METHODOLOGY FOR CALLABLE SWAPS AND BERMUDAN "EXERCISE INTO" SWAPTIONS

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Abstract. Here we present a methodology for obtaining quick decent prices for callable swaps and Bermudan "exercise into" swaps using the LGM model.

Key words. Bermudans, callable swaps

1. Introduction. This is part of three related papers: *Evaluating and hedging exotic swap instruments via LGM* explains the theory and usage of the LGM model in detail. This paper, *Methodology for Callable Swaps and Bermudan "Exercise Into" swaptions,* details the methodology, including all steps of the pricing procedure. Finally, *Procedure for pricing Bermudans and callable swaps,* breaks down the method into a procedure and set of algorithms.

This paper has three appendices. The first appendix discusses handling Bermudan options on amortizing swaps (as opposed to bullet swaps). Amortizers require a slightly more sophisticated *deal characterization* step, which results in selecting a different set of vanilla instruments for calibration. Once the deals are selected, the calibration and evaluation steps are identical to those of the bullet Bermudans. The second appendix discusses American swaptions. With the appropriate pre-processing step, American swaptions can be priced by by using the Bermudan pricing engine. The third appendix is used to point out the modifications that are needed if the two legs are in different currencies.

1.1. Notation. In our notation today is always $t = 0$, and

$$D(T) = \text{today's discount factor for maturity } T. \tag{1.1a}$$

For any date t in the future, let $Z(t;T)$ be the value of \$1 to be delivered at a later date T,

$$Z(t;T) = \text{zero coupon bond, maturity } T\text{, as seen at } t. \tag{1.1b}$$

These discount factors and zero coupon bonds are the ones obtained from the currency's swap curve. Clearly $D(T) = Z(0;T)$. We use distinct notation for discount factors and zero coupon bonds to remind ourselves that discount factors $D(T)$ are *not* random; we can always obtain the current discount factors from the stripper. Zero coupon bonds $Z(t;T)$ *are* random, at least until time catches up to date t.

Also, we use $\mathfrak{N}(z)$ and $G(z)$ to be the standard (cumulative) normal distribution and Gaussian density, respectively:

$$\mathfrak{N}(z) = \int_{-\infty}^{z} G(z')dz', \qquad G(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \tag{1.2}$$

2. Deal definition and representation. Bermudans arise mainly from two sources. The first is a direct Bermudan swaption, also called an "exercise into" Bermudan. The other (more common) source is a cancellable swap, which is invariably priced as a swap plus a Bermudan swaption to enter the opposite swap. Bermudans from both sources (and virtually any other Bermudan that arises) fit into following deal structure. After defining this deal structure, we will show how to fit the most common types of Bermudans into the structure.

Our Bermudan structure contains the following information:

Payment information:

(2.1a) $t[0, 1, 2, ..., n] =$ paydates

(2.1b) $C[*, 1, 2, ..., n] =$ full payments for each interval

(2.1c) $N[*, 1, 2, ..., n] =$ notionals for each interval

Exercise information:

(2.1d) $PorR =$ payer or receiver flag

(2.1e) $t^{ex}[*1, 2, ..., J] =$ exercise (notification) dates

(2.1f) $t^{set}[*, 1, 2, ..., J] =$ settlement date if exercised at τ_j^{ex}

(2.1g) $i^{first}[*, 1, 2, ..., J] =$ first coupon payment received if exercised at τ_j^{ex}

(2.1h) $fee[*, 1, 2, ..., J] =$ exercise fee (paid on t_j^{set})

(2.1i) $rfp[*, 1, 2, ..., J] =$ reduction in first coupon payment received if exer at τ_j^{ex}

(In my notation, $*$ means this element of the array is not used. In my opinion, the indexing is simpler and less confusing if we waste the first entry in all the vectors except t, but this is only a personal preference.)

The Bermudan can be exercised on any of the notification dates t_j^{ex} for $j = 1, 2, ..., J$. Suppose first the $PorR$ flag is set to "receiver." Then, if the Bermudan is exercised at date t_j^{ex}, the owner receives all the payments starting with payment $i = i_j^{first}$. However, the first payment received is reduced by rfp_j (which may be zero):

(2.2a) $C_i - rfp_j$ received at t_i for $i = i_j^{first}$,

(2.2b) C_i received at t_i for $i = i_j^{first} + 1, ..., n$.

In return, the owner pays the notional plus the exercise fee at the settlement date

(2.2c) $N_{i_j^{first}} + fee_j$ paid at t_j^{set}.

The full payments C_i include the fixed leg's interest, notional payments and prepayments, as well as adjustments for basis spreads and any margins. The floating leg is mainly accounted for by paying the notional N_j on settlement.

Suppose now the $PorR$ flag is set to "payer." If the Bermudan is exercised at date t_j^{ex}, one receives the payment

(2.3a) $N_{i_j^{first}} - fee_j$ received at t_j^{set}.

and makes the payments

(2.3b) $C_i - rfp_j$ paid at t_i for $i = i_j^{first}$,

(2.3c) C_i paid at t_i for $i = i_j^{first} + 1, ..., n$.

In the next section we show how real deals, both the "exercise into" and callable swap Bermudans, can be put into the above deal structure. From then on we work exclusively with deal structure.

2.1. Swap. Let us first define the swap, and then define the exercise features of the two types of Bermudan. We assume that the swap exchanges a fixed leg against a standard floating leg plus a margin; we also assume that the legs are *in the same currency*. (This latter assumption is dropped in Appendix C).

2.1.1. Fixed leg. Let

$$t_0^{th} < t_1^{th} < t_2^{th} \cdots < t_{n-1}^{th} < t_n^{th} \tag{2.4a}$$

$$t_0 < t_1 < t_2 \cdots < t_{n-1} < t_n \tag{2.4b}$$

be the fixed leg's theoretical and actual dates. In our notation,

$$t_{i-1} < t \le t_i \tag{2.5}$$

is period i, and

$$N_i = \text{notional for period } i, \tag{2.6a}$$

$$R_i^{fix} = \text{fixed rate for period } i, \tag{2.6b}$$

$$a_i = \text{cvg}(t_{i-1}, t_i, \beta^{fix}) = \text{day count fraction for period } i. \tag{2.6c}$$

The fixed leg payments are

$$N_i \alpha_i R_i^{fix} \qquad \text{paid at } t_i, \qquad \text{for } i = 1, 2, ..., n \tag{2.6d}$$

2.1.2. Funding (floating) leg. Let the floating leg's theoretical and actual dates be

$$\tau_0^{th} < \tau_1^{th} < \tau_2^{th} \cdots < \tau_{n-1}^{th} < \tau_m^{th} \tag{2.7a}$$

$$\tau_0 < \tau_1 < \tau_2 \cdots < \tau_{n-1} < \tau_m \tag{2.7b}$$

where the beginning and end dates of the two legs must agree:

$$t_0^{th} = \tau_0^{th}, \qquad t_n^{th} = \tau_m^{th}, \tag{2.7c}$$

$$t_0 = \tau_0, \qquad t_n = \tau_m. \tag{2.7d}$$

Let the j^{th} floating period be $\tau_{j-1} < t < \tau_j$, and let

$$N_j^{flt} = \text{notional for the } j^{th} \text{ period}, \tag{2.8a}$$

$$m_j = \text{margin for the } j^{th} \text{ period} \tag{2.8b}$$

$$\text{bs}_j = \text{floating rate's basis spread for } j^{th} \text{ period} \tag{2.8c}$$

$$\tilde{\alpha}_j = \text{cvg}(\tau_{j-1}, \tau_j, \beta^{flt}) = \text{day count fraction for period } j \tag{2.8d}$$

The floating leg pays the floating rate plus a margin,

$$N_j^{flt} \tilde{\alpha}_j [r_j^{flt} + m_j^{orig}] \qquad \text{paid at } \tau_j, \qquad j = 1, 2, ..., m. \tag{2.9}$$

Prior to fixing, the j^{th} floating leg payment is worth the same as the payments

$$N_j^{flt} = \qquad \text{paid at } \tau_{j-1}, \tag{2.10a}$$

$$\{-1 + \tilde{\alpha}_j [\text{bs}_j + m_j]\} N_j^{flt} \qquad \text{paid at } \tau_j, \tag{2.10b}$$

for $j = 1, 2, ..., m$. This is just the definition of the (forward) basis spread bs.

2.1.3. Bond model of a swap. Floating leg dates often occur with a different frequency (usually more frequent) than the fixed leg dates. We are going to replace the floating leg payments with the equivalent payments based on the fixed rate schedule. Unless the basis spreads and margins are identically zero, this will result in an invisibly small approximation.

Suppose first that the floating leg intervals are equal to or shorter than the fixed leg intervals. Based on the *theoretical dates*, we can assign every floating leg interval j to a fixed leg interval

$$j \in I_i \qquad \text{if and only if } t_{i-1}^{th} < \tau_j^{th} \le t_i^{th}. \tag{2.11a}$$

It makes no sense for the floating leg notional to change when the fixed rate notional does not change. We restrict ourselves to deals whose floating rate notional N_j^{flt} *is constant and equal to the fixed rate notional* N_i *within each fixed rate interval*:

$$N_j^{flt} = N_i \qquad \text{for all } j \in I_i. \tag{2.11b}$$

The net swap payments (fixed minus floating) for interval i are:

$$-N_i \qquad \text{paid at } t_{i-1}, \tag{2.12a}$$

$$N_i \alpha_i R_i^{fix} + N_i \qquad \text{paid at } t_i, \tag{2.12b}$$

$$N_i \tilde{\alpha}_j [\text{bs}_j + m_j^{orig}] \qquad \text{paid at } \tau_j \qquad \text{for all } j \in I_i \tag{2.12c}$$

We move the basis spread and margin to the fixed leg, approximating the swap payments for interval i as

$$-N_i \qquad \text{paid at } t_{i-1}, \tag{2.13a}$$

$$N_i \alpha_i R_i^{eff} + N_i \qquad \text{paid at } t_i, \tag{2.13b}$$

for $i = 1, 2, ..., n$. Here the effective fixed rate for interval i is:

$$R_i^{eff} = R_i^{fix} - \frac{\sum_{j \in I_i} \tilde{\alpha}_j [\text{bs}_j + m_j] D(\tau_j)}{\alpha_i D(t_i)}. \tag{2.13c}$$

Suppose now that the floating leg intervals occur less frequently than the fixed leg intervals. Based on the *theoretical dates*, we again assume that we can assign every fixed leg interval i to a floating leg leg interval

$$i \in I_j \qquad \text{if and only if } \tau_{j-1}^{th} < t_i^{th} \le \tau_j^{th}. \tag{2.14a}$$

We again assume that the fixed rate notionals N_i *are constant and equal to the floating rate notional* N_j^{flt} *within each floating rate interval*:

$$N_i = N_j^{flt} \qquad \text{for all } i \in I_j. \tag{2.14b}$$

We move the basis spread and margin to the fixed leg. This once again leads to approximating the swap payments for interval i as

$$-N_i \qquad \text{paid at } t_{i-1}, \tag{2.15a}$$

$$N_i \alpha_i R_i^{eff} + N_i \qquad \text{paid at } t_i, \tag{2.15b}$$

for $i = 1, 2, ..., n$. Here the effective fixed rate for interval i is:

$$R_i^{eff} = R_i^{fix} - \frac{\tilde{\alpha}_j [\text{bs}_j + m_j] D(\tau_j)}{\sum_{i \in I_j} \alpha_i D(t_i)}. \tag{2.15c}$$

2.1.4. Optionality: "Exercise into" Bermudan swaptions. Let us first consider an "exercise into" Bermudan option.. It is not uncommon for a Bermudan to be exercisable more frequently than once a period, as in a semi-pay, monthly call deal. So we need to allow for intra-period exercises. The optionality can be defined by
(i) a payer/receiver flag

$$PorR, \tag{2.16a}$$

(ii) a set of notification dates,

$$t_1^{ex}, t_2^{ex}, ..., t_J^{ex} \tag{2.16b}$$

(iii) a set of theoretical and actual settlement (start)-upon-exercise dates,

$$t_1^{th,set}, t_2^{th,set}, ..., t_J^{th,set} \tag{2.16c}$$

$$t_1^{set}, t_2^{set}, ..., t_J^{set} \tag{2.16d}$$

(iv) a set of exercise fees

$$fee_1, fee_2, ..., fee_J. \tag{2.16e}$$

Suppose the payer/receiver flag is "receive." Then if the deal is exercised on the notification date τ_j^{ex}, the owner receives the swap starting from the settlement date t_j^{set}. Specifically, define i_j^{first} as the i with

$$t_{i-1}^{th} \le t_j^{th,set} < t_i^{th}. \tag{2.17a}$$

For the first payment, the owner receives the interest that accrues from the settlement date t_j^{set} to the first coupon date t_i at i_j^{first}. This is less that the full coupon if the settlement date t_j^{set} is after the interval starts at t_{i-1}. So if the deal is exercised at t_{ex}^j, then the owner of the option gets

$$N_i\alpha_j^{first}R_i^{eff} + N_i - N_{i-1} \qquad \text{at } t_i \qquad \text{for } i = i_j^{first} \tag{2.17b}$$

$$N_i\alpha_i R_i^{eff} + N_i - N_{i+1} \qquad \text{at } t_i \qquad \text{for } i = i_j^{first}+1, ..., n-1 \tag{2.17c}$$

$$N_n\alpha_n R_n^{eff} + N_n \qquad \text{at } t_n \qquad \text{for } i = n \tag{2.17d}$$

where

$$\alpha_j^{first} = \text{cvg}(t_j^{set}, t_i) \qquad \text{with } i = i_j^{first}. \tag{2.17e}$$

In return, the owner pays

$$N_i + fee_j \qquad \text{at } t_j^{set} \qquad \text{with } i = i_j^{first}. \tag{2.17f}$$

If the payer/receiver flag is "payer." Then if the deal is exercised at t_j^{ex}, the owner recieves

$$N_i - fee_j \qquad \text{at } t_j^{set} \qquad \text{with } i = i_j^{first}, \tag{2.18a}$$

and pays

$$N_i\alpha_j^{first}R_i^{eff} + N_i - N_{i-1} \qquad \text{at } t_i \qquad \text{for } i = i_j^{first} \tag{2.18b}$$

$$N_i\alpha_i R_i^{eff} + N_i - N_{i+1} \qquad \text{at } t_i \qquad \text{for } i = i_j^{first}+1, ..., n-1 \tag{2.18c}$$

$$N_n\alpha_n R_n^{eff} + N_n \qquad \text{at } t_n \qquad \text{for } i = n \tag{2.18d}$$

We can fit this deal into the above Bermudan structure by defining the full payments

$$C_i = N_i \alpha_i R_i^{eff} + N_i - N_{i+1} \qquad \text{for } i = 1, ..., n-1, \tag{2.19a}$$

$$C_n = N_n \alpha_n R_n^{eff} + N_n, \tag{2.19b}$$

and defining the i_j^{first} as the index i for which

$$t_{i-1}^{th} \leq t_j^{th,set} < t_i^{th} \qquad \text{for } j = 1, ..., J \tag{2.19c}$$

and defining the reduction in the first payment as

$$rfp_j = (\alpha_i - \alpha_j^{first}) N_i R_i^{eff} = N_i R_i^{eff} \left\{ \text{cvg}(t_{i-1}, t_i) - \text{cvg}(t_j^{set}, t_i) \right\} \qquad \text{for } j = 1, ..., J \tag{2.19d}$$

with $i = i_j^{first}$.The notionals N_i, pay/rec flag $PorR$, exercise fees fee_j, and exercise and settlement dates, τ_j^{ex} and τ_j^{set}, are copied into the structure unchanged.

Aside. Best practices is for a deal's confirm to specify
i) the theoretical settlement-upon-exercise dates $t_j^{th,set}$;
ii) the business day rules and holiday calendars needed to obtain the actual settlement dates from the theoretical dates (these should be identical to the rules for the fixed leg), and
iii) that the notification date must occur at least N business days (or calendar days) before the actual settlement date.
Then regardless of whether holidays are added or subtracted after the deal is struck, the settlement dates always relate to the payment dates in the same way without one day gaps opening up.

Confusingly, the settlement (start)-upon-exercise dates are often called the "exercise" dates and the exercise (notification) dates are simply known as notification dates..

2.1.5. Optionality: callable swaps. Let us now a callable swap. Again, Bermudans may be callable more frequently than once a period. If a swap is called mid-period, the fixed and floating leg's accrued interest must be paid, as well as any exercise fee, on the settlement date. Then no further payments are received. This is equivalent to a non-callable swap, plus a Bermudan swpation to enter the opposite swap.

Consider a callable swap. Let it be a payer or receiver, according to

$$PorR. \tag{2.20a}$$

The callability is defined by
(i) a set of notification dates,

$$t_1^{ex}, t_2^{ex}, ..., t_J^{ex} \tag{2.20b}$$

(iii) a set of theoretical and actual settlement-upon-exercise dates,

$$t_1^{th,set}, t_2^{th,set}, ..., t_J^{th,set} \tag{2.20c}$$

$$t_1^{set}, t_2^{set}, ..., t_J^{set} \tag{2.20d}$$

(iii) a set of exercise fees

$$fee_1, fee_2, ..., fee_J. \tag{2.20e}$$

The value of the cancellable swap is the value of the full (non-cancellable) swap plus the value of the cancelation feature. We assume that the non-cancellable swap is priced elsewhere. Here we only price the canacellation feature.

Suppose the payer/receiver flag is "payer," and suppose the cancellation feature is exercised on the notification date τ_j^{ex}. Define i_j^{first} as the first coupon after the settlement-upon-exercise date:

$$t_{i-1}^{th} \leq t_j^{th,set} < t_i^{th}. \tag{2.21a}$$

Cancelling the payer swap is equivalent to recieving all the fixed rate payments, and making all the floating leg payments, starting with payment $i - i_j^{firxt}$. So the owner receives the fixed leg payments

$$N_i \alpha_i R_i^{eff} + N_i - N_{i+1} \qquad \text{at } t_i \qquad \text{for } i = i_j^{first}, ..., n-1 \tag{2.21b}$$

$$N_n \alpha_n R_n^{eff} + N_n \qquad \text{at } t_n \qquad \text{for } i = n, \tag{2.21c}$$

and makes the floating leg payments, which are equivalent to

$$N_i \qquad \text{at } t_{i-1}. \tag{2.21d}$$

At the settlement date, the owner also pays the accrued fixed leg interest, receives the accrued floating rate interest, and pays any exercise fee. So the owner must also pay

$$fee_j + N_i \alpha_j^{set} \left(R_i^{eff} - r_i^{true} \right) \qquad \text{at } t_j^{set} \tag{2.22a}$$

with

$$\alpha_j^{set} = \text{cvg}(t_{i-1}, t_j^{set}) \qquad \text{with } i = i_j^{first}. \tag{2.22b}$$

Here we use the true rate

$$r_i^{true} = \frac{Z(t, t_{i-1}) - Z(t, t_i)}{\alpha_i Z(t, t_i)} \tag{2.22c}$$

for interval i, instead of the forward floating rate, because the basis spread is already incorpoarted into R_i^{eff}. The floating rate payment at t_{i-1} along with the settlement payments are now equivalent to

$$N_i + fee_j + N_i \alpha_j^{set} \left\{ R_i^{eff} - \Delta r_j^{flt} \right\} \qquad \text{at } t_j^{set} \tag{2.23a}$$

where

$$\Delta r_j^{flt} = \frac{Z(t, t_{i-1}) - Z(t, t_i)}{\alpha_i Z(t, t_i)} - \frac{Z(t, t_{i-1}) - Z(t, t_j^{set})}{\alpha_j^{set} Z(t, t_j^{set})} \tag{2.23b}$$

is the difference between the rates for the full and partial intervals. Virtually every desk neglect this correction. We can do better by estimating this difference from today's curve:

$$\Delta r_j^{flt} = \frac{D(t_{i-1}) - D(t_i)}{\alpha_i D(t_i)} - \frac{D(t_{i-1}) - D(t_j^{set})}{\alpha_j^{set} D(t_j^{set})}. \tag{2.24}$$

In summary, if the owner of a payer swap cancels (by providing notification at t_j^{ex}), the cancellation is equivalent to receiving the payments

$$N_i \alpha_i R_i^{eff} + N_i - N_{i+1} \qquad \text{at } t_i \qquad \text{for } i = i_j^{first}, ..., n-1 \tag{2.25a}$$

$$N_n \alpha_n R_n^{eff} + N_n \qquad \text{at } t_n \qquad \text{for } i = n, \tag{2.25b}$$

and making the payment

$$N_i + fee_j + N_i\alpha_j^{set}\left\{R_i^{eff} - \Delta r_j^{flt}\right\} \qquad \text{at } t_j^{set}, \tag{2.25c}$$

where

$$\Delta r_j^{flt} = \frac{D(t_{i-1}) - D(t_i)}{\alpha_i D(t_i)} - \frac{D(t_{i-1}) - D(t_j^{set})}{\alpha_j^{set} D(, t_j^{set})}. \tag{2.25d}$$

Similarly, suppose the owner of a receiver swap cancels (by providing notification at t_j^{ex}). Cancellation is equivalent to making receving the payment

$$N_i - fee_j + N_i\alpha_j^{set}\left\{R_i^{eff} - \Delta r_j^{flt}\right\} \qquad \text{at } t_j^{set}, \tag{2.26a}$$

and the payments

$$N_i\alpha_i R_i^{eff} + N_i - N_{i+1} \qquad \text{at } t_i \qquad \text{for } i = i_j^{first}, ..., n-1 \tag{2.26b}$$

$$N_n\alpha_n R_n^{eff} + N_n \qquad \text{at } t_n \qquad \text{for } i = n, \tag{2.26c}$$

As before,

$$\Delta r_j^{flt} = \frac{D(t_{i-1}) - D(t_i)}{\alpha_i D(t_i)} - \frac{D(t_{i-1}) - D(t_j^{set})}{\alpha_j^{set} D(t_j^{set})}. \tag{2.26d}$$

As previously stated, we assume that the value of the non-cancellable swap is calculated elsewhere, and here only price the cancellation feature. We can fit this cancellation feature into our Bermudan structure by defining the full payments to be

$$N_i\alpha_i R_i^{eff} + N_i - N_{i+1} \qquad \text{at } t_i \qquad \text{for } i = 1, ..., n-1 \tag{2.27a}$$

$$N_n\alpha_n R_n^{eff} + N_n \qquad \text{at } t_n \qquad \text{for } i = n. \tag{2.27b}$$

by defining the payer/receiver flag to be *receiver* if a payer swap is cancellable, and to be *payer* if a receiver swap is cancellable, by defining i_j^{first} so that

$$t_{i-1}^{th} \le t_j^{th,set} < t_i^{th}, \tag{2.27c}$$

by defining the exercise fee to be

$$fee_j = fee_j \pm N_i\alpha_j^{set}\left\{R_i^{eff} - \frac{D(t_{i-1}) - D(t_i)}{\alpha_i D(t_i)} + \frac{D(t_{i-1}) - D(t_j^{set})}{\alpha_j^{set} D(t_j^{set})}\right\} \qquad \text{for } j = 1, ..., J. \tag{2.27d}$$

Here the “ + ȷ sign is to be taken for callable payer swaps, and the “ − ȷ sign for callable receivers. For callable swaps, the reduction in the first payment to be zero

$$rfp_j = 0 \qquad \text{for } j = 1, ..., J \tag{2.27e}$$

and the notionals N_i, the fixed leg pay dates t_i, the execise and settlement dates t_j^{ex} and t_j^{set} are to be copied into the structure without change.with $i = i_j^{first}$.The notionals N_i, pay/rec flag $PorR$, exercise fees fee_j, and exercise and settlement dates, τ_j^{ex} and τ_j^{set}, are copied to the structure unchanged.

Aside. Best practices is for a deal's confirm to specify
i) the theoretical settlement-upon-exercise dates (call dates) $t_j^{th,set}$;
ii) the business day rules and holiday calendars needed to obtain the actual call dates from the theoretical dates, and
iii) that the notification date must occur at least N business days (or calendar days) before the actual settlement date.

The settlement-upon-call dates are often called the "call" dates and the exercise (notification) dates are simply known as notification dates.

3. The LGM (Linear Gauss Markov) model.

3.1. Basic LGM. We value these deals using calibrated LGM models. This model is chosen because it is very reliable as well as being very easy to work with. As explained fully in *Evaluating and hedging exotic swap instruments via LGM*, the one factor LGM model has a single state variable x, and uses the numeraire

$$N(t,x) = \frac{1}{D(t)} e^{+H(t)x+\frac{1}{2}H^2(t)\zeta(t)} \tag{3.1a}$$

Let $V^{full}(t,x)$ be the actual value of any deal. Throughout we one use only the *reduced* value

$$V(t,x) = \frac{V^{full}(t,x)}{N(t,x)}. \tag{3.1b}$$

At $t=0, x=0$, the numeraire is 1, so today's full values and reduced values are identical. As we shall see, the full values at other dates are not relevent.

The LGM model can be summarized in two relations: First, the (reduced) value $V(t,x)$ of any deal can be determined from its value at any later date T via the expected value

$$V(t,x) = \frac{1}{\sqrt{2\pi\Delta\zeta}} \int_{-\infty}^{\infty} e^{-(X-x)^2/2\Delta\zeta} V(T,X)dX, \tag{3.2a}$$

where

$$\Delta\zeta = \zeta(T) - \zeta(t). \tag{3.2b}$$

Second, the (reduced) value of a zero coupon bond with maturity t_i is

$$Z(t,x;t_i) = D(t_i) e^{-H(t_i)x-\frac{1}{2}H^2(t_i)\zeta(t)}, \tag{3.2c}$$

as can be determined by substituting $V(T,X) = 1/N(T,X)$ in the expected value. Here the functions $H(T)$ and $\zeta(t)$ are found by the calibration step. They are equivalent to the mean reversion parameters $\kappa(t)$ and the local vol $\sigma(t)$ in the Hull-White model.

3.2. Invariances. Recall from *Evaluating and hedging exotic swap instruments via LGM* that all deal prices remain the same if we replace

$$H(T) \longrightarrow H(T) + C, \qquad \zeta(t) \longrightarrow \zeta(t) \tag{3.3a}$$

$$H(T) \longrightarrow KH(T), \qquad \zeta(t) \longrightarrow \zeta(t)/K^2 \tag{3.3b}$$

for any constants C and K. These invariances need to be recognized (and exploited!) in the calibration step.

3.3. Swaption value. Calibration is a procedure for choosing the functions $H(T)$ and $\zeta(t)$ so that the LGM prices match the actual market prices for a selected set of swaptions, caplets, and floorlets. Here we obtain a closed form expression for the LGM price of these instruments.

Consider a (receiver) swap with start date t_0, fixed leg pay dates $t_1, t_2, \ldots, t_n$, and fixed rate R^{fix}. The fixed leg makes the payments

$$\alpha_i R^{fix} \qquad \text{paid at } t_i \qquad \text{for } i = 1, 2, \ldots, n-1, \tag{3.4a}$$

$$1 + \alpha_n R^{fix} \qquad \text{paid at } t_n, \tag{3.4b}$$

where $\alpha_i = \text{cvg}(t_{i-1}, t_i, \beta)$ is the coverage for period i according to the fixed leg's day count basis β. On any given day t, the fixed leg's value is

$$V_{fix}(t, x) = R^{fix} \sum_{i=1}^{n} \alpha_i Z(t, x; t_i) + Z(t, x; t_n) \tag{3.5a}$$

As discussed in, *Evaluating and hedging exotic swap instruments via LGM*, the value of the floating leg is

$$V_{flt}(t, x) = Z(t, x; t_0) + \sum_{i=1}^{n} \alpha_i S_i Z(t, x; t_i), \tag{3.5b}$$

where S_i is the floating rate's basis spread, adjusted to the fixed legs day count basis and frequency. The value of the receiver swap is

$$V_{rec}(t, x) = \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i \right) Z(t, x; t_i) + Z(t, x; t_n) - Z(t, x; t_0). \tag{3.5c}$$

where the strike R_{fix} and effective spread S_i are known constants.

Consider a European option on this swap (a swaption), and let τ_{ex} be the exercise date. Under the one factor LGM model, today's value for the swaption is

$$V_{rec}^{opt}(0, 0) = \frac{1}{\sqrt{2\pi\zeta_{ex}}} \int_{-\infty}^{\infty} e^{-X^2/2\zeta_{ex}} \left\{ \begin{array}{cc} V_{rec}(\tau_{ex}, X) & \text{if positive} \\ 0 & \text{if negative} \end{array} \right\} dX, \tag{3.6}$$

where $\zeta_{ex} = \zeta(\tau_{ex})$ Integrating yields the exact pricing formulas

$$V_{rec}^{opt}(0, 0) = \sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i \right) D_i \mathfrak{N} \left(\frac{y^* + [H_i - H_0] \zeta_{ex}}{\sqrt{\zeta_{ex}}} \right) + D_n \mathfrak{N} \left(\frac{y^* + [H_n - H_0] \zeta_{ex}}{\sqrt{\zeta_{ex}}} \right) - D_0 \mathfrak{N} \left(\frac{y^*}{\sqrt{\zeta_{ex}}} \right) \tag{3.7a}$$

where y^* is obtained by solving

$$\sum_{i=1}^{n} \alpha_i \left(R^{fix} - S_i \right) D_i e^{-(H_i - H_0) y^* - \frac{1}{2} (H_i - H_0)^2 \zeta_{ex}} + D_n e^{-(H_n - H_0) y^* - \frac{1}{2} (H_n - H_0)^2 \zeta_{ex}} = D_0. \tag{3.7b}$$

Observe that the swaption value depends on $\zeta(t)$ only through $\zeta_{ex} = \zeta(\tau_{ex})$, and on $H(T)$ only through

$$\Delta H_i = H_i - H_0 = H(T_i) - H(T_0). \tag{3.8}$$

at the pay dates. Using Newton's method in the calibration procedure requires the derivatives of the prices with respect to the model parameters. We observe that

$$\frac{\partial}{\partial\sqrt{\zeta_{ex}}}\hat{V}^{opt}_{rec}(0,0) = \sum_{i=1}^{n} [H_i - H_0]\,\alpha_i\left(R^{fix} - S_i\right) D_i G\left(\frac{y^* + [H_i - H_0]\,\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right) + [H_n - H_0]\, D_n G\left(\frac{y^* + [H_n - H_0]\,\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right) \tag{3.9a}$$

$$\frac{\partial}{\partial H_0}\hat{V}^{opt}_{rec}(0,0) = -\sqrt{\zeta_{ex}}\sum_{i=1}^{n} \alpha_i\left(R^{fix} - S_i\right) D_i G\left(\frac{y^* + [H_i - H_0]\,\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right) - \sqrt{\zeta_{ex}} D_n G\left(\frac{y^* + [H_n - H_0]\,\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right) \tag{3.9b}$$

$$\frac{\partial}{\partial H_i}\hat{V}^{opt}_{rec}(0,0) = \sqrt{\zeta_{ex}}\alpha_i\left(R^{fix} - S_i\right) D_i G\left(\frac{y^* + [H_i - H_0]\,\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right) \tag{3.9c}$$

$$\frac{\partial}{\partial H_n}\hat{V}^{opt}_{rec}(0,0) = \sqrt{\zeta_{ex}}\left[1 + \alpha_n\left(R^{fix} - S_n\right)\right] D_n G\left(\frac{y^* + [H_n - H_0]\,\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right) \tag{3.9d}$$

3.4. Bermudan payoff. Recall the Bermudan structure has the payment information

$$t[0,1,2,...,n] = \text{paydates,} \tag{3.10a}$$

$$C[*,1,2,...,n] = \text{full payments for each interval,} \tag{3.10b}$$

$$N[*,1,2,...,n] = \text{notionals for each interval,} \tag{3.10c}$$

and the exercise information:

$$PorR = \text{payer or receiver flag} \tag{3.10d}$$

$$t^{ex}[*1,2,...,J] = \text{exercise (notification) dates} \tag{3.10e}$$

$$t^{set}[*,1,2,...,J] = \text{settlement date if exercised at } \tau^{ex}_j \tag{3.10f}$$

$$i^{first}[*,1,2,...,J] = \text{first coupon payment received if exercised at } \tau^{ex}_j \tag{3.10g}$$

$$fee[*,1,2,...,J] = \text{exercise fee (paid on } t^{set}_j \text{)} \tag{3.10h}$$

$$rfp[*,1,2,...,J] = \text{reduction in first coupon payment received if exer at } \tau^{ex}_j \tag{3.10i}$$

If the $PorR$ flag is set to "receiver.", then the payoff on the j^{th} exercise date is

$$P_j(t^{ex}_j, x) = (C_{i_0} - rfp_j)\, Z(t^{ex}_j, x; t_{i_0}) + \sum_{i_0+1}^{n} C_i Z(t^{ex}_j, x; t_i) - (N_{i_0} + fee_j)\, Z(t^{ex}_j, x; t^{set}_j) \tag{3.11a}$$

where $i_0 = i^{first}_j$ for simplicity. Similarly, if the $PorR$ flag is set to "payer." then the j^{th} payoff is

$$P_j(t^{ex}_j, x) = -(C_{i_0} - rfp_j)\, Z(t^{ex}_j, x; t_{i_0}) - \sum_{i_0+1}^{n} C_i Z(t^{ex}_j, x; t_i) + (N_{i_0} - fee_j)\, Z(t^{ex}_j, x; t^{set}_j) \tag{3.11b}$$

Since we know the value of the (reduced) zero coupon bond,

$$Z(t, x; T) = D(T) e^{-H(T)x - \frac{1}{2}H^2(T)\zeta(t)}, \tag{3.11c}$$

we can write these payoffs explicitly. For receivers,
(3.12a)

$$P_j(t_j^{ex},x) = (C_{i_0} - rfp_j)\, D_{i_0} e^{-H_{i_0}x - \frac{1}{2}H_{i_0}^2\zeta_j} + \sum_{i_j^0+1}^{n} C_i D_i e^{-H_i x - \frac{1}{2}H_i^2\zeta_j} - \left(N_{i_j^0} + fee_j\right) D_j^s e^{-H_j^s x - \frac{1}{2}\left(H_j^s\right)^2\zeta_j},$$

and for payers,
(3.12b)

$$P_j(t_j^{ex},x) = -\,(C_{i_0} - rfp_j)\, D_{i_0} e^{-H_{i_0}x - \frac{1}{2}H_{i_0}^2\zeta_j} - \sum_{i_j^0+1}^{n} C_i D_i e^{-H_i x - \frac{1}{2}H_i^2\zeta_j} + \left(N_{i_j^0} - fee_j\right) D_j^s e^{-H_j^s x - \frac{1}{2}\left(H_j^s\right)^2\zeta_j}.$$

Here

(3.12c) $$D_i = D(t_i), \qquad D_j^s = D(t_j^{set})$$

(3.12d) $$\zeta_j = \zeta(t_j^{ex}), \qquad H_i = H(t_i), \qquad H_j^s = H(t_j^{set})$$

4. Evaluating the deal.

4.1. Rollback. Let us assume that the calibration procedure has given us $\zeta(t)$ and $H(T)$. We now show how to evaluate the Bermudan.

For each exercise j we break x into a grid of points,

(4.1a) $$x_k^{(j)} = h_j(k - m_x) \qquad \text{for } k = 0, 1, ..., 2m_x.$$

(Below, we show how to choose the spacing h_j and width $\pm h_j m_x$ of the grid). We define

(4.1b) $$P_{j,k} = P(t_j^{ex}, x_k^{(j)})$$

as the payoff if the deal is exercised at t_j^{ex}. If $PorR$ is "receiver," eqs. 3.12a - 3.12d allow us to calculate the payoff as
(4.2a)

$$P_{j,k} = (C_{i_0} - rfp_j)\, D_{i_0} e^{-H_{i_0}x_k^{(j)} - \frac{1}{2}H_{i_0}^2\zeta_j} + \sum_{i_j^0+1}^{n} C_i D_i e^{-H_i x_k^{(j)} - \frac{1}{2}H_i^2\zeta_j} - \left(N_{i_j^0} + fee_j\right) D_j^s e^{-H_j^s x_k^{(j)} - \frac{1}{2}\left(H_j^s\right)^2\zeta_j}$$

at each $k = 0, 1, ..., 2m_x$. Similarly, if $PorR$ is "payer," we calculate
(4.2b)

$$P_{j,k} = -\,(C_{i_0} - rfp_j)\, D_{i_0} e^{-H_{i_0}x - \frac{1}{2}H_{i_0}^2\zeta_j} - \sum_{i_j^0+1}^{n} C_i D_i e^{-H_i x - \frac{1}{2}H_i^2\zeta_j} + \left(N_{i_j^0} - fee_j\right) D_j^s e^{-H_j^s x_k^{(j)} - \frac{1}{2}\left(H_j^s\right)^2\zeta_j}.$$

at each $k = 0, 1, ..., 2m_x$.

Rollback is a backwards induction scheme. We first use 4.2a - 4.2bto obtain the payoff $P_{J,k}$ at the last exercise date. Then

(4.3) $$V_{J,k} = V(\tau_J^{ex}, x_k) = \max\{P_{J,k}, 0\} \qquad \text{at each } k = 0, 1, ..., 2m_x,$$

is the value of the deal on the last exercise at t_J^{ex}, assuming that it has not been exercised at an earlier exercise date.

Now suppose that we know the value of the deal at some exercise date t_j^{ex}, assuming that it was not exercised on any of the exercise dates before t_j^{ex}. That is, we know

$$V_{j,k} = V(t_j^{ex}, x_k^{(j)}), \qquad \text{where } x_k^{(j)} = h_j(k - m_x) \qquad \text{for } k = 0, 1, ..., 2m_x. \tag{4.4}$$

We now go to $j-1$. We first break x into a grid of points (see below)

$$x_k^{(j-1)} = h_{j-1}(k - m_x) \qquad \text{for } k = 0, 1, ..., 2m_x. \tag{4.5}$$

We use the Gaussian convolution formula to find the value of the deal at each node $x_k^{(j-1)}$ at t_{j-1}^{ex}:

$$V^+(t_{j-1}^{ex}, x_k^{(j-1)}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} V_j(t_j^{ex}, x_k^{(j-1)} + y\sqrt{\zeta_j - \zeta_{j-1}}) dy. \tag{4.6}$$

This is the deal's value at node $x_k^{(j-1)}$ on t_{j-1}^{ex} assuming it has not been exercised at t_{j-1}^{ex} or at any earlier exercise. We calculate this integral as the weighted sum,

$$V^+(t_{j-1}^{ex}, x_k^{(j-1)}) = V_{j-1,k}^+ = \sum_{i=0}^{2m_y} w_i V_{j,k'(i)} \tag{4.7a}$$

with

$$k'(i) = \frac{h_{j-1}(k - m_x) + y_i\sqrt{\zeta_j - \zeta_{j-1}}}{h_j} + m_x, \tag{4.7b}$$

where the weights w_i and y_i will be specified shortly. Since k' will not be an integer, one should use piecewise linear interpolation (with flat extrapolation) on $V_{j,k}$ to get the $V_{j,k'}$. Note that this sum over i has to be done for each node k, for $k = 0, 1, ..., 2m_x$.

Now $V_{j-1,k}^+ = V^+(t_{j-1}^{ex}, x_k^{(j-1)})$ is the value of the deal at t_{j-1}^{ex} assuming that the deal has not been exercised at t_{j-1}^{ex} or earlier. We now include the value of the exercise at t_{j-1}^{ex}. If the deal is exercised at t_{j-1}^{ex}, one gets the payoff $P_{j-1,k}$ given by 4.2a - 4.2b with $j \longrightarrow j-1$. Taking the maximum at each x,

$$V_{j-1,k} = \max\left\{P_{j-1,k}, V_{j-1,k}^+\right\} \qquad \text{for } k = 0, 1, ..., 2m_x \tag{4.7c}$$

now provides the the deal's value at t_{j-1}^{ex}, including the exercise at t_{j-1}^{ex}.

By looping over the rollback step, one obtains the value of the deal on the first exercise date, $V_{1,k} = V_1(t_1^{ex}, x_k)$. A final integration gives today's value of the deal:

$$V(0,0) = \sum_{i=0}^{2m_y} w_i V_{1,k'(i)} \tag{4.8a}$$

with

$$k'(i) = \frac{y_i\sqrt{\zeta_1}}{h_1} + m_x \tag{4.8b}$$

4.2. European options. Traditionally, Bermudan pricers also output the values of the European options that make up the Bermudan. This helps traders understand which exercise dates are the most valuable, and how much extra they are paying for the Bermudan over the most expensive European. Since we typically calibrate to these swaptions, the value of the European option should be the same as the market value. So for our case it is just a useful double check.

The payoff of the European option is

$$V_{j,k}^{eur} = \max\{P_j(\tau_j, x_k), 0\}, \tag{4.9a}$$

and a single integration gives today's value of the j^{th} European option of the range note

$$V_j^{eur}(0,0) = \sum_{i=0}^{2m_y} w_i V_{j,k'(i)}^{eur} \tag{4.9b}$$

with

$$k'(i) = \frac{y_i\sqrt{\zeta_j}}{h_j} + m. \tag{4.9c}$$

4.3. Discretization and weights. One usually sets the x grid to be set number of points per standard deviation, with the width of the grid being a set multiple of the standard deviation. Recall that at t_j^{ex} the variable x has mean 0 and variance $\zeta_j = \zeta(t_j^{ex})$. Setting the discretization as λ_x points per standard deviation, and extending the grid to $\pm N_x$ standard deviations, we have

$$x_k^{(j)} = h_j(k - m_x) \qquad \text{for } k = 0, 1, ..., 2m_x \tag{4.10a}$$

with

$$h_j = \sqrt{\zeta_j}/\lambda_x, \qquad m_x = w_x\lambda_x. \tag{4.10b}$$

Although some experimentation may be needed, typically $N_x = 4$ to 5.5 and $\lambda_x = 18$ to 32 work well.

To discretize the Gaussian and find the weights w_i, one again chooses the number of standard deviations and the number of points per standard deviation:

$$y_i = h_y(i - m_y) \qquad \text{for } i = 0, 1, ..., 2m_y, \tag{4.11a}$$

$$h_y = 1/\lambda_y, \qquad m_y = N_y\lambda_y, \tag{4.11b}$$

Typically $N_y = 4$ to 5.5 and $\lambda_y = 10$ to 16 work well.

One then generates a preliminary set of weights from

$$\begin{aligned} w_i &= \int_{y_i-h_y}^{y_i+h_y} \left(1 - \frac{|y_i - y|}{h_y}\right) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy \\ &= \left(1 + \frac{y_i}{h_y}\right) \mathcal{N}(y_i + h_y) - 2\frac{y_i}{h_y}\mathcal{N}(y_i) + \left(1 - \frac{y_i}{h_y}\right) \mathcal{N}(y_i - h_y) \\ &\quad + \frac{1}{h_y}\{G(y_i + h_y) - 2G(y_i) + G(y_i - h_y)\} \end{aligned} \tag{4.12a}$$

for $i = 1, 2, ..., 2m_y - 1$. Here $\mathcal{N}(y)$ is the standard cumulative normal distribution, and $G(y)$ is the Gaussian density,

$$G(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}}, \qquad \mathcal{N}(y) = \int_{-\infty}^{y} G(y)dy. \tag{4.12b}$$

For $i = 0$ and $i = 2m_y$, we have special weights,

$$w_{2m_y} = w_0 = \int_{y_0}^{y_0+h_y} \left(1 - \frac{|y_0 - y|}{h_y}\right) \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy + \int_{-\infty}^{y_0} \frac{e^{-y^2/2}}{\sqrt{2\pi}} dy$$
$$= \left(1 + \frac{y_0}{h_y}\right) \mathcal{N}(y_0 + h_y) - \frac{y_0}{h_y} \mathcal{N}(y_0) + \frac{1}{h_y} \left\{G(y_0 + h_y) - G(y_0)\right\}. \tag{4.12c}$$

4.3.1. Normalization of the weights. Once these weights are generated, one usually normalizes the weights,

$$w_i^{new} = (A + By_i^2) w_i, \tag{4.13a}$$

where the A and B are chosen so that

$$\sum_{i=0}^{2m_y} w_i^{new} = 1, \qquad \sum_{i-0}^{2m_y} y_i^2 w_i^{new} = 1. \tag{4.13b}$$

(By symmetry, all the odd moments are already zero.) If one calculates the moments with the original weights,

$$M_0 = \sum_{i=0}^{2m_y} w_i, \qquad M_2 = \sum_{i=0}^{2m_y} y_i^4 w_i, \qquad M_2 = \sum_{i=0}^{2m_y} y_i^4 w_i, \tag{4.13c}$$

we see that

$$A = \frac{M_4 - M_2}{M_0 M_4 - M_2^2}, \qquad B = -\frac{M_2 - M_0}{M_0 M_4 - M_2^2}. \tag{4.13d}$$

4.3.2. Partial sums of the weights. We can speed up our integration routine if we have the weight generation routine also return a vector of partial sums,

$$S_i = \sum_{k=0}^{i} w_k. \tag{4.14}$$

Recall that the integration step

$$V_{j-1,k}^{+} = \sum_{i=0}^{2m_y} w_i V_{jk'(i)} \tag{4.15a}$$

with

$$k'(i) = \frac{h_j (k - m_x) + y_i \sqrt{\zeta_{j+1} - \zeta_j}}{h_{j+1}} + m_x \tag{4.15b}$$

uses flat-linear-flat interpolation on the $V_{jk'(i)}$. We can replace the sum over the $i's$ with $k'(i) < 0$ and with $k'(i) > 2m_x$:

$$V_{j-1,k}^{+} = \sum_{i=i^*+1}^{i^{**}-1} w_i V_{jk'(i)} + V_{j,0} S_{i^*} + V_{j,2m_x} (1 - S_{i^{**}}) \tag{4.16a}$$

where i^* is the largest i with

(4.16b) $$k'(i^*) < 0,$$

and i^{**} is the smallest i with

(4.16c) $$k'(i^{**}) > 2m_x.$$

4.4. Accounting for the kinks. The $V_j(x)$ in the integrand has a discontinuous first derivative where the max switches from $V_j^+(x)$ to $P_j(\tau_j^{ex}, x)$. This "kink" in the integrand is the dominant error, and by eliminating this error, we can gain almost a full order of accuracy.

Recall that in each step we take the maximum at each x_k

(4.17a) $$V_{j,k} = \max\left\{P_{j,k}, V_{j,k}^+\right\} \qquad \text{for } k = 0, 1, ..., 2m_x,$$

where $V_{j,k}^+$ should be set identically zero for the last exercise $j = J$. We then evaluate the integral

(4.17b) $$V_{j-1,k}^+ = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} V_j \left(\frac{h_{j-1}(k - m_x) + y\sqrt{\zeta_j - \zeta_{j-1}}}{h_j} + m_x \right) dy$$

as the weighted sum,

(4.18a) $$V_{j-1,k}^+ = \sum_{i=0}^{2m_y} w_i V_{jk'(i)}$$

(4.18b) $$k'(i) = \frac{h_j(k - m_x) + y_i\sqrt{\zeta_{j+1} - \zeta_j}}{h_{j+1}} + m_x$$

where piecewise linear interpolation is used to obtain $V_{jk'(i)}$ from the grid points.

Suppose that when we are taking the max, $V_{j,k} = \max\left\{P_{j,k}, V_{j,k}^+\right\}$ for all k, we record where the payoff curve $P_{j,k}$ and the curve $V_{j,k}^+$ cross. Suppose that these curves cross in the interval $K < k < K+1$. Then $P_{j,K} - V_{j,K}^+$ and $P_{j,K+1} - V_{j,K+1}^+$ have opposite signs. Define

(4.19a) $$m_K = \min\left\{P_{j,K}, V_{j,K}^+\right\}, \qquad M_K = \max\left\{P_{j,K}, V_{j,K}^+\right\},$$

(4.19b) $$m_{K+1} = \min\left\{P_{j,K+1}, V_{j,K+1}^+\right\}, \qquad M_{K+1} = \max\left\{P_{j,K+1}, V_{j,K+1}^+\right\}$$

Using a linear approximation (this is all we need to make the correction), these curves cross at the point

(4.19c) $$k^* = K + \frac{M_K - m_K}{M_{K+1} - m_{K+1} + M_K - m_K} = K + 1 - \frac{M_{K+1} - m_{K+1}}{M_{K+1} - m_{K+1} + M_K - m_K}$$

in the interval. Our integration scheme is linear, so our base integration routine approximates the integrand as

(4.20a) $$V_{j,k} = M_K + (k - K)(M_{K+1} - M_K) \qquad \text{for } K < k < K+1.$$

If we approximate both $P_j(\tau_j, x_k)$ and $V_{j,k}^+$ as linear in k, then we would obtain

(4.20b) $$V_{j,k} = \begin{cases} M_K + (k-K)(m_{K+1} - M_K) & \text{for } K < k < k^* \\ m_K + (k-K)(M_{K+1} - m_K) & \text{for } k^* < k < K+1 \end{cases}.$$

The error in the integrand is

$$E(k) = \begin{cases} -(k-K)(M_{K+1} - m_{K+1}) & \text{for } K < k < k^* \\ -(K+1-k)\,(M_K - m_K) & \text{for } K < k < k^* \end{cases} . \tag{4.20c}$$

We need to make the correction to $V^+_{j-1,k}$ of

$$C^+_{j-1,k} = \frac{1}{\sqrt{2\pi}} \int_{k(y)=K}^{k(y)=K+1} e^{-y^2/2} E(k(y))dy, \tag{4.21a}$$

where

$$k(y) = \frac{h_{j-1}\,(k - m_x) + y\sqrt{\zeta_j - \zeta_{j-1}}}{h_j} + m_x. \tag{4.21b}$$

The average value of the error over the interval is

$$E_{avg} = -\tfrac{1}{2} \frac{(M_K - m_K)\,(M_{K+1} - M_K)}{M_{K+1} - m_{K+1} + M_K - m_K}. \tag{4.22a}$$

If $h_j/\sqrt{\zeta_j - \zeta_{j-1}}$isn't too large, say, $h_j/\sqrt{\zeta_j - \zeta_{j-1}} \leq 1$, we can correct the majority of the numerical error arising from the "kink" by evaluating the Gaussian at the midpoint and using the average. Thus, we should add the correction

$$C^+_{j-1,k} = \frac{h_j E_{avg}}{\sqrt{\zeta_j - \zeta_{j-1}}} G\left(\frac{h_j(K + \frac{1}{2} - m_x) - h_{j-1}(k - m_x)}{\sqrt{\zeta_j - \zeta_{j-1}}} \right) \qquad \text{if } h_j \leq \sqrt{\zeta_j - \zeta_{j-1}} \tag{4.22b}$$

to $V^+_{j-1,k}$ for each k. On rare occasions, $h_j/\sqrt{\zeta_j - \zeta_{j-1}}$ may be too large to evaluate the Gaussian at the midpoint. For these cases, one should add the correction

$$C^+_{j-1,k} = E_{avg}\mathcal{N}\left(\frac{h_j(K + 1 - m_x) - h_{j-1}(k - m_x)}{\sqrt{\zeta_j - \zeta_{j-1}}} \right) - E_{avg}\mathcal{N}\left(\frac{h_j(K - m_x) - h_{j-1}(k - m_x)}{\sqrt{\zeta_j - \zeta_{j-1}}} \right) \qquad \text{if } h_j > \sqrt{\zeta_j - \zeta_{j-1}}. \tag{4.22c}$$

Of course, the kinks should be corrected when evaluating the European options as well as the Bermudan option.nstruments, and then calibrating the model so that it matches the model prices against the market prices for these instruments.

4.5. Exotics evaluator. *The evaluation step can be written in a way which is completely independent of the deal and the LGM model.* Suppose we provide the evaluation routine with the following as inputs:
(1) the number of exercises J and the values $\zeta_1, \zeta_2, ..., \zeta_J$,
(2) a pointer to a function which calculates the payoff $P_{j,k} = P_j(x_k)$,
(3) the density of points $1/\lambda_x$ and width N_x of the x grid to be used
(4) the density of points $1/\lambda_y$ and width N_y of the y discretization to be used
The evaluation routine depends on no other input. This means that we can use the same evaluation function for different deal types just by writing new payoff functions.

4.6. Writing the payoff function. Recall that the payoff functions are

(4.23a)

$$P_{j,k} = (C_{i_0} - rfp_j)\, D_{i_0} e^{-H_{i_0} x_k^{(j)} - \frac{1}{2} H_{i_0}^2 \zeta_j} + \sum_{i_j^0+1}^{n} C_i D_i e^{-H_i x_k^{(j)} - \frac{1}{2} H_i^2 \zeta_j} - \left(N_{i_j^0} + fee_j\right) D_j^s e^{-H_j^s x_k^{(j)} - \frac{1}{2}\left(H_j^s\right)^2 \zeta_j}$$

if $PorR$ is "receiver," and

(4.23b)

$$P_{j,k} = -\,(C_{i_0} - rfp_j)\, D_{i_0} e^{-H_{i_0} x - \frac{1}{2} H_{i_0}^2 \zeta_j} - \sum_{i_j^0+1}^{n} C_i D_i e^{-H_i x - \frac{1}{2} H_i^2 \zeta_j} + \left(N_{i_j^0} - fee_j\right) D_j^s e^{-H_j^s x_k^{(j)} - \frac{1}{2}\left(H_j^s\right)^2 \zeta_j}.$$

if $PorR$ is "payer." Calculating these payoffs can be the most compute-intensive part of the calculation. (Calculating discount factors is especially worrisome since it is beyond our control.

We can ameliorate this by ensuring that there are as few redundant calculations as possible. Before reaching the evaluator, one usually creates a second structure out of the Bermudan structure. The second structure contains the following vectors:

(i) the first payment upon each exercise,

$$iFirst[*, 1, .., J] : i_0 = i_j^{first} \qquad \text{for } j = 1, ..., J \tag{4.24a}$$

(ii) the discounted full payments,

$$DisPay[*, 1, ..., n] : C_i D_i = C_i D(t_i) \qquad \text{for } i = 1, ..., n \tag{4.24b}$$

(iii) the discounted amount exchanged for the fixed leg at each exercise,

$$DisExPrice[*, 1, 2, ..., J] : \left(N_{i_j^0} \pm fee_j\right) D_j^s = \left(N_{i_j^0} \pm fee_j\right) D(t_j^{set}) \qquad \text{for } j = 1, ..., J \tag{4.24c}$$

(where the "$+$ȷ sign is for receivers, and the "$-$ȷ sign is for payers),

(iv) the mean reversion function on each pay date and on each settlement date,

$$H[*, 1, 2, ..., n] : H_i = H(t_i) \qquad \text{for } i = 1, ..., n \tag{4.24d}$$

$$Hset[*, 1, 2, ..., J\} : H_j^s = H(t_j^{set}) \qquad \text{for } j = 1, ..., J, \tag{4.24e}$$

(v) finally the value ζ at the exercise dates:

$$\zeta[*, 1, 2, ..., J) : \zeta_j = \zeta(t_j^{ex}) \qquad \text{for } j = 1, ..., J. \tag{4.24f}$$

For completeness, the structure also contains

(vi) the number of exercises and number of paydates:

$$J = \text{number of exercises} \tag{4.24g}$$

$$n = \text{number of payments} \tag{4.24h}$$

The payoff functions can be calculated entirely from these pre-calculated vectors. Besides making the code more efficient, this enhances the soundness of the code because neither the discount curve $D(t)$ nor the model parmeters $\zeta(t)$ and $H(T)$, nor the original Bermudan structure needs to be passed down any further into the evaluator. The only thing inputs needed for the core evaluation routine are the new structure, a pointer to the function which calculates the payoffs $P_{j,k}$ from the new structure, the vector $\zeta[*, 1, .., J]$ of variances, and the dicretization variables λ_x, N_x, λ_y, and N_y.

(The $\zeta[*, 1, .., J]$ vector *should* be passed outside of the second structure because the evaluator should just pass the structure to the payoff function without relying on what's inside; the $\zeta[*, 1, .., J]$ vector should also be passed inside the structure so that the payoff can be constructed solely from information stored within the structure.)

One other comment about efficiency. Normally one calls the payoff function with the entire vector $x[0, 1, ..., 2m_x]$ and it returns the vector $P_{j,k}$ for $k = 0, 1, 2, ..., 2m_x$. For many deal types, the payoff vector can be caculated more efficiently than the individual payoffs.

5. Calibration. The calibration procedure consists of three steps. First is to characterize the deal by extracting its essential features. Second is to select a set of vanilla calibration instruments based on the characterization and an over-all calibration strategy. The last part is applying the algorithms that choose $\zeta(t)$ and $H(T)$ to match the LGM and market prices of the calibration instruments.

Careful inspection will show that only the characterization step depends on the exotic being a Bermudan; the remaining two steps depend only on the features extracted by the the characterization step. This means that to handle the calibration step for other deal types (callable inverse floaters, callable capped floaters, callable range notes, ...) we just need to re-write the deal characterization part of the routine.

5.1. Deal characterization. We characterize deals by three quantities for each exercise. The first is the exercise (notification) date itself,

$$t_j^{ex} \qquad \text{for } j = 1, 2, ..., J. \tag{5.1a}$$

The second quantity is the length of the swap obtained upon exercise,

$$\ell_j = t_n - t_j^{set} \qquad \text{for } j = 1, 2, ..., J. \tag{5.1b}$$

The last last piece of information determines how far the underlying is from being at-the-money for each exercise. There are several different measures of this distance. The one I prefer is to determine the parallel shift γ_j needed,

$$D(t_i) \longrightarrow D(t_i)e^{-\gamma_j t_i} \tag{5.2}$$

so that today's value of the j^{th} payoff is at-the-money.

Suppose that if the deal is exercised at t_j^{ex}. The receiver gets

$$C_{i_0} - rfp_j \qquad \text{paid at } t_{i_0}, \tag{5.3a}$$

$$C_i \qquad \text{paid at } t_i \qquad \text{for } i = i_0 + 1, ..., n, \tag{5.3b}$$

and in return pays

$$N_{i_j^0} + fee_j \qquad \text{paid at } t_j^{set} \qquad \text{if } PorR \text{ is reciever,} \tag{5.3c}$$

$$N_{i_j^0} - fee_j \qquad \text{paid at } t_j^{set} \qquad \text{if } PorR \text{ is payer.} \tag{5.3d}$$

Here we are using the abbreviation $i_0 = i_j^{first}$ for the first paydate after settlement. Clearly this payoff is at the money when

$$(C_{i_0} - rfp_j)\, D(t_{i_0})e^{-\gamma_j\left(t_{i_0} - t_j^{set}\right)} + \sum_{i=i_j^0+1}^{n} C_i D(t_i)e^{-\gamma_j\left(t_i - t_j^{set}\right)} = \left(N_{i_j^0} \pm fee_j\right) D(t_j^{set}). \tag{5.4}$$

The idea behind characterization is that the "most natural" set of vanilla instruments for representing the Berumdan are the swaptions (one for each exercise date) which

(a) have the same exercise date,
(b) have the same length of the underlying swap,
(c) are at-the-money for the same parallel shift of the yield curve.
Using these swaptions in calibration implies that our vega risks will be to these swaptions, and in the normal course of events, our Bermudan would then be hedged by a linear combination of these swaptions.

This is eminently reasonable for Bermudan options on bullet swaps (and like-shaped underlyings). It is less reasonable for Bermudan options on amortizing swaps, and perhaps for zero coupon swaps. Would a 10 year option on a 20 year amorting swap be better represented by an 10 into 20 bullet swaption, or a 10 into 10 bullet? In appendix A we develop a more robust method of characterizing the option based on the duration and convexity of the payoff. This method should be used for options on amortizers or zero-coupon swaps. Here we calibrate based on the above characterization. In Appendix A we point out the differences needed for amortizers.

5.2. Calibration instruments.

5.2.1. Diagonal swaptions. Most decent calibration methods use the Bermudan's diagonal swaptions, which we construct here. For each of the exercise dates t_j^{ex}, let T_j^{set} be the currency's standard spot date:

$$T_j^{set} = \text{SpotDate}(t_j^{ex}, ccy) \qquad \text{for } j = 1, 2, ..., J. \tag{5.5}$$

Let t_{end}^{th} and t_{Berm}^{act} be the theoretical and actual end dates of the Bermudan. The diagonal swaptions are the swaptions with exercise date t_j^{ex}, start date T_j^{set}, and the end date T_n for $j = 1, 2, ..., J$. It remains to choose the strike R_j^{diag} of these swaptions and to construct the payments.

Let us create a standard fixed leg and floating leg schedules based on the theoretical end date t_{end}^{th}:

$$T_0, T_1, T_2, ..., T_n = t_{end}^{act}. \tag{5.6a}$$

$$T_0^{flt}, T_1^{flt}, T_2^{flt}, ..., T_m^{flt} = t_{end}^{act}. \tag{5.6b}$$

The longest diagonal swap is $j = 1$, which starts at T_1^{set}. The schedule should be carried back far enough so that T_0 and T_0^{flt} are on or before this start date:

$$T_0 \le T_1^{set} < T_1, \qquad T_0^{flt} \le T_1^{set} < T_1^{flt} \tag{5.7}$$

For each swaption j, let i_j^1 be the index of the first pay date after T_j^{set}:

$$T_{i_j^1 - 1} \le T_j^{set} < T_{i_j^1}. \tag{5.8}$$

Then the fixed leg payments for swaption j are:

$$\tilde{\alpha}_j R_j^{diag} \qquad \text{at } T_i \qquad \text{for } i = i_j^1 \tag{5.9a}$$

$$\alpha_i R_j^{diag} \qquad \text{at } T_i \qquad \text{for } i = i_j^1 + 1, i_j^1 + 2, ..., n-1 \tag{5.9b}$$

$$1 + \alpha_n R_j^{diag} \qquad \text{at } T_n \qquad \text{for } i = n \tag{5.9c}$$

Here,

$$\alpha_i = \text{cvg}(T_{i-1}, T_i, \text{fix}) \qquad \text{for } i = 1, 2, .., n \tag{5.9d}$$

is the fixed leg day count fraction for the full periods, and

$$\tilde{\alpha}_j = \text{cvg}(T_j^{set}, T_{i_j^1}, \text{fix}) \tag{5.9e}$$

is the day count fraction for the first period, which is may be a stub. (The argument "fix" means to used the fixed leg's day count basis).

We now constuct the floating leg, converting the basis spreads from the floating leg's frequency and basis to the fixed leg's frequency and basis. Consider a floating leg that starts at, say, k_0. This floating leg is equivalent to

$$\text{(5.10a)} \qquad 1 \qquad \text{at } T_{k_0}^{flt}$$

$$\text{(5.10b)} \qquad \beta_k \, \text{bs}_k \qquad \text{at } T_k^{flt} \qquad \text{for } k = k_0+1, ..., m.$$

Here, bs_k is the basis spread for the period beginning at T_{k-1}^{flt} and ending at T_k^{flt}, and

$$\text{(5.10c)} \qquad \beta_k = \text{cvg}(T_{k-1}^{flt}, T_k^{flt}, \text{flt}) \qquad \text{for } k = 1, 2, .., m$$

is the day count fraction for the full floating point period. We convert the basis spreads to the fixed leg's frequency and day count basis in the usual way.

If the floating leg frequency is the shorter than, or equal to, the fixed leg frequency, define

$$\text{(5.11a)} \qquad S_i = \frac{\sum_{k \in I_i} \beta_k \, \text{bs}_k \, D(T_k^{flt})}{\alpha_i D(T_i)}$$

where $k \in I_i$ are the floating leg intervals that are part of the i^{th} fixed leg interval:

$$\text{(5.11b)} \qquad k \in I_i \qquad \text{if and only if } T_{i-1}^{th} < T_k^{flt,th} \leq T_i^{th}.$$

If the floating leg frequency is longer than the fixed leg frequency (this is rare), define

$$\text{(5.12a)} \qquad S_i = \frac{\beta_k \, \text{bs}_k \, D(T_k^{flt})}{\sum_{i \in I_k} \alpha_i D(T_i)}$$

where $i \in I_k$ are the fixed leg intervals that are part of the k^{th} floating leg interval:

$$\text{(5.12b)} \qquad i \in I_k \qquad \text{if and only if } T_{k-1}^{flt,th} < T_i^{th} \leq T_k^{flt,th}.$$

We the j^{th} swaption, we approximate the floating leg payments as being equivalent to

$$\text{(5.13a)} \qquad 1 \qquad \text{at } T_j^{set}$$

$$\text{(5.13b)} \qquad \tilde{\alpha}_j S_i \qquad \text{at } T_i \qquad \text{for } i = i_j^1$$

$$\text{(5.13c)} \qquad \alpha_i S_i \qquad \text{at } T_i \qquad \text{for } i = i_j^1+1, i_j^1+2, ..., n$$

The net payments for the swaption are

$$\text{(5.14a)} \qquad -1 \qquad \text{at } T_j^{set}$$

$$\text{(5.14b)} \qquad \tilde{\alpha}_j \left(R_j^{diag} - S_i\right) \qquad \text{at } T_i \qquad \text{for } i = i_j^1$$

$$\text{(5.14c)} \qquad \alpha_i \left(R_j^{diag} - S_i\right) \qquad \text{at } T_i \qquad \text{for } i = i_j^1+1, i_j^1+2, ..., n$$

$$\text{(5.14d)} \qquad 1 + \alpha_n \left(R_j^{diag} - S_n\right) \qquad \text{at } T_n \qquad \text{for } i = n$$

We now choose the strikes of the diagonal swaptions. The strike swaption j is set so that the swaption is in the money at the same shift as the Bermudan:

$$R_j^{diag} = \frac{D_j^{set} - D_n e^{-\gamma_j (T_n - T_j^{set})} + \tilde{\alpha}_j S_{i_j^1} D_{i_j^1} e^{-\gamma_j (T_{i_j^1} - T_j^{set})} + \sum_{i=i_j^1+1}^{n} \alpha_i S_i D_i e^{-\gamma_j (T_i - T_j^{set})}}{\tilde{\alpha}_j D_{i_j^1} e^{-\gamma_j (T_{i_j^1} - T_j^{set})} + \sum_{i=i_j^1+1}^{n} \alpha_i D_i e^{-\gamma_j (T_i - T_j^{set})}}, \tag{5.15a}$$

where

$$D_j^{set} = D(T_j^{set}), \qquad D_i = D(T_i), \qquad \text{etc.} \tag{5.15b}$$

After constructing the diagonal swaptions, we obtain their market price via Black's formula,

$$\text{Mkt}_j^{diag} = \left\{ \tilde{\alpha}_j D_{i_j^1} + \sum_{i=i_j^1+1}^{n} \alpha_i D_i \right\} \left\{ R_j^{diag} \mathcal{N}(d_1) - R_j^{sw} \mathcal{N}(d_2) \right\}, \tag{5.16a}$$

where R_j^{sw} is the (break even) swap rate for the j^{th} diagonal swap,

$$R_j^{sw} = \frac{D_j^{set} - D_n + \tilde{\alpha}_j S_{i_j^1} D_{i_j^1} + \sum_{i=i_j^1+1}^{n} \alpha_i S_i D_i}{\tilde{\alpha}_j D_{i_j^1} + \sum_{i=i_j^1+1}^{n} \alpha_i D_i}, \tag{5.16b}$$

and where

$$d_{1,2} = \frac{\log R_j^{diag}/R_j^{sw} \pm \frac{1}{2}\sigma^2 t_j^{ex}}{\sigma \sqrt{t_j^{ex}}}. \tag{5.16c}$$

Here σ is the log normal volatility obtained from, for example, the *GetVol* function.

5.2.2. Row swaptions. Some calibration methods use the Bermudan's "row" swaptions. Let t_1^{ex} be the earliest exercise date of the Bermudan, and let T_1^{set} be the corresponding spot date. The j^{th} row swaption is the swaption with start date T_1^{set} and end date T_j. It's equivalent payments are:

(5.17a)	-1	at T_1^{set}	
(5.17b)	$\tilde{\alpha}_1 \left(R_j^{row} - S_i \right)$	at T_i	for $i = i_1^1$
(5.17c)	$\alpha_i \left(R_j^{row} - S_i \right)$	at T_i	for $i = i_1^1 + 1, i_1^1 + 2, ..., j-1$
(5.17d)	$1 + \alpha_j \left(R_j^{row} - S_j \right)$	at T_j	for $i = j$

Here the dates T_i, day count fractions $\tilde{\alpha}_j, \alpha_i$ and equivalent basis spreads S_i are the precisely the same quantities calculated for the diagonal swaptions.

If the exercise date t_1^{ex} is too near today, say less than 3 months, then one should choose replace it with the first exercise date t_j^{ex} which is, say, at least 3 months from today.

The diagonal swaptions are defined for $j = i_{\min}, i_{\min} + 1, ..., n$ where $T_{i_{\min}} - T_1^{set}$ is the shortest inteval which makes a decent swap (say 10 months).

We choose the strike :

$$R_j^{row} = \frac{D_1^{set} - D_j e^{-\gamma_1 (T_j - T_1^{set})} + \tilde{\alpha}_1 S_{i_1^1} D_{i_1^1} e^{-\gamma_1 (T_{i_1^1} - T_1^{set})} + \sum_{i=i_1^1+1}^{j} \alpha_i S_i D_i e^{-\gamma_1 (T_i - T_1^{set})}}{\tilde{\alpha}_1 D_{i_1^1} e^{-\gamma_1 (T_{i_1^1} - T_1^{set})} + \sum_{i=i_1^1+1}^{j} \alpha_i D_i e^{-\gamma_1 (T_i - T_1^{set})}}, \tag{5.18}$$

These strikes are all at the money at the same parallel shift γ_1 at the Bermudan's *first* payoff for t_{ex}^1.

The market value of these swaptions are

$$\text{Mkt}_j^{row} = \left\{ \tilde{\alpha}_1 D_{i_1^1} + \sum\nolimits_{i=i_1^1+1}^{j} \alpha_i D_i \right\} \left\{ R_j^{row} \mathcal{N}(d_1) - R_j^{sw} \mathcal{N}(d_2) \right\}, \tag{5.19a}$$

where R_j^{sw} is the (break even) swap rate for the j^{th} row swap,

$$R_j^{sw} = \frac{D_1^{set} - D_j + \tilde{\alpha}_1 S_{i_1^1} D_{i_1^1} + \sum_{i=i_1^1+1}^{j} \alpha_i S_i D_i}{\tilde{\alpha}_1 D_{i_1^1} + \sum_{i=i_1^1+1}^{j} \alpha_i D_i}, \tag{5.19b}$$

and where

$$d_{1,2} = \frac{\log R_j^{row}/R_j^{sw} \pm \frac{1}{2}\sigma^2 t_j^{ex}}{\sigma\sqrt{t_j^{ex}}}. \tag{5.19c}$$

Again the implied vol σ needs to be obtained from, *e.g.*, GetVol.

5.2.3. Column swaptions. For the calibration strategies which use a column of swaptions, we choose the swaptions which have exercise date t_j^{ex}, start date T_j^{set}, and end date $T_{i_j^{end}}$ where i_j^{end} is the first index such that $T_{i_j^{end}} - T_j^{set}$ makes a decent swap (is at least, say, 10 months long). For each $j = 1, 2, ..., J$, the equivalent payments for swaptions j is:

$$-1 \qquad \text{at } T_j^{set} \tag{5.20a}$$

$$\tilde{\alpha}_j \left(R_j^{col} - S_i \right) \qquad \text{at } T_i \qquad \text{for } i = i_j^1 \tag{5.20b}$$

$$\alpha_i \left(R_j^{col} - S_i \right) \qquad \text{at } T_i \qquad \text{for } i = i_1^1 + 1, ..., i_j^{end} - 1 \tag{5.20c}$$

$$1 + \alpha_i \left(R_j^{col} - S_i \right) \qquad \text{at } T_i \qquad \text{for } i = i_j^{end} \tag{5.20d}$$

Here the dates T_i, day count fractions $\tilde{\alpha}_j, \alpha_i$ and equivalent basis spreads S_i are the precisely the same quantities calculated for the diagonal swaptions. We choose the strike R_j^{col} so that each swaption is at the money for the same parallel shift as the Bermudan,

$$R_j^{col} = \frac{D_j^{set} - D_{i_j^{end}} e^{-\gamma_j (T_{i_j^{end}} - T_j^{set})} + \tilde{\alpha}_j S_{i_j^1} D_{i_j^1} e^{-\gamma_j (T_{i_j^1} - T_j^{set})} + \sum_{i=i_j^1+1}^{i_j^{end}} \alpha_i S_i D_i e^{-\gamma_j (T_i - T_j^{set})}}{\tilde{\alpha}_j D_{i_j^1} e^{-\gamma_j (T_{i_j^1} - T_j^{set})} + \sum_{i=i_j^1+1}^{i_j^{end}} \alpha_i D_i e^{-\gamma_j (T_i - T_j^{set})}}, \tag{5.20e}$$

After constructing the column swaptions, we obtain their market price via Black's formula,

$$\text{Mkt}_j^{col} = \left\{ \tilde{\alpha}_j D_{i_j^1} + \sum\nolimits_{i=i_j^1+1}^{i_j^{end}} \alpha_i D_i \right\} \left\{ R_j^{col} \mathcal{N}(d_1) - R_j^{sw} \mathcal{N}(d_2) \right\}, \tag{5.21a}$$

where R_j^{sw} is the (break even) swap rate for the j^{th} column swap,

$$R_j^{sw} = \frac{D_j^{set} - D_{i_j^{end}} + \tilde{\alpha}_j S_{i_j^1} D_{i_j^1} + \sum_{i=i_j^1+1}^{i_j^{end}} \alpha_i S_i D_i}{\tilde{\alpha}_j D_{i_j^1} + \sum_{i=i_j^1+1}^{i_j^{end}} \alpha_i D_i}, \tag{5.21b}$$

and where

$$d_{1,2} = \frac{\log R_j^{col}/R_j^{sw} \pm \frac{1}{2}\sigma^2 t_1^{ex}}{\sigma\sqrt{t_1^{ex}}}. \tag{5.21c}$$

Here σ is the log normal volatility obtained from, for example, the *GetVol* function.

5.2.4. Caplets. For the calibration strategies which use caplets (floorlets), we choose the swaptions which have exercise date t_j^{ex}, start, start date T_j^{set}, and end date T_j^{end}, where the end date is either 3 months or 6 months from the start date, depending on the currency. For each $j = 1, 2, ..., J$, the equivalent payments for caplet j is:

(5.22a) $$-1 \qquad \text{at } T_j^{set}$$

(5.22b) $$1 + \beta_j \left(R_j^{cap} - \text{bs}_j \right) \qquad \text{at } T_j^{end}$$

where

(5.22c) $$\beta_j = \text{cvg}(T_j^{set}, T_j^{end}, \text{flt}) \qquad \text{for } j = 1, 2, .., J$$

is the appropriate day count fraction. Here bs_j is the basis spread for the floating rate set for start date T_j^{set}. We choose the strike R_j^{cap} so that each swaption is at the money for the same parallel shift as the Bermudan,

(5.22d) $$R_j^{cap} = \frac{D_j^{set} + \left(1 - \beta_j \, \text{bs}_j\right) D_j^{end} e^{-\gamma_j (T_j^{end} - T_j^{set})}}{\beta_j D_j^{end} e^{-\gamma_j (T_j^{end} - T_j^{set})}},$$

After constructing the column swaptions, we obtain their market price via Black's formula,

(5.23a) $$\text{Mkt}_j^{cap} = \left\{ \beta_j D_j^{end} \right\} \left\{ R_j^{cap} \mathcal{N}(d_1) - R_j^{FRA} \mathcal{N}(d_2) \right\},$$

where R_j^{FRA} is the (break even) rate for the j^{th} diagonal swap is

(5.23b) $$R_j^{FRA} = \frac{D_j^{set} - \left(1 - \beta_j \, \text{bs}_j\right) D_j^{end}}{\beta_j D_j^{end}},$$

and where

(5.23c) $$d_{1,2} = \frac{\log R_j^{cap} / R_j^{FRA} \pm \frac{1}{2}\sigma^2 t_j^{ex}}{\sigma \sqrt{t_j^{ex}}}.$$

Here σ is the log normal caplet volatility obtained from, for example, the *GetVol* function.

5.3. Calibration to the diagonal swaptions. Having constructed the universe of possible calibration instruments, we now go through the calibration strategies and algorithms one by one. Pricing Bermudans accurately requires calibrating the model to the diagonal swaptions. For if our model doesn't correctly price the European swaptions that make up the Bermudan, how could we believe the price obtained for the Bermudan? In this section we present the strategies for calibrating on diagonal swaptions. These are: calibration to the diagonal with a constant mean revesion κ; calibration to the diagonal with a known function $H(T)$; calibration to the diagonal with a linear $\zeta(t)$, and calibration to the diagonal with a known $\zeta(T)$.

For instruments other than Bermudans, it may be appropriate to calibrate to other series of vanilla instruments. So following are sections devoted to calibating on a series of caplets, to calibrating on a column of swaptions, and to calibrating to a row of swaptions

Since the LGM model has two model "parameters," $\zeta(t)$ and $H(T)$, we can calibrate jointly to two distinct series of vanilla instrumetns. In the final section we present calibration strategies which calibrate jointly to the diagonal swaptions plus another series of instruments. These are: calibration to the diagonal swaptions and a row swaptions, calibration to the diagonal swaptions and a column swaptions, and calibrating to the diagonal swaptions and to caplets. For completeness, we also calibrate on a row and column of swaptions, on a row of swaptions and to caplets.

5.3.1. Calibration to the diagonal swaptions with constant κ. For this calibration strategy, the mean reversion coefficient κ is a user-supplied constant (Where to obtain good wake-up values for κ is discussed below. Empirically κ is usually between -1% and $+5\%$..

Recall that $H"(T)/H(T) = -\kappa$, so that $H(T) = Ae^{-\kappa T} + B$ for some constants A and B. At this point we use the model invariants $H(T) \longrightarrow CH(T)$ and $H(T) \longrightarrow H(T) + K$ to set

$$H(T) = \frac{1 - e^{-\kappa T}}{\kappa}, \tag{5.24}$$

without loss of generality, where T is measured in years. With $H(T)$ known, we compute

$$H_j^s = H(T_j^{set}) = \frac{1 - e^{-\kappa T_j^{set}}}{\kappa} \qquad \text{for } j = 1, 2, ..., J \tag{5.25a}$$

$$H_i = H(T_i) = \frac{1 - e^{-\kappa T_i}}{\kappa} \qquad \text{for } i = 1, 2, ..., n. \tag{5.25b}$$

We now determine $\zeta_j = \zeta(t_j^{ex})$ for each j by calibrating to diagonal j.

Recall that if the j^{th} diagonal swaption is exercised at its notification date t_j^{ex}, the payments are

$$-1 \qquad \text{at } T_j^{set} \tag{5.26a}$$

$$\tilde{\alpha}_j \left(R_j^{diag} - S_i\right) \qquad \text{at } T_i \qquad \text{for } i = i_j^1 \tag{5.26b}$$

$$\alpha_i \left(R_j^{diag} - S_i\right) \qquad \text{at } T_i \qquad \text{for } i = i_j^1 + 1, i_j^1 + 2, ..., n \tag{5.26c}$$

$$1 + \alpha_n \left(R_j^{diag} - S_n\right) \qquad \text{at } T_n \qquad \text{for } i = n, \tag{5.26d}$$

Under the LGM model, the value of this swaption is thus

$$\begin{aligned} V_j^{diag}(0,0) = {} & \tilde{\alpha}_j \left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} \mathfrak{N}\left(\frac{y^* + \Delta H_{i_j^1}\zeta_j}{\sqrt{\zeta_j}}\right) \\ & + \sum_{i=i_j^1+1}^{n} \alpha_i \left(R_j^{diag} - S_i\right) D_i \mathfrak{N}\left(\frac{y^* + \Delta H_i \zeta_j}{\sqrt{\zeta_j}}\right) \\ & + D_n \mathfrak{N}\left(\frac{y^* + \Delta H_n \zeta_j}{\sqrt{\zeta_j}}\right) - D_j^{set} \mathfrak{N}\left(\frac{y^*}{\sqrt{\zeta_j}}\right) \end{aligned} \tag{5.27a}$$

where y^* is obtained by solving

$$\begin{aligned} & \tilde{\alpha}_j \left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} e^{-\Delta H_{i_j^1} y^* - \frac{1}{2}\Delta H_{i_j^1}^2 \zeta_j} + \sum_{i=i_j^1+1}^{n} \alpha_i \left(R_j^{diag} - S_i\right) D_i e^{-\Delta H_i y^* - \frac{1}{2}\Delta H_i^2 \zeta_j} \\ & \qquad + D_n e^{-\Delta H_n y^* - \frac{1}{2}\Delta H_n^2 \zeta_j} = D_j^{set}, \end{aligned} \tag{5.27b}$$

and where we have used

$$\Delta H_i = H_i - H_j^{set} = H(T_i) - H(T_j^{set}) \tag{5.27c}$$

We also have a formula for the derivative

$$\frac{\partial}{\partial\sqrt{\zeta_j}} V_j^{diag}(0,0) = \Delta H_{i_j^1}\tilde{\alpha}_j \left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} G\left(\frac{y^* + \Delta H_{i_j^1}\zeta_j}{\sqrt{\zeta_j}}\right) + \sum_{i=i_j^1+1}^{n} \Delta H_i \alpha_i \left(R_j^{diag} - S_i\right) D_i G\left(\frac{y^* + \Delta H_i \zeta_j}{\sqrt{\zeta_j}}\right) + \Delta H_n D_n G\left(\frac{y^* + \Delta H_n \zeta_j}{\sqrt{\zeta_j}}\right). \tag{5.28}$$

We can use a global Newton's scheme to compute the value of $\sqrt{\zeta_j}$ which sets the theoretical price to the market price:

$$V_j^{diag}(0,0) = \mathrm{Mkt}_j^{diag} \tag{5.29}$$

Repeating for all j gives us $\zeta(0) = 0$ and $\zeta(t_j^{ex})$ for $j = 1, 2, ..., J$. We use piecewise linear interpolation to get values of $\zeta(t)$ at other values of t. It should be noted that evaluating the Bermudan *does not* require $\zeta(t)$ at any other dates.

Re-scaling $H(T)$ and $\zeta(t)$. At this point we have both $H(T)$ and $\zeta(t)$. Many firms find it convenient to use a standard scaling for $H(T)$ and $\zeta(t)$, to aid intuition if for no other reason. One can use the invariances

$$H(T) \longrightarrow H(T) + C, \qquad \zeta(t) \longrightarrow \zeta(t) \tag{5.30a}$$

$$H(T) \longrightarrow KH(T), \qquad \zeta(t) \longrightarrow \zeta(t)/K^2 \tag{5.30b}$$

to re-scale these quantities, if desired. For example, many people choose to set $H(0) = 0$ and $H(t_{end}) = t_{end}$, where t_{end} is the final pay date of the deal in years.

Aside: Initial guess. An accurate initial guess for $\sqrt{\zeta_j}$ can be found from the equivelent vol formula. This yields

$$\sqrt{\frac{\zeta_j}{\sigma t_j^{ex} R_j^{sw} R_j^{diag}}} \approx \frac{\tilde{\alpha}_j D_{i_j^1} + \sum_{i=i_j^1+1}^{n} \alpha_i D_i}{\tilde{\alpha}_j \left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} \Delta H_{i_j^1} + \sum_{i=i_j^1+1}^{n} \alpha_i \left(R_j^{diag} - S_i\right) D_i \Delta H_i + D_n \Delta H_n} \tag{5.31}$$

where σ is the swaptions implied vol from the marketplace.

Aside: Global Newton's method for one parameter fits. Suppose one is trying to solve

$$f(z) = \text{target} \tag{5.32a}$$

for z. Normally one starts from an intial guess z_0, and expands $f(z_{n+1}) = f(z_n + \delta z) \approx f(z_n) + f'(z_n)\delta z$ to obtain a Newton's method:

$$\delta z = z_{n+1} - z_n = \frac{\text{target} - f(z_n)}{f'(z_n)}. \tag{5.32b}$$

Provided this algorithm converges, it converges very rapidly. Unfortunately, this algorithm sometimes diverges.

The global Newton method differs in only one respect: after calculating the Newton step δz, one checks to see if taking the step decreases the error. If it does, one accepts the step. If it does not, then one cuts the step in half, and then again checks to see if the error decreases. Eventually the error will decrease, and the step is accepted. The next Newton step is then calculated.

Aside: Infeasible market prices. Since

$$\zeta(t) = \int_0^t \alpha^2(t')dt', \tag{5.33}$$

clearly $\zeta(t)$ must be an increasing function of t :

$$0 = \zeta(0) \le \zeta_1 \le \zeta_2 \le \cdots \le \zeta_J. \tag{5.34}$$

Since each ζ_j is calibrated seperately, it may happen that $\zeta_j < \zeta_{j-1}$. (In practice this happens very, very rarely, but it *does* happen). One should test to see that the condition $\zeta_j \ge \zeta_{j-1}$is true after each ζ_j is found, and when this condition is violated, one should replace ζ_j by ζ_{j-1}, its minimum feasible value:

$$\zeta_j \longrightarrow \zeta_{j-1} \qquad \text{if } \zeta_j < \zeta_{j-1}. \tag{5.35}$$

This means that the j^{th} swaption will be priced at the closest possible price to the market price attainable within the calibrated LGM model, but it will not match the price exactly.

Aside: Where do the κ's come from?. Suppose we set κ, calibrate the model to the diagonal, and then price the Bermudan. The resulting Bermudan price is a slightly increasing function of κ. Selecting the right κ ensures that we match the market price for the Bermudan. Desks often use a matrix to keep track of the κ needed to price a "y NC x" Bermudan correctly. That is, they fill in the κ's for the liquid Bermudans, and use "continuity" obtain the other entries in the matrix. Empirically, the κ change very, very slowly. market makers keep track of the mean reversion κ.

We should plan to have a matrix of "wake-up" values, perhaps by currency, for this strategy. I can obtain the current κ matrix.

5.3.2. Calibration to the diagonal swaptions with $H(T)$ specified. Suppose that $H(T)$ is specified *a priori*. (A possible source of such curves $H(T)$ is indicated below). Typically $H(T)$ is given at discrete points $H(T_1), H(T_2), \ldots, H(T_N)$, and piecewise linear interpolation is used between nodes. Piecewise linear interpolation is equivalent to assuming that all shifts of the forward rate curve are by piecewise constant curves.

With $H(T)$ set, we can use the preceding procedure and formulas to calibrate on the diagonal swaptions. This determines the value of $\zeta(t)$ at $t_1^{ex}, t_2^{ex}, \ldots, t_J^{ex}$. As above, one adds the point $\zeta(0) = 0$, one ensures that the $\zeta_j = \zeta(t_j^{ex})$ are increasing, and one re-scales $\zeta(t), H(T)$ to taste. If one needs $\zeta(t)$ for other values of t, one uses piecewise linear interpolation.

Origin of the $H(T)$. Suppose one had the set of 30 NC 20, 30 NC 15, 30 NC 10, 30 NC 5 and 30 NC 1 Bermudan swaptions. Wouldn't it be nice if the same curve $H(T)$ were used for each of these Bermudans? The 30 NC 10 Bermudan includes the 30 NC 15 and the 30 NC 20 Bermudans. It would be satisfying if our valuation procedure for the 30 NC 15 and 30 NC 20 assigned the same price to these Bermudans regardless of whether they were individual deals or part of a larger Bermudan.

One could arrange this by first using a constant κ, let's call it κ_4, to calibrate and price the 30 NC 20 Bermudan. Without loss of generality, we could select

$$\begin{aligned} H'(T) &= e^{\kappa_4(T_{30}-T)} \\ H(T) &= -\tfrac{e^{\kappa_4(T_{30}-T)}-1}{\kappa_4} \end{aligned} \qquad \text{for } T_{20} \le T \le T_{30}. \tag{5.36a}$$

We would calibrate on the diagonal to find $\zeta(t)$ at expiry dates $\tau_m, \tau_{m+1}, \ldots$ beyond 20 years, and then price the 30 NC 20 Bermudan. Selecting the right value of κ_4 would match the Bermudan price to its market value. Neither the swaption prices nor the Bermudan prices depend on $H(T)$ or $\zeta(t)$ for dates before the 20 year point.

To price the 30 NC 15, one could use the $H(T)$ obtained from κ_4 for years 20 to 30, and choose a different kappa, say κ_3, for years 15 to 20:

$$H'(T) = e^{\kappa_3(T_{20}-T)} e^{\kappa_4(T_{30}-T_{20})}$$

$$H(T) = -\frac{e^{\kappa_3(T_{20}-T)} - 1}{\kappa_3} e^{\kappa_4(T_{30}-T_{20})} - \frac{e^{\kappa_4(T_{30}-T_{20})} - 1}{\kappa_4} \qquad \text{for } T_{15} \le T \le T_{20}. \tag{5.36b}$$

Calibrating would produce the same $\zeta(t)$ values for years 20 to 30 as before. In addition, for each κ_3 it would determine $\zeta(t)$ for years 15 to 20. By selecting the right κ_3, one could match the 30 NC 15 Bermudan's market price.

Continuing in this way, one produces the values of $\zeta(t)$ and $H(T)$ for years 10 to 15, for years 5 to 10, and finally for years 1 to 5. This $\zeta(t)$ and $H(T)$ would then yield a model which matches all the diagonal swaptions and happens to correctly price all the liquid, 30y co-terminal Bermudans. These $\kappa(t)$'s turn out to be extremely stable, only varying very rarely, and then by small amounts. Typically a desk would remember the $\kappa(t)$'s as a function of the co-terminal points, relying on the same $\kappa(t)$'s for years. I will obtain the current $\kappa t)$'s to use for the wakeup value for this strategy.

In general, if T_n is the co-terminal point and $T_0, T_1, \ldots, T_{n-1}$ are the "no call" points, then $H(T)$ is:

$$H(T) = -\frac{e^{\kappa_j(T_j-T)} - 1}{\kappa_j} \prod_{i=j+1}^{n} e^{\kappa_i(T_i-T_{i-1})} - \sum_{k=j+1}^{n} \frac{e^{\kappa_k(T_k-T_{k-1})} - 1}{\kappa_k} \prod_{i=k+1}^{n} e^{\kappa_i(T_i-T_{i-1})} \qquad \text{for } T_{j-1} \le T \le T_j \tag{5.37}$$

5.3.3. Calibration to the diagonal swaptions with linear $\zeta(t)$. This is an idea pioneered by Solomon brothers. Let us use a constant local volatility. Then

$$\zeta(t) = \int_0^t \alpha^2 dt' = \alpha_0^2 t \tag{5.38a}$$

is linear. By using the invariance $\zeta(t) \longrightarrow \zeta(t)/K^2, H(T) \longrightarrow KH(T)$ we can choose α_0 to be any arbitrary constant without affecting any prices. So we choose

$$\zeta(t) = \alpha_0^2 t, \tag{5.38b}$$

where t is measured in years, and the dimensionless constant α_0 is, say,

$$\alpha_0 = 10^{-2}. \tag{5.38c}$$

We use the second invariance to set $H_n = H(T_n) = 0$. We shall calibrate the diagonal swaptions to determine the values of $H(T)$ on the settlement dates, $H_j^{set} = H(T_j^{set})$. For other values of T, we assume that $H(T)$ is piecewise linear:

$$H(T) = H_1^{set} + \frac{T - T_1^{set}}{T_2^{set} - T_1^{set}} \left(H_2^{set} - H_1^{set}\right) \qquad \text{for } T \le T_j^{set}, \tag{5.39a}$$

$$H(T) = H_{j-1}^{set} + \frac{T - T_{j-1}^{set}}{T_j^{set} - T_{j-1}^{set}} \left(H_j^{set} - H_{j-1}^{set}\right) \qquad \text{for } T_{j-1}^{set} \le T \le T_j^{set} \tag{5.39b}$$

$$H(T) = H_J^{set} + \frac{T - T_J^{set}}{T_n - T_J^{set}} \left(H_n - H_J^{set}\right) \qquad \text{for } T_J^{set} \le T \tag{5.39c}$$

The value of the j^{th} diagonal swaption can be written as

$$(5.40a)\quad V_j^{diag}(0,0) = \tilde{\alpha}_j \left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} \mathfrak{N}\left(\frac{q + h_{i_j 1}\zeta_j}{\sqrt{\zeta_j}}\right) + \sum_{i=i_j^1+1}^{n} \alpha_i \left(R_j^{diag} - S_i\right) D_i \mathfrak{N}\left(\frac{q + h_i\zeta_j}{\sqrt{\zeta_j}}\right) + D_n \mathfrak{N}\left(\frac{q}{\sqrt{\zeta_j}}\right) - D_j^{set} \mathfrak{N}\left(\frac{q + h_j^{set}\zeta_j}{\sqrt{\zeta_j}}\right)$$

where

$$(5.40b)\quad h_i = H(T_i) - H(T_n) \qquad \text{for } i = 1, 2, ..., n$$

$$(5.40c)\quad h_j^{set} = H(T_j^{set}) - H(T_n) \qquad \text{for } j = 1, 2, ..., J$$

and where q is determined implicitly by

$$(5.40d)\quad \tilde{\alpha}_j \left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} e^{-h_{i_j^1} q - \frac{1}{2} h_{i_j^1}^2 \zeta_j} + \sum_{i=i_j^1+1}^{n} \alpha_i \left(R_j^{diag} - S_i\right) D_i e^{-h_i q - \frac{1}{2} h_i^2 \zeta_j} + D_n = D_j^{set} e^{-h_j^{set} q - \frac{1}{2} h_i^2 \zeta_j}.$$

The last swaption J only depends on h_J^{set} and on h_i for the paydates after T_J^{set}. Since $H_n = 0$, these values are given in terms of h_J^{set}

$$(5.41a)\quad h_i = h_J^{set} \frac{T_n - T}{T_n - T_J^{set}} \qquad \text{for } i \geq i_J^1.$$

There is a unique value of h_J^{set} which matches the LGM price to the market price for the last swaption. This can be easily found by a global Newton's scheme, since we have the derivative

$$(5.41b)\quad \frac{1}{\sqrt{\zeta_J}} \frac{\partial V_J^{diag}}{\partial h_J^{set}} = \tilde{\alpha}_J \left(R_J^{diag} - S_{i_J^1}\right) \frac{T_n - T_{i_J^1}}{T_n - T_J^{set}} D_{i_J^1} G\left(\frac{q + h_{i_J^1}\zeta_J}{\sqrt{\zeta_J}}\right) + \sum_{i=i_J^1+1}^{n-1} \alpha_i \left(R_J^{diag} - S_i\right) \frac{T_n - T_i}{T_n - T_J^{set}} D_i G\left(\frac{q + h_i\zeta_J}{\sqrt{\zeta_J}}\right) - D_J^{set} G\left(\frac{q + h_j^{set}\zeta_J}{\sqrt{\zeta_J}}\right)$$

analytically.

Now suppose that we have calibrated all the swaptions after j to obtain $h_{j+1}^{set}, h_{j+2}^{set}, ..., h_J^{set}$. Since we are using piecewise interpolation, this determines $H(T)$ for all $T \geq T_{j+1}^{set}$. We now calibrate on swaption j to obtain h_j^{set}. The value of this swaption depends on h_j^{set} and any paydates between T_j^{set} and T_{j+1}^{set}:

$$(5.42a)\quad h_i = \frac{T_{j+1}^{set} - T}{T_{j+1}^{set} - T_j^{set}} h_j^{set} + \frac{T - T_j^{set}}{T_{j+1}^{set} - T_j^{set}} h_{j+1}^{set} \qquad \text{for } i_j^1 \leq i \leq i_{j+1}^1$$

Since h_i for $i \geq i_{j+1}^1$ are known from previous steps in the calibration, the only unknown parameter is h_j^{set}. A global Newton's scheme can be used to efficiently determin thunique value of this parameter which matches the j^{th} swaption's LGM price to its market price. Note that the derivative of the value with respect to h_j is

(5.42b)
$$\frac{1}{\sqrt{\zeta_j}}\frac{\partial V_j^{diag}}{\partial h_j^{set}} = \tilde{\alpha}_j\left(R_j^{diag} - S_{i_J^1}\right)\frac{T_{j+1}^{set} - T_{i_j^1}}{T_{j+1}^{set} - T_j^{set}} D_{i_J^1} G\left(\frac{q + h_{i_J^1}\zeta_j}{\sqrt{\zeta_j}}\right)$$
$$+ \sum_{i=i_j^1+1}^{i_{j+1}^1-1} \alpha_i\left(R_j^{diag} - S_i\right)\frac{T_{j+1}^{set} - T_i}{T_{j+1}^{set} - T_j^{set}} D_i G\left(\frac{q + h_i\zeta_j}{\sqrt{\zeta_j}}\right)$$
$$-D_j^{set} G\left(\frac{q + h_j^{set}\zeta_j}{\sqrt{\zeta_j}}\right)$$

Continuing, we can calibrate on the swaptions one at a time (backwards) to obtain

(5.43)
$$H_1^{set}, H_2^{set}, ..., H_J^{set}, H_n$$

on the dates $t_1^{set}, ..., t_J^{set}, t_n$. One uses linear interpolation/extrapolation to get $H(t)$ at other values of t. Of course, after finding the $\zeta(t), H(T)$, one can use the invariances to scale them to taste.

Infeasible values. In deriving the swaption formulas, we assumed that $H(T)$ was an increasing function of T. Since we are calibrating the H_j^{set}'s seperately, it may happen that H_j^{set} may exceed H_{j+1}^{set}. (In practice, this has never happened to my knowledge. Still one must be prepared.) After each H_j^{set} is found, one should check to see that

(5.44)
$$H_j^{set} \leq H_{j+1}^{set}.$$

If this condition is violated, one should reset $H_{j-1} = H_j$. This means the j^{th} swaption would not match its market price exactly. Instead it would be the closest feasible price.

Initial guess. The equivalent vol techniques yields

(5.45)
$$\sqrt{\frac{\sigma t_j^{ex} R_j^{sw} R_j^{diag}}{\zeta_j}}$$
$$\approx \frac{\tilde{\alpha}_j\left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} h_{i_j^1} + \sum_{i=i_j^1+1}^{n} \alpha_i\left(R_j^{diag} - S_i\right) D_i h_i - D_j^{set} h_j^{set}}{\tilde{\alpha}_j D_{i_j^1} + \sum_{i=i_j^1+1}^{n} \alpha_i D_i}$$

where R_j^{sw} is the swap rate and σ is the swaption's implied vol from the marketplace. Since this is linear in the h's, one can solve to get a decent initial guess for h_j^{set}.

5.3.4. Calibration to diagonal swaptions with prescribed $\zeta(t)$. The preceding calibration procedure did *not* depend on $\zeta(t)$ being linear; it just depended on $\zeta(t)$ being known. So suppose that $\zeta(t)$ is a known function which is increasing and has $\zeta(0) = 0$. We could carry out the preceding calibration procedure to determine $H(T)$ from the diagonal swaptions.

5.4. Calibration to caplets. There are many exotic structures which are more naturally priced and hedged in terms of caplets. Autocaps and revolvers, for example. Even though these calibration methods shouldn't be used for pricing Bermudans, we present them here for completeness. We will also make use of these calibration methods later for joint calibrations to the diagonal swaptions and caplets.

For each $j = 1, 2, ..., J$, the equivalent payments for caplet j are:

(5.46a)
$$-1 \qquad \text{at } T_j^{set}$$

(5.46b)
$$1 + \beta_j\left(R_j^{cap} - \mathrm{bs}_j\right) \qquad \text{at } T_j^{end}.$$

Here

$$\beta_j = \text{cvg}(T_j^{set}, T_j^{end}, \text{flt}) \qquad \text{for } j = 1, 2, .., J \tag{5.46c}$$

is the appropriate day count fraction and bs_j is the basis spread for the floating rate set for start date T_j^{set}.

Caplet and floorlets are one period swaptions. If we specialize the swaption formulas 3.7a, 3.7b to one period, we find that the LGM price for the caplet is

$$V_j^{cap}(0,0) = D_j^{end}\left[1 + \beta_j\left(R_j^{cap} - \text{bs}_j\right)\right]\mathcal{N}(d_1^{lgm}) - D_j^{end}\left[1 + \beta_j\left(R_j^{FRA} - \text{bs}_j\right)\right]\mathcal{N}\left(d_2^{lgm}\right) \tag{5.47a}$$

where R_j^{FRA} is the break-even caplet rate

$$R_j^{FRA} = \frac{D_j^{set} - \left(1 - \beta_j\, \text{bs}_j\right) D_j^{end}}{\beta_j D_j^{end}}, \tag{5.47b}$$

and where d_1^{lgm} and d_2^{lgm} are given by

$$d_{1,2}^{lgm} = \frac{\log \dfrac{1 + \beta_j\left(R_j^{cap} - \text{bs}_j\right)}{1 + \beta_j\left(R_j^{FRA} - \text{bs}_j\right)} \pm \frac{1}{2}\left(H_j^{end} - H_j^{set}\right)^2 \zeta_j}{\left(H_j^{end} - H_j^{set}\right)\sqrt{\zeta_j}}. \tag{5.47c}$$

Here,

$$H_j^{set} = H(T_j^{set}), \qquad H_j^{end} = H(T_j^{end}), \qquad \zeta_j = \zeta(t_j^{ex}). \tag{5.47d}$$

We observe this is Black's formula for a European option on an asset with forward price,

$$F = 1 + \beta_j\left(R_j^{cap} - \text{bs}_j\right), \tag{5.48a}$$

with strike

$$K = 1 + \beta_j\left(R_j^{FRA} - \text{bs}_j\right) = D_j^{set}/D_j^{end}, \tag{5.48b}$$

and with settlement date T_j^{end}. Suppose we use an implied volatility routine to find the implied (price) vol $\sigma_j^{cap,price}$ which matches this caplet to its market value. Then

$$\left(H_j^{end} - H_j^{set}\right)\sqrt{\zeta_j} = \sigma_j^{cap,price}\sqrt{t_{ex}}. \tag{5.48c}$$

5.4.1. Calibration to caplets with constant mean reversion κ. For this calibration strategy, the mean reversion coefficient κ is a user-supplied constant. Recall that $H"(T)/H(T) = -\kappa$, so that $H(T) = Ae^{-\kappa T} + B$ for some constants A and B. At this point we use the model invariants to set

$$H(T) = \frac{1 - e^{-\kappa T}}{\kappa}, \tag{5.49a}$$

without loss of generality, where T is measured in years. With $H(T)$ known, matching the caplets to their market price requires

$$\sqrt{\zeta_j} = \frac{\sigma_j^{cap,price}\sqrt{t_{ex}}}{H_j^{end} - H_j^{set}} \qquad \text{for } j = 1, 2, .., J \tag{5.49b}$$

This determines $\zeta(t)$ at the exercise dates $t_1^{ex}, t_2^{ex}, ..., t_J^{ex}$. Again, it may happen that $\zeta_j < \zeta_{j-1}$ for some j, in which case we need to make the replacement

$$\zeta_j \longrightarrow \zeta_{j-1} \qquad \text{if } \zeta_j < \zeta_{j-1}. \tag{5.50}$$

As usual, we append $\zeta(0) = 0$ and use piecewise linear interpolation to obtain $\zeta(t)$ at other dates. Having found $\zeta(t)$ and $H(T)$, one can use the invariances to normalize them according to test.

5.4.2. Calibration to caplets with $H(T)$ specified. The above calibration procedure does not depend on κ being constant. It depends only on $H(T)$ being known. If $H(T)$ is an externally supplied function, then we can carry out the same calibration to obtain $\zeta(t)$.

5.4.3. Calibration to caplets with linear $\zeta(t)$. For this calibration procedure, we assume the local volatility α is constant

$$\zeta(t) = \int_0^t \alpha^2 dt' = \alpha_0^2 t. \tag{5.51a}$$

is linear. By using the multiplicative invariance $\zeta(t) \longrightarrow \zeta(t)/K^2, H(T) \longrightarrow KH(T)$ we can choose

$$\zeta(t) = \alpha_0^2 t, \tag{5.51b}$$

where t is measured in years and α_0 is, say,

$$\alpha_0 = 10^{-2}. \tag{5.51c}$$

Matching the caplets to their market prices requires

$$H_j^{end} - H_j^{set} = \frac{\sigma_j^{cap,price}}{\alpha_0} \qquad \text{for } j = 1, 2, .., J. \tag{5.52}$$

We now use the additive invariance to set $H(T_n) = 0$ and take $H(T)$ to be piecewise linear

$$H(T) = H_1^{set} + \frac{T - T_1^{set}}{T_2^{set} - T_1^{set}} \left(H_2^{set} - H_1^{set}\right) \qquad \text{for } T \le T_j^{set}, \tag{5.53a}$$

$$H(T) = H_{j-1}^{set} + \frac{T - T_{j-1}^{set}}{T_j^{set} - T_{j-1}^{set}} \left(H_j^{set} - H_{j-1}^{set}\right) \qquad \text{for } T_{j-1}^{set} \le T \le T_j^{set} \tag{5.53b}$$

$$H(T) = H_J^{set} + \frac{T - T_J^{set}}{T_n - T_J^{set}} \left(H_n - H_J^{set}\right) \qquad \text{for } T_J^{set} \le T \tag{5.53c}$$

Starting at the last caplet $j = J$, we see that we must choose

$$H_J^{set} = -\frac{\sigma_J^{cap,price}}{\alpha_0} \frac{T_n - T_J^{set}}{T_J^{end} - T_J^{set}} \tag{5.54a}$$

since $H(T_n) = 0$. Suppose our calibration procedure has produced $H_{j+1}^{set}, H_{j+2}^{set}, ..., H_J^{set}$, and H_n. We now find H_j^{set}. First, if $T_j^{end} \le T_{j+1}^{set}$, then

$$H_j^{set} = H_{j+1}^{set} - \frac{\sigma_j^{cap,price}}{\alpha_0} \frac{T_{j+1}^{set} - T_j^{set}}{T_j^{end} - T_j^{set}} \qquad \text{if } T_j^{end} \le T_{j+1}^{set}. \tag{5.54b}$$

On the other hand, if $T_j^{end} > T_{j+1}^{set}$, then $H(T_j^{end})$ is in the already-calibrated region of the curve, and can be found by piecewise linear interpolation on $H_{j+1}^{set}, H_{j+2}^{set}, ..., H_n$. In this case, we use

$$H_j^{set} = H_j^{end} - \frac{\sigma_j^{cap,price}}{\alpha_0} \qquad \text{if } T_j^{end} > T_{j+1}^{set}. \tag{5.54c}$$

to set H_j^{set}. Continuting backwards in this way, we obtain H_j^{set} at all the settlement dates j. Having $\zeta(t)$ and $H(T)$, we can now normalize them to taste.

5.4.4. Calibration to caplets with prescribed $\zeta(t)$. The above procedure for determining $H(T)$ did not depend on $\zeta(t)$ be linear in t; it only relied on $\zeta(t)$ being a known function. Suppose that $\zeta(t)$ is an externally supplied function. Then we can use the above procedure to find $H(T)$ provided we make the replacement

$$\frac{\sigma_j^{cap,price}}{\alpha_0} \longrightarrow \sigma_j^{cap,price} \frac{\sqrt{t_j^{ex}}}{\sqrt{\zeta(t_j^{ex})}} \tag{5.55}$$

5.5. Calibration to a column of swaptions. Recall that tthe equivalent payments for the j^{th} column swaptions j are

$$-1 \qquad \text{at } T_j^{set} \tag{5.56a}$$

$$\tilde{\alpha}_j\left(R_j^{col} - S_i\right) \qquad \text{at } T_i \qquad \text{for } i = i_j^1 \tag{5.56b}$$

$$\alpha_i\left(R_j^{col} - S_i\right) \qquad \text{at } T_i \qquad \text{for } i = i_1^1 + 1, ..., i_j^{end} - 1 \tag{5.56c}$$

$$1 + \alpha_i\left(R_j^{col} - S_i\right) \qquad \text{at } T_i \qquad \text{for } i = i_j^{end} \tag{5.56d}$$

Under the LGM model, the value of this swaption is thus

$$\begin{aligned} V_j^{col}(0,0) = {} & \tilde{\alpha}_j\left(R_j^{col} - S_{i_j^1}\right) D_{i_j^1} \mathfrak{N}\left(\frac{y^* + \Delta H_{i_j^1}\zeta_j}{\sqrt{\zeta_j}}\right) \\ & + \sum_{i=i_j^1+1}^{i_j^{end}} \alpha_i\left(R_j^{col} - S_i\right) D_i \mathfrak{N}\left(\frac{y^* + \Delta H_i\zeta_j}{\sqrt{\zeta_j}}\right) \\ & + D_{i_j^{end}} \mathfrak{N}\left(\frac{y^* + \Delta H_{i_j^{end}}\zeta_j}{\sqrt{\zeta_j}}\right) - D_j^{set} \mathfrak{N}\left(\frac{y^*}{\sqrt{\zeta_j}}\right) \end{aligned} \tag{5.57a}$$

where y^* is obtained by solving

$$\begin{aligned} \tilde{\alpha}_j\left(R_j^{coil} - S_{i_j^1}\right) D_{i_j^1} e^{-\Delta H_{i_j^1} y^* - \frac{1}{2}\Delta H_{i_j^1}^2 \zeta_j} + \sum_{i=i_j^1+1}^{i_j^{end}} \alpha_i\left(R_j^{col} - S_i\right) D_i e^{-\Delta H_i y^* - \frac{1}{2}\Delta H_i^2 \zeta_j} \\ + D_{i_j^{end}} e^{-\Delta H_{i_j^{end}} y^* - \frac{1}{2}\Delta H_{i_j^{end}}^2 \zeta_j} = D_j^{set}, \end{aligned} \tag{5.57b}$$

and where we have used

$$\Delta H_i = H_i - H_j^{set} = H(T_i) - H(T_j^{set}) \tag{5.57c}$$

These formulas are identical to the formulas for the diagonal swaptions, provided one replaces R_j^{diag} with R_j^{row} and replaces n with i_j^{end} for each swaption j. With a little tinkering, one can use the same software to calibrate each column swaption as used for the corresponding diagonal swaption. This gives us the methods

- Calibration to a column of swaptions with constant mean reversion
- Calibration to a column of swaptions with $H(T)$ specified
- Calibration to a column of swaptions with linear $\zeta(t)$
- Calibration to a column of swaptions with prescribed $\zeta(t)$.

Written properly, these routines should work with an arbitrary i_j^{end}, so one does not have to limit oneself to a column of swaptions. Instead one can use any sequence of swaptions which has an increasing set of exercise dates t_j^{ex} and settlement dates T_j^{set}.

5.6. Calibration to a row of swaptions. Recall that the j^{th} row swaption is the swaption with start date T_1^{set} and end date T_j. It's equivalent payments are:

$$-1 \quad \text{at } T^{set} \tag{5.58a}$$

$$\tilde{\alpha}_1\left(R_j^{row}-S_i\right) \quad \text{at } T_i \quad \text{for } i=i_1 \tag{5.58b}$$

$$\alpha_i\left(R_j^{row}-S_i\right) \quad \text{at } T_i \quad \text{for } i=i_1+1, i_1+2, ..., j-1 \tag{5.58c}$$

$$1+\alpha_j\left(R_j^{row}-S_j\right) \quad \text{at } T_j \quad \text{for } i=j. \tag{5.58d}$$

Here the dates T_i, day count fractions $\tilde{\alpha}_j, \alpha_i$ and equivalent basis spreads S_i are the precisely the same quantities calculated for the diagonal swaptions. We also abbreviate $i_1 = i_1^1$ for the index of the first paydate after T_1^{set}. Under the LGM model, the value of the j^{th} row swaption is

$$\begin{aligned} V_j^{row}(0,0) = {} & \tilde{\alpha}_j\left(R_j^{row}-S_{i_1}\right) D_{i_1} \mathfrak{N}\left(\frac{y^*+\Delta H_{i_1}\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right) \\ & + \sum_{i=i_1+1}^{j} \alpha_i\left(R_j^{row}-S_i\right) D_i \mathfrak{N}\left(\frac{y^*+\Delta H_i\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right) \\ & + D_j \mathfrak{N}\left(\frac{y^*+\Delta H_j\zeta_{ex}}{\sqrt{\zeta_{ex}}}\right) - D^{set}\mathfrak{N}\left(\frac{y^*}{\sqrt{\zeta_{ex}}}\right) \end{aligned} \tag{5.59a}$$

where y^* is obtained by solving

$$\begin{aligned} \tilde{\alpha}_j\left(R_j^{row}-S_{i_1}\right) D_{i_1} e^{-\Delta H_{i_1} y^*-\frac{1}{2}\Delta H_{i_1}^2\zeta_{ex}} + \sum_{i=i_1+1}^{j} \alpha_i\left(R_j^{row}-S_i\right) D_i e^{-\Delta H_i y^*-\frac{1}{2}\Delta H_i^2\zeta_{ex}} \\ + D_j e^{-\Delta H_j y^*-\frac{1}{2}\Delta H_j^2\zeta_{ex}} = D^{set}, \end{aligned} \tag{5.59b}$$

and where we have used

$$D^{set} = D(T_1^{set}), \qquad \zeta_{ex} = \zeta(t_1^{ex}), \qquad \Delta H_i = H_i - H^{set} = H(T_i) - H(T_1^{set}), \tag{5.59c}$$

Since all these swaptions have the same exercise date, they depend only on a single value of $\zeta(t)$, namely ζ_{ex}. It makes no sense to calibrate $\zeta(t)$ from these swaptions. This leaves two natural methods for calibrating a row of swaptions:

- Calibration to a row of swaptions with linear $\zeta(t)$

- Calibration to a row of swaptions with prescribed $\zeta(t)$.

In the first case, we can use the multiplicative invariance to set

$$\zeta_{ex} = \alpha_0^2 t_1^{ex} \tag{5.60a}$$

without loss of generality, where $\alpha_0 = 10^{-2}$. This puts us in the second case where ζ_{ex} is prescribed as an input.

Since ζ_{ex} is known, we only need to find $H(T)$ via calibration. We use the second invariance to set $H^{set} = 0$, and prescribe $H(T)$ to be piecewise linear with nodes at the end dates $T_{i_1}, T_{i_1+1}, ..., T_n$:

$$H(T) = \frac{T - T_1^{set}}{T_{i_1} - T_1^{set}} H_{i_1} \qquad \text{for } T \le T_{i_1}, \tag{5.60b}$$

$$H(T) = H_{i-1} + \frac{T - T_{i-1}}{T_i - T_{i-1}} (H_i - H_{i-1}) \qquad \text{for } T_{i-1} \le T \le T_i, \qquad i = i_1 + 1, .., n-1 \tag{5.60c}$$

$$H(T) = H_{n-1} + \frac{T - T_{n-1}}{T_n - T_{n-1}} (H_n - H_{n-1}) \qquad \text{for } T_{n-1}^{set} \le T \tag{5.60d}$$

We note that the first swaption $j = i_1$ depends only on $H(T)$ that are determined by H_{i_1}. A global Newton scheme suffices to find H_{i_1} by matching this swaption against its market value. The next swaption depends on the same $H(T)$ values as before, along with one new value, $H_j = H(T_j)$ with $j = i_1^1 + 1$. Again H_j can be found by calibrating the j^{th} row swaption. Iterating, we can determine H at all the nodes by calibrating to successive swaptions. Again, it's concievable that one of these H_j is not increasing. In that case we have to replace it to ensure that $H(T)$ is non-decreasing:

$$H_j \longrightarrow H_{j-1} \qquad \text{if } H_j < H_{j-1}. \tag{5.61}$$

5.7. Calibraton to two series of vanilla instruments. Since the LGM model has two model "parameters," $\zeta(t)$ and $H(T)$, we can calibrate to two series of vanilla instruments. Following are the most popular strategies

5.7.1. Calibration to diagonal swaptions and a row of swaptions. Recall that a row of swaptions is a set of swaptions, all with the same exercise date t_1^{ex} and same start date T_1^{set}, but with varying end dates. We use the multiplicative invariance to set

$$\zeta_{ex} = \alpha_0^2 t_1^{ex} \tag{5.62}$$

without loss of generality, where $\alpha_0 = 10^{-2}$. Since this is the only value of $\zeta(t)$ used by the row swaptions, we can now use the "calibration to a row of swaptions with prescribed $\zeta(t)$" routine described above to find $H(T)$. Knowing $H(T)$, we can use the "calibration to the diagonal swaptions with $H(T)$ specified" described above to find $\zeta(t)$. At this point we can normalize $\zeta(t), H(T)$ to taste.

After calibrating to a row of swaptions to determine $H(T)$, one does not have to use the diagonal swaptions to find $\zeta(t)$. Instead one could calibrate on the caplets or a column of swaptions. This gives us the methods

5.7.2. Calibration to caplets and a row of swaptions. After calibrating to the row of swaptions to determine $H(T)$, one can use the "calibration to caplets with $H(T)$ specified" routine described above to find $\zeta(t)$.

5.7.3. Calibration to a column and row of swaptions. After calibrating to the row of swaptions to determine $H(T)$, one can use the "calibration to a column of swaptions with $H(T)$ specified " routine described above to find $\zeta(t)$.

5.7.4. Calibration to diagonal swaptions and a column of swaptions. This calibration method simultaneously calibrating the j^{th} diagonal and the j^{th} column swaption to determine both $\zeta_j = \zeta(t_j^{ex})$ and $H(T_j^{set})$. One starts at the last pair, $j = J$, and works backward.

Recall that the j^{th} diagonal and j^{th} column swaption share identical exercise dates t_j^{ex} and settlement dates T_j^{set}. They differ only in the end date: the diagonal swaption goes all the way to T_n , while the column swaption stops at $T_{i_j^{end}}$.

For the last pair of swaption, $j = J$, we usually have $i_J^{end} = n$, and the two swaptions are identical. Even if they are not identical, we should exclude the last column swap as being too similar to the diagonal swap.

The value of the j^{th} diagonal swaption can be written as

$$V_j^{diag}(0,0) = \tilde{\alpha}_j \left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} \mathfrak{N}\left(\frac{q + h_{i_{j^1}}\zeta_j}{\sqrt{\zeta_j}}\right) + \sum_{i=i_j^1+1}^{n} \alpha_i \left(R_j^{diag} - S_i\right) D_i \mathfrak{N}\left(\frac{q + h_i\zeta_j}{\sqrt{\zeta_j}}\right) + D_n \mathfrak{N}\left(\frac{q}{\sqrt{\zeta_j}}\right) - D_j^{set} \mathfrak{N}\left(\frac{q + h_j^{set}\zeta_j}{\sqrt{\zeta_j}}\right) \tag{5.63a}$$

where

$$\tilde{\alpha}_j \left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} e^{-h_{i_j^1} q - \frac{1}{2} h_{i_j^1}^2 \zeta_j} + \sum_{i=i_j^1+1}^{n} \alpha_i \left(R_j^{diag} - S_i\right) D_i e^{-h_i q - \frac{1}{2} h_i^2 \zeta_j} + D_n = D_j^{set} e^{-h_j^{set} q - \frac{1}{2} h_i^2 \zeta_j}. \tag{5.63b}$$

Similarly, the value of the j^{th} column swaption is

$$V_j^{col}(0,0) = \tilde{\alpha}_j \left(R_j^{col} - S_{i_j^1}\right) D_{i_j^1} \mathfrak{N}\left(\frac{u + h_{i_j^1}\zeta_j}{\sqrt{\zeta_j}}\right) + \sum_{i=i_j^1+1}^{i_j^{end}} \alpha_i \left(R_j^{col} - S_i\right) D_i \mathfrak{N}\left(\frac{u + h_i\zeta_j}{\sqrt{\zeta_j}}\right) + D_{i_j^{end}} \mathfrak{N}\left(\frac{u + h_{i_j^{end}}\zeta_j}{\sqrt{\zeta_j}}\right) - D_j^{set} \mathfrak{N}\left(\frac{u + h_j^{set}\zeta_j}{\sqrt{\zeta_j}}\right) \tag{5.63c}$$

where

$$\tilde{\alpha}_j \left(R_j^{col} - S_{i_j^1}\right) D_{i_j^1} e^{-h_{i_j^1} u - \frac{1}{2} h_{i_j^1}^2 \zeta_j} + \sum_{i=i_j^1+1}^{i_j^{end}} \alpha_i \left(R_j^{col} - S_i\right) D_i e^{-h_i u - \frac{1}{2} h_i^2 \zeta_j} + D_{i_j^{end}} e^{-h_{i_j^{end}} u - \frac{1}{2} h_{i_j^{end}}^2 \zeta_j} = D_j^{set} e^{-h_j^{set} u - \frac{1}{2} h_i^2 \zeta_j}. \tag{5.63d}$$

Here we are using

$$h_i = H(T_i) - H(T_n) \qquad \text{for } i = 1, 2, ..., n \tag{5.63e}$$

$$h_j^{set} = H(T_j^{set}) - H(T_n) \qquad \text{for } j = 1, 2, ..., J \tag{5.63f}$$

We use piecewise linear interpolation for $H(T)$, with nodes at the start dates T_j^{set} and the final end date T_n:

$$H(T) = \frac{T_2^{set} - T}{T_2^{set} - T_1^{set}} H_1^{set} + \frac{T - T_1^{set}}{T_2^{set} - T_1^{set}} H_2^{set} \qquad \text{for } T \le T_j^{set}, \tag{5.64a}$$

$$H(T) = \frac{T_j^{set} - T}{T_j^{set} - T_{j-1}^{set}} H_{j-1}^{set} + \frac{T - T_{j-1}^{set}}{T_j^{set} - T_{j-1}^{set}} H_j^{set} \qquad \text{for } T_{j-1}^{set} \le T \le T_j^{set} \tag{5.64b}$$

$$H(T) = \frac{T_n - T}{T_n - T_J^{set}} H_J^{set} + \frac{T - T_J^{set}}{T_n - T_J^{set}} H_n \qquad \text{for } T_J^{set} \le T \tag{5.64c}$$

Without loss of generality, we choose $H(T)$ to be 0 at the final pay date. This means that $H(T)$ and $h(T)$ are identical in the above formulas. We use the second invariance to set the slope of $H(T)$ to be 1 in the final interval:

$$H_J^{set} = H(T_J^{set}) = T_J^{set} - T_n \qquad H_n = H(T_n) = 0 \tag{5.65}$$

This determines all the values of $H(T)$ for $T \ge T_J^{set}$, so the last swaption depends only on one unknown parameter, $\zeta_J = \zeta(t_J^{ex})$. We use our standard global Newton scheme to determine ζ_J.

Suppose that we have already found $H(T)$ for $T \ge T_{j+1}^{set}$ for some j. We now find $H_j = H(T_j)$ and ζ_j by matching the j^{th} diagonal and j^{th} column swaption. These swaptions depend on $\zeta_j = \zeta(t_j^{ex})$ (which is unknown), $H(T)$ for $T \ge T_{j+1}^{set}$ (which is known), and on

$$H(T) = \frac{T_{j+1}^{set} - T}{T_{j+1}^{set} - T_j^{set}} H_j^{set} + \frac{T - T_j^{set}}{T_{j+1}^{set} - T_j^{set}} H_{j+1}^{set} \qquad \text{for } T_j^{set} \le T \le T_j^{set}, \tag{5.66}$$

(which is determined by H_j^{set}, which is unknown). So there are two parameters to fit, H_j^{set} and ζ_j and two swaption values to set to their market prices. We will use a global multi-factor Newton's method to find these parameters. This requires differentiating the swaption values:

$$\frac{\partial V_j^{diag}}{\partial \sqrt{\zeta_j}} = h_{i_j^1} \tilde{\alpha}_j \left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} G\left(\frac{q + h_{i_j^1}\zeta_j}{\sqrt{\zeta_j}}\right) + \sum_{i=i_j^1+1}^{n} h_i \alpha_i \left(R_j^{diag} - S_i\right) D_i G\left(\frac{q + h_i \zeta_j}{\sqrt{\zeta_j}}\right) - h_j^{set} D_j^{set} G\left(\frac{q + h_j^{set}\zeta_j}{\sqrt{\zeta_j}}\right) \tag{5.67a}$$

$$\frac{\partial V_j^{col}}{\partial \sqrt{\zeta_j}} = h_{i_j^1} \tilde{\alpha}_j \left(R_j^{col} - S_{i_j^1}\right) D_{i_j^1} G\left(\frac{u + h_{i_j^1}\zeta_j}{\sqrt{\zeta_j}}\right) + \sum_{i=i_j^1+1}^{i_j^{end}} h_i \alpha_i \left(R_j^{col} - S_i\right) D_i G\left(\frac{u + h_i \zeta_j}{\sqrt{\zeta_j}}\right) + h_{i_j^{end}} D_{i_j^{end}} G\left(\frac{u + h_{i_j^{end}}\zeta_j}{\sqrt{\zeta_j}}\right) - h_j^{set} D_j^{set} G\left(\frac{u + h_j^{set}\zeta_j}{\sqrt{\zeta_j}}\right) \tag{5.67b}$$

and

$$\text{(5.67c)} \qquad \frac{1}{\sqrt{\zeta_j}}\frac{\partial V_j^{diag}}{\partial H_j^{set}} = \frac{T_{j+1}^{set} - T_{i_j^1}}{T_{j+1}^{set} - T_j^{set}}\tilde{\alpha}_j \left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} G\left(\frac{q + h_{i_{j^1}}\zeta_j}{\sqrt{\zeta_j}}\right)$$

$$+ \sum_{i=i_j^1+1}^{T_i \leq T_{j+1}^{set}} \frac{T_{j+1}^{set} - T_i}{T_{j+1}^{set} - T_j^{set}} \alpha_i \left(R_j^{diag} - S_i\right) D_i G\left(\frac{q + h_i\zeta_j}{\sqrt{\zeta_j}}\right)$$

$$- D_j^{set} G\left(\frac{q + h_j^{set}\zeta_j}{\sqrt{\zeta_j}}\right)$$

$$\text{(5.67d)} \qquad \frac{1}{\sqrt{\zeta_j}}\frac{\partial V_j^{col}}{\partial H_j^{set}} = \frac{T_{j+1}^{set} - T_{i_{j^1}}}{T_{j+1}^{set} - T_j^{set}}\tilde{\alpha}_j \left(R_j^{col} - S_{i_j^1}\right) D_{i_j^1} G\left(\frac{u + h_{i_j^1}\zeta_j}{\sqrt{\zeta_j}}\right)$$

$$+ \sum_{i=i_j^1+1}^{i_j^{end}} \max\left\{\frac{T_{j+1}^{set} - T_i}{T_{j+1}^{set} - T_j^{set}}, 0\right\} \alpha_i \left(R_j^{col} - S_i\right) D_i G\left(\frac{u + h_i\zeta_j}{\sqrt{\zeta_j}}\right)$$

$$+ \max\left\{\frac{T_{j+1}^{set} - T_{i_j^{end}}}{T_{j+1}^{set} - T_j^{set}}, 0\right\} D_{i_j^{end}} G\left(\frac{u + h_{i_j^{end}}\zeta_j}{\sqrt{\zeta_j}}\right) - D_j^{set} G\left(\frac{u + h_j^{set}\zeta_j}{\sqrt{\zeta_j}}\right)$$

From 5.63b, 5.63d, we deduce that

$$\text{(5.68a)} \qquad \tilde{\alpha}_j \left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} G\left(\frac{q + h_{i_{j^1}}\zeta_j}{\sqrt{\zeta_j}}\right) + \sum_{i=i_j^1+1}^{n} \alpha_i \left(R_j^{diag} - S_i\right) D_i G\left(\frac{q + h_i\zeta_j}{\sqrt{\zeta_j}}\right)$$

$$+ D_n G\left(\frac{q + h_n\zeta_j}{\sqrt{\zeta_j}}\right) = D_j^{set} G\left(\frac{q + h_j^{set}\zeta_j}{\sqrt{\zeta_j}}\right),$$

$$\text{(5.68b)} \qquad \tilde{\alpha}_j \left(R_j^{col} - S_{i_j^1}\right) D_{i_j^1} G\left(\frac{u + h_{i_{j^1}}\zeta_j}{\sqrt{\zeta_j}}\right) + \sum_{i=i_j^1+1}^{i_j^{end}} \alpha_i \left(R_j^{col} - S_i\right) D_i G\left(\frac{u + h_i\zeta_j}{\sqrt{\zeta_j}}\right)$$

$$+ D_{i_j^{end}} G\left(\frac{u + h_{i_j^{end}}\zeta_j}{\sqrt{\zeta_j}}\right) = D_j^{set} G\left(\frac{u + h_j^{set}\zeta_j}{\sqrt{\zeta_j}}\right),$$

This shows that the four derivatives can be wriiten as:

$$\text{(5.69a)} \qquad \frac{\partial V_j^{diag}}{\partial \sqrt{\zeta_j}} = \left(H_{i_{j^1}} - H_j^{set}\right)\tilde{\alpha}_j \left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} G\left(\frac{q + h_{i_{j^1}}\zeta_j}{\sqrt{\zeta_j}}\right)$$

$$+ \sum_{i=i_j^1+1}^{n} \left(H_i - H_j^{set}\right) \alpha_i \left(R_j^{diag} - S_i\right) D_i G\left(\frac{q + h_i\zeta_j}{\sqrt{\zeta_j}}\right)$$

$$+ \left(H_n - H_j^{set}\right) D_n G\left(\frac{q + h_n\zeta_j}{\sqrt{\zeta_j}}\right)$$

$$\frac{\partial V_j^{col}}{\partial\sqrt{\zeta_j}} = \left(H_{i_{j^1}} - H_j^{set}\right)\tilde{\alpha}_j\left(R_j^{col} - S_{i_j^1}\right)D_{i_j^1}G\left(\frac{u + h_{i_j^1}\zeta_j}{\sqrt{\zeta_j}}\right) + \sum_{i=i_j^1+1}^{i_j^{end}}\left(H_i - H_j^{set}\right)\alpha_i\left(R_j^{col} - S_i\right)D_iG\left(\frac{u + h_i\zeta_j}{\sqrt{\zeta_j}}\right) + \left(H_{i_j^{end}} - H_j^{set}\right)D_{i_j^{end}}G\left(\frac{u + h_{i_j^{end}}\zeta_j}{\sqrt{\zeta_j}}\right) \tag{5.69b}$$

and

$$\frac{1}{\sqrt{\zeta_j}}\frac{\partial V_j^{diag}}{\partial H_j^{set}} = -\min\left\{\frac{T_{i_j^1} - T_j^{set}}{T_{j+1}^{set} - T_j^{set}}, 1\right\}\tilde{\alpha}_j\left(R_j^{diag} - S_{i_j^1}\right)D_{i_j^1}G\left(\frac{q + h_{i_{j^1}}\zeta_j}{\sqrt{\zeta_j}}\right) - \sum_{i=i_j^1+1}^{n}\min\left\{\frac{T_i - T_j^{set}}{T_{j+1}^{set} - T_j^{set}}, 1\right\}\alpha_i\left(R_j^{diag} - S_i\right)D_iG\left(\frac{q + h_i\zeta_j}{\sqrt{\zeta_j}}\right) - D_nG\left(\frac{q + h_n\zeta_j}{\sqrt{\zeta_j}}\right) \tag{5.69c}$$

$$\frac{1}{\sqrt{\zeta_j}}\frac{\partial V_j^{col}}{\partial H_j^{set}} = -\min\left\{\frac{T_{i_{j^1}} - T_j^{set}}{T_{j+1}^{set} - T_j^{set}}, 1\right\}\tilde{\alpha}_j\left(R_j^{col} - S_{i_j^1}\right)D_{i_j^1}G\left(\frac{u + h_{i_j^1}\zeta_j}{\sqrt{\zeta_j}}\right) - \sum_{i=i_j^1+1}^{i_j^{end}}\min\left\{\frac{T_i - T_j^{set}}{T_{j+1}^{set} - T_j^{set}}, 1\right\}\alpha_i\left(R_j^{col} - S_i\right)D_iG\left(\frac{u + h_i\zeta_j}{\sqrt{\zeta_j}}\right) - \min\left\{\frac{T_{i_j^{end}} - T_j^{set}}{T_{j+1}^{set} - T_j^{set}}, 1\right\}D_{i_j^{end}}G\left(\frac{u + h_{i_j^{end}}\zeta_j}{\sqrt{\zeta_j}}\right) \tag{5.69d}$$

Since $H(T)$ is increasing, the value of the diagonal and column swaption both go up with $\sqrt{\zeta_j}$, with the value of the diagonal swaption going up faster than the column swaption by roughly the ratio $\frac{H_{i_j^{end}} - H_j^{set}}{H_n - H_j^{set}}$. The value of the diagonal and column swaption both increase at roughtly the same rate as H_j^{set} *decreases.*

With the derivatives in hand, we can use a global Newton scheme to find ζ_j and H_j^{set}. We require that

$$\zeta_j < \zeta_{j+1}, \qquad H_j^{set} < H_{j+1}^{set}. \tag{5.70}$$

Suppose we start our search at the corner $\zeta_j = \zeta_{j+1}, H_j^{set} = H_{j+1}^{set}$. If $V_j^{diag} > \text{Mkt}_j^{diag}$, then we decrease ζ_j, keeping H_j^{set} at H_{j+1}^{set}, until we match $V_j^{diag} = \text{Mkt}_j^{diag}$. If $V_j^{diag} < \text{Mkt}_j^{diag}$, then we decrease H_j, keeping $\zeta_j = \zeta_{j+1}$ until $V_j^{diag} = \text{Mkt}_j^{diag}$. Let us now imagine decreasing both ζ_j and H_j on a trajectory such that V_j^{diag} remains equal to Mkt_j^{diag}. On this trajectory V_j^{col} increases. So if we start with $V_j^{col} \leq \text{Mkt}_j^{col}$, then a unique solution exists. We use a global Newton scheme to find it. Alternatively, if $V_j^{col} > \text{Mkt}_j^{col}$, we can do no better than keeping the current ζ_j and H_j. (These are the parameters which fit the diagonal swaptions exactly, and come as close as possible to fitting the column swaptions).

Once we've found both ζ_j, H_j^{set} we step back to $j-1$, etc. until we've found ζ_j, H_j^{set} for $j = 1, 2, ..., J$. In the usual way, we use piecewise linear interpolation between the known values of $\zeta(t)$ and $H(T)$, and use the invariances to rescale ζ, H to taste.

5.7.5. Calibration to two columns of swaptions. The above calibration techniques does not depend on one set of swaptions being the diagonal swaptions. It just relies on there being J swaption pairs, with both members of each pair sharing the same exercise date t_j^{ex} and start date T_j^{set}, and having distinctly different end dates. For some exotics, like MBS traunches, it makes more sense to calibrate on two columns of swaptions, say the 1y and 10y tenors. With only trivial modifications, the algorithm described above will calibrate these more general sets of swaption pairs..

5.7.6. Calibration to diagonal swaptions and caplets. This calibration method simultaneously calibrates the j^{th} diagonal swaption and the j^{th} caplet to determine both $\zeta_j = \zeta(t_j^{ex})$ and $H(T_j^{set})$. As in the preceding case, one starts at the last pair, $j = J$, and works backward. For parameter stability, we *do not* calibrate to the final caplet, since in our view it may not be sufficiently "different" from the last diagonal swaption.

Recall that the j^{th} diagonal swaption and j^{th} caplet share identical exercise (fixing) dates t_j^{ex} and settlement dates T_j^{set}. They differ only in the end date: the diagonal swaption goes all the way to T_n , while the caplet stops at T_j^{end}. As above, the value of the j^{th} diagonal swaption can be written as

(5.71a)
$$\begin{aligned} V_j^{diag}(0,0) = {} & \tilde{\alpha}_j \left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} \mathfrak{N}\left(\frac{q + h_{i_{j^1}}\zeta_j}{\sqrt{\zeta_j}}\right) \\ & + \sum_{i=i_j^1+1}^{n} \alpha_i \left(R_j^{diag} - S_i\right) D_i \mathfrak{N}\left(\frac{q + h_i\zeta_j}{\sqrt{\zeta_j}}\right) \\ & + D_n \mathfrak{N}\left(\frac{q}{\sqrt{\zeta_j}}\right) - D_j^{set} \mathfrak{N}\left(\frac{q + h_j^{set}\zeta_j}{\sqrt{\zeta_j}}\right) \end{aligned}$$

where

(5.71b)
$$\begin{aligned} & \tilde{\alpha}_j \left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} e^{-h_{i_j^1} q - \frac{1}{2} h_{i_j^1}^2 \zeta_j} + \sum_{i=i_j^1+1}^{n} \alpha_i \left(R_j^{diag} - S_i\right) D_i e^{-h_i q - \frac{1}{2} h_i^2 \zeta_j} \\ & \qquad + D_n = D_j^{set} e^{-h_j^{set} q - \frac{1}{2} h_i^2 \zeta_j}. \end{aligned}$$

Recall that the value of the j^{th} caplet matches its market value when

(5.71c)
$$\sqrt{\zeta_j} = \frac{\sigma_j^{cap,price} \sqrt{t_{ex}}}{h_j^{end} - h_j^{set}}$$

where $\sigma_j^{cap,price}$ is the implied price vol defined earlier. Here we are using

(5.72a)
$$h_i = H(T_i) - H(T_n) \qquad \text{for } i = 1, 2, ..., n$$

(5.72b)
$$h_j^{set} = H(T_j^{set}) - H(T_n) \qquad \text{for } j = 1, 2, ..., J$$

(5.72c)
$$h_j^{end} = H(T_j^{end}) - H(T_n) \qquad \text{for } j = 1, 2, ..., J$$

We use piecewise linear interpolation for $H(T)$, with nodes at the start dates T_j^{set} and the final end date

T_n:

(5.73a) $$H(T) = \frac{T_2^{set} - T}{T_2^{set} - T_1^{set}} H_1^{set} + \frac{T - T_1^{set}}{T_2^{set} - T_1^{set}} H_2^{set} \qquad \text{for } T \le T_j^{set},$$

(5.73b) $$H(T) = \frac{T_j^{set} - T}{T_j^{set} - T_{j-1}^{set}} H_{j-1}^{set} + \frac{T - T_{j-1}^{set}}{T_j^{set} - T_{j-1}^{set}} H_j^{set} \qquad \text{for } T_{j-1}^{set} \le T \le T_j^{set}$$

(5.73c) $$H(T) = \frac{T_n - T}{T_n - T_J^{set}} H_J^{set} + \frac{T - T_J^{set}}{T_n - T_J^{set}} H_n \qquad \text{for } T_J^{set} \le T$$

Without loss of generality, we choose $H(T)$ to be 0 at the final pay date. Then $H(T)$ and $h(T)$ are equal in the above formulas. We use the second invariance to set the slope of $H(T)$ to be 1 in the final interval:

(5.74) $$H_J^{set} = H(T_J^{set}) = T_J^{set} - T_n \qquad H_n = H(T_n) = 0$$

This determines all the values of $H(T)$ for $T \ge T_J^{set}$, so the last swaption depends only on one unknown parameter, $\zeta_J = \zeta(t_J^{ex})$. We use our standard global Newton scheme to determine ζ_J.

Suppose that we have already found $H(T)$ for $T \ge T_{j+1}^{set}$ for some j. Consider the j^{th} diagonal swaption. It depends on $\zeta_j = \zeta(t_j^{ex})$ and $H(T)$ for $T \ge T_{j+1}^{set}$ and on

(5.75) $$H(T) = \frac{T_{j+1}^{set} - T}{T_{j+1}^{set} - T_j^{set}} H_j^{set} + \frac{T - T_j^{set}}{T_{j+1}^{set} - T_j^{set}} H_{j+1}^{set} \qquad \text{for } T_j^{set} \le T \le T_j^{set}.$$

As before, the differentiating the diagonal swaption value eventually yields

(5.76a) $$\frac{\partial V_j^{diag}}{\partial \sqrt{\zeta_j}} = \left(H_{i_{j^1}} - H_j^{set}\right) \tilde{\alpha}_j \left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} G\left(\frac{q + h_{i_{j^1}}\zeta_j}{\sqrt{\zeta_j}}\right) + \sum_{i=i_j^1+1}^{n} \left(H_i - H_j^{set}\right) \alpha_i \left(R_j^{diag} - S_i\right) D_i G\left(\frac{q + h_i\zeta_j}{\sqrt{\zeta_j}}\right) + \left(H_n - H_j^{set}\right) D_n G\left(\frac{q + h_n\zeta_j}{\sqrt{\zeta_j}}\right)$$

(5.76b) $$\frac{1}{\sqrt{\zeta_j}} \frac{\partial V_j^{diag}}{\partial H_j^{set}} = -\min\left\{\frac{T_{i_j^1} - T_j^{set}}{T_{j+1}^{set} - T_j^{set}}, 1\right\} \tilde{\alpha}_j \left(R_j^{diag} - S_{i_j^1}\right) D_{i_j^1} G\left(\frac{q + h_{i_{j^1}}\zeta_j}{\sqrt{\zeta_j}}\right) - \sum_{i=i_j^1+1}^{n} \min\left\{\frac{T_i - T_j^{set}}{T_{j+1}^{set} - T_j^{set}}, 1\right\} \alpha_i \left(R_j^{diag} - S_i\right) D_i G\left(\frac{q + h_i\zeta_j}{\sqrt{\zeta_j}}\right) - D_n G\left(\frac{q + h_n\zeta_j}{\sqrt{\zeta_j}}\right)$$

The value of the swaption increases as $\sqrt{\zeta_j}$ increases and increases as H_j decreases.

Fitting the caplet requires

(5.77a) $$H_j^{end} - H_j^{set} = \frac{\sigma_j^{cap,price}\sqrt{t_j^{ex}}}{\sqrt{\zeta_j}}$$

If $T_j^{end} \le T_{j+1}^{set}$, then H_j^{end} is known in terms of H_j^{set},

$$H_j^{end} = H_j^{set} + \left(H_{j+1}^{set} - H_j^{set}\right) \frac{T_j^{end} - T_j^{set}}{T_{j+1}^{set} - T_j^{set}} \qquad \text{if } T_j^{end} \le T_{j+1}^{set}. \tag{5.77b}$$

If $T_j^{end} > T_{j+1}^{set}$, then H_j^{end} is known from preceding calibraton step. Fitting to the caplet thus requires

$$H_j^{set}(\sqrt{\zeta_j}) = H_{j+1}^{set} - \frac{\sigma_j^{cap,price}\sqrt{t_j^{ex}}}{\sqrt{\zeta_j}} \frac{T_{j+1}^{set} - T_j^{set}}{T_j^{end} - T_j^{set}} \qquad \text{if } T_j^{end} \le T_{j+1}^{set}, \tag{5.78a}$$

$$H_j^{set}(\sqrt{\zeta_j}) = H_j^{end} - \frac{\sigma_j^{cap,price}\sqrt{t_j^{ex}}}{\sqrt{\zeta_j}} \qquad \text{if } T_j^{end} > T_{j+1}^{set}. \tag{5.78b}$$

These formulas describe the trajectory $\sqrt{\zeta_j}, H_j^{set}(\sqrt{\zeta_j})$ on which the caplet matches it's market value. With $H_j^{set} = H_j^{set}(\sqrt{\zeta_j})$, we use a 1 parameter global Newton method (starting at $\zeta_j = \zeta_{j+1}$) to choose ζ_j to match $V_j^{diag} = \text{Mkt}_j^{diag}$. Note that along this trajectory,

$$\frac{dV_j^{diag}}{d\sqrt{\zeta_j}} = \frac{\partial V_j^{diag}}{\partial\sqrt{\zeta_j}} + \left(H_{j+1}^{set} - H_j^{set}\right) \frac{1}{\sqrt{\zeta_j}} \frac{\partial V_j^{diag}}{\partial H_j^{set}} \qquad \text{if } T_j^{end} \le T_{j+1}^{set} \tag{5.79a}$$

$$\frac{dV_j^{diag}}{d\sqrt{\zeta_j}} = \frac{\partial V_j^{diag}}{\partial\sqrt{\zeta_j}} + \left(H_j^{end} - H_j^{set}\right) \frac{1}{\sqrt{\zeta_j}} \frac{\partial V_j^{diag}}{\partial H_j^{set}} \qquad \text{if } T_j^{end} > T_{j+1}^{set} \tag{5.79b}$$

Once we've found both ζ_j, H_j^{set} we step back to $j-1$, etc. until we've found ζ_j, H_j^{set} for $j = 1, 2, ..., J$. In the usual way, we use piecewise linear interpolation between the known values of $\zeta(t)$ and $H(T)$, and use the invariances to rescale ζ, H to taste.

Appendix A. Bermudans on amortizing swaps.

The notional of an amortizing swap steadily declines over the life of the swap. Given a calibrated LGM model, one can evaluate a Bermudan amortizer in exactly the same way as a Bermudan bullet swap. Since the final price of a deal is determined largely by which instruments are used in calibration, the question is which instruments should be used to get the most adroit pricing and hedging? If we have an 5 year option on a 20 year amortizing swap, surely it will behave more like the 5 into 10 or a 5 into 12 vanilla swaption instead of a 5 into 20 swaption.

There are two main approaches to calibrating the model for amortizing Bermudans:
(A) For each exercise date, select the vanilla swaption whose behavior matches the behaviour of the Bermudan payoff as closely as possible. These swaptions then replace the "diagonal" swaptions.
(B) For each exercise date, calibrate the model to *European options* on the amortizing swap. To obtain the European option's price, we construct a construct a basket of swaps that exactly reproduces the Bermudan's payoff; we then use LGM itself to value European option on the basket.

A.1. Calibrating to the "equivalent vanilla swaption". We need to select the vanilla swaption whose behaviour most nearly matches the Bermudan's payoff for each exercise. Consider the exercise at t_j^{ex}. The Bermudan's fixed leg recieves

$$C_i - rfp_j \qquad \text{at } t_i \qquad \text{for } i = i_j^{first}, \tag{A.1a}$$

$$C_i \qquad \text{at } t_i \qquad \text{for } i = i_j^{first}+1, ..., n, \tag{A.1b}$$

The Bermudan's floating leg receives payments equivalent to

$$N_{i_j^{first}} \pm fee_j \qquad \text{at } t_j^{set}, \tag{A.1c}$$

where the " + ȷ sign is used if the fixed leg (receiver) has the exercise privilege, and the " − ȷ sign is used if the payer has the option. Let us abbreviate

$$M_j = N_{i_j^{first}} \pm fee_j. \tag{A.2}$$

At any date t, the value of these legs is

$$V_j^{fix}(t) = \left(C_{i_j^{first}} - rfp_j\right) Z(t, t_{i_j^{first}}) + \sum_{i=i_j^{first}+1}^{n} C_i Z(t, t_i), \tag{A.3a}$$

$$V_j^{flt}(t) = M_j Z(t, t_j^{set}). \tag{A.3b}$$

A.1.1. Ratio matching. There are two main ideas for picking the "most similar" swaption, *ratio* matching and *payoff* matching. Consider the ratio

$$\text{Ratio} = \frac{V_j^{fix}(t_j^{ex})}{V_j^{flt}(t_j^{ex})} = \frac{C_{i_j^{first}} - rfp_j}{M_j} \frac{Z(t_j^{ex}, t_{i_j^{first}})}{Z(t_j^{ex}, t_j^{set})} + \sum_{i=i_j^{first}+1}^{n} \frac{C_i}{M_j} \frac{Z(t_j^{ex}, t_i)}{Z(t_j^{ex}, t_j^{set})}. \tag{A.4}$$

This ratio represents the dollars received per dollar spent upon exercising the option. Suppose we *model* the yield curve as being today's yield curve plus parallel shifts and tilts. Then,

$$\begin{aligned} \frac{Z(t,T)}{Z(t,t_j^{set})} &\longrightarrow \frac{D(T)}{D(t_j^{set})} e^{-\gamma(T-t_{j^{set}})-\delta(T-t_j^{set})^2+\cdots} \\ &= \frac{D(T)}{D(t_j^{set})} \left\{1 - \gamma(T - t_{j^{set}}) - \left(\delta - \tfrac{1}{2}\gamma^2\right)(T - t_j^{set})^2 + \cdots\right\}, \end{aligned} \tag{A.5}$$

where γ and δ are the amount of the parallel shift and tilt, respectively. Under these movements, the ratio becomes

$$\text{Ratio} = \text{Moneyness} - \gamma\, \text{Sensitivity} - (\delta - \tfrac{1}{2}\gamma^2)\, \text{Convexity} + \cdots, \tag{A.6a}$$

where

$$\text{Moneyness} = \left(C_{i_j^{first}} - rfp_j\right) \frac{D(t_{i_j^{first}})}{D(t_j^{set})} + \sum_{i=i_j^{first}+1}^{n} C_i \frac{D(t_i)}{D(t_j^{set})} \tag{A.6b}$$

$$\text{Sensitivity} = \left(t_{i_j^{first}} - t_j^{st}\right)\left(C_{i_j^{first}} - rfp_j\right) \frac{D(t_{i_j^{first}})}{D(t_j^{set})} + \sum_{i=i_j^{first}+1}^{n} \left(t_i - t_j^{st}\right) C_i \frac{D(t_i)}{D(t_j^{set})} \tag{A.6c}$$

$$\text{Convexity} = \left(t_{i_j^{first}} - t_j^{st}\right)^2 \left(C_{i_j^{first}} - rfp_j\right) \frac{D(t_{i_j^{first}})}{D(t_j^{set})} + \sum_{i=i_j^{first}+1}^{n} \left(t_i - t_j^{st}\right)^2 C_i \frac{D(t_i)}{D(t_j^{set})} \tag{A.6d}$$

Consider a standard bullet swap with exercise date $t_j^{ex}$and with a start date $T_j^{ref\ st}$ which is spot-of-t_j^{ex}. Let the $T_j^{ref\ end}$ be the theoretical end date, and let the strike be R_j^{ref}. Assume that the swap has K periods, the first of which is generally a stub, and let the fixed rate dates be $s_0, s_1, \ldots, s_K$. Clearly $s_0 = T_j^{ref\ st}$. Then the ratio of the swap's fixed leg to the floating leg is

$$\text{Ratio}_j^{ref} = \sum_{k=1}^{K} \text{cvg}(s_{k-1}, s_k) \left(R_j^{ref} - S_k\right) \frac{Z(t_j^{ex}, s_k)}{Z(t_j^{ex}, s_0)} + \frac{Z(t_j^{ex}, s_K)}{Z(t_j^{ex}, s_0)}, \tag{A.7}$$

where S_k is the floating leg's basis spread adjusted to the fixed leg's frequency and day count basis. Under the same set of yield curve movements as before, this ratio becomes

$$\text{Ratio}_j^{ref} = \text{Moneyness}_j^{ref} - \gamma\,\text{Sensitivity}_j^{ref} - (\delta - \tfrac{1}{2}\gamma^2)\,\text{Convexity}_j^{ref} + \cdots, \tag{A.8a}$$

where

$$\text{Moneyness}_j^{ref} = \sum_{k=1}^{K} \text{cvg}(s_{k-1}, s_k) \left(R_j^{ref} - S_k\right) \frac{D(s_k)}{D(s_0)} + \frac{D(s_K)}{D(s_0)} \tag{A.8b}$$

$$\text{Sensitivity}_j^{ref} = \sum_{k=1}^{K} (s_k - s_0)\,\text{cvg}(s_{k-1}, s_k) \left(R_j^{ref} - S_k\right) \frac{D(s_k)}{D(s_0)} + (s_K - s_0)\frac{D(s_K)}{D(s_0)} \tag{A.8c}$$

$$\text{Convexity}_j^{ref} = \sum_{k=1}^{K} (s_k - s_0)^2\,\text{cvg}(s_{k-1}, s_k) \left(R_j^{ref} - S_k\right) \frac{D(s_k)}{D(s_0)} + (s_K - s_0)^2\frac{D(s_K)}{D(s_0)} \tag{A.8d}$$

If the ratio of the reference swap matched the ratio of the amortizing swap reference under all possible movements of the yield curve, then clearly the value of the European option on the amortizer would equal the value of the vanilla swaption, With two free variables R_j^{ref} and $T_j^{ref\ end}$, however, we can only choose the swaption which matches the moneyness and the sensitivity of the Bermudan's payoff. The two fixed legs match only for parallel shifts, so although we still argue that the values should be nearly the same, we incur some risk in doing so. From a trader's perspective, if we went long (short) the amortizing swaption and short (long) the bullet, we would be neutral for parallel shifts, but exposed to tilts. If the prices on the two were significantly different, it would tempt enough traders to take the tilt risk, eliminating the mis-balance.

A.1.2. Payoff matching. In ratio matching, we essentially matched the floating leg exactly and matched the fixed leg as well as possible. To get the option's value right, however, all we have to do is mimic the forward value of the net payoff (fixed minus floating correctly). This allows us one more variable: the notional. A \$2 swap with a 3y tenor may have the same sensitivity as a \$1 swap with a 7y tenor.By varying the notional, we can match both the sensitivity *and* the convexity. Let

$$s_0 = T_j^{ref\ st} \tag{A.9}$$

be the standard spot date for t_j^{ex}. Consider the forward value of the payoff for date s_0 as seen at date t_j^{ex} :

$$\text{Payoff}(t_j^{ex}) = \left(C_{i_j^{first}} - rfp_j\right) \frac{Z(t_j^{ex}, t_{i_j^{first}})}{Z(t_j^{ex}, s_0)} + \sum_{i=i_j^{first}+1}^{n} C_i \frac{Z(t_j^{ex}, t_i)}{Z(t_j^{ex}, s_0)} - M_j \frac{Z(t_j^{ex}, t_j^{set})}{Z(t_j^{ex}, s_0)}. \tag{A.10}$$

Once again we suppose that the yield curve undergoes parallel shifts and tilts:

$$\begin{aligned} \frac{Z(t_{ex}^j, T)}{Z(t_{ex}^j, t_0)} &= \frac{D(T)}{D(s_0)} e^{-\gamma(T-s_0)-\delta(T-s_0)^2} \\ &= \frac{D(T)}{D(s_0)} \left\{1 - \gamma(T - s_0) - (\delta - \tfrac{1}{2}\gamma^2)(T - s_0)^2 + \cdots\right\}. \end{aligned} \tag{A.11}$$

Then the forward value of the payoff becomes

(A.12a) $$\text{Payoff}(t_j^{ex}) = \text{FwdVal} - \gamma(T - t_0)\,\text{Sensitivity} - (\delta - \tfrac{1}{2}\gamma^2)\,\text{Convexity} + \cdots$$

where

(A.12b) $$\text{FwdVal} = \left(C_{i_j^{first}} - rfp_j\right)\frac{D(t_{i_j^{first}})}{D(s_0)} + \sum_{i=i_j^{first}+1}^{n} C_i \frac{D(t_i)}{D(s_0)} - M_j \frac{D(t_j^{set})}{D(s_0)},$$

(A.12c) $$\text{Sensitivity} = \left(t_{i_j^{first}} - s_0\right)\left(C_{i_j^{first}} - rfp_j\right)\frac{D(t_{i_j^{first}})}{D(s_0)} + \sum_{i=i_j^{first}+1}^{n} (t_i - s_0)\, C_i \frac{D(t_i)}{D(s_0)} - \left(t_j^{set} - s_0\right) M_j \frac{D(t_j^{set})}{D(s_0)},$$

(A.12d) $$\text{Convexity} = \left(t_{i_j^{first}} - s_0\right)^2\left(C_{i_j^{first}} - rfp_j\right)\frac{D(t_{i_j^{first}})}{D(s_0)} + \sum_{i=i_j^{first}+1}^{n} (t_i - s_0)^2\, C_i \frac{D(t_i)}{D(s_0)} - \left(t_j^{set} - s_0\right)^2 M_j \frac{D(t_j^{set})}{D(s_0)}.$$

Now consider a standard bullet swap with notional M_j^{ref}, start date $s_0 = T_j^{ref\ st}$, theoretical end date $T_j^{ref\ end}$, and strike R_j^{ref}. Assume that the swap has K periods, the first of which is generally a stub, and let the fixed rate dates be $s_0, s_1, \ldots, s_K$. Clearly $s_0 = t_0$. With three free variables (M, R^{ref}, and t_{end}^{th}), we can match the forward value, the sensitivity, *and* the convexity of the amortizing swap:

(A.13a) $$\text{FwdVal}_j^{ref} = M_j^{ref}\left\{\sum_{k=1}^{K} \text{cvg}(s_{k-1}, s_k)\left(R^{ref} - S_k\right)\frac{D(s_k)}{D(s_0)} + \frac{D(s_K)}{D(s_0)} - 1\right\},$$

(A.13b) $$\text{Sensitivity}_j^{ref} = M_j^{ref}\left\{\sum_{k=1}^{K} (s_k - s_0)\,\text{cvg}(s_{k-1}, s_k)\left(R^{ref} - S_k\right)\frac{D(s_k)}{D(s_0)} + (s_K - s_0)\frac{D(s_K)}{D(s_0)}\right\},$$

(A.13c) $$\text{Convexity}_j^{ref} = M_j^{ref}\left\{\sum_{k=1}^{K} (s_k - s_0)^2\,\text{cvg}(s_{k-1}, s_k)\left(R^{ref} - S_k\right)\frac{D(s_k)}{D(s_0)} + (s_K - s_0)^2\frac{D(s_K)}{D(s_0)}\right\},$$

The forward value of the amortizing swap and the reference swap are now the same under reasonably large parallel shifts of the yield curve, and under not-too-large tilts of the yield curve. If we went long (short) the amortizing swaption and short (long) the bullet swaption, we would be delta *and* gamma neutral for parallel shifts. We also be delta neutral for tilts. We assume that this combination should be priced at zero, since even a small difference would tempt traders to take on the residual yield curve risk. If we were agressive, we would claim that the value of the this reference swaption is the same as the value of the European option on the amortizing swap. Here we are less aggressive, and simply insist that we calibrate the LGM model to these reference swaptions in lieu of the diagonal swaptions.

A.2. Matching by baskets. Recall that if the Bermudan is exercised at t_j^{ex}, the payoff's fixed leg is

$$C_i - rfp_j \qquad \text{at } t_i \qquad \text{for } i = i_j^{first}, \tag{A.14a}$$

$$C_i \qquad \text{at } t_i \qquad \text{for } i = i_j^{first}+1, ..., n, \tag{A.14b}$$

and the payoff's floating leg is equivalent to the payment M_j at t_j^{set}. We approximate the floating leg payment as a payment of

$$M_j \frac{D(t_j^{set})}{D(T_j^{set})} \qquad \text{at } T_j^{set}, \tag{A.14c}$$

where T_j^{set} is the standard spot-of-t_j^{ex} for the currency.

Suppose we knew the market price of the European option on this amortizing swap. Then we would use our favorite calibration strategy (constant κ + diagonals, caplets+diagonals, ...), and calibrate the model to the European option on the amortizing swap in lieu of the "diagonal" swaptions. Unfortunately, there are no liquid quotes for the prices of European options on amortizers. Instead we are going to reconstruct the amortizing swap as a linear combination of standard bullet swaps; i.e., express the amortizing swap as a basket of ordinary swaps. We then use the LGM model itself to find the value of a European option on the basket in terms of the market values of each swaptions. This European value is then to be fed into our calibration scheme.

A.2.1. Constructing the basket. Define swap k to be the bullet swap which starts on date T_j^{set}, and ends on the k^{th} paydate t_k of the Bermudan's payoff. We *assume* (or approximate) the bullet swap's pay dates as being the same as the Bermudan pay dates. So we assume that swap k has fixed leg pay dates

$$t_i \qquad \text{for } i = i_j^{first}, i_j^{first}+1, \ldots, k \tag{A.15}$$

and start date T_j^{set}. Let M_k^{ref} and R_k^{ref} be the notional and strike of the k^{th} swap. Then its fixed leg payments are

$$M_k^{ref} \beta_i \left(R_k^{ref} - S_i\right) \qquad \text{at } t_i \qquad \text{for } i = i_j^{first}, 2, \ldots, k-1 \tag{A.16a}$$

$$M_k^{ref} \left\{1 + \beta_k \left(R_k^{ref} - S_i\right)\right\} \qquad \text{at } t_k \tag{A.16b}$$

and it floating leg payments are equivalent to

$$M_k^{ref} \qquad \text{at } T_j^{set}. \tag{A.16c}$$

Here S_i is the basis spread for the i^{th} interval, adjusted to the fixed leg's frequency and day count basis is the usual way, and

$$\beta_i = \text{cvg}(T_j^{set}, t_i) \qquad \text{for } i = i_j^{first}, \tag{A.16d}$$

$$\beta_i = \text{cvg}(t_{i-1}, t_i) \qquad \text{for } i = i_j^{first}+1, 2, \ldots, k-1. \tag{A.16e}$$

We wish to choose the notionals M_k^{ref} and strikes R_k^{ref} so that the sum of all the payments of these reference swaps equals the payments in the Bermudan's payoff. Equating the i^{th} payment of all the swaps in the basket to the i^{th} payment of the amortzing swap yields

$$\sum_{k=i}^{n} M_k^{ref} \beta_i \left(R_k^{ref} - S_i\right) + M_i^{ref} = C_i - rfp_j \qquad \text{for } i = i_j^{first} \tag{A.17a}$$

$$\sum_{k=i}^{n} M_k^{ref} \beta_i \left(R_k^{ref} - S_i\right) + M_i^{ref} = C_i \qquad \text{for } i = i_j^{first}+1, 2, \ldots, n \tag{A.17b}$$

Equating the floating legs yields

$$\text{(A.17c)} \qquad \sum_{k=i_j^{first}}^{n} M_k^{ref} = M_j \frac{D(t_j^{set})}{D(T_j^{set})}.$$

Now, if all the strikes R_k^{ref} were specified, then we could work backwards. We would first determine the notional M_k^{ref} as $k=n$ needed to match the last payment, then the notional for $k=n-1$ needed to match the next to last payment, etc. Proceeding in this way, we would match all the fixed leg payments, but the sum of these notionals M_k^{ref} would *not* (unless we were very lucky) match $M_j D(t_j^{set})/D(T_j^{set})$. We use our freedom to choose the strikes to get one more degree of freedom in choosing the notionals.

Strike choice A. There are two obvious methods for choosing the reference strikes. The first is setting each swap's strike equally far from the money, so that the same parallel shift is needed to bring each to the money:

$$\text{(A.18a)} \qquad R_k^{ref} = R_k^{sw} + \lambda \qquad \text{for } k = i_j^{first}, i_j^{first}+1, \ldots, n$$

where R_k^{sw} is the forward (break-even) fixed rate for swap k. Solving

$$\text{(A.18b)} \qquad \beta_i \sum_{k=i}^{n} M_k^{ref}\left(R_k^{sw} - S_i\right) + \lambda\beta_i \sum_{k=i}^{n} M_k^{ref} + M_i^{ref} = C_i - rfp_j \qquad \text{for } i = i_j^{first}$$

$$\text{(A.18c)} \qquad \beta_i \sum_{k=i}^{n} M_k^{ref}\left(R_k^{sw} - S_i\right) + \lambda\beta_i \sum_{k=i}^{n} M_k^{ref} + M_i^{ref} = C_i \qquad \text{for } i = i_j^{first}+1, 2, \ldots, n$$

determines the notionals $M_k^{ref}(\lambda)$ in terms of λ. We then need to find which λ enables the floating leg to be matched:

$$\text{(A.18d)} \qquad \sum_{k=i_j^{first}}^{n} M_k^{ref}(\lambda) = M_j \frac{D(t_j^{set})}{D(T_j^{set})}.$$

This can be done by a quick global Newton's scheme, starting from $\lambda = 0$.

Strike choice B. The second method is a variant of this scheme. It sets the strikes of the reference swaps to be the same number of standard deviations from the money:

$$\text{(A.19a)} \qquad R_k^{ref} = R_k^{sw} + \lambda\sigma_k^{atm} \qquad \text{for } k = i_j^{first}, i_j^{first}+1, \ldots, n.$$

Here R_k^{sw} is again the swap rate, and now σ_k^{atm} is the at-the-money swaption volatility for the swaption with exercise date t_j^{ex}, start date T_j^{set}, and end date t_k. Solving

$$\text{(A.19b)} \qquad \beta_i \sum_{k=i}^{n} M_k^{ref}\left(R_k^{sw} - S_i\right) + \lambda\beta_i \sum_{k=i}^{n} M_k^{ref}\sigma_k^{atm} + M_i^{ref} = C_i - rfp_j \qquad \text{for } i = i_j^{first}$$

$$\text{(A.19c)} \qquad \beta_i \sum_{k=i}^{n} M_k^{ref}\left(R_k^{sw} - S_i\right) + \lambda\beta_i \sum_{k=i}^{n} M_k^{ref}\sigma_k^{atm} + M_i^{ref} = C_i \qquad \text{for } i = i_j^{first}+1, 2, \ldots, n$$

determines the notionals $M_k^{ref}(\lambda)$ in terms of λ. We can then use a quick global Newton's scheme to find which λ enables the floating leg to be matched:

$$\text{(A.20)} \qquad \sum_{k=i_j^{first}}^{n} M_k^{ref}(\lambda) = M_j \frac{D(t_j^{set})}{D(T_j^{set})}.$$

Which strike method to use. We don't have enough market experience to know whether the second method of choosing strikes offeres significantly better pricing/hedging. If it does not, then we should use the simpler method, method A. Note that these can be programmed together, since it is just a matter of inserting the weights σ_k^{atm} into the problem.

A.2.2. Pricing the European option on the basket. Now that we have replicated the amortizing swap as a basket of bullet swaps, we price the European option on the basket. We first calibrate LGM model to reproduce the market price of each swaption in the basket, and then use the calibrated LGM model to price the European option on the basket. *Once we obtain the price of the European option on the basket, we throw this calibration away. This calibration has no role in pricing our original Bermudan* except *to obtain the value of the European option on the amortizing swap.*

Recall that the k^{th} swap has the fixed leg payments,

$$M_k^{ref}\beta_i\left(R_k^{ref}-S_i\right) \qquad \text{at } t_i \qquad \text{for } i=i_j^{first},2,\ldots,k-1 \tag{A.20a}$$

$$M_k^{ref}\left\{1+\beta_k\left(R_k^{ref}-S_i\right)\right\} \qquad \text{at } t_k \tag{A.20b}$$

and it floating leg payments are equivalent to

$$M_k^{ref} \qquad \text{at } T_j^{set}. \tag{A.20c}$$

The LGM value of receiver swaption k is

$$V_{basket}^{LGM}=\sum_{i=i_j^{first}}^{k}\beta_i\left(R_k^{ref}-S_i\right)D_i\mathcal{N}\left(\frac{y_k+\Delta H_i\zeta_j^{ex}}{\sqrt{\zeta_j^{ex}}}\right)+D_K\mathcal{N}\left(\frac{y_k+\Delta H_i\zeta_j^{ex}}{\sqrt{\zeta_j^{ex}}}\right)-D_0\mathcal{N}\left(\frac{y_k}{\sqrt{\zeta_j^{ex}}}\right) \tag{A.21a}$$

whose y_k is determined implicitly by solving,

$$\sum_{i=i_j^{first}}^{k}\beta_i\left(R_k^{ref}-S_i\right)D_ie^{-\Delta H_iy_k-\frac{1}{2}\Delta H_i^2\zeta_j^{ex}}+D_ke^{-\Delta H_ky_k-\frac{1}{2}\Delta H_k^2\zeta_j^{ex}}=D_j^{set}, \tag{A.21b}$$

and where we have used

$$\Delta H_i=H_i-H_j^{set}=H(t_i)-H(T_j^{set}). \tag{A.21c}$$

We calibrate these swaptions by using the "calibration of a row of swaptions" technique described above. (More to the point, we can use the same routines). We first set

$$\zeta_j^{et}=\zeta(t_j^{ex})=10^{-4}*t_j^{ex}, \tag{A.22a}$$

$$H(T_j^{set})=0 \tag{A.22b}$$

without loss of generality. We then assume that $H(T)$ is piecewise linear with nodes at t_i for $i=i_j^{first},i_j^{first}+1,\ldots,n$. We first calibrate on the swaption $k=i_j^{first}$, which determines $H(t_i)$ for $i=i_j^{first}$. We then calibrate on swaption $k=i_j^{first}+1$, which determines the next $H(t_i)$. Continuing gives us all the values of $H(t_i)$. Once we have calibrate $H(t)$, then the value of the European option on the basket is

(A.23a)
$$V^{LGM} = \left(C_{i_j^{first}} - rfp_j\right) D_{i_j^{first}} \mathcal{N}\left(\frac{y + \Delta H_i \zeta_j^{ex}}{\sqrt{\zeta_j^{ex}}}\right) + \sum_{i=i_j^{first}+1}^{n} C_i D_i \mathcal{N}\left(\frac{y_k + \Delta H_i \zeta_j^{ex}}{\sqrt{\zeta_j^{ex}}}\right)$$
$$- M_j D(t_j^{set}) \mathcal{N}\left(\frac{y_k}{\sqrt{\zeta_j^{ex}}}\right)$$

where y is the unique solution of:
(A.23b)
$$\left(C_{i_j^{first}} - rfp_j\right) D_{i_j^{first}} \exp\{-\Delta H_{i_j^{first}} y_k - \tfrac{1}{2}\Delta H^2_{i_j^{first}} \zeta_j^{ex}\} + \sum_{i=i_j^{first}+1}^{n} C_i D_i e^{-\Delta H_i y_k - \frac{1}{2}\Delta H_i^2 \zeta_j^{ex}} = M_j D(t_j^{set}).$$

Once we have the value V^{LGM}_{basket} of the European option on the amortizing swap, we can use this as the market price of the amortizing swap in our calibration.

Appendix B. American swaptions.

Appendix C. Cross-currency swaptions.