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Markov interest rate models

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A general procedure for creating Markovian interest rate models is presented. The models created by this procedure automatically fit within the HJM framework and fit the initial term structure exactly. Therefore they are arbitrage free. Because the models created by this procedure have only one state variable per factor, two- and even three-factor models can be computed efficiently, without resorting to Monte Carlo techniques. This computational efficiency makes calibration of the new models to market prices straightforward. Extended Hull–White, extended CIR, Black–Karasinski, Jamshidian's Brownian path independent models, and Flesaker and Hughston's rational log normal models are one-state variable models which fit naturally within this theoretical framework. The 'separable' n-factor models of Cheyette and Li, Ritchken, and Sankarasubramanian – which require $n(n+3)/2$ state variables – are degenerate members of the new class of models with $n(n+3)/2$ factors. The procedure is used to create a new class of one-factor models, the 'β–η models.' These models can match the implied volatility smiles of swaptions and caplets, and thus enable one to eliminate smile error. The β–η models are also exactly solvable in that their transition densities can be written explicitly. For these models accurate – but not exact – formulas are presented for caplet and swaption prices, and it is indicated how these closed form expressions can be used to efficiently calibrate the models to market prices.

1. Introduction

Heath, Jarrow and Morton (1990, 1992) created a broad framework for developing arbitrage-free term structure models. In general, HJM models are non-Markovian and require extensive Monte Carlo simulation to calibrate model parameters to market prices, to value contingent claims, and to determine hedges. This computational burden can be greatly reduced by using special cases of the HJM models which are Markovian. Notable among these special cases are the 'separable' n-factor models of Cheyette (1992) and Li *et al*. (1995) and Ritchken and Sankarasubramanian (1995). Although the 'separable' n factor models are Markovian, they require $n(n+3)/2$ state variables, which still imposes a stiff computational burden. Moreover, to date there has been no systematic procedure for finding HJM models which are Markovian.

Here we use the method of the undetermined numeraire to develop a general procedure for creating Markovian term structure models. The resulting models automatically fit within the HJM framework and match the initial discount curve. Thus they are arbitrage free. Extended Hull–White (1990a, b), extended CIR (Cox *et al*., 1985), Black-Karasinski (1991), Jamshidian's (1991) Brownian path independent models, and Flesaker and Hughston's rational log normal models (1996) all fit naturally within our theoretical framework.

Unlike the 'separable' models, the new models require only n state variables for an n-factor

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model. This greatly reduces their computational complexity, allowing two- and even three-factor models to be used without requiring Monte Carlo simulations. Although we can re-create the separable models within our framework as degenerate models with $n(n+3)/2$ factors and state variables, we can also create models which are direct analogues of the separable models and require only n factors and state variables. These analogues are asymptotically equivalent to their separable counterparts in the limit of small volatilities, and, at normal market volatilities, the two sets of models should give essentially identical results.

We use our procedure to create a new class of one factor models, the 'β–η models.' Unlike other one-factor models, these models can match the implied volatility smiles of swaptions and caplets, and thus enable us to largely eliminate smile error from our books. The β–η models are also exactly solvable in that their transition densities (Green's functions) can be written exactly. For these models we present accurate – but not exact – formulas for caplet and swaption prices, and indicate how these closed form expressions can be used to efficiently calibrate the models to exact market prices.

2. Derivation

Consider a continuous trading economy on the interval $[0, T_{max}]$. To fix notation, let $f_0(T)$ be today's instantaneous forward rate for date T for this economy. Then the current value of a zero coupon bond which pays \$1 at maturity T is

$$D(0, T) \equiv \mathrm{e}^{-\int_0^T f_0(T')\,\mathrm{d}T'} \tag{2.1}$$

and the current discount factor from time t to T is $D(t, T) \equiv D(0, T)/D(0, t)$.

To model this economy, recall that arbitrage-free term structure models have three essential elements. First is a set of stochastic processes which drive the evolution of interest rates. Second is a numeraire, a tradable instrument with positive value and no cash flows. Commonly the money market account or a specific zero coupon bond is used as the numeraire. The final element is a valuation formula which states that, in the absence of any cash flows, the value of any tradable instrument[1] is a Martingale when expressed in units of the numeraire (Harrison and Pliska, 1981; Harrison and Kreps, 1979).

In a Markovian model, the entire term structure at any time t must be determined solely by the values $X_1, X_2, \ldots, X_n$ of a finite set of random state variables $X_1(t), X_2(t), \ldots, X_n(t)$. We assume that these state variables evolve according to the Itö processes

$$\mathrm{d}X_i(t) = \mu_i(t, \mathbf{X})\,\mathrm{d}t + \sigma_i(t, \mathbf{X})\,\mathrm{d}\widehat{W}_i(t), \qquad X_i(0) = 0, \qquad i = 1, 2, \ldots, n \tag{2.2a}$$

Here $\widehat{W}_1(t), \widehat{W}_2(t), \ldots, \widehat{W}_n(t)$ represent n Brownian motions, which can be correlated

$$\mathrm{d}\widehat{W}_i(t)\,\mathrm{d}\widehat{W}_j(t) = \rho_{ij}(t)\,\mathrm{d}t \tag{2.2b}$$

To obtain a Markovian model, we need to choose a numeraire whose value N depends only on the values $x_1, x_2, \ldots, x_n$ of the state variables: $N \equiv \mathrm{e}^{H(t,\mathbf{x})}$. We stress that at this point, the function

[1] The term 'tradable instruments' is used to refer to both fundamental instruments that are actively traded and all derivatives and synthetic instrument that can be created.

$H(t, \boldsymbol{x})$ is arbitrary and we do *not* need to know which financial instrument is represented by the numeraire. Simply requiring $N(t, \boldsymbol{x})$ to be a market instrument (with no cash flows) will enable us to derive all interest rates.

To express the valuation formula, consider a tradable instrument which has the value $V(T, \boldsymbol{X}(T))$ at some date T and pays a cash flow $C(t, \boldsymbol{X}(t))$. The value of the instrument expressed in units of the numeraire is $V(t, \boldsymbol{x})/N(t, \boldsymbol{x}) \equiv V(t, \boldsymbol{x})\mathrm{e}^{-H(t,\boldsymbol{x})}$. Requiring $V(t, \boldsymbol{x})/N(t, \boldsymbol{x})$ to be a Martingale in the absence of any cash flows then yields

$$V(t, \boldsymbol{x}) = \mathrm{e}^{H(t,\boldsymbol{x})}\widehat{E}\left\{ V(T, \boldsymbol{X}(T))\,\mathrm{e}^{-H(T,\boldsymbol{X}(T))} + \int_t^T C(t', \boldsymbol{X}(t'))\,\mathrm{e}^{-H(t',\boldsymbol{X}(t'))}\,\mathrm{d}t' \;\middle|\; \boldsymbol{X}(t) = \boldsymbol{x} \right\}, \tag{2.3}$$

where $\boldsymbol{x} = \boldsymbol{X}(t)$ is the state of the economy at time t. Equation 2.3 gives the value of the instrument at any time $t < T$. Note that $\widehat{E}$ is the expected value under the probability measure[2] determined by Equation 2.2.

Without loss of generality, we re-write the numeraire as

$$N(t, \boldsymbol{x}) = \frac{1}{D(0, t)}\,\mathrm{e}^{h(t,\boldsymbol{x})+A(t)} \tag{2.4a}$$

where

$$h(t, \boldsymbol{0}) \equiv 0, \qquad A(0) = 0 \tag{2.4b}$$

Then the valuation formula becomes

$$V(t, \boldsymbol{x}) = \mathrm{e}^{h(t,\mathbf{x})+A(t)}\widehat{E}\Big\{ V(T, \boldsymbol{X}(T))D(t, T)\,\mathrm{e}^{-h(T,\boldsymbol{X}(T))-A(T)} + \int_t^T C(t', \boldsymbol{X}(t'))D(t, t')\,\mathrm{e}^{-h(t',\boldsymbol{X}(t'))-A(t')}\,\mathrm{d}t' \;\Big|\; \boldsymbol{X}(t) = \boldsymbol{x} \Big\} \tag{2.5}$$

The valuation formula (2.5), with the state variables $\boldsymbol{X}(t)$ determined by the stochastic process (2.2), defines a term structure model. As we shall see, if this model is consistent with the initial discount curve then it is arbitrage free. In particular, this model automatically satisfies the crucial 'forward rate drift restriction' of HJM (1992) theory.[3] We stress that apart from some mathematical regularity conditions, the drift rates $\mu_i(t, \boldsymbol{x})$, the volatilities $\sigma_i(t, \boldsymbol{x})$, and the function $h(t, \boldsymbol{x})$ are arbitrary. The remaining function $A(t)$ will be determined by requiring the valuation formula (2.5) to be consistent with the initial discount curve.

To impose consistency, define $Z(t, \boldsymbol{x}; T)$ to be the value at $t, \boldsymbol{x}$ of a zero coupon bond which pays \$1 at maturity T. From (2.5),

[2] Equation 2.2 thus defines the risk neutral probability measure induced by the numeraire $N(t, \boldsymbol{x})$, and does not represent 'real world' probabilities.

[3] This is *not* surprising: (2.2) and (2.5) self-consistently define the probability measure of the risk neutral-world under a specific (if unknown) numeraire $N(t, \boldsymbol{x})$; this then uniquely defines the risk neutral probability measure under every other numeraire, including the money market numeraire; and HJM (1992) showed that for any self-consistent risk-neutral world, the instantaneous forward rates must satisfy the forward drift restriction under the probability measure induced by the money market numeraire.

$$Z(t,\ x;\ T) = D(t,\ T)\,e^{h(t,x)+A(t)-A(T)}S(t,\ x;\ T) \tag{2.6a}$$

where

$$S(t,\ x;\ T) \equiv \widehat{E}\{e^{-h(T,X(T))} \mid X(t) = x\} \tag{2.6b}$$

As we are taking $X(0) = \mathbf{0}$, consistency with the initial discount curve requires $Z(0,\ \mathbf{0};\ T) \equiv D(0,\ T)$. Since $h(t,\ \mathbf{0}) = A(0) = 0$, this requirement becomes

$$A(T) = \log S(0,\ \mathbf{0};\ T) = \log \widehat{E}\{e^{-h(T,X(T))} \mid X(0) = \mathbf{0}\} \text{ for all } T > 0 \tag{2.7}$$

Thus consistency with the initial discount curve requires choosing $A(T)$ so that the expected value of $e^{-h(T,X(T))-A(T)}$ is 1 for all $T > 0$.

Note that the current discount factors $D(0,\ T)$ do *not* appear in (2.7); choosing $A(T)$ so that the model is consistent with the initial term structure is *independent* of the actual initial term structure. Once $A(T)$ has been selected according to (2.7), then *if the initial term structure changes, all we need do is replace the discount factors $D(t,\ T)$ with the new discount factors*, and the model will be consistent with the new initial term structure. This is useful during calibration since it separates the volatility functions $\sigma_i(t,\ X)$, $\mu_i(t,\ X)$, and $h(t,\ X)$ endogenous to the model from the current discount factors $D(t,\ T)$, which are exogenous.

With the economy in state $X(t) = x$ at date t, the instantaneous forward rates $f(t,\ x;\ T)$ are defined implicitly by $Z(t,\ x;\ T) = \exp\{-\int_t^T f(t,\ x;\ T')\,dT'\}$. From (2.6),

$$\int_t^T f(t,\ x;\ T')\,dT' = \int_t^T f_0(T')\,dT' - h(t,\ x) - A(t) + A(T) - \log S(t,\ x;\ T) \tag{2.8}$$

Hence the forward rate curve at t, x is

$$f(t,\ x;\ T) = f_0(T) + A'(T) - S_T(t,\ x;\ T)/S(t,\ x;\ T) \tag{2.9}$$

where we are using subscripts to denote partial derivatives. The short rate $r(t,\ x)$ is defined as $f(t,\ x;\ t)$, so we have

$$r(t,\ x) = f_0(t) + A'(t) - S_T(t,\ x;\ t)/S(t,\ x;\ t) \tag{2.10a}$$

In Appendix A it is shown that this expression for the short rate is equivalent to

$$r(t,\ x) = f_0(t) + A'(t) + h_t(t,\ x) + \sum_i \mu_i(t,\ x)h_{x_i}(t,\ x)$$
$$+\frac{1}{2}\sum_{ij} \rho_{ij}(t)\sigma_i(t,\ x)\sigma_j(t,\ x)[h_{x_ix_j}(t,\ x) - h_{x_i}(t,\ x)h_{x_j}(t,\ x)] \tag{2.10b}$$

With $A(t)$ defined by (2.7), the valuation formula (2.5) and random processes (2.2) uniquely define a term structure model which is consistent with the initial discount curve. The instantaneous forward interest rates and short rate for this model are given by (2.9) and (2.10), respectively. At this point it would be natural to use Martingale theory (Harrison and Pliska, 1981; Harrison and Kreps, 1979) to show that this model is arbitrage free. Instead, in Appendix A we show that this model fits within the HJM framework. There it is found that *using the money market as a numeraire*, the process for the instantaneous forward rate, $F(t,\ T) \equiv f(t,\ X(t);\ T)$, satisfies

$$dF(t, T) = \sum_{ij} \rho_{ij}(t) a_T^i(t, \mathbf{X}; T) a^j(t, \mathbf{X}; T)\, dt - \sum_i a_T^i(t, \mathbf{X}; T)\, d\widetilde{W}_i(t) \tag{2.11a}$$

in the risk-neutral probability world, with

$$a^i(t, \mathbf{x}; T) = \sigma_i(t, \mathbf{x}) \left(\frac{S_{x_i}(t, \mathbf{x}; T)}{S(t, \mathbf{x}; T)} - \frac{S_{x_i}(t, \mathbf{x}; t)}{S(t, \mathbf{x}; t)} \right) \qquad i = 1, 2, \ldots, n. \tag{2.11b}$$

Inspection shows that the drift terms in (2.11a) satisfy the forward drift restriction of HJM (1992) theory. Consequently, any term structure model given by (2.5), (2.2), and (2.7) fits within the HJM framework, and provided it satisfies innocuous regularity conditions, the model is arbitrage free.[4]

Equation 2.5 can be used to directly value any contingent claim whose 'payoff' $V(T, \mathbf{X}(T))$ and cash flow $C(t, \mathbf{X}(t))$ can be expressed as explicit functions of the state variables. Since (2.9) expresses the entire forward rate curve $f(t, \mathbf{x}; T)$ explicitly, this includes all instruments whose payoff and cash flow depend only on the then current yield curve. This includes European, Bermudan, and American swaptions, caps, and most yield curve options. However, there is no guarantee that instruments whose payoff or cash flow are path dependent (e.g. instruments with payments determined by the average short rate over the preceding period) can be evaluated without adding extra state variables or resorting to path dependent valuation techniques.

3. One factor models

We now use the general framework for arbitrage-free Markovian models developed in Section 2 to create a special class of one factor models. In Section 4 we will further specialize this class of models to obtain the β–η models.

Although one-factor models are usually expressed in terms of the money market numeraire using the short rate r as the state variable, any such 'short rate model' can be re-cast within the framework of Section 2 by using a zero coupon bond as the numeraire. Conversely, any one-factor model within the framework of Section 2 can be transformed into a short rate model. These transformations are carried out in Appendix B.

Within the framework of Section 2, a one-factor model is defined by the stochastic process for the state variable,

$$dX(t) = \mu(t, X)\, dt + \sigma(t, X)\, d\widehat{W}(t) \qquad X(0) = 0 \tag{3.1}$$

and the numeraire

$$N(t, x) = \frac{1}{D(0, t)}\, e^{h(t,x)+A(t)} \tag{3.2}$$

which creates the link between the state variable and financial markets.

[4] We do not restate these conditions here; we refer the reader to HJM (1992). Although these conditions are innocuous mathematically, each points out a more stringent business condition. For example, the mathematical requirement 'market completeness' points out that risks must be hedgeable with liquid instruments with minimal transaction costs.

We argue that with our current knowledge about interest rates, we are not justified in using both a complicated function $h(t, x)$ in the numeraire and a complicated stochastic process for $X(t)$. Instead we should choose either a relatively simple function $h(t, \boldsymbol{x})$ or a relatively simple stochastic process. We arbitrarily choose to use a simple numeraire

$$N(t, x) = \frac{1}{D(0, t)} \mathrm{e}^{\lambda(t)x + A(t)} \tag{3.3}$$

for our model. The consistency condition (2.7) then becomes

$$A(t) = \log \widehat{E}\{\mathrm{e}^{-\lambda(t)X(t)} \mid X(0) = 0\} \tag{3.4}$$

Now consider the drift term, $\mu(t, X)\,\mathrm{d}t$. Suppose the model had, say, a positive drift so that $X(t)$ increased on average. Then $A(t)$ would be negative and would effectively cancel out the average effect of the drift term in the numeraire. Similarly, if the drift term was negative, $A(t)$ would be positive, again canceling out most of the drift. Indeed (3.4) can be written as

$$\widehat{E}\{\mathrm{e}^{-\lambda(t)X(t) - A(t)} \mid X(0) = 0\} = 1 \tag{3.5}$$

Since $A(t)$ cancels out the main effects of the drift term, any residual influence of the drift term $\mu(t, X)\,\mathrm{d}t$ on the model would be quite subtle. Therefore, in the interests of simplicity, we take our model to be driftless.

So consider the special class of one factor models

$$\mathrm{d}X(t) = \sigma(t, X)\,\mathrm{d}\widehat{W}(t) \qquad X(0) = 0 \tag{3.6}$$

with the numeraire (3.3). As shown in Appendix C, this class of models includes the extended Hull–White, the extended CIR, and 'Black–Karasinski like' models. However, the Black–Karasinski model itself cannot be included without adding a drift term to (3.6).

With the numeraire (3.3), the valuation formula becomes

$$V(t, x) = \mathrm{e}^{\lambda(t)x + A(t)} \widehat{E}\left\{ V(T, X(T))D(t, T)\,\mathrm{e}^{-\lambda(T)X(T) - A(T)} \right.$$
$$\left. + \int_t^T C(t', X(t'))D(t, t')\,\mathrm{e}^{-\lambda(t')X(t') - A(t')}\,\mathrm{d}t' \;\middle|\; X(t) = x \right\} \tag{3.7}$$

so the value of a zero coupon bond becomes

$$Z(t, x; T) = D(t, T)\,\mathrm{e}^{-[\lambda(T) - \lambda(t)]x - [A(T) - A(t)] + M(t, x; T)} \tag{3.8}$$

Here we have defined

$$M(t, x; T) = \log \widehat{E}\{\mathrm{e}^{-\lambda(T)[X(T) - x]} \mid X(t) = x\} = \lambda(T)x + \log S(t, x, T) \tag{3.9a}$$

and the consistency condition is now

$$A(T) = M(0, 0; T) \qquad \text{for all } T \tag{3.9b}$$

It is instructive to examine the forward rate curve. Since $Z(t, x; T) = \exp\{-\int_t^T f(t, x; T')\mathrm{d}T'\}$, Equation 3.8 shows that if the economy is in state $X(t)$ at time t, then the instantaneous forward rate for date T would be

$$f(t, X(t); T) = f_0(T) + \lambda'(T)X + A'(T) - M_T(t, X; T) \tag{3.10a}$$

The corresponding short rate would be

$$r(t, X(t)) = f_0(t) + \lambda'(t)X + A'(t) - M_T(t, X; t) \tag{3.10b}$$

For the moment, let us neglect the last two terms in (3.10a) and (3.10b). Then the forward rate curve is just the curve $\lambda'(T)X$ added to the original forward curve $f_0(T)$. Note that $\lambda'(T)$ is a 'gearing' factor: if the state variable $X(t)$ is shifted by an amount Δ, then the short rate would shift by $\delta r(t) = \lambda'(t)\Delta$ and the forward rates $f(t, X; T)$ would shift by $\delta f = \lambda'(T)\Delta = \lambda'(T)\delta r/\lambda'(t)$. Since we expect shocks to affect short maturities more than longer maturities, we expect $\lambda'(T)$ to be a decreasing function of T.

Note that Equation 3.6 implies that $X(t)$ is a Martingale, so for all times $T > t$, the expected value of $X(T)$ is just $X(t)$. Thus, if $X(t)$ was shifted by an amount Δ, then for all later times T the expected value of $X(T)$ would also shift by Δ Consequently, if a shock shifted the short rate by $\delta r(t) = \lambda'(t)\Delta$, then for all later times T the expected value of the short rate $r(T, X(T))$ would shift by $\lambda'(T)\delta r/\lambda'(t)$. As T increases, the expected value of the short rate returns to the initial forward curve $f_0(T)$ at a rate determined by how rapidly $\lambda'(T)$ decreases. So mean reversion of interest rates is directly determined by the function $\lambda(T)$.

With $f(t, X; T) = f_0(T) + \lambda'(T)X + \ldots,$ the distribution of forward rates is determined by the distribution of $X(t)$. The width of this distribution is determined mainly by the overall magnitude of $\sigma(t, x)$, and the shape of this distribution (i.e. the deviation from Gaussian or log normal behavior) is determined mainly by the functional form of the dependence of $\sigma(t, x)$ on x. As noted above, mean reversion of interest rates is determined mainly by the function $\lambda(T)$. Since skews and smiles (the changes in an option's implied volatility as the strike changes) depend mainly on the shape of the distribution, and since the change in the implied volatility as the duration of the underlying instrument changes depends mainly on mean reversion, this separation makes it easy to calibrate these models to both at-the-money and off-market instruments of differing durations. This makes these models very useful for pricing, hedging, and understanding instruments in the presence of implied volatility skews and smiles.

The third term $A'(T)$ in (3.10) embodies the consistency requirement, and is *much* smaller than the first two terms. Even though (3.6) implies that $X(t)$ has mean zero, the convex relation between bond prices and interest rates would cause bond prices to drift, on average, as the distribution of $X(t)$ spreads out. The term $A'(T)$ cancels the expected value of this convexity effect. This is why the 'consistency' term $A(T)$ depends on the functions $\lambda(T)$ and $\sigma(t, x)$, but not on the initial term structure $f_0(T)$. The last term $M_T(t, x; T)$ arises because as $X(t)$ evolves away from $X(0) = 0$, the expected value of the convexity effect changes.

We can clarify the relation between the distribution of $X(T)$ and the convexity terms $A(T)$ and $M(t, x; T)$ by expanding (3.9) in powers of λ. Define the moments

$$m_k(t, x; T) = \widehat{E}\{[X(T) - x]^k \mid X(t) = x\} \qquad k = 1, 2, \ldots \tag{3.11}$$

and note that $m_1(t, x; T) = 0$. Expanding (3.9) yields

$$M(t, x; T) = \tfrac{1}{2}\lambda^2(T)m_2(t, x; T) - \tfrac{1}{6}\lambda^3(T)m_3(t, x; T) + \tfrac{1}{24}\lambda^4(T)[m_4(t, x; T) - 3m_2^2(t, x; T)] + \ldots \tag{3.12a}$$

$$A(T) = \tfrac{1}{2}\lambda^2(T)m_2(0, 0; T) - \tfrac{1}{6}\lambda^3(T)m_3(0, 0; T) + \tfrac{1}{24}\lambda^4(T)[m_4(0, 0; T) - 3m_2^2(0, 0; T)] + \ldots \tag{3.12b}$$

Our experience calibrating one-factor models to US swaption and caplet markets shows that the magnitude of the quadratic term $\frac{1}{2}\lambda^2(T)m_2(t, x; T)$ is roughly $(T - t)T^2 \times 10^{-4}$, where time is measured in years. The cubic and quartic terms, which arise from the skewness and kurtosis of the distribution, are much smaller and have magnitudes of roughly $(T - t)^2T^3 \times 10^{-8}$ and $(T - t)^2T^4 \times 10^{-9}$, respectively.[5] We conclude that it is safe to truncate after the quadratic term except for unusually sensitive instruments, unusually long durations, or unusually competitive markets.

Truncating after the quadratic terms yields the forward rates

$$f(t, x; T) = f_0(T) + \lambda'(T)x + \frac{1}{2}\frac{\partial}{\partial T}\{\lambda^2(T)[m_2(0, 0; T) - m_2(t, x; T)]\} + \ldots \tag{3.13}$$

Consequently, the expected forward rate is

$$\widehat{E}\{f(t, X(t); T) \mid X(0) = 0\} = f_0(T) + \lambda'(T)\lambda(T)m_2(0, 0; t) + \ldots \tag{3.14a}$$

Note that $m_2(0, 0; t)$ is the variance of $X(t)$, not $X(T)$. The covariance between forward rates at different maturities is

$$\mathrm{Cov}\{f(t, X(t); T_1), f(t, X(t); T_2) \mid X(0) = 0\} = \lambda'(T_1)\lambda'(T_2)m_2(0, 0; t) + \ldots \tag{3.14b}$$

showing that the two rates are perfectly correlated through this order.

Before continuing, let us briefly address the ‘unknown numeraire’ in (3.2). By construction, $N(t, x)$ is always a valid numeraire.[6] Suppose we arbitrarily set $\lambda(T_{ref}) = 0$ for some date T_{ref}. Then (3.9) would imply that $A(T_{ref}) = 0$ and $M(t, x; T_{ref}) = 0$, so the value of the zero coupon bond of maturity T_{ref} would be $D(t, T_{ref})e^{\lambda(t)+A(t)}$ (see (3.8)). So *if* we made this choice, then the ‘unknown numeraire’ $N(t, x)$ would represent a zero coupon bond paying $1/D(0, T_{ref})$ dollars at maturity T_{ref} – at least for times $t \leqslant T_{ref}$. Although $N(t, x)$ would remain a valid numeraire for times $t > T_{ref}$, it would no longer represent any simple security. In the same spirit, we could restrict the $\lambda(t)$ curve so that $\lambda(t) \to 0$ as $t \to \infty$; then the numeraire would correspond to Flesaker and Hughston’s (1996) ‘absolute terminal measure’.

4. The β–η models

We now specialize to volatilities of the form $\sigma(t, x) = \alpha(t)[1 + \beta X(t)]^\eta$, where β and η are constant, and define the β–η models by

$$dX(t) = \alpha(t)[1 + \beta X(t)]^\eta\, d\widehat{W}(t) \qquad X(0) = 0 \tag{4.1a}$$

[5] This is equivalent to the change in interest rates over a period Δt having a standard deviation of size $O(100\ bps\ [\Delta t/yrs]^{1/2})$, variance of size $O(10^{-4}\Delta t/yrs)$, kurtosis of size $O(1)$, and skewness of size $O(10^{-2}[\Delta t/yrs]^{1/2})$.
[6] Substituting $V(t, x) \equiv N(t, x)$ and $C(t, x) \equiv 0$ into the valuation formula (2.5) shows that for any $T_{final} > 0$, $N(t, x)$ is the value of the interest rate option which has no cash flow and has the ‘payoff’ $N(T_{final}, X(T_{final}))$ at T_{final}.

with the numeraire

$$N(t, x) = \frac{1}{D(0, t)} e^{\lambda(t)x + A(t)} \tag{4.1b}$$

Without loss of generality, we normalize $\lambda(t)$ so that $\lambda'(0) = 1$.

A key advantage of β–η models is that there is a closed form solution for the transition density

$$p(t, x; \bar{t}, \bar{x})\,d\bar{x} = \text{Prob}\{\bar{x} < X(\bar{t}) < \bar{x} + d\bar{x} \mid X(t) = x\} \tag{4.2}$$

The transition density p satisfies the backward equation

$$p_t + \tfrac{1}{2}\alpha^2(t)|1 + \beta x|^{2\eta} p_{xx} = 0 \qquad t < \bar{t} \tag{4.3a}$$

$$p \longrightarrow \delta(x - \bar{x}) \text{ as } t \longrightarrow \bar{t} \tag{4.3b}$$

In terms of the new variables

$$\tau(t) = \int_0^t \alpha^2(t')\,dt' \qquad y(x) = \frac{|1 + \beta x|^{1-\eta}}{\beta(1 - \eta)} \tag{4.4}$$

this becomes

$$p_\tau + \frac{1}{2}\left(p_{yy} - \frac{2\nu - 1}{y} p_y\right) = 0 \qquad \tau < \bar{\tau} \tag{4.5a}$$

$$p \longrightarrow [\beta(1 - \eta)\bar{y}]^{-2\nu+1}\delta(y - \bar{y}) \text{ as } \tau \longrightarrow \bar{\tau} \tag{4.5b}$$

with $\nu = 1/(2 - 2\eta)$. In particular, $\tau(t)$ is roughly the variance of $X(t)$, which is around $10^{-4}t$ in US dollar markets, where t is measured in years.

Equation 4.5 can be solved by a combination of Laplace transform and Green's function techniques. For exponents $0 < \eta < 1/2$ this yields

$$p(t, x; \bar{t}, \bar{x}) = (y/\bar{y})^\nu \frac{\bar{y}}{\bar{\tau} - \tau}\{\tfrac{1}{2}I_{-\nu}(y\bar{y}/(\bar{\tau} - \tau)) + \tfrac{1}{2}I_\nu(y\bar{y}/(\bar{\tau} - \tau))\}\, e^{-(\bar{y}^2 + y^2)/2(\bar{\tau} - \tau)} \frac{d\bar{y}}{d\bar{x}}$$

$$\text{for } \bar{x} > -1/\beta \tag{4.6a}$$

$$p(t, x; \bar{t}, \bar{x}) = \frac{\sin \pi\nu}{\pi}(y/\bar{y})^\nu \frac{\bar{y}}{\bar{\tau} - \tau} K_\nu(y\bar{y}/(\bar{\tau} - \tau))\, e^{-(\bar{y}^2 + y^2)/2(\bar{\tau} - \tau)} \frac{d\bar{y}}{d\bar{x}} \qquad \text{for } \bar{x} < -1/\beta \tag{4.6b}$$

Here

$$\frac{d\bar{y}}{d\bar{x}} = [(1 - \eta)\beta\bar{y}]^{-2\nu+1} \tag{4.7}$$

and K_ν and $I_{\pm\nu}$ are the modified Bessel functions.

The β–η models have an artificial barrier at $x = -1/\beta$, where the volatility $|1 + \beta x|^\eta$ goes to zero. For exponents $\eta < 1/2$, the volatility does not decay fast enough as $y \longrightarrow 0$ to prevent y from becoming negative. If we wished to avoid negative values of y, we could put a reflecting boundary condition at $y = 0$; this would eliminate (4.6b) and change the Bessel functions in (4.6a) from $\frac{1}{2}(I_{-\nu} + I_\nu)$ to $I_{-\nu}$.

The probability of y approaching zero is extremely small, of order $e^{-1/\beta^2(1-\eta)^2\tau}$. This is much too small to affect the pricing of realistic products. Since β and η can be chosen to reproduce the volatility smile reasonably well, indicating that the model is reproducing the right shape of the risk-neutral probability distribution near the distribution's centre, we argue that the benefit of increased accuracy near the distribution's centre far outweighs the liability of having an artificial barrier in the extreme tails of the distribution. In fact, we do not expect any model to be valid in the extreme tails of the distribution where there is no relevant market experience.

For exponents $1/2 \leqslant \eta < 1$ we have

$$p(t, x; \bar{t}, \bar{x}) = (y/\bar{y})^{\nu} \frac{\bar{y}}{\bar{\tau} - \tau} I_{\nu}(y\bar{y}/(\bar{\tau} - \tau))\, \mathrm{e}^{-(\bar{y}^2 + y^2)/2(\bar{\tau} - \tau)} \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} + \frac{\Gamma(\nu, y^2/2(\bar{\tau} - \tau))}{\Gamma(\nu)} \delta(\bar{x} + 1/\beta) \tag{4.8}$$

where $\Gamma(\nu, \theta)$ is the incomplete gamma function. For these exponents the volatility declines fast enough to prevent y from becoming negative, but it doesn't prevent y from reaching the origin. This causes a very small, but finite, probability of y being exactly zero. As before, this probability is much too small to affect the pricing of realistic deals.

For both sets of exponents, the transition probability can be expanded as

$$p(t, x; \bar{t}, \bar{x}) = \frac{(y/\bar{y})^{\nu - 1/2}}{[2\pi(\bar{\tau} - \tau)]^{1/2}}\, \mathrm{e}^{-(\bar{y} - y)^2/2(\bar{\tau} - \tau)} \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}} \left\{ 1 - \frac{4\nu^2 - 1}{8y\bar{y}}(\bar{\tau} - \tau) + \dots \right\} \tag{4.9}$$

for small values of the variance $\bar{\tau} - \tau$.

Finally, for exponent $\eta = 1$ we need to re-define y as

$$y(x) = \beta^{-1} \log(1 + \beta x) \tag{4.10a}$$

Then the transition density is simply

$$p(t, x; \bar{t}, \bar{x}) = \frac{\mathrm{e}^{-\beta\bar{y}}}{[2\pi(\bar{\tau} - \tau)]^{1/2}}\, \mathrm{e}^{-(\bar{y} - y + \beta(\bar{\tau} - \tau)/2)^2/2(\bar{\tau} - \tau)} \tag{4.10b}$$

With the transition density in hand, $M(t, x; T)$ can be found by integration,

$$M(t, x; T) = \log\left\{ \int p(t, x; T, \bar{x})\, \mathrm{e}^{-\lambda(T)(\bar{x} - x)}\, \mathrm{d}\bar{x} \right\} \tag{4.11a}$$

Close inspection of (4.5) and (4.11a) reveals that $M(t, x; T)$ is of the form

$$M(t, x; T) \equiv M(S, \Theta) \tag{4.11b}$$

where

$$S = \beta^2(\bar{\tau} - \tau)/(1 + \beta x)^{2(1-\eta)} \qquad \Theta = \lambda(T)(1 + \beta x)/\beta \tag{4.11c}$$

This is useful since for each exponent η, the function $M(S, \Theta)$ can be calculated and tabulated once, eliminating most of the computational burden of the β–η model. Once these tables have been calculated, the values $Z(t, x; T)$ of zero coupons bonds, the forward rate curve $f(t, x; T)$, and the short rate $r(t, x)$ can be obtained directly from (3.8)–(3.10).

Three exponents merit special attention for their simplicity: $\eta = 0$, $\eta = 1/2$, and $\eta = 1$. In Appendix C we analyse the β–η model for these exponents, allowing β to depend on time:

$$dX(t) = \alpha(t)[1 + \beta(t)X(t)]^{\eta}\, d\widehat{W}(t) \tag{4.12}$$

Here we quote only the results with β constant.

When $\eta = 0$, the β–η model becomes the extended Hull–White model. In this case $A(T)$ and $M(t, x; T)$ are given by

$$A(t) = \tfrac{1}{2}\lambda^2(T)\overline{\tau} \qquad M(t, x; T) = \tfrac{1}{2}\lambda^2(T)(\overline{\tau} - \tau) \tag{4.13}$$

where

$$\overline{\tau} = \int_0^T \alpha^2(t')\, dt' \qquad \tau = \int_0^t \alpha^2(t')\, dt' \tag{4.14}$$

Equations 3.8–3.10 now yield explicit expressions for the value of zero coupon bonds, the forward rate curve, and the short rate.

Similarly, $A(T)$ and $M(t, x; T)$ can be written explicitly when $\eta = 1/2$,

$$A(T) = \frac{\lambda^2(T)\overline{\tau}}{2 + \beta\lambda(T)\overline{\tau}} \qquad M(t, x; T) = \frac{(1 + \beta x)\lambda^2(T)(\overline{\tau} - \tau)}{2 + \beta\lambda(T)(\overline{\tau} - \tau)} \tag{4.15}$$

where $\overline{\tau}$ and τ are given by (4.14). This again enables $Z(t, x; T)$, $f(t, x; T)$ and $r(t, x)$ to be obtained from (3.8)–(3.10). Besides deriving the corresponding formulas when $\beta = \beta(t)$, in Appendix C we find that $\beta(t)$ can be chosen so that this model is exactly the CIR model. Alternatively, one can use the extra freedom of choosing $\beta(t)$ to calibrate the model to a series of off-market instruments as well as a series of at-the-money instruments, and thus reproduce the volatility smile.

The exponent $\eta = 1$ (Black–Karasinski like models) is more difficult. Although we have been unable to derive explicit expressions for $A(T)$ and $M(t, x; T)$, the moments $m_k(t, x; T)$ can be written explicitly. When β is constant,

$$m_2(t, x; T) = [(1 + \beta x)/\beta]^2[e^{\beta^2(\overline{\tau} - \tau)} - 1] \tag{4.16a}$$

$$m_3(t, x; T) = [(1 + \beta x)/\beta]^3[e^{3\beta^2(\overline{\tau} - \tau)} - 3\, e^{\beta^2(\overline{\tau} - \tau)} + 2] \tag{4.16b}$$

$$m_4(t, x; T) = [(1 + \beta x)/\beta]^4[e^{6\beta^2(\overline{\tau} - \tau)} - 4\, e^{3\beta^2(\overline{\tau} - \tau)} + 6\, e^{\beta^2(\overline{\tau} - \tau)} - 3] \tag{4.16c}$$

We can use the first few terms in expansion (3.12) for $M(t, x; T)$ and $A(T)$, and substitute these expansions into (3.8) and (3.10) to obtain $Z(t, x; T)$, $f(t, x; T)$, and $r(t, x)$. Even if only the second moment is used, this should be accurate enough to price all but the most sensitive instruments.

4.1. *European option prices*

A vanilla receiver swaption is a European option which gives the holder the right to receive a predetermined series of cash payments C_i on dates t_i, $i = 1, 2, \ldots, n$, in return for essentially[7] paying K on the settlement date t_s. A payor swaption is a European option which gives the

[7] We are assuming that the floating leg re-values to K at the settlement date.

holder the right to receive the strike K at settlement in return for making a predetermined series of payments. Similarly, a caplet or floorlet is a European option which gives the holder the right to exchange specific payments with the issuer.

Consider a receiver swaption with exercise date t_e. Since $N(0, 0) = 1$, the value of the option today is

$$V(0, 0) = D(0, t_e) \int p(0, 0; t_e, x_e) Q(t_e, x_e) e^{-\lambda(t_e)x_e - A(t_e)} dx_e \tag{4.17a}$$

Here

$$Q(t_e, x_e) = \left\{ \sum_i C_i Z(t_e, x_e; t_i) - KZ(t_e, x_e; t_s) \right\}^+ \tag{4.17b}$$

is the value of the payoff if the economy is in state $X(t_e) = x_e$ at expiry. Substituting (3.8) for the zero coupon bonds gives

$$V(0, 0) = \int p(0, 0; t_e, x_e) \left\{ \sum_i C_i D(0, t_i) e^{-\lambda_i x_e - A_i + M(t_e, x_e; t_i)} - KD(0, t_s) e^{-\lambda_s x_e - A_s + M(t_e, x_e; t_s)} \right\}^+ dx_e \tag{4.18}$$

where $\lambda_i = \lambda(t_i)$, $\lambda_s = \lambda(t_s)$, etc. Since the transition densities $p(0, 0; t_e, x_e)$ are known and the function $M(t, x; T)$ is pre-tabulated, these options can be valued exactly with a single integration.

Let F_0 be today's forward value of the cash flow at settlement,

$$F_0 = \sum_i C_i D(t_s, t_i) \tag{4.19}$$

The implied *price* vol of the option[8] is defined as the volatility σ_B for which Black's formula,

$$V_0 = D(0, t_s)[F_0 \mathcal{N}(d_1) - K \mathcal{N}(d_2)] \tag{4.20a}$$

$$d_{1,2} = \frac{\log F_0/K \pm \frac{1}{2}\sigma_B^2 t_e}{\sigma_B \sqrt{t_e}} \tag{4.20b}$$

yields the correct price V_0 given by (4.18).

In Hagan and Woodward (in preparation) singular perturbation techniques are used to obtain accurate approximations to the option price (4.18), with the results stated in terms of implied price vols. There it is found that to leading order the option price is

$$\sigma_B = \left(\frac{1}{t_e} \int_0^{t_e} \alpha^2(t') dt' \right)^{1/2} \Lambda_1 (1 + \beta x_0)^\eta + \dots \tag{4.21a}$$

where x_0 is defined implicitly by

[8] The *price* vol defined here should not be confused with the more commonly quoted *rate* vol.

$$\tfrac{1}{2}(F_0 + K) = \sum_i C_i D(t_s, t_i)\, e^{-(\lambda_i - \lambda_s)x_0} \tag{4.21b}$$

Here

$$\Lambda_1 \equiv \frac{\sum_i C_i D(t_s, t_i)(\lambda_i - \lambda_s)\, e^{-\lambda_i x_0}}{\sum_i C_i D(t_s, t_i)\, e^{-\lambda_i x_0}} \tag{4.21c}$$

A much more accurate formula is quoted in Appendix F.

Equation 4.21a illustrates the roles played by local volatility, mean reversion, and skew in pricing. The local volatility $\alpha(t)$ only enters (4.21a), which shows that the implied price vol σ_B is proportional to the mean-square-average of the local volatility between today and the expiration date.

Recall that $\lambda'(t)$ is a decreasing function, and that 'mean reversion' refers to how rapidly $\lambda'(t)$ decreases (see (3.10) et seq). As mean reversion increases, $\lambda(t_i) - \lambda(t_s) \equiv \lambda_i - \lambda_s$ decreases, which decreases the value of Λ_1. Thus, as mean reversion increases, the price vol σ_B of the swaption decreases, as expected. Increasing the mean reversion also increases the importance of the earlier payments C_i relative to the later payments.

Equations 4.19 and 4.21b imply that $x_0 = 0$ when the option is struck at-the-money ($K = F_0$), and that x_0 is a decreasing function of the strike K. Thus, the price of an at-the-money swaption is independent of β and η, at least to within the approximations used to obtain (4.21). If β or η is zero, the price vol is independent of the strike K; as β and η increase from zero, the implied price vol becomes a decreasing function of the strike K.

Besides giving insight into how the model parameters affect option prices, Equation 4.21 can be used effectively in model calibration. Calibration is the process of choosing the model parameters $\alpha(t)$ and $\lambda(t)$ to minimize or eliminate the errors between the model's predicted prices and the actual prices of a set of standard market instruments. This is typically an iterative process, with most of the computational time spent calculating derivatives of the predicted prices with respect to variations in the model parameters. By using the approximate formula (4.21) to obtain a shrewd initial guess for the model parameters, and then using the derivatives of (4.21) in place of the actual derivatives, one can greatly speed up the calibration process.

5. Conclusions.

We used HJM theory to develop a framework for creating arbitrage-free Markovian term structure models. This framework makes designing arbitrage-free interest rate models which match the initial discount curve a straightforward task, freeing the designer to concentrate on imbuing the model with less fundamental attributes. Not only are the resulting models Markovian, but they have no more state variables than are needed to handle the random factors. This parsimony opens up new methods for evaluating and using multi-factor models: trees, explicit and implicit finite difference schemes, and Green's function techniques can be used as well as Monte Carlo methods.

All arbitrage-free Markovian term structure models we are aware of fit within this framework.

Appendices B and C explicitly demonstrate that the extended Hull–White (1990a, b), extended CIR (Cox *et al*. 1985), Black–Karasinski (1991), and Jamshidian's (1991) Brownian path independent models fit within this framework, and Appendix D shows that Flesaker and Hughston's (1996) rational interest rate models also fit within the framework. The separable n-factor models are more difficult since they require $n(n+3)/2$ state variables. Appendix E shows that these models can be constructed as degenerate $n(n+3)/2$ factor models. There we also create n-factor models which are analogues of the separable models and are asymptotically equivalent to their separable counterparts in the limit of small volatilities.

In Section 4 we used this framework to create the β–η models, a class of one-factor models defined by the process

$$\mathrm{d}X(t) = \alpha(t)[1+\beta X(t)]^{\eta}\,\mathrm{d}\widehat{W}(t) \qquad X(0)=0 \tag{5.1}$$

and the numeraire

$$N(t,\,x) = \frac{1}{D(0,\,t)}\,\mathrm{e}^{\lambda(t)x+A(t)} \tag{5.2}$$

These models enable us to account for local volatility, mean reversion, and skew, and allow both at-the-money and off-market instruments of varying maturities to be priced simultaneously. Their analytic tractability is an added benefit.

Describing local volatility, mean reversion, and skew is as much as can be expected from a one-factor model. However, some products are sensitive to risks which can not be described by a one-factor model. The two most common risks are sensitivity to rate decorrelation/curve flexing and forward/stochastic volatilities. Decorrelation risk occurs because forward rates become progressively less correlated as the difference between the maturity dates (and start dates) increases; this decorrelation tends to make the forward rate curve 'flex'. In contrast, all forward rates in a one-factor model are essentially perfectly correlated. Forward volatility risk occurs in deals which contain an 'option on an option'. Examples are captions (options on caps), and Bermudan and American swaptions. In each case exercising the option involves either receiving or giving up an option, so the exercise decision must balance the intrinsic value of the payoff against the value of the option received or lost, which is determined by the volatility environment that pertains at the exercise date.

The theoretical framework makes it easy to extend popular one-factor models to include these risks. For example, we can convert the β–η model to a stochastic volatility model by changing the equation for the state variable to

$$\mathrm{d}X(t) = \alpha(t)[1+\beta(t)X(t)]^{\eta}Y(t)\,\mathrm{d}\widehat{W}_1(t) \qquad X(0)=0 \tag{5.3a}$$

$$\mathrm{d}Y(t) = \gamma(t)Y(t)\,\mathrm{d}\widehat{W}_2(t) \qquad Y(0)=1 \tag{5.3b}$$

and using the same numeraire as before,

$$N(t,\,x) = \frac{1}{D(0,\,t)}\,\mathrm{e}^{\lambda(t)x+A(t)} \tag{5.3c}$$

Finally, note that the stochastic differential equations for the state variables play only a peripheral role in our theoretical framework; one could choose to directly model the risk neutral transition densities $p(t, x; T; X)$, dispensing with these equations entirely.

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Appendix

A *Connection with HJM*

Here we show that the random process followed by the forward rate

$$F(t;T) \equiv f(t, \mathbf{X}(t); T) = f_0(T) + A'(T) - S_T(t, \mathbf{X}(t); T)/S(t, \mathbf{X}(t); T) \tag{A.1}$$

satisfies the forward rate drift restriction of HJM.

The backward Kolmogorov equation for (2.6b) implies that $S(t, \boldsymbol{x}; T)$ satisfies the PDE

$$S_t + \sum_i \mu_i(t, \boldsymbol{x}) S_{x_i} + \frac{1}{2}\sum_{ij} \rho_{ij}(t)\sigma_i(t, \boldsymbol{x})\sigma_j(t, \boldsymbol{x}) S_{x_i x_j} = 0 \qquad 0 < t < T \tag{A.2a}$$

$$S(T, \boldsymbol{x}; T) = \mathrm{e}^{-h(T, \boldsymbol{x})} \tag{A.2b}$$

Applying Ito's lemma to (A.1), and using (A.2) to simplify the result, then yields

$$\mathrm{d}F(t;T) = \frac{1}{2}\sum_{ij} \rho_{ij}(t)\sigma_i(t, \mathbf{X})\sigma_j(t, \mathbf{X})(S_{x_i}S_{x_j}/S^2)_T\,\mathrm{d}t - \sum_i \sigma_i(t, \mathbf{X})(S_{x_i}/S)_T\,\mathrm{d}\widehat{W}_i(t) \tag{A.3}$$

Equation A.3 appears to violate the forward drift restriction of HJM (1992) theory. However, this restriction is phrased in terms of the risk-neutral probability measure *in the money market numeraire*. To switch to the money market numeraire, we apply the backwards Kolmogorov equation to the valuation formula

$$V(t, \boldsymbol{x}) = \mathrm{e}^{h(t,\boldsymbol{x})+A(t)}\widehat{E}\left\{ V(T, \mathbf{X}(T))D(t,T)\,\mathrm{e}^{-h(T,\mathbf{X}(T))-A(T)} + \int_t^T C(t', \mathbf{X}(t'))D(t,t')\,\mathrm{e}^{-h(t',\mathbf{X}(t'))-A(t')}\,\mathrm{d}t' \;\middle|\; \mathbf{X}(t) = \boldsymbol{x} \right\} \tag{A.4}$$

This shows that the value of a marketable instrument expressed in units of the numeraire,

$$\widehat{V}(t, \boldsymbol{x}) \equiv V(t, \boldsymbol{x})\,\mathrm{e}^{-H(t,\boldsymbol{x})} = V(t, \boldsymbol{x})\,\mathrm{e}^{-\int_0^t f_0(T')\,\mathrm{d}T' - h(t,\boldsymbol{x}) - A(t)} \tag{A.5a}$$

satisfies the partial differential equation

$$\widehat{V}_t + \sum_i \mu_i(t, \boldsymbol{x})\widehat{V}_{x_i} + \frac{1}{2}\sum_{ij}\rho_{ij}(t)\sigma_i(t, \boldsymbol{x})\sigma_j(t, \boldsymbol{x})\widehat{V}_{x_i x_j} = -C(t, \boldsymbol{x})\,\mathrm{e}^{-\int_0^t f_0(T')\,\mathrm{d}T' - h(t,\boldsymbol{x}) - A(t)} \tag{A.5b}$$

Consequently, $V(t, \boldsymbol{x})$ satisfies the partial differential equation

$$V_t + \sum_i \tilde{\mu}_i(t, \boldsymbol{x})V_{x_i} + \frac{1}{2}\sum_{ij}\rho_{ij}(t)\sigma_i(t, \boldsymbol{x})\sigma_j(t, \boldsymbol{x})V_{x_i x_j} - r(t, \boldsymbol{x})V = -C(t, \boldsymbol{x}) \tag{A.6a}$$

where

$$\tilde{\mu}_i(t, \boldsymbol{x}) = \mu_i(t, \boldsymbol{x}) - \sigma_i(t, \boldsymbol{x})\sum_j \rho_{ij}(t)\sigma_j(t, \boldsymbol{x})h_{x_j}(t, \boldsymbol{x}) \tag{A.6b}$$

and

$$r(t, \boldsymbol{x}) = f_0(t) + A'(t) + h_t(t, \boldsymbol{x}) + \sum_i \mu_i(t, \boldsymbol{x}) h_{x_i}(t, \boldsymbol{x})$$

$$+ \frac{1}{2} \sum_{ij} \rho_{ij}(t) \sigma_i(t, \boldsymbol{x}) \sigma_j(t, \boldsymbol{x}) [h_{x_i x_j}(t, \boldsymbol{x}) - h_{x_i}(t, \boldsymbol{x}) h_{x_j}(t, \boldsymbol{x})]. \tag{A.6c}$$

Expression (A.6c) for the short rate appears to be different from (2.10a). We can show that these two expressions are equivalent by applying Ito's lemma to $e^{-h(T, X(T))}$ in (2.6b), taking the expected value, and letting $T \to t$. This yields

$$-\frac{S_T(t, \boldsymbol{x}; t)}{S(t, \boldsymbol{x}; t)} = h_t + \sum_i \mu_i(t, \boldsymbol{x}) h_{x_i} + \frac{1}{2} \sum_{ij} \rho_{ij}(t) \sigma_i(t, \boldsymbol{x}) \sigma_j(t, \boldsymbol{x}) [h_{x_i x_j} - h_{x_i} h_{x_j}] \tag{A.7a}$$

as is needed for (A.6c) to agree with (2.10a). For future reference note that

$$-\frac{S_{x_i}(t, \boldsymbol{x}; t)}{S(t, \boldsymbol{x}; t)} = h_{x_i}(t, \boldsymbol{x}) \tag{A.7b}$$

Define the money market numeraire $B(t)$ by

$$\frac{dB(t)}{B(t)} = r(t, \boldsymbol{X}(t))\, dt \tag{A.8}$$

Inspection of (A.6a) reveals that it is the backwards Kolmogorov equation for

$$V(t, \boldsymbol{x}) = \widetilde{E} \left\{ \frac{B(t)}{B(T)} V(T, \boldsymbol{X}(T)) + \int_t^T \frac{B(t)}{B(t')} C(t', \boldsymbol{X}(t'))\, dt' \;\middle|\; \boldsymbol{X}(t) = \boldsymbol{x} \right\} \tag{A.9}$$

where $\widetilde{E}$ refers to the expected value in a probability measure for which

$$dX_i(t) = \tilde{\mu}_i(t, \boldsymbol{X}(t))\, dt + \sigma_i(t, \boldsymbol{X}(t))\, d\widetilde{W}_i(t) \qquad i = 1, 2, \ldots, \tag{A.10a}$$

Here $\widetilde{W}_1(t), \widetilde{W}_2(t), \ldots, \widetilde{W}_n(t)$ are Brownian motions with the same correlation structure as before:

$$d\widetilde{W}_i(t)\, d\widetilde{W}_j(t) = \rho_{ij}(t)\, dt \tag{A.10b}$$

Clearly, (A.9) is the valuation formula using the money market as the numeraire.

Comparing (A.10) with (2.2) shows that changing from $N(t, \boldsymbol{x})$ to the money market numeraire is equivalent to using Girsanov's theorem to change the probability measure so that

$$d\widetilde{W}_i(t) = d\widehat{W}_i(t) + \sum_j \rho_{ij}(t) \sigma_j(t, \boldsymbol{X}) h_{x_j}(t, \boldsymbol{X})\, dt \qquad i = 1, 2, \ldots, n, \tag{A.11}$$

are n Brownian motions with the same correlation structure as before. Substituting (A.11) into the forward rate process (A.3) now yields

$$dF(t, T) = \sum_{ij} \rho_{ij}(t) a_T^i(t, \boldsymbol{X}; T) a^j(t, \boldsymbol{X}; T)\, dt - \sum_i a_T^i(t, \boldsymbol{X}; T)\, d\widetilde{W}_i(t) \tag{A.12a}$$

with

$$a^i(t, x; T) = \sigma_i(t, x)\left(\frac{S_{x_i}(t, x; T)}{S(t, x; T)} - \frac{S_{x_i}(t, x; t)}{S(t, x; t)}\right) \tag{A.12b}$$

where we have used (A.7b) to replace $h_{x_i}(t, x)$. Inspection of the drift term in (A.12a) shows that the forward rate process satisfies the HJM restriction. See Equation 18 of HJM (1992).

B *Short rate models*

Arbitrage free short rate models are expressed in the money market numeraire,

$$V(t, r) = \widetilde{E}\left\{ e^{-\int_t^T R(t')\,dt'} V(T, R(T)) + \int_t^T C(t', R(t'))\, e^{-\int_t^{t'} R(t'')\,dt''}\, dt' \;\middle|\; R(t) = r \right\} \tag{B.1a}$$

and assume that the risk neutral process for the short rate $R(t)$ is of the form

$$dR(t) = [\theta(t) + \mu(t, R)]\,dt + s(t, R)\,d\widetilde{W}(t) \tag{B.1b}$$

under this numeraire. Here the function $\theta(t)$ must be selected to match the initial discount curve. With some ingenuity, $\theta(t)$ can be found analytically for some models; otherwise a forward induction scheme can be used (Jamshidian, 1991).

B.1. *Recasting short rate models*

We first recast these models in the framework of Section 2 by choosing a zero coupon bond as the numeraire. Equations B.1a and B.1b imply that $V(t, r)$ satisfies

$$V_t + [\theta(t) + \mu(t, r)]V_r + \tfrac{1}{2}s^2(t, r)V_{rr} - rV = -C(t, r) \tag{B.2}$$

Let us select a maturity T_{ref}, and define the zero coupon bond

$$Z(t, r; T_{ref}) = \widetilde{E}\{e^{-\int_t^{T_{ref}} R(t')\,dt'} \mid R(t) = r\} \tag{B.3}$$

Then Z solves the partial differential equation

$$Z_t + [\theta(t) + \mu(t, r)]Z_r + \tfrac{1}{2}s^2(t, r)Z_{rr} - rZ = 0 \qquad t < T_{ref} \tag{B.4a}$$

$$Z(T_{ref}, r; T_{ref}) = 1 \tag{B.4b}$$

Suppose we denominate the value $V(t, r)$ of tradable instruments in units of $Z(t, r; T_{ref})$,

$$\widehat{V}(t, r) = \frac{V(t, r)}{Z(t, r; T_{ref})} \tag{B.5}$$

Substituting (B.5) into (B.2) then yields

$$\widehat{V}_t + [\theta(t) + \hat{\mu}(t, r)]\widehat{V}_r + \tfrac{1}{2}s^2(t, r)\widehat{V}_{rr} = -\frac{C(t, r)}{Z(t, r; T_{ref})} \tag{B.6a}$$

where

$$\hat{\mu}(t, r) = \mu(t, r) + s^2(t, r)\frac{Z_r(t, r; T_{ref})}{Z(t, r; T_{ref})} \tag{B.6b}$$

This is equivalent to

$$V(t, r) = Z(t, r; T_{ref})\widehat{E}\left\{\frac{V(T, R(T))}{Z(T, R(T); T_{ref})} + \int_t^T \frac{C(t', R(t'))}{Z(t', R(t'); T_{ref})}\mathrm{d}t' \;\middle|\; R(t) = r\right\} \tag{B.7a}$$

where $R(t)$ evolves according to

$$\mathrm{d}R(t) = [\theta(t) + \hat{\mu}(t, r)]\,\mathrm{d}t + s(t, r)\,\mathrm{d}\widehat{W}(t) \tag{B.7b}$$

as can be seen by applying the backwards Kolmogorov equation to (B.7a). Note that this argument represents an extension of Jamshidian's (1991) Brownian path independent models.

B.2 *Recasting one-factor models as short rate models*

We now consider the general class of one factor models,

$$\mathrm{d}X(t) = \mu(t, X(t))\,\mathrm{d}t + \sigma(t, X(t))\,\mathrm{d}\widehat{W}(t) \tag{B.8a}$$

under the numeraire

$$N(t, x) = \frac{1}{D(0, t)}\,\mathrm{e}^{h(t,x)+A(t)} \tag{B.8b}$$

To match the initial term-structure, $A(t)$ must be chosen as

$$A(t) = \log \widehat{E}\{\mathrm{e}^{-h(t,X(t))} \mid X(0) = 0\} \tag{B.8c}$$

Once this is done, the short rate is given by

$$r(t, x) = f_0(t) + A'(t) + h_t(t, x) + \mu(t, x)h_x(t, x) + \tfrac{1}{2}\sigma^2(t, x)[h_{xx}(t, x) - h_x^2(t, x)] \tag{B.9}$$

See (A.6c).

To re-write this as a short-rate model requires switching to the money market numeraire. Using (A.11), the risk-neutral process for $X(t)$ is

$$\mathrm{d}X(t) = [\mu(t, X) - \sigma^2(t, X)h_x(t, X)]\,\mathrm{d}t + \sigma(t, X)\,\mathrm{d}\widetilde{W}(t). \tag{B.10}$$

under the money market numeraire. Applying Ito's lemma to (B.9) now shows that the process for the short rate, $R(t) \equiv r(t, X(t))$, is

$$\mathrm{d}R(t) = [r_t + (\mu - \sigma^2 h_x)r_x + \tfrac{1}{2}\sigma^2 r_{xx}]\,\mathrm{d}t + \sigma r_x\,\mathrm{d}\widetilde{W}(t). \tag{B.11}$$

We see that expressing an arbitrary one-factor model (B.8a)–(B.8c) as a short rate model requires inverting the relationship $R = r(t, x)$ given in (B.9) to obtain $x = x(t, R)$, and then substituting this into (B.11) to obtain the drift and volatility terms as functions of t and R. This is a tedious (and needless) process, but it simplifies significantly for several classes of models. For example, consider the class of models examined in Section 3. For these models $\mu(t, x) \equiv 0$ and $h(t, x) \equiv \lambda(t)x$, so the corresponding short rate models are

$$dR(t) = [r_t - \lambda(t)\sigma^2(t, x)r_x + \tfrac{1}{2}\sigma^2(t, x)r_{xx}]\,dt + \sigma(t, x)r_x\,d\widetilde{W}(t) \tag{B.12a}$$

where

$$r(t, x) = f_0(t) + A'(t) + \lambda'(t)x - \tfrac{1}{2}\lambda^2(t)\sigma^2(t, x) \tag{B.12b}$$

C *Special exponents*

The β–η models are defined by the numeraire

$$N(t, x) = \frac{1}{D(0, t)}\,e^{\lambda(t)x + A(t)} \tag{C.1a}$$

and the process

$$dX(t) = \alpha(t)[1 + \beta(t)X(t)]^{\eta}\,d\widehat{W}(t) \tag{C.1b}$$

where $A(t) = M(0, 0; T)$. Here $\lambda(t)$, $\alpha(t)$, and $\beta(t)$ are arbitrary model parameters that can be used to calibrate the model to the prices of market instruments. We now consider the exponents $\eta = 0$, $\eta = 1/2$, and $\eta = 1$ and solve for

$$M(t, x; T) = \log \widehat{E}\{e^{-\lambda(T)[X(T)-x]} \mid X(t) = x\} \tag{C.2}$$

The value $Z(t, x; T)$ of zero coupon bonds, the instantaneous forward rates $f(t, x; T)$, and the short rate $r(t, x)$ can then be read off from (3.8) and (3.10).

C.1. *Extended Hull-White ($\eta = 0$)*

When $\eta = 0$, the transition density $p(t, x; \bar{t}, \bar{x})$ is Gaussian,

$$p(t, x; \bar{t}, \bar{x}) = \frac{e^{-(\bar{x}-x)^2/2(\bar{\tau}-\tau)}}{[2\pi(\bar{\tau} - \tau)]^{1/2}} \tag{C.3}$$

where

$$\bar{\tau} = \int_0^{\bar{t}} \alpha^2(t')\,dt' \qquad \tau = \int_0^{t} \alpha^2(t')\,dt' \tag{C.4}$$

Integrating to compute the expected value in (C.2) yields

$$M(t, x; T) = \tfrac{1}{2}\lambda^2(T)(\bar{\tau} - \tau) \qquad A(T) = \tfrac{1}{2}\lambda^2(T)\bar{\tau} \tag{C.5}$$

To show that this is identical to the extended Hull–White model, note that (3.10) yields

$$r(t, x) = f_0(t) + \lambda'(t)x + \lambda'(t)\lambda(t)\tau(t) \tag{C.6}$$

Equation B.12 now shows that the equivalent short rate model is

$$dR(t) = [\theta(t) - \mu(t)R(t)]\,dt + s(t)\,d\widetilde{W}(t) \tag{C.7a}$$

under the money market numeraire, where

$$\mu(t) = -\lambda''(t)/\lambda'(t) \qquad s(t) = \lambda'(t)\alpha(t) \qquad \theta(t) = f_0'(t) + \mu(t)f_0(t) + [\lambda'(t)]^2\tau(t) \tag{C.7b}$$

The extended Hull–White model is usually expressed as the short rate process (C.7a), where $\mu(t)$ and $s(t)$ are arbitrary model parameters and $\theta(t)$ is chosen as

$$\theta(t) = f_0'(t) + \mu(t)f_0(t) + \int_0^t s^2(t')\,\mathrm{e}^{-2\int_{t'}^t \mu(t'')\,\mathrm{d}t''}\,\mathrm{d}t' \tag{C.8}$$

to match the initial discount curve. Note that mean reversion is expressed through the parameter $\mu(t)$ in the extended Hull–White model, while it is expressed through the 'gearing factor' $\lambda(t)$ in the equivalent β–η model.

C.2. *CIR-like models* ($\eta = \frac{1}{2}$)

When $\eta = \frac{1}{2}$ the state variable obeys

$$\mathrm{d}X(t) = \alpha(t)[1 + \beta(t)X(t)]^{1/2}\,\mathrm{d}\widehat{W}(t) \tag{C.9}$$

Consequently, the transition density $p(t, x; \bar{t}, \bar{x})$ satisfies the *forward* Kolmogorov equation

$$p_{\bar{t}} = \tfrac{1}{2}\alpha^2(\bar{t})([1 + \beta(\bar{t})\bar{x}]p)_{\bar{x}\bar{x}} \qquad \bar{t} > t \tag{C.10a}$$

$$p \longrightarrow \delta(\bar{x} - x) \text{ as } \bar{t} \longrightarrow t \tag{C.10b}$$

To obtain $M(t, x; T)$ we take the two-sided Laplace transform

$$P(t, x; \bar{t}, \lambda) \equiv \int_{-\infty}^{\infty} \mathrm{e}^{-\lambda\bar{x}}\, p(t, x; \bar{t}, \bar{x})\,\mathrm{d}\bar{x} \tag{C.11}$$

which yields

$$P_{\bar{t}} = \tfrac{1}{2}\alpha^2(\bar{t})\lambda^2\{P - \beta(\bar{t})P_\lambda\} \qquad \bar{t} > t \tag{C.12a}$$

$$P = \mathrm{e}^{-\lambda x} \text{ at } \bar{t} = t \tag{C.12b}$$

This is a first-order hyperbolic equation which can be solved by the method of characteristics. We obtain

$$\log P(t, x; \bar{t}, \lambda) = -\lambda x + \lambda^2\left\{\frac{x[B(\bar{t}) - B(t)]}{2 + \lambda[B(\bar{t}) - B(t)]} + \int_t^{\bar{t}} \frac{2\alpha^2(t')\,\mathrm{d}t'}{(2 + \lambda[B(\bar{t}) - B(t')])^2}\right\} \tag{C.13a}$$

where

$$B(t) = \int_0^t \alpha^2(t')\beta(t')\,\mathrm{d}t' \tag{C.13b}$$

Consequently, $M(t, x; T)$ and $A(T)$ are

$$M(t, x; T) = \lambda^2(T)\left\{\frac{x[B(T) - B(t)]}{2 + \lambda(T)[B(T) - B(t)]} + \int_t^T \frac{2\alpha^2(t')\,\mathrm{d}t'}{(2 + \lambda(T)[B(T) - B(t')])^2}\right\} \tag{C.14a}$$

$$A(T) = \lambda^2(T)\int_0^T \frac{2\alpha^2(t')\,\mathrm{d}t'}{(2+\lambda(T)[B(T)-B(t')])^2} \tag{C.14.b}$$

The values of zero coupon bonds, the forward rate curve, and the short rate can now be obtained directly from (3.8) and (3.10). In particular, the short rate is

$$r(t, x) = f_0(t) + K(t)[x + C(t)] \tag{C.15a}$$

with

$$K(t) = \lambda'(t) - \tfrac{1}{2}\lambda^2(t)\alpha^2(t)\beta(t) \qquad C(t) = \lambda(t)\int_0^t \frac{\alpha^2(t')\,\mathrm{d}t'}{(1+\lambda(t)[B(t)-B(t')]/2)^3} \tag{C.15b}$$

Let us cast this model as the equivalent short rate model. Using (B.12) shows that this model is

$$\mathrm{d}R(t) = [\theta(t) - \mu(t)R(t)]\,\mathrm{d}t + s(t)\sqrt{\mathrm{d}(t) + R(t)}\,\mathrm{d}\widetilde{W}(t) \tag{C.16a}$$

under the money market numeraire, where

$$\mu(t) = -[K'(t) - \lambda(t)\alpha^2(t)\beta(t)]/K(t) \tag{C.16.b}$$

$$\theta(t) = f_0'(t) + \mu(t)f_0(t) + K(t)C'(t) - \lambda(t)\alpha^2(t)K(t)[1-\beta(t)C(t)] \tag{C.16c}$$

$$s(t) = \alpha(t)\sqrt{K(t)\beta(t)} \qquad \mathrm{d}(t) = K(t)/\beta(t) - f_0(t) - K(t)C(t) \tag{C.16d}$$

The extended CIR model is usually expressed as

$$\mathrm{d}R(t) = [\theta(t) - \mu(t)R(t)]\,\mathrm{d}t + s(t)\sqrt{R(t)}\,\mathrm{d}\widetilde{W}(t) \tag{C.17}$$

Here $\mu(t)$ and $s(t)$ are arbitrary model parameters, and $\theta(t)$ must be chosen to fit the initial term structure. To state this requirement explicitly, recall that the value of a zero coupon bond is

$$Z(t, r; T) = \widetilde{E}\{\mathrm{e}^{-\int_t^T R(t')\,\mathrm{d}t'} \mid R(t) = r\} \tag{C.18}$$

Equivalently, $Z(t, r; T)$ is the solution of the backwards equation

$$Z_t + [\theta(t) - \mu(t)r]Z_r + \tfrac{1}{2}s^2(t)rZ_{rr} - rZ = 0 \qquad t < T \tag{C.19a}$$

with

$$Z(T, r; T) = 1 \tag{C.19b}$$

The solution of (C.19) is

$$Z(t, r; T) = \mathrm{e}^{-B(t,T)r - A(t,T)} \tag{C.20a}$$

where $B(t, T)$ satisfies the Ricatti equation

$$B_t = \mu(t)B + \tfrac{1}{2}s^2(t)B^2 - 1 \qquad t < T \tag{C.20b}$$

with the boundary condition $B(T, T) = 0$, and

$$A(t, T) = \int_t^T \theta(t')B(t', T)\,\mathrm{d}t' \tag{C.20c}$$

Matching the initial discount curve requires $Z(0, 0; T) = D(0, T) \equiv e^{-\int_0^T f_0(T')\,dT'}$, which requires $\theta(t)$ to be chosen so that

$$\int_0^T \theta(t')B(t', T)\,dt' = \int_0^T f_0(T')\,dT' - B(0, T)f_0(0) \tag{C.21}$$

We see that matching the extended CIR model to the initial discount curve requires solving a Ricatti equation to obtain the zero coupon bond values. This can be done efficiently, but it is difficult to understand the qualitative impact of yield curve shifts on pricing.

Comparing (C.16) and (C.17) shows that the extended CIR model is a special case of the β–η model with $\eta = 1/2$ and with $\beta(t)$ chosen so that $d(t) = 0$. Unsurprisingly, making this choice of $\beta(t)$ also requires solving a Ricatti equation.

A simpler alternative is to use the $\eta = 1/2$ model without requiring $d(t) \equiv 0$. The $\eta = 1/2$ model then gives the zero coupon bond values directly, without solving a Ricatti equation. In addition, $\beta(t)$ can be used to match the implied volatility smile by calibrating the model to the prices of off-market instruments. Although the CIR model is more aesthetically appealing, with its natural barrier occuring at $R(t) = 0$ and its simple relation between the volatility and the short rate, we stress that neither model can be expected to be valid near $R(t) = 0$. Since the barrier in the $\eta = 1/2$ model is irrelevant for pricing commonly traded US instruments, it seems better to use $\beta(t)$ to match the implied volatility smile.

C.3. *Black–Karasinski-like models ($\eta = 1$)*

When $\eta = 1$ we have been unable to find the Laplace transform of the transition density, and thus $M(t, x; T)$. However, we can find the moments $m_k(t, x; T)$ explicitly, and then use the expansions

$$M(t, x; T) = \tfrac{1}{2}\lambda^2(T)m_2(t, x; T) - \tfrac{1}{6}\lambda^3(T)m_3(t, x; T) + \ldots \tag{C.22a}$$

$$A(T) = \tfrac{1}{2}\lambda^2(T)m_2(0, 0; T) - \tfrac{1}{6}\lambda^3(T)m_3(0, 0; T) + \ldots \tag{C.22b}$$

Equations 3.8 and 3.10 then determine the zero coupon values, the forward rate curve, and the short rate as before. Although these expressions are only approximate, using the first term suffices to price all but the most sensitive instruments.

With the exponent $\eta = 1$,

$$dX(T) = \alpha(T)[1 + \beta(T)X(T)]\,d\widehat{W}(T) \tag{C.23}$$

and applying Ito's lemma yields

$$\begin{aligned} d[X(T) - x]^k = {} & \tfrac{1}{2}k(k-1)\alpha^2(T)[1 + \beta(T)X]^2[X - x]^{k-2}\,dT \\ & + k\alpha(T)[1 + \beta(T)X][X - x]^{k-1}\,d\widehat{W}(T) \end{aligned} \tag{C.24}$$

Taking the expected value then yields equations for the moments $m_k(t, x; T)$:

$$\frac{dm_k}{dT} = \tfrac{1}{2}k(k-1)\alpha^2(T)\{(1 + \beta x)^2 m_{k-2} + 2\beta(1 + \beta x)m_{k-1} + \beta^2 m_k\} \tag{C.25}$$

See (3.11). Clearly at $T = t$ we have $m_0(t, x; t) = 1$ and $m_k(t, x; t) = 0$ for all $k > 1$. Solving these equations successively yields

$$m_0(t, x; T) \equiv 1 \qquad m_1(t, x; T) \equiv 0 \tag{C.26a}$$

$$m_2(t, x; T) = \int_t^T \alpha^2(t')[1 + \beta(t')x]^2 \exp\left\{\int_{t'}^T \alpha^2(T')\beta^2(T')\,\mathrm{d}T'\right\} \mathrm{d}t' \tag{C.26b}$$

$$m_3(t, x; T) = 6\int_t^T \int_t^{t'} \alpha^2(t')\alpha^2(t'')\beta(t')[1 + \beta(t')x][1 + \beta(t'')x]^2$$

$$\cdot \exp\left\{3\int_{t'}^T \alpha^2(T')\beta^2(T')\,\mathrm{d}T' + \int_{t''}^{t'} \alpha^2(T')\beta^2(T')\,\mathrm{d}T'\right\} \mathrm{d}t''\,\mathrm{d}t' \tag{C.26c}$$

As earlier, Equation B.12 can be used to obtain the equivalent short rate model. Even though the distribution of $X(t)$ is roughly lognormal, the short rate model is not the Black–Karasinski model regardless of how $\beta(t)$ is chosen. The Black–Karasinski model requires adding a drift term to the stochastic differential equation for $X(t)$.

D Rational interest rate models

Consider the interest rate model defined by the valuation formula

$$V(t, \boldsymbol{x}) = N(t, \boldsymbol{x})\widehat{E}\left\{\frac{V(T, \boldsymbol{X}(T))}{N(T, \boldsymbol{X}(T))} + \int_t^T \frac{C(t', \boldsymbol{X}(t'))}{N(t', \boldsymbol{X}(t'))}\,\mathrm{d}t' \;\middle|\; \boldsymbol{X}(t) = \boldsymbol{x}\right\} \tag{D.1a}$$

with the numeraire

$$N(t, \boldsymbol{x}) = \frac{1}{D(0, t) + \boldsymbol{b}(t)\cdot \boldsymbol{x}} \tag{D.1b}$$

and assume that the state variables $\boldsymbol{X}(t)$ evolve according to

$$\mathrm{d}X_i(t) = \sigma_i(t, \boldsymbol{X})\,\mathrm{d}\widehat{W}_i(t) \qquad X_i(0) = 0 \qquad i = 1, 2, \ldots, n \tag{D.1c}$$

under this numeraire in the risk-neutral world. Noting that $\boldsymbol{X}(t)$ is a Martingale, we find that the zero coupon bond prices are

$$Z(t, \boldsymbol{x}; T) = \widehat{E}\left\{\frac{D(0, T) + \boldsymbol{b}(T)\cdot \boldsymbol{X}(T)}{D(0, t) + \boldsymbol{b}(t)\cdot \boldsymbol{x}} \;\middle|\; \boldsymbol{X}(t) = \boldsymbol{x}\right\} = \frac{D(0, T) + \boldsymbol{b}(T)\cdot \boldsymbol{x}}{D(0, t) + \boldsymbol{b}(t)\cdot \boldsymbol{x}} \tag{D.2}$$

Consequently, the instantaneous forward rate curve is

$$f(t, \boldsymbol{x}; T) = -\frac{D_T(0, T) + \boldsymbol{b}'(T)\cdot \boldsymbol{x}}{D(0, T) + \boldsymbol{b}(T)\cdot \boldsymbol{x}} \tag{D.3}$$

Flesaker and Hughston's (1996) rational lognormal model is the special case $\sigma_i(t, \boldsymbol{x}) = \alpha_i(t)(1 + x_i)$. These models have the advantage that all forward rates $f(t, \boldsymbol{x}; T)$ are guaranteed to remain positive provided that $b_i'(T) \leqslant 0$ for each i and $D_T(0, T) < \sum b_i'(T)$ for all T.

Moreover, the one-factor version of this model has closed-form prices for swaptions and caplets (Flesaker and Hughston, 1996).

E *Separable models*

The general n-factor HJM model for the forward rate is

$$\mathrm{d}F(t,\omega(t);T)=\sum_{ij}\rho_{ij}(t)a_T^i(t,\omega(t);T)a^j(t,\omega(t);T)\,\mathrm{d}t-\sum_i a_T^i(t,\omega(t);T)\,\mathrm{d}\widetilde{W}_i(t) \qquad \text{(E.1)}$$

in the risk-neutral world under the money market numeraire:

$$V(t)=\widetilde{E}\left\{\mathrm{e}^{-\int_t^T R(t'',\omega'')\,\mathrm{d}t''}V(T,\omega(T))+\int_t^T C(t',\omega(t'))\,\mathrm{e}^{-\int_t^{t'} R(t'',\omega'')\,\mathrm{d}t''}\;\middle|\;\mathrm{d}t'\mathscr{F}_t\right\} \qquad \text{(E.2)}$$

Here $\omega(t)$ indicates that the volatilities a_T^i, payoffs, and cash flows can depend on all measurable events that can be resolved by time t.

The separable models (Li *et al*., 1995; Ritchken and Sankarasubramanian, 1995; Cheyette, 1992) are derived by presuming that the volatilities have the form

$$a_T^i(t,\omega(t);T)=\lambda_i'(T)\sigma_i(t,\omega(t)) \qquad i=1,2,\ldots,\mathrm{n} \qquad \text{(E.3a)}$$

so that

$$a^i(t,\omega(t);T)=[\lambda_i(T)-\lambda_i(t)]\sigma_i(t,\omega(t)) \qquad i=1,2\ldots,\mathrm{n} \qquad \text{(E.3b)}$$

Then the stochastic process for the forward rates becomes

$$\mathrm{d}F(t,\omega(t);T)=\sum_{ij}\lambda_i'(T)[\lambda_j(T)-\lambda_j(t)]\rho_{ij}(t)\sigma_i(t,\omega(t))\sigma_j(t,\omega(t))\,\mathrm{d}t$$
$$-\sum_i\lambda_i'(T)\sigma_i(t,\omega(t))\,\mathrm{d}\widetilde{W}_i(t) \qquad \text{(E.4)}$$

Integrating over t now yields

$$F(t,\omega;T)=f_0(T)+\sum_i\lambda_i'(T)X_i(t,\omega)+\sum_{ij}\lambda_i'(T)\lambda_j(T)Y_{ij}(t,\omega) \qquad \text{(E.5)}$$

where

$$X_i(t,\omega)=-\int_0^t\sigma_i(t',\omega')\,\mathrm{d}\widetilde{W}_i(t')-\sum_j\int_0^t\lambda_j(t')\rho_{ij}(t')\sigma_i(t',\omega')\sigma_j(t',\omega')\,\mathrm{d}t' \qquad i=1,\ldots,\mathrm{n} \qquad \text{(E.6a)}$$

$$Y_{ij}(t,\omega)=\int_0^t\rho_{ij}(t')\sigma_i(t',\omega')\sigma_j(t',\omega')\,\mathrm{d}t' \qquad i,j=1,\ldots,\mathrm{n} \qquad \text{(E.6b)}$$

One now assumes that the coefficients $\sigma_i(t,\omega(t))$ depend on the path taken by the Brownian motion only through the value of the state variables $\mathbf{X}(t)$, $\mathbf{Y}(t)$ at time t. This is realistic since (E.5) expresses the entire term structure in terms of these variables. With this assumption, (E.6) becomes

$$\mathrm{d}X_i(t) = -\sigma_i(t, \mathbf{X}, \mathbf{Y})\left\{\mathrm{d}\widetilde{W}_i(t) + \sum_j \lambda_j(t)\rho_{ij}(t)\sigma_j(t, \mathbf{X}, \mathbf{Y})\,\mathrm{d}t\right\} \qquad X_i(0) = 0 \qquad i = 1, \ldots, \mathrm{n} \tag{E.7a}$$

$$\mathrm{d}Y_{ij}(t) = \rho_{ij}(t)\sigma_i(t, \mathbf{X}, \mathbf{Y})\sigma_j(t, \mathbf{X}, \mathbf{Y})\,\mathrm{d}t \qquad \mathbf{Y}_{ij}(0) = 0 \qquad i, j = 1, \ldots, \mathrm{n} \tag{E.7b}$$

and the instantaneous forward rate curve is

$$f(t, \boldsymbol{x}, \boldsymbol{y}; T) = f_0(T) + \sum_i \lambda_i'(T)x_i + \sum_{ij} \lambda_i'(T)\lambda_j(T)y_{ij} \tag{E.8}$$

where $x_i = X_i(t)$ and $y_{ij} = Y_{ij}(t)$ are the values of the state variables at time t.

Under the money market numeraire, the general separable model is defined by the forward rate curve (E.8) and the stochastic processes (E.7). We can put this model into the context of Section 2 by switching to the numeraire

$$N(t, \boldsymbol{x}, \boldsymbol{y}) = \frac{1}{D(0, t)}\exp\left\{\sum_i \lambda_i(t)x_i + \tfrac{1}{2}\sum_{ij} \lambda_i(t)\lambda_j(t)y_{ij}\right\} \tag{E.9}$$

Under this numeraire the value of a tradable instrument is

$$V(t, \boldsymbol{x}, \boldsymbol{y}) = N(t, \boldsymbol{x}, \boldsymbol{y})\widehat{E}\left\{\frac{V(T, \mathbf{X}(T), \mathbf{Y}(T))}{N(T, \mathbf{X}(T), \mathbf{Y}(T))} + \int_t^T \frac{C(t', \mathbf{X}(t'), \mathbf{Y}(t'))}{N(t', \mathbf{X}(t'), \mathbf{Y}(t'))}\,\mathrm{d}t' \;\middle|\; \mathbf{X}(t) = \boldsymbol{x}, \mathbf{Y}(t) = \boldsymbol{y}\right\} \tag{E.10}$$

Reversing the steps in Appendix A, we find that the state variables evolve according to

$$\mathrm{d}X_i(t) = -\sigma_i(t, \mathbf{X}, \mathbf{Y})\,\mathrm{d}\widehat{W}_i(t) \qquad X_i(0) = 0 \qquad i = 1, \ldots, \mathrm{n} \tag{E.11a}$$

$$\mathrm{d}Y_{ij}(t) = \rho_{ij}(t)\sigma_i(t, \mathbf{X}, \mathbf{Y})\sigma_j(t, \mathbf{X}, \mathbf{Y})\,\mathrm{d}t \qquad Y_{ij}(0) = 0 \qquad i, j = 1, \ldots, \mathrm{n} \tag{E.11b}$$

under the new numeraire. Finally, integrating the forward rate curve (E.8) with respect to the maturity T yields the zero coupon bond prices:

$$Z(t, \boldsymbol{x}, \boldsymbol{y}; T) = D(t, T)\exp\left\{-\sum_i [\lambda_i(T) - \lambda_i(t)]x_i - \frac{1}{2}\sum_{ij}[\lambda_i(T)\lambda_j(T) - \lambda_i(t)\lambda_j(t)]y_{ij}\right\} \tag{E.12}$$

Note that $Z(0, \mathbf{0}, \mathbf{0}; T) = D(0, T)$, so the separable models are automatically consistent with the initial term structure.

Since $Y_{ij}(t) = Y_{ji}(t)$, the n-factor separable model generally requires $n(n+3)/2$ state variables. As the random processes $Y_{ij}(t)$ are not being driven directly by Brownian motions, we can only view the separable models within the framework of Section 2 by considering (E.11) as a highly degenerate $n(n+3)/2$ factor model.

The zero coupon bond prices in (E.12) show that the extra state variables y_{ij} are needed to account for convexity effects in an arbitrage-free manner. Since convexity effects are quite small, and since (E.11b) shows that the $Y_{ij}(t)$ are only driven indirectly by the stochastic processes $d\widehat{W}_i(t)$, the distributions of $Y_{ij}(t)$ are highly peaked about their means, which are near zero. Specifically, as the volatilities σ_i decrease, the means of the variables $Y_{ij}(t)$ scale like $\sigma^2 t$ and the variances of $Y_{ij}(t)$ scale like $\sigma^6 t^3$. This suggests that the convexity effects can be accounted for by replacing the distributions of $Y_{ij}(t)$ by δ-functions, selecting the position of the δ-functions to make the theory arbitrage free. Carrying this idea through leads to the arbitrage-free n-state variable model defined by the valuation formula

$$V(t, \boldsymbol{x}) = N(t, \boldsymbol{x})\widehat{E}\left\{ \frac{V(T, \boldsymbol{X}(T))}{N(T, \boldsymbol{X}(T))} + \int_t^T \frac{C(t', \boldsymbol{X}(t'))}{N(t', \boldsymbol{X}(t'))}\, dt' \;\middle|\; \boldsymbol{X}(t) = \boldsymbol{x} \right\}$$

the numeraire

$$N(t, \boldsymbol{x}) = \frac{1}{D(0, t)}\, e^{\boldsymbol{\lambda}(t)\cdot \boldsymbol{x} + A(t)}$$

and the risk neutral processes

$$dX_i(t) = -\sigma_i(t, \boldsymbol{X})\, d\widehat{W}_i(t) \qquad X_i(0) = 0 \qquad i = 1, \ldots, n$$

under this numeraire. Here $A(T)$ is defined by

$$A(T) = \log \widehat{E}\{e^{-\boldsymbol{\lambda}(T)\cdot \boldsymbol{X}(T)} | \boldsymbol{X}(0) = 0\}$$

This model is the multi-factor generalization of the one-factor models in Section 3. Following the analysis there shows that the zero coupon bond prices are

$$Z(t, \boldsymbol{x}; T) = D(t, T) e^{-[\boldsymbol{\lambda}(T) - \boldsymbol{\lambda}(t)]\cdot \boldsymbol{x} - A(T) + A(t) + M(t, \boldsymbol{x}; T)}$$

where

$$M(t, \boldsymbol{x}; T) = \log \widehat{E}\{e^{-\boldsymbol{\lambda}(T)\cdot[\boldsymbol{X}(T) - \boldsymbol{x}]} \mid \boldsymbol{X}(t) = \boldsymbol{x}\}$$

Consequently the instantaneous forward rates are

$$f(t, \boldsymbol{x}; T) = f_0(T) + \boldsymbol{\lambda}'(T)\cdot \boldsymbol{x} + A'(T) - M_T(t, \boldsymbol{x}; T)$$

Typically $\lambda_i'(T)\sigma_i$ is around 10^{-2} in US dollar markets. At these volatilities the n-state variable model (E.13) is virtually identical to the analogous separable model in (E.9)–(E.11). Although the n-state variable model has the disadvantage of requiring computation of $M(t, \boldsymbol{x}; T)$ to obtain the zero coupon bond prices, this is usually outweighed by having many fewer state variables than the separable models.

F *Approximate prices of European options*

As in Section 4.1, let t_e be the expiration date of a European option to receive the cash flows C_i on dates t_i, $i = 1, 2, \ldots, n$ in return for paying the strike K on the settlement date t_s. Also define

$$F_0 = \sum_i C_i D(t_s, t_i) \qquad \tau_e \equiv \int_0^{t_e} \alpha^2(t')\,\mathrm{d}t' \tag{F.1}$$

as before. To express the option prices succinctly, define t_0 by the relation

$$\int_0^{t_0} \alpha^2(t')\,\mathrm{d}t' = \tau_e/2 \tag{F.2}$$

define x^* by the implicit relation

$$\tfrac{1}{2}(F_0 + K) = \sum_i C_i D(t_s, t_i)\,\mathrm{e}^{-[\lambda_i - \lambda_s]x^* - A_i + A_s + M(t_0, x^*; t_i) - M(t_0, x^*; t_s)} \tag{F.3}$$

and define Λ_k by

$$\Lambda_k = \frac{\sum_i C_i D(t_s, t_i)[\lambda_i - \lambda_s - M_x(t_0, x^*; t_i) + M_x(t_0, x^*; t_s)]^k\,\mathrm{e}^{-\lambda_i x^* - A_i + M(t_0, x^*; t_i)}}{\sum_i C_i D(t_s, t_i)\,\mathrm{e}^{-\lambda_i x^* - A_i + M(t_0, x^*; t_i)}} \tag{F.4}$$

Then the implied price vol of the option is (Hagan and Woodward, in preparation)

$$\sigma_B = \Lambda_1(\tau_e/t_e)^{1/2}(1 + \beta x^*)^{\eta}\left\{1 + \tfrac{1}{24}(1 + \beta x^*)^{2\eta}\tau_e[\Lambda_1^2 - \frac{\eta(2-\eta)\beta^2}{(1+\beta x^*)^2} + 2\frac{\Lambda_3}{\Lambda_1} - 3\frac{\Lambda_2^2}{\Lambda_1^2}] \right.$$
$$\left. + \frac{(F_0 - K)^2}{6\Lambda_1^2(F_0 + K)^2}[2\Lambda_1^2 - \frac{\eta(1+\eta)\beta^2}{(1+\beta x^*)^2} + \frac{3\eta\beta\Lambda_2}{(1+\beta x^*)\Lambda_1} + \frac{\Lambda_3}{\Lambda_1} - 3\frac{\Lambda_2^2}{\Lambda_1^2}] + \ldots\right\} \tag{F.5}$$

This expression yields prices that are usually accurate to within a tenth of a basis point, although the error can be several times larger in unusual situations.