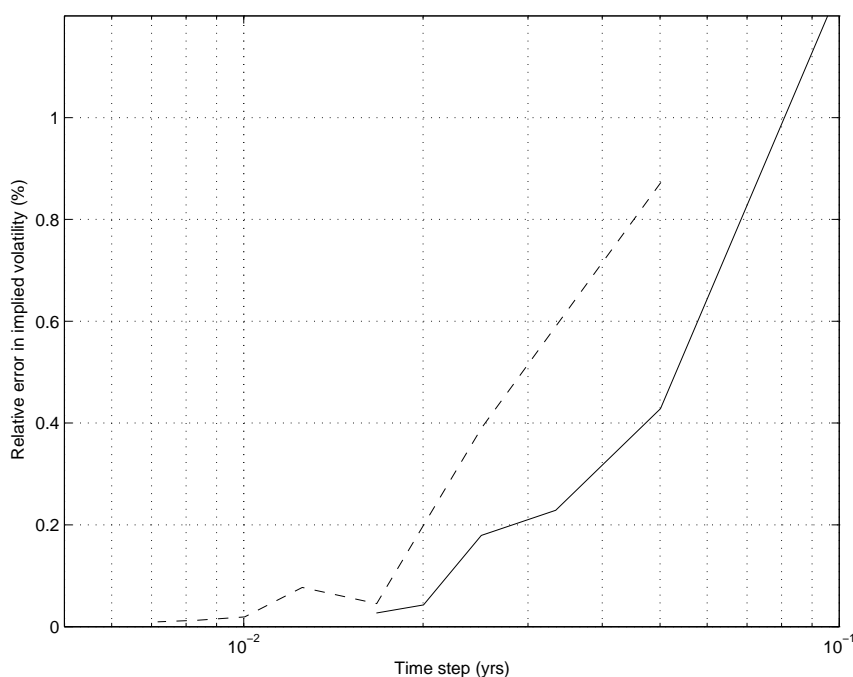


Figure 7.3: Convergence of the calibration procedure measured as relative error in implied volatility when approximating the short rate process with a varying time discretization. The solid curve corresponds to results obtained when calibrating to the price of the 10 year contract and the dashed curve when calibrating to the price of the 5 year contract.

using the price of the 5 year swaption. The complexity of the method is as in the case of the pricing method quadratic, although with a significantly larger growth factor.

7.3 Sensitivity measures

In the following section, we present some results on the pricing method's sensitivity in changes in input data. As the results are more or less identical for both swaptions, we only present the sensitivity measures obtained when working with the 10 year swaption in favour of brevity. All calculations have been performed using the equidistant short rate discretization $\delta = 0.01$.

7.3.1 Price sensitivity w.r.t. the short rate volatility

In order to get an understanding of what effect the relatively slow convergence of the calibration procedure has on the price of the contract, it is interesting to study the difference in valuation produced by the pricing procedure when vary-

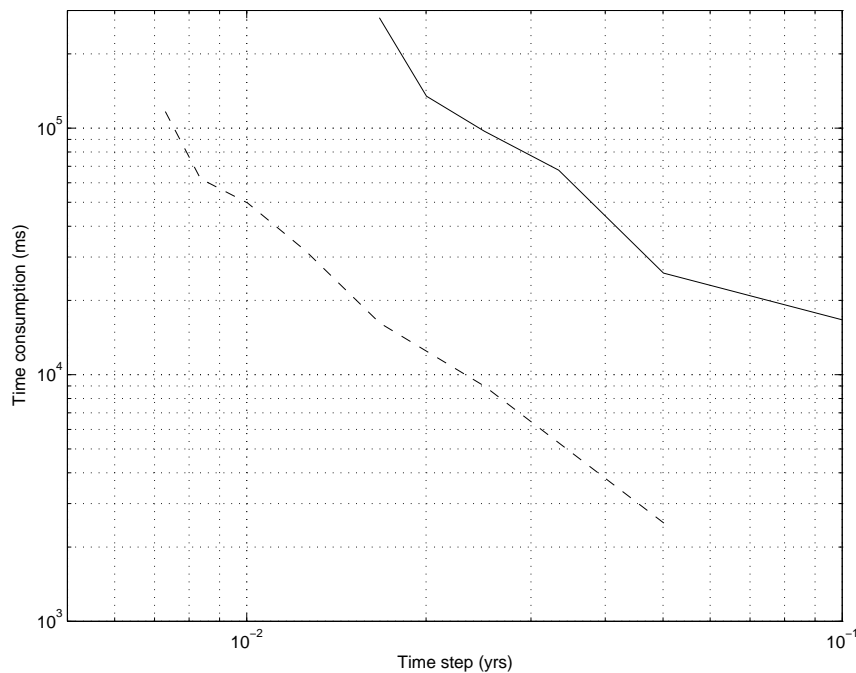


Figure 7.4: Time consumption of the calibration procedure measured in ms when approximating the short rate process with a varying time discretization. The solid curve corresponds to results obtained when calibrating to the price of the 10 year contract and the dashed curve when calibrating to the price of the 5 year contract.

ing the short rate volatility. Figure 7.5 depicts this change in value measured as per cent of the notional amount of the underlying IRS when varying the volatility using $\sigma = 0.0075$ as reference. By studying the figure it becomes clear that the valuation calculated by the pricing procedure varies linearly with the change in volatility. This result furthermore suggests that the calibration equation (6.11) should be fairly straightforward to solve using numerical methods, e.g. the secant method proposed earlier.

Knowing the accuracy and speed of the calibration method as well as the pricing procedure's sensitivity to changes in the short rate volatility, this sensitivity measure could be used as an indicator suggesting whether a holder should or should not use the proposed pricing and calibration method depending on the required accuracy in pricing.

7.3.2 Price sensitivity w.r.t. the initial term structure

Another sensitivity measure often of interest to owners of Bermudan swaptions is the so called Δ -measure², which calculates the change in the price of the

²This measure is not to be confused with the Δ -measure used in the context of pricing stock options (the interested reader is referred to [2]).

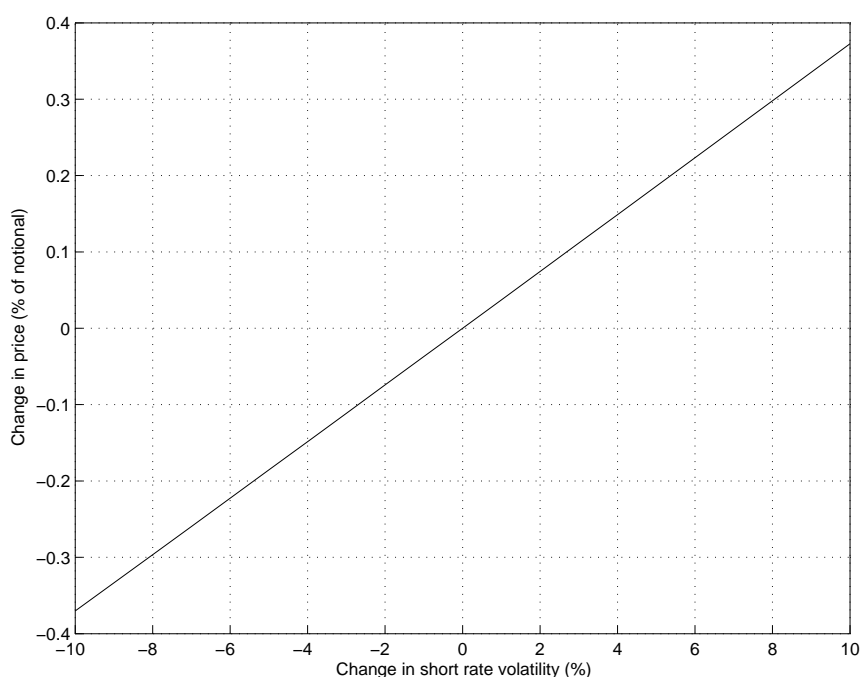


Figure 7.5: Change in price of the 10 year contract measured as per cent of the notional amount, κ , of the underlying IRS when varying the short rate volatility using $\sigma = 0.0075$ as reference.

option due to a perturbation of the effective zero rates given by the initial term structure. Typically the perturbation consists of a constant additive increment³ in the range of a few basis points, where a basis point is defined as 10^{-4} . Figure 7.6 shows this change in price when varying the increment using the short rate volatility $\sigma = 0.0075$. As is the case when measuring the sensitivity in price towards changes in volatility, the figure reveals a linear relation between the Δ -value and the size of the increment.

The Δ -measure plays a key role for the owner of a Bermudan swaption as it is used for both producing risk estimates as well as constructing hedging positions. Knowing that the value of the short rate volatility very well may alter over time due to market changes and that the calibration procedure converges quite slowly (and therefore most likely will produce implied volatilities associated with a non-negligible error) it is therefore of interest to study how the Δ -value varies when pricing the swaption using different short rate volatilities. In figure 7.7 the change in price due to a lift corresponding to one basis point is plotted for different choices of short rate volatilities. As can be observed in the figure, the absolute difference in Δ -value increases slightly as the difference in short rate volatility becomes larger.

³Letting $y^*(0, T)$ denote a market observed zero rate with maturity T , the Δ -value corresponding to an increment α measures the change in price of the Bermudan swaption when letting $y^*(0, T) \leftarrow y^*(0, T) + \alpha$ for all $T \geq 0$.

More specifically, the change in the Δ -measure is in the magnitude of 0.001% of the notional amount and is thus quite minor.

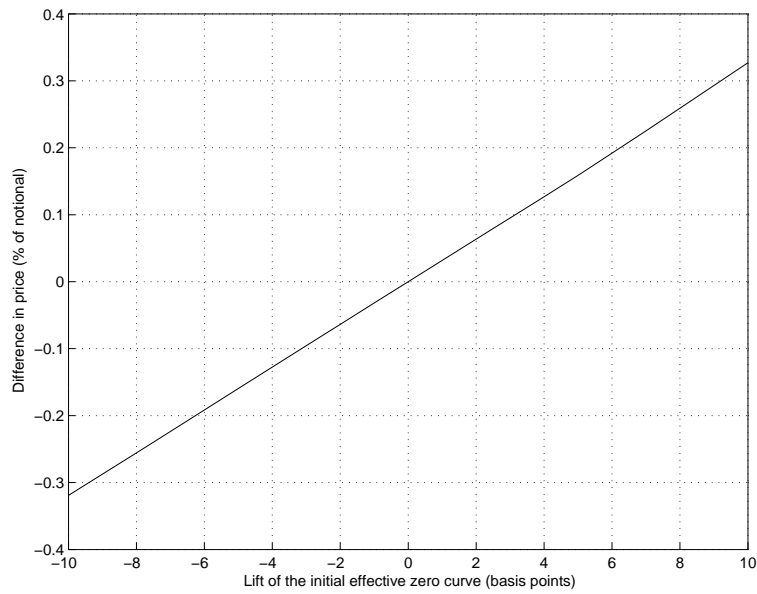


Figure 7.6: Change in price of the 10 year contract measured as per cent of the notional amount, κ , of the underlying IRS when adjusting the initial effective zero curve with a constant increment measured in basis points.

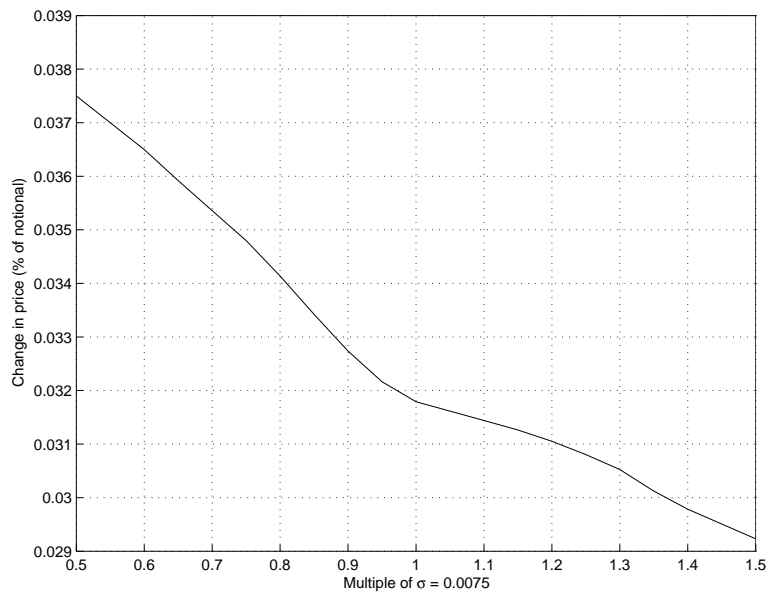


Figure 7.7: Change in price of the 10 year contract measured as per cent of the notional amount, κ , of the underlying IRS when adjusting the initial effective zero curve with a constant increment of one basis point. The difference in price is plotted versus the short rate volatility used by the pricing procedure.

Chapter 8

Summary and conclusions

This thesis has focused on both theoretical and practical aspects of pricing Bermudan swaptions. First of all, an exact pricing formula in terms of a Snell envelope has been derived by solving the optimal stopping problem of pricing the contract through risk neutral valuation. As this formula has no immediate use in practical situations due to the fact that it relies on the evaluation of multidimensional integral expressions, a general approximate valuation procedure based on a recombining binomial tree approximation of the short rate process has been proposed.

Secondly, an explicit pricing formula suited for producing estimate prices of Bermudan swaptions has been proposed by specializing the approximate valuation procedure to the Ho-Lee short rate model. Apart from being computationally fast, this procedure also has the benefit of being easily calibrated to market data as its only required input consists of observed market prices of zero coupon bonds (an observed discount function) as well as a previous valuation of a Bermudan swaption contract. These properties make the pricing procedure particularly useful in situations where estimate prices need to be obtained quickly using limited market data. In situations such as these, methods used for pricing Bermudan swaptions with high accuracy usually require computationally intense procedures when calibrating their parameters to market data. In a worst case scenario some of these methods might not even be able to produce any price at all due to the lack of input needed for calibration.

Although some results concerning the behaviour of the specialized pricing method have been presented, a topic that remains to be studied is the accuracy of this method in terms of calculated price and sensitivity measures compared to more sophisticated pricing procedures using closer approximations of the pricing problem as well as larger sets of observed market data when calibrating. Knowing the limitations of the Ho-Lee model (cf. the discussion in Chapter 6), it is however reasonable to expect that the prices rendered by the suggested pricing procedure only should be considered as estimates of the true value.

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Appendix A

Pricing zero coupon bonds using the Ho-Lee binomial tree

In the following chapter, we provide a derivation of the valuation of zero coupon bonds when specifying the dynamics of the short rate process to that suggested by the Ho-Lee model. Using risk neutral valuation, the value, $\hat{p}_{i,j}^m$, at the time T_i and state j of a zero coupon bond having maturity T_m is calculated as

$$\begin{aligned}\hat{p}_{i,j}^m &= \mathbf{E}^Q \left[e^{-\left(\hat{r}_i(T_{i+1}-T_i) + \hat{r}_{i+1}(T_{i+2}-T_{i+1}) + \dots + \hat{r}_{m-1}(T_m - T_{m-1})\right)} \middle| \mathcal{F}_{T_i} \right] \\ &= \mathbf{E}^Q \left[e^{-\delta(\hat{r}_i + \hat{r}_{i+1} + \dots + \hat{r}_{m-1})} \middle| \mathcal{F}_{T_i} \right]\end{aligned}$$

where the relation between the indices i , j and m is given by

$$\begin{aligned}i, m &= 0, 1, \dots, N \\ i &\leq m \\ j &= -i, -i + 2, \dots, i - 2, i.\end{aligned}$$

Using the definition of the process \hat{r} given by equation (6.1), we furthermore get

$$\begin{aligned}
 \hat{p}_{i,j}^m &= \mathbf{E}^Q \left[e^{-\delta \sum_{k=i}^{m-1} \theta_k} e^{-\delta^{3/2} \sigma \left(\sum_{k=1}^i B_k^Q + \sum_{k=1}^{i+1} B_k^Q + \dots + \sum_{k=1}^{m-1} B_k^Q \right)} \middle| \mathcal{F}_{T_i} \right] \\
 &= e^{-\delta \sum_{k=i}^{m-1} \theta_k} \mathbf{E}^Q \left[e^{-\delta^{3/2} \sigma \left(\sum_{k=1}^i B_k^Q + \sum_{k=1}^{i+1} B_k^Q + \dots + \sum_{k=1}^{m-1} B_k^Q \right)} \middle| \mathcal{F}_{T_i} \right] \\
 &= e^{-\delta \sum_{k=i}^{m-1} \theta_k} \mathbf{E}^Q \left[e^{-\delta^{3/2} \sigma (m-i) \sum_{k=1}^i B_k^Q} e^{-\delta^{3/2} \sigma \left(\sum_{k=i+1}^{i+1} B_k^Q + \dots + \sum_{k=i+1}^{m-1} B_k^Q \right)} \middle| \mathcal{F}_{T_i} \right].
 \end{aligned}$$

Since the value process at the time T_i is in state j , it must necessarily hold that

$$\sum_{k=1}^i B_k^Q = j \quad Q\text{-a.s.}$$

and that

$$\left\{ \sum_{k=1}^i B_k^Q = j \right\} \in \mathcal{F}_{T_i}.$$

which implies that the exponent storing this sum may be moved outside of the expectation, i.e.

$$\hat{p}_{i,j}^m = e^{-\delta \sum_{k=i}^{m-1} \theta_k - \delta^{3/2} \sigma j (m-i)} \mathbf{E}^Q \left[e^{-\delta^{3/2} \sigma \left(\sum_{k=i+1}^{i+1} B_k^Q + \dots + \sum_{k=i+1}^{m-1} B_k^Q \right)} \middle| \mathcal{F}_{T_i} \right].$$

We proceed with the calculation by only focusing on the expectation. Restructuring the summation results in

$$\mathbf{E}^Q \left[e^{-\delta^{3/2} \sigma \left(\sum_{k=i+1}^{i+1} B_k^Q + \dots + \sum_{k=i+1}^{m-1} B_k^Q \right)} \middle| \mathcal{F}_{T_i} \right] = \mathbf{E}^Q \left[e^{-\delta^{3/2} \sigma \sum_{k=i+1}^{m-1} (m-k) B_k^Q} \middle| \mathcal{F}_{T_i} \right]$$

As the stochastic sequence $(B_k)_{k=i+1}^{m-1}$ by definition consists of IID symmetric Bernoulli random variables, the expectation is finally calculated as the product

$$\begin{aligned}
 \mathbf{E}^Q \left[e^{-\delta^{3/2} \sigma \sum_{k=i+1}^{m-1} (m-k) B_k^Q} \middle| \mathcal{F}_{T_i} \right] &= \prod_{k=i+1}^{m-1} \mathbf{E}^Q \left[e^{-\delta^{3/2} \sigma (m-k) B_k^Q} \middle| \mathcal{F}_{T_i} \right] \\
 &= \prod_{k=i+1}^{m-1} \cosh \left(\delta^{3/2} \sigma (m-k) \right)
 \end{aligned}$$

which in turn determines the value of the process \hat{p}^m at the time T_i and state j as

$$\hat{p}_{i,j}^m = e^{-\delta \sum_{k=i}^{m-1} \theta_k - \delta^{3/2} \sigma j(m-i)} \prod_{k=i+1}^{m-1} \cosh(\delta^{3/2} \sigma(m-k))$$

where the indices i , j and m are given according to the relation

$$\begin{aligned} i, m &= 0, 1, \dots, N \\ i &\leq m \\ j &= -i, -i+2, \dots, i-2, i. \end{aligned}$$