Valuation of Inflation-Indexed American Cap

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Synopsis

This thesis derives three pricing formulae for inflation-indexed American cap under the Jarrow and Yildirim (2003) model, a major model in the inflation modeling literature. Three approaches are used: analytical method, approximate dynamic programming and duality theory. The thesis first examines the pricing of several vanilla inflation-indexed swaps and caplets. Then it introduces the notion of stopping time and shows that the price of the cap can be deduced in a way that is similar to the pricing of an European call under the Black Scholes formula. The thesis also shows that the cap price can be reasonably approximated by first forming basis functions for the Q-Factor and then performing the least square Monte Carlo routine. Finally, the thesis uses the concepts of supremum and infimum in duality theory and derives the conditions in which the cap price equates with its upper bound.

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Notation

- LFM = Lognormal Forward-LIBOR model
- HJM = Heath-Jarrow-Merton model
- JY model = Jarrow and Yildirim model
- SV = Stochastic volatility
- SDE = Stochastic differential equation
- PDE = Partial differential equation
- IICapFloor = Inflation-indexed cap/floor
- IICplt = Inflation-indexed caplet/floorlet
- ZCIIS = Zero coupon inflation-indexed swap
- YYIIS = Year-on-year inflation-indexed swap
- LSMC = Least Square Monte Carlo
- IIACap = Inflation-indexed American cap/floor
- ADP = Approximate dynamic programming
- CPI = Consumer price index
- GDP = Gross domestic product
- ZCB = Zero coupon bond
- r(t) or r_t: Instantaneous spot real interest rate at time t
- n(t) : Instantaneous spot nominal interest rate at time t
- σ_n: Volatility of nominal rate
- σ_r: Volatility of real rate
- $\rho_{r,n}$: Correlation between nominal and real rates
- $P_n(t,T)$: ZCB price in the nominal economy
- $P_r(t,T)$: ZCB price in the real economy
- E_n : Expectation with respect to the nominal economy
- E_r : Expectation with respect to the real economy
- I(t) : CPI at time t
- N(·) = Normal cumulative density function
- Q^T: T-forward measure
- Q^B: Risk neutral measure with bank account as numeraire
- B(t): Money market account
- φ_i : Floating-leg year-fraction

Chapter 1

Introduction

1.1 Background and Literature Survey

Everyone concerns about inflation, whether they are rich or poor, young or old. The affluent people seek investments to maintain their wealth whereas the needy individuals worry about the rising living expenses and the stagnating wages in the economy. Hyper-inflation could even cause riots and political turmoil. Retirees and individuals close to retirement would hope that their pensions could give them the living standards they had anticipated. So it is natural that one would seek ways to protect against inflation.

Some of them go into the stock markets, some go into the real estates, some go to the commodities like gold and silver, and yet some go for inflation-protected bonds issued by the governments (i.e. US TIPS, UK Glit and the latest Hong Kong iBond). Depending on the risk preference and the investment goals of investors, an asset class may be preferred to another asset classes. The inflation-indexed bonds in the bond market are known to offer strong protection against inflation. But the con of buying a real bond is that the investors have to use a large amount of money to buy the bond. In practice, this is not a big problem because the investors can easily transfer their saving in banks to do the transaction.

Inflation derivative, on the other hand, is a sophisticated class of asset class, with its payoff linked to the inflation index rate overtime. This class of investment requires little upfront and exchange payments only when the actual inflation rate deviates from the pre-agreed rate. When the inflation exposure is very large for some big corporations, this style of hedging inflation risk is much preferred to buying the real bond alone. This is because, comparing to the real bonds, the derivative helps to lower transaction fees, rebalancing efforts, liquidity needs and more importantly it saves cost of capital for the companies.

Based on foreign currency analogy, Barone and Castagna (1997) and Jarrow and Yildrim (2003) developed similar frameworks to price inflation derivatives. The JY model has become the theoretical backbone model in inflation modeling because it closely resembles the HJM model in the interest rate derivatives literature. But the JY model is difficult to calibrate. To overcome this issue, Mercurio (2005) later developed his first market model based on the major assumptions that the forward inflation rate is lognormal and the real and nominal interest rate follow a LFM. Still, it is hard to estimate the real rate volatility.

This problem is alleviated after Kazilla (1999), Belgrade, Benhamou and Koehler (2004) and Mercurio (2005) independently developed the second market model. This model uses the fact that the forward inflation rate is a martingale under a forward risk neutral measure and then derives the dynamics of the lagged forward inflation rate to obtain a price for YYIIS. Macrina (2006) offers a radically different pricing framework for inflation derivatives. His framework focuses on filtration specifications and discrete time asset pricing. He found that the prices of many inflation derivatives would depend on the stochastic processes of money supply and aggregate consumption and etc. His work has sparked a new direction in inflation modeling recently.

1.2 Motivation and Objectives

Motivated by the effectiveness of inflation derivatives in hedging inflation, in this paper we focus on how to price major vanilla swaps and caps in the inflation market and then explore the pricing of a highly flexible exotic inflation derivative called inflation-indexed American cap (IIACap), all under the arbitrage-free framework in the mathematical finance literature.

The IIACap is a stream of inflation-indexed American caplet (an American option on the inflation rate). The IIACap is chosen to be discussed and analyzed over the other similar exotic inflation derivative, namely, the inflation-indexed Bermudan swaption. This is because the IIACap is less explored in the inflation modeling literature than the related Bermudan swaption. The reasons could be that the Bermudan swaptions are traded and priced more frequently and the trend that the swaption is a more favorable mean of financing and hedging for large corporations.

Despite of the popularity of Bermudan swaption, the IIACap has three advantages over the swaption. Firstly, the cap gives the holder more flexibility in option exercises and offers more protection over a surge in the inflation rate throughout the entire cap contract period. But the Bermudan swaption can only be exercised once, hence it offers less protection when the inflation curve is either W or M shaped. Secondly, caps will realize tremendous gain if all caplets are exercised early. This occurs when the inflation rate reaches its historical peak. Thirdly, the cap holder has none to pay the counterparty while the swaption holder starts to make netted payments to the other party upon the exercise of the option. Hence, the swaption poses greater liquidity risk and default risk than the cap. All these advantages motivate us to explore the pricing of an IIACap in details.

1.3 Plan

We start by introducing inflation rate and the actual inflation markets in chapter 2. In Chapter 3, we will review the key tools used in mathematical finance and introduces several important fixed income measure used in interest rate and inflation rate modeling. In chapter 4 we will discuss the major models being used in inflation modelling. The chapter also illustrates inflation modeling under the information-based pricing model, a new pricing framework in mathematical finance. Next we rigorously derive pricing formulae for two major vanilla swaps under the JY model in chapter 5. In chapter 6, we will deduce a condensed formula for American caplet and cap and explain the roles of both approximate dynamic programming method (a.k.a the Least Square Monte Carlo technique) and duality theory in approximating the true IIACap price. Finally, we conclude all our results in chapter 7 and give personal comments in chapter 8.

Chapter 2

Inflation

2.1 Inflation Overview

Prices of good and service in an economy never stay the same over time. Our food prices are generally four times more expensive than those 40 years ago. If the young men and women invested nothing in these 40 years, they would have struggled to keep up with the rising shelter and food expenses nowadays. Formally, the term inflation (deflation) is used to describe the increasing (decreasing) prices of goods and service in an economy in general.

The goods accounted in the inflation can range from books, bulbs to toilet paper and etc. Service could be haircut, concerts, comedy and so on. Since there are more than millions of goods in an economy, governments hire statisticians and economists to collect data and make appropriate adjustments (i.e. seasonality) in order to provide an accurate inflation rate to the citizens. The official inflation rate can effect citizens' saving and spending decisions. In some extreme cases, the number can even stir up dissatisfaction toward wages and cause labor strikes and public protests in the economy. Hence, almost everyone would show concerns about the inflation rates in the past, present and future.

There are two ways to measure inflation in an economy. One is the GDP deflator and the other is the consumer price index (CPI). These two indices can be easily found in the websites of central banks and trusted government agencies. In economics, gross domestic product (GDP) is the market value of all the final goods and service produced within a country in a certain period. Current price level is used to measure the nominal GDP while the base year price level is used to calculate the real GDP. The GDP deflator is the ratio of nominal GDP to the real GDP multiplied by

100. The formula is

$$\text{GDP Deflator} = 100(\frac{Nominal\ GDP}{Real\ GDP}).$$

CPI is a measure of the average change of the market basket over time. The basket is based on the goods and services consumed in urban areas within the country. The CPI formula for a particular good is

$$\text{CPI} = 100(\frac{new\ cost}{old\ cost}).$$

For n goods in the market basket, the CPI formula is

$$\text{CPI} = \sum_{i=1}^{n} CPI_i\, weight_i.$$

In this paper, we assume to use the CPI formula above when calculating our inflation rate and denote the CPI Index function I(t) as the CPI index at time t, for some t≥0. Hence at t=0, I(0)=1. The inflation rate is the percentage change of the CPI index over the time interval [t, t+h], h>0. Its formula is given by

$$\text{Infl(t,t+h)}=\frac{I(t+h)}{I(t)} - 1.$$

The inflation rate is regarded as the floating rate in an inflation rate swap. A party pays the inflation rate for a pre-agreed rate while the other party does the opposite.

2.2 Nominal Rates, Real Rates and Expected Inflation

It is very important to make distinction between nominal rate and real rate and inflation rate. The nominal rate is the rate told and promised to the investor while the real rate is the return an investor receives, net of inflation. A positive real rate return in an investment represents an increase in the purchasing power. But in practice the inflation rate is not known when an investment is made. So the ante real rate would be the nominal rate minus the expected inflation. This gives rise to the famous Fisher effect in economics

$$i \approx r + \pi,$$

$$\text{or } i = r + \pi^e,$$

where i is the nominal interest rate, r is the real interest rate and π is the actual inflation rate, and $+\pi^e$ is the ex-ante or expected inflation rate.

2.3 Inflation Market

2.3.1 Indexed Bonds

An indexed bond aims to protect the purchasing power of the savings of investors by linking the bond's cash flows to fluctuations of some pre-agreed price indices. The country CPI index is typically used in an indexed bond. Other indices include, but not limited to, GDP deflator and retail price index (RPI). We give a list of indexed bonds offered in the global inflation market in Table 1.

OECD countries with public inflation-indexed bonds

Country	Issue Date (latest)	Index Used
Australia	1983-2003	CPI
Austria	2003-	European CPI,US CPI
Canada	1991-	CPI
Chile	1966-	CPI
Czech Republic	1997	CPI
Denmark	1982-	CPI
Finland	1945-1967	Wholesale Price
France	1998- 2001-	Domestic CPI European CPI
Germany	2002,2003	European CPI

Greece	2003-	European CPI
Hungry	1995-1999	CPI
Iceland	1995-	CPI
Ireland	1983-1985	CPI
Israel	1955-	CPI
Italy	2003-	European CPI
Mexico	1989-	CPI
New Zealand	1995-1999	CPI
Norway	1982	CPI
Peru	1999-	CPI
Poland	1992-	CPI
Sweden	1994-	CPI
Turkey	1997-	CPI
United Kingdom	1981-	CPI
United States	1997-	CPI

Source: Table 1.1 in Deacon, Derry and Mirfendereski (2004)

Table 1

According to Deacon, Derry and Mirfendereski (2004), the capital-indexed form is by far the most common form of an indexed bond. The capital-indexed structure allows inflation adjustments to all the coupons and principal payments. The value of an indexed bond is influenced by two factors, namely, the real rate return and the compensation for the purchasing power degrade due to inflation. In this sense, the real return is certain throughout the life of the bond.

2.3.2 Derivatives

Derivatives are typically designed to accommodate the needs of investors, hedgers, speculators and securities issuers. The global market for the inflation-indexed has become more mature and established since 2000. This motivates the activities and trades in the parallel inflation derivative market. Pension funds, fixed income funds and hedge funds have long been interested in this class of derivative to better manage the inflation risks across countries in their investment portfolios. The vanilla inflation derivatives are zero-coupon inflation-indexed swaps, year on year inflation-indexed swaps and inflation-indexed caps and floors.

Chapter 3

Mathematical Background

In this chapter, we will review some definitions and notations used in interest rate modeling and then illustrate some important theorems that will aid us in pricing inflation derivatives. Most definitions, theorems and proofs are drawn from Shreve (2004) Andersen and Piterbarg (2010), Brigo and Mercurio (2006), and Bertsekas (2005). Firstly, we introduce the meaning of no arbitrage and the fundamental theorem of asset pricing. Secondly, we state Ito's lemma, Girsanov theorem and change of numeraire technique. Then the rollout algorithm in control theory will be introduced. This will form the theoretical background and justification for the Q-value iteration routine in chapter 6. Lastly, we state some common notations for fixed income securities that will be used extensively throughout the paper.

3.1 Fundamentals of Mathematical Finance

Definition 3.1.1 (Trading Strategy)

Assume an investor can engage in a trading strategy involving p assets $X_1,\ldots,X_p$where p is an integer and p≥1. Let $\emptyset_i$(t,w) represents the holdings of the i-th asset at time t. Then the value of a trading strategy at time t is thus

$$\mu(t) = \emptyset(t)^T X(t) \tag{3.1}$$

Definition 3.1.2 (Arbitrage)

An arbitrage opportunity is a self-financing strategy $\emptyset$ for some time t$\in [0, T]$,

$\mu(0) \geq 0$ almost surely, and P($\mu(0) > 0$) >0,

where u(t) is given by (3.1).

We wish to know when the price of an asset will be free of arbitrage. And we could state such conditions in term of the equivalent martingale measure.

Definition 3.1.3 (Equivalent Measure)

Two measures P and Q on the same measure space $(\Omega, \mathcal{F})$ are said to be equivalent measures (i.e. P ~ Q) if they have the same null sets, i.e. P(A)=0 $\Leftrightarrow$ Q(A)=0, $\forall A \in \mathcal{F}$.

Definition 3.1.4 (Attainability)

Assume a contingent claim in the market has payoff V_T depending on the i-th asset X_i over time interval [0, T]. Then, the contingent claim is said to be attainable if there is a permissible trading strategy $\emptyset$ such that $\mu(\emptyset) = V_T$.

Definition 3.1.5 (Complete Market)

A financial market is complete if all contingent claims in the market are attainable.

Theorem 3.1.1 (The First Fundamental Theorem of Asset Pricing)

A financial market is arbitrage free if and only if there is an equivalent martingale measure P ~ Q.

Theorem 3.1.2 (The Second Fundamental Theorem of Asset Pricing)

The equivalent measure Q is unique and hence the price of a contingent claim is also unique if and only if the underlying financial market is complete.

Theorem 3.1.3 (Ito's lemma)

Assume an integer n≥1. Consider a system of n stochastic differential equations $\mathrm{d}X_t$=$u_t dt + \sigma_t dB_t$ where X_t=$(X_{t,1}, \ldots X_{t,n})$, B_t is a n-dimensional Brownian motion, and both u_t and σ_t are nx1 matrices. Let f=f(t, X_t). The Ito's lemma states that

$$\mathrm{df}(\mathrm{t}, X_t) = \frac{\partial \mathrm{f}(\mathrm{t}, X_t)}{\partial t} dt + \nabla_{X_t}^T \mathrm{f} \cdot dX_t + \frac{1}{2} d\mathrm{X}_t^T \cdot \nabla_{X_t}^2 \mathrm{f} \cdot dX_t,$$

where dot products are used in the second and third terms and $\nabla_{X_t}^T \mathrm{f}$ and $\nabla_{X_t}^2 \mathrm{f}$ are the gradient of $\mathrm{f}(\mathrm{t}, X_t)$ and n x n Hessian matrix of $\mathrm{f}(\mathrm{t}, X_t)$, respectively.

Theorem 3.1.4 (Girsanov theorem)

On a probability space $(\Omega, \mathrm{F}, \mathrm{P})$, let X_t=$(X_{t,1}, \ldots X_{t,n})$ be a n-dimensional adapted process and T>0. Define

$$Z(t)=\exp\left\{-\int_0^t X(u)\cdot dW(u) - \frac{1}{2}\int_0^t ||X(u)||^2 du\right\},$$

$$\widetilde{W}(t) = W(t) + \int_0^t X(u)du.$$

Set Z=Z(T) and further assume that

$$E\int_0^t ||X(u)||^2 Z^2(u)du<\infty.$$

Then E[Z]=1. The new probability measure $\tilde{P}$ is then given by

$$\tilde{P}(A) = \int_A Z(\omega)dP(\omega),\ \forall \mathrm{A} \in \mathrm{F},$$

with d-dimensional Brownian motion $\widetilde{W}(t)$.

Theorem 3.1.5 (Martingale representation theorem)

On a probability space $(\Omega, \mathrm{F}, \mathrm{P})$, assume T>0 and 0≤t≤T the filtration F(t) is generated by a n-dimensional Brownian motion W(t). Allow M(t) be a martingale with respect to F(t) under probability measure P. Then there exists an adapted n-dimensional process Y_s=$(Y_{s,1}, \ldots Y_{s,n})$, 0≤s≤T, such that

$$M(t)=M(0)+\int_0^t Y(u)\cdot dW(u).$$

The same result holds for the martingale $\widetilde{M}(t)$ under $\tilde{P}$ after applying the Girsanov theorem.

Theorem 3.1.6 (Change of numeraire)

Assume there exists a probability measure Q^N for a numeraire N and $Q^N \sim Q$, an initial measure. The price of any traded asset V relative to the numeraire N in the market is a martingale under Q^N. Mathematically, we write

$$\frac{V(t)}{N(t)}=E^{Q^N}\left[\frac{V(T)}{N(T)}|F_t\right].$$

Let B be an arbitrary numeraire with Q^B as the new probability measure and $Q^B \sim Q$ .Then,

$$\frac{V(t)}{B(t)}=E^{Q^B}\left[\frac{V(T)}{B(T)}|F_t\right].$$

Proof:

Knowing that $\frac{V(t)}{N(t)}$ is a martingale under Q^N, we write,

$$\frac{V(0)}{N(0)} = E^{Q^N}\left[\frac{V(T)}{N(T)}|F_t\right] = E^{Q^B}\left[\frac{B(0)}{N(0)}\frac{V(T)}{B(T)}|F_t\right]=E^{Q^N}\left[\frac{V(T)}{N(T)}\frac{dQ^N}{dQ^B}|F_t\right] = E^{Q^N}\left[\frac{V(T)}{N(T)}Z|F_t\right],$$

where Z is the Radon-Nikodym derivative.

Using an abstract Bayes' rule on the expression above, we obtain that $\frac{V(t)}{B(t)}$ is a martingale under Q^B. The more complicated use of this technique, called the change of numeraire toolkit is given in Brigo and Mercurio (2006)

3.2 Real Analysis and Control Theory

3.2.1 Real Analysis

Definition 3.2.1 (Infimum)

Let A be a partially ordered set and $\mathrm{B} \subseteq \mathrm{A}$. The infimum of B is the greatest element of A such that it is also less than or equal to any element in B. The infimum of B is also called the greatest lower bound of B. We denote infimum with the operator "inf".

Definition 3.2.2 (Supremum)

Let A be a partially ordered set and $\mathrm{B} \subseteq \mathrm{A}$. The supremum of B is the smallest element of A such that it is also greater than or equal to any element in B. The supremum of B is also called the least upper bound of B. We denote supremum with the operator “sup”.

Definition 3.2.3 (Monotonically Decreasing function)

If $x \leq y$, then $f(x) \geq f(y)$ for all x and y. Then the function f(·) is a monotonically decreasing function.

3.2.2 Control Theory

This section provides the theoretical foundation for the least square Monte Carlo algorithm outlined in chapter 6. We first introduce a basic dynamic programming problem and then show how to solve the problem with Q-Factor simulation.

Definition 3.2.4 (Dynamic Programming Algorithm)

Consider $\pi^* := \{u_0^*, u_1^*, \dots, u_{N-1}^*\}$ be an optimal policy for a N-period optimization problem, where u_i^*, i=0,…, N-1, is the control or decision variable to be selected at time i. The problem is to minimize a cost-to-go function from time 0 to time N. Mathematically, we seek π^* such that

$$\min_{u_k \in U_k(x_k)} \mathrm{E}\left\{g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k(x_k), w_k)\right\},$$

where $U_k(x_k)$ is the feasible set of policy under given constraints, x_k is the state variable, $g(\cdot)$ is the cost function, and w_k is the disturbance variable.

Now consider a sub-problem going from time i to time N. By the well-known principle of optimality, the truncated policy $\{u_i^*, \dots, u_{N-1}^*\}$ is optimal for the sub-problem

$$\min_{u_k \in U_k(x_k)} \mathrm{E}\left\{g_N(x_N) + \sum_{k=i}^{N-1} g_k(x_k, u_k(x_k), w_k)\right\},\ i \geq 0.$$

Definition 3.2.5 (Rollout Algorithm)

The rollout algorithm is a one-step lookahead dynamic programming problem

$$\min_{u_k \in U_k(x_k)} \mathrm{E}\{g_k(x_k, u_k(x_k), w_k) + \tilde{J}_{k+1}(x_{k+1})\},$$

where $\tilde{J}_{k+1}$ is an approximation of the true cost-to-go function J_{k+1}.

The underlying idea is that at each state x_k, we minimize the cost at that state and also the cost of moving x_k into its next state. Intuitively, this problem can be applied to price American option. We switch the minimization problem into a problem in which we choose $u_k \in U_k(x_k)$ (a trading strategy) to maximize the expected option payoff under an appropriate risk neutral measure. The objective function (the function inside the expectation) will be the maximum between option intrinsic value and its expected value. The expected value will take all the possible next state x_{k+1} into consideration if we use the rollout-algorithm above. So rather than using an approximate cost-to-go function $\tilde{J}_{k+1}$ in a minimization problem, we would use an approximate "payoff-to-go" function in pricing American option. The discussion above gives a general and yet informal way to approach the problem of pricing American option dynamically. We will have more to say about the conceptual "payoff-to-go" function above and then formally derive a procedure to value an IIACap.

3.3 Bond Mathematics

This section provides definitions of zero-coupon bond, LIBOR rate, forward rate, swap rate and annuity factors. Then we introduce some common fixed income probability measures such as spot measure, terminal measure, swap measure and T-forward measure.

Definition 3.3.1 (Zero-Coupon Bond)

Consider a bond paying 1 unit at a fixed maturity T. The bond is called a T-bond and its price at time t, given T >t, is denoted as P(t,T). Then P(t, T) is the discounted factor, telling us how much the $1 cash flow at time T is worth at time t. Using the definition further, we have two useful relations.

$$P(t,t)=1 \tag{3.8}$$

$$P(t,S)\ P(S,T)=P(t,T),\ T>S>t \tag{3.9}$$

(3.8) is intuitive because the discount factor used to discount current value is always one. (3.9) says that discounting $1 from T to t is the same as discounting $1 from T to S first then discount again from S to t. If (3.9) is violated, there will be arbitrage opportunities.

Definition 3.3.2 (LIBOR Rate)

The LIBOR rate is called London Interbank offered rate. It is the rate available at time t that banks borrow and lend to each other in a time period [T, T+τ], where T+τ> T > t and $\tau > 0.\tau$ is the length of the loan. Denote the rate as L(t, T, T+τ). Expressed mathematically,

$$\mathrm{L}(\mathrm{t},\mathrm{T},\mathrm{T}+\tau)=\frac{1}{\tau}[\frac{P(t,T)}{P(t,T+\tau)}-1]$$

Given t=T and use (3.8) and (3.9) we get,

$$L(t, T, T+\tau) = L(T, T, T+\tau) = \frac{1}{\tau}\left[\frac{1}{P(T, T+\tau)} - 1\right] \quad (3.10)$$

Definition 3.3.3 (Forward Bond Price)

Forward bond price is the discounted bond price to purchase a bond at time T, with payoff of \$1 at time T+τ. Mathematically,

$$P(t, T)P(t, T, T+\tau) = P(t, T+\tau), \ \tau > 0. \quad (3.11)$$

Denote y(t, T ,T+τ) as continuously compounded forward yield.

$$P(t, T, T+\tau) = e^{-y(t, T, T+\tau)}. \quad (3.12)$$

Definition 3.3.4 (Forward Rate)

Forward rate is the yield of the forward bond price. The usual market quotes involve discrete-time compounding. So we define a discrete simple forward rate with tenor τ as L(t, T, T+τ). Its relation to forward bond price is

$$1+\tau L(t, T, T+\tau) = 1/P(t, T, T+\tau). \quad (3.13)$$

Expressing (3.13) in terms of bond price, we have

$$L(t, T, T+\tau) = \frac{1}{\tau}\frac{P(t, T+\tau) - P(t, T)}{P(t, T)}. \quad (3.14)$$

In the limit $\tau \to 0$, $L(T, T, T+\tau) \to f(t, T)$. The quantity f(t, T) is called instantaneous forward rate from time t to time T. Analogous to (3.14), we have

$$f(t, T) = -\frac{\partial lnP(t, T)}{\partial T}. \quad (3.15)$$

Using (3.12), we obtain a relation between bond price and forward rates

$$P(t, T, T+\tau) = \exp\left(-\int_T^{T+\tau} f(t, u)du\right), \quad (3.16)$$

where f(t, T) is given in (3.15).

Definition 3.3.5 (Annuity Factor)

Given a tenor structure, 0≤T_0<T_1<…<T_N, for any two integers k, m satisfying conditions 0≤ $\mathrm{k} < \mathrm{N}$, m>0 and k+m≤N, we define an annuity factor $A_{k,m}$ by

$$A_{k,m}(t) = \sum_{n=k}^{k+m-1} P(t, T_{n+1})\tau_n,\ \ \tau_n = T_{n+1} - T_n. \tag{3.17}$$

The annuity factor is extremely useful in evaluating swap that has discrete and fixed cash flows over the length of the swap contract. Notice that $A_{k,m}(t)$ depends on k, m ,and time t. The variable k denotes the beginning of the cash flow series whereas the variable m denotes the number of payment in the series. Hence, at time k+m, the cash flow ends. We could denote T_{k+m} as maturity as well.

Definition 3.3.6 (Swap Rates)

Follow the setup in Definition 3.3.4, denote $S_{k,m}(t)$ the time t swap rate.

$$S_{k,m}(t) = (P(t, T_k) - P(t, T_{k+m}))/\, A_{k,m}(t),\ t \le T_k. \tag{3.18}$$

Or equivalently, by (3.14),

$$S_{k,m}(t) = [\sum_{n=k}^{k+m-1} \tau_n P(t, T_{n+1}) L(t, T_k, T_k)]/\, A_{k,m}(t), \quad t \le T_k. \tag{3.19}$$

3.4 Common Risk Neutral Measures

T-Forward Measure

T-Forward Measure Q^Tuses a T-maturity zero-coupon bond as the numeraire asset. From the risk neutral formula, we write,

$$V(t)/P(t,T)=E_t^T(V(T)/P(T,T)) , \quad t \le T. \tag{3.20}$$

The super subscript T denotes the expectation under Q^T and function V is the payoff function of a contingent claim. Rewriting (3.20), we get a more convenient form

$$V(t)=P(t,T)\ E_t^T(V(T)). \tag{3.21}$$

Next we show that the forward LIBOR rate L(T,T,T+τ) is a martingale under the T+τ forward measure $Q^{T+\tau}$. That is

$$L(T,T,T+\tau)= E_t^{T+\tau}(L(T,T,T+\tau)), \quad t \le T. \tag{3.22}$$

By definition of LIBOR rate and (3.10),

$$L(t, T, T+\tau) = \frac{1}{\tau}\left[\frac{P(t,T)}{P(t,T+\tau)} - 1\right].$$

For L(T,T,T+τ) to be a martingale under $Q^{T+\tau}$, we need $P(t,T)/P(t,T+\tau)$ to be a martingale under $Q^{T+\tau}$. Treat P(t,T) as V(t) and P(t, T+τ) as the numeraire. Then, $V(t)/P(t,T+\tau)=E_t^{T+\tau}(V(T)/P(T,T+\tau))$ and thus $P(t,T)/P(t,T+\tau)$ is a martingale.

Spot Measure

Given the usual tenor structure $0=T_0<T_1<\ldots<T_N=T$. Consider a trading strategy where we invest in a money market account φ from time 0 to the next time index in the tenor structure T_1 and then reinvest the proceeds over and over again until T. For some t>0 and t<T, let B(t) be the rolling money account balance. Using (3.13), we write

$$B(t)=\prod_{n=0}^{i}(1+\tau_n L_n(T_n))P(t,T_{i+1}), \qquad T_i < t \le T_{i+1}, \tag{3.23}$$

where we write $L_n(T_n)$=L(t, T_n, T_{n+1}) for convenience.

We call the measure induced by the numeraire B(t) as the spot measure Q^B with its associated expectation notation as $E_t^B(\cdot)$. The corresponding risk neutral valuation formula is

$$V(t)= E_t^B\left(V(T)\frac{B(t)}{B(T)}\right). \tag{3.24}$$

Terminal Measure

The terminal measure Q^{T_N} is similar to the T-forward measure except that the numeraire P(t,T_N) will always be observable. This provides eases in pricing in some circumstances. The corresponding risk neutral valuation is,

$$V(t)= P(t,T_N)E_t^{T_N}(V(T)P(T,T_N)). \tag{3.25}$$

Swap Measure

Swap measure $Q^{k,m}$ is the measure induced by the annuity factor $A_{k,m}(t)$. Because of this choice of numeraire, the swap measure is also called the annuity measure. With no arbitrage, we have pricing formula

$$V(t)= A_{k,m}(t)E_t^{k,m}(V(T)/A_{k,m}(t)). \tag{3.26}$$

It is easy to show the swap rate is a martingale under $Q^{k,m}$. Recall the definition of swap rate in (3.18)

$$S_{k,m}(t) = (P(t,T_k) - P(t,T_{k+m}))/ A_{k,m}(t),\ t\leq T_k.$$

Notice both $P(t,T_k)/ A_{k,m}(t)$ and $P(t,T_{k+m})/ A_{k,m}(t)$ are martingales under $Q^{k,m}$. So their difference must also be martingale and so does the swap rate. The swap measure is generally the measure of choice in pricing swaptions. However, T-forward measure is often used in pricing vanilla inflation-indexed swaps.

Chapter 4

Inflation Models

We start our chapter by first introducing the HJM model and the Log-Normal LIBOR model in interest rate modeling. These two models form the foundations for several other inflation models, i.e. the JY model and the two market models.

4.1 The HJM Framework

The HJM approach (see Heath, Jarrow & Morton 1992) assumes all economic information is originated in a finite number of Brownian motions. The HJM model also generates a very board class of models. So here we introduce the bond price dynamics and forward rate dynamics and then show how to derive the log-normal HJM models, the class of models where the Lognormal Forward LIBOR Model (LFM) and the second market model belong.

4.1.1 Bond Price and Forward Rate Dynamics

Assume we have finite time frame [0,T], T<∞. Let W(t) be an adapted n-dimensional Q-Brownian motion, where the risk neutral measure Q is assumed to exist and be unique. Using the continuously rolling money market bank account B(t) as numeraire, we know the deflated bond price P_B(t,T) must be a martingale in Q. Formally, we write

$$\mathrm{d}P_B(t,T) = -P_B(t,T)\sigma_P(t,T)^T dW(t), t \leq T, \tag{4.1}$$

where the bond price volatility $\sigma_P(t,T)$ is a n-dimensional adapted stochastic process to the filtration generated by W(t).

We also assume $\sigma_P(t,T)$ is regular such that the deflated bond process satisfies the square-integrable condition, such that its variance is finite. Plus, $\sigma_P(T,T)$ =0 so that the condition P(T,T)=1 holds at maturity. Using Ito's lemma, we can rewrite (4.1) as a geometric Brownian motion with drift

$$\frac{\mathrm{dP(t,T)}}{\mathrm{P(t,T)}} = \mathrm{r(t)dt} - \sigma_P(t,T)^T dW(t) \quad (4.2)$$

The drift term r(t) is the short rate process. (4.2) is the general form of HJM models for bond price dynamics. To obtain the forward rate dynamics, we simply use Ito's lemma (the usual log-transformation) again on (4.2), we obtain

$$\mathrm{dlnP(t,T)} = \mathrm{O(dt)} - \sigma_P(t,T)^T dW(t), \quad (4.3)$$

where O(dt) means the term with order dt.

Recall from (3.15), we have a formula of forward rate

$$\mathrm{f(t,T)} = -\frac{\partial lnP(t,T)}{\partial T}.$$

Using (3.15) and differentiating both sides of (4.3) with respect to T and then changing signs, we obtain a new formula for forward rate consistent with the HJM framework

$$\mathrm{df(t,T)} = u_f(t,T)dt + \sigma_f(t,T)^T dW(t), \quad (4.4)$$

where

$$\sigma_f(t,T) = \frac{\partial}{\partial T}(\sigma_P(t,T)) \quad . \quad (4.5)$$

Furthermore, in the risk neutral measure Q, we have

$$u_f(t,T) = \sigma_f(t,T)^T \sigma_P(t,T) \quad . \quad (4.6)$$

Using the change of numeraire technique, (4.4) can be rewritten as a SDE without drift in the T-forward measure as

$$\mathrm{df(t,T)} = \sigma_f(t,T)^T dW^T(t) \quad (4.7)$$

4.1.2 Lognormal HJM Model

Because we know interest rate shall be positive in general, we usually avoid using the Gaussian HJM models. To ensure positive interest rate, we use lognormal type of HJM models. The specification is

$$\sigma_f(\mathrm{t,T}) = \mathrm{f(t,T)\sigma(t,T)} \quad (4.8)$$

Substituting (4.8) into the forward rate dynamics (4.7) in the T-forward measure, we get

$$\mathrm{df(t,T)} = \mathrm{f(t,T)}\sigma(\mathrm{t,T})^{T} dW^{T}(t) \tag{4.9}$$

The forward rate f(t,T) in (4.9) is obviously lognormal. This specification of HJM model provides solid background to further derive the well-known LIBOR model and some general inflation market models. We now show the link between LIBOR model and the HJM framework.

4.2 Lognormal Forward LIBOR Model (LFM)

Recall the definition of the LIBOR forward rate $L_n(t)$ in (3.13). Since (3.13) contains the forward bond price P(t, T_n, T_{n+1}). We first apply Ito's lemma on the bond price dynamics in (4.2) to obtain a forward price dynamics. The forward price dynamics can be rewritten as

$$\frac{d\mathrm{P}(\mathrm{t},T_n,T_{n+1})}{\mathrm{P}(\mathrm{t},T_n,T_{n+1})} = O(dt) - (\sigma_P(t,T_{n+1})^T - \sigma_P(t,T_n)^T)dW(t). \tag{4.10}$$

Now using the definition of $L_n(t)$ in (3.13) and perform a change of variables on (4.10), we have

$$\mathrm{d}L_n(t) = O(dt) + \tau_n^{-1}(1 + \tau_n\, L_n(t))[\int_{T_n}^{T_{n+1}} \sigma_f(t,\mathrm{s})^T)ds]\; dW(t). \tag{4.11}$$

We denote the diffusion term in (4.11) as $\sigma_n(t)$. As the time increment approaches to zero, we see that $\sigma_n(t) \rightarrow \sigma_f(f,T_n)$, as expected. (4.11) also tells us that the specification of the forward rate volatility $\sigma_f(t,\mathrm{s})$ would give a unique LIBOR forward rate $\sigma_n(t)$, but not the vice versa.

4.3 Jarrow and Yildirim Model (2003)

The foreign-currency analogy is often used in inflation modeling. The framework assumes real rates are the interest rate in the real economy whereas nominal rates are the interest rate in the nominal economy. Then, the CPI can be interpreted as the exchange rate between the two economies. Because such inflation model is similar to the modeling of domestic and foreign interest rate dynamics, the framework is thus named as foreign-currency analogy (FCA).

Denote I(t) as the CPI at time t. So at time 0, our base CPI will be I(0). Normalize I(0) to one. Then, a unit of currency at time 0 will be worth 1/I(t) at time t. This represents the exchange rate between the nominal and real currencies. Because FCA involves two interest rates, we have two forward LIBOR rates. Let the subscripts n and r represent nominal and real economies, respectively. Rewriting (3.14), we have

$$L_x(t, T, T+\tau)=\frac{1}{\tau}\frac{P_x(t,T+\tau)-P_x(t,T)}{P_x(t,T)}, \quad x\in\{n,r\}. \tag{4.12}$$

The instantaneous forward LIBOR rates in the two economies will be

$$f_x(t, T)=-\frac{\partial \ln P_x(t,T)}{\partial T}, \quad x\in\{n,r\}. \tag{4.13}$$

The short rate function will be

$$n(t)= f_n(T, T),$$

$$r(t)= f_r(T, T).$$

The forward CPI from time t to T will be denoted as,

$$f_I(t, T)=I(t)\frac{P_r(t,T)}{P_n(t,T)}.$$

Given a probability space (Ω, $\boldsymbol{\mathcal{F}}$, P) with filtration $\mathcal{F}_t$. The JY model is a three factor model consisting of three SDEs

$$df_n(t,T)=\alpha_n(t,T)dt+\theta_n(t,T)dW_n^P(t),$$

$$df_r(t,T)=\alpha_r(t,T)dt+\theta_r(t,T)dW_r^P(t),$$

$$dI(t)=I(t)u(t)dt+\sigma_I I(t)dW_I^P(t),$$

where I(0)>0 and initial forward rate $f_r(0,T)=f_x^M(0,T)$, $x\in\{n,r\}$, and the three Brownian motions are assumed to have correlations $\rho_{n,r}, \rho_{r,I}, \rho_{n,I}$.

The functions α_n, α_r, and u are adapted process to the filtration $\mathcal{F}_t$.The diffusion terms θ_n and θ_r are deterministic functions and the inflation volatility σ_I is taken to be a positive constant. The setting of the JY model allows us to deduce analytical formulas for swap, cap and floor easily. The drawback of the model is that it is difficult to calibrate with the historical data. To overcome the parameter estimation issues, several inflation market models have been proposed.

4.4 Mercurio Market Models

Mercurio (2005) gave two market models for inflation-indexed swaps

and options and we will call these as the first and second inflation market models. In his first market model, he assumed a LFR model on both nominal and real rates and a lognormal dynamics for the forward inflation rate. After using the change of numeraire toolkit, the SDE system under the T_i-forward measure $Q_n^{T_i}$ is

$$\mathrm{d}F_n(t;T_{i-1},T_i) = \sigma_{n,i}F_n(t;T_{i-1},T_i)dW_i^n(t),$$

$$\mathrm{d}F_r(t;T_{i-1},T_i) = F_r(t;T_{i-1},T_i)[-\rho_{I,r,i}\sigma_{I,i}\sigma_{r,i}dt + \sigma_{r,i}dW_i^r(t),$$

where W_i^n and W_i^r are two Brownian motions with instantaneous correlation ρ_i, $\sigma_{n,i}$ and $\sigma_{r,i}$ are positive constants, and $\rho_{I,r,i}$ is the correlation between the forward inflation rate $\zeta_i(\cdot)$ and $F_r(\cdot;T_{i-1},T_i)$.

The drawback of this first market model is that the volatility of real rates $\sigma_{r,i}$ could be very hard to estimate using historical data. To overcome this disadvantage, in the second market model he used the fact that $\zeta_i(\cdot)$ is a martingale under $Q_n^{T_i}$. Then he applied the change of numeraire toolkit and derived d $\zeta_{i-1}(\mathrm{t})$ so that the expectation term $E_n^{T_i}[\frac{\zeta_i(T_{i-1})}{\zeta_{i-1}(T_{i-1})} - 1|\mathcal{F}_t]$ can be easily evaluated when pricing vanilla inflation-indexed derivatives. Mathematically,

$$\mathrm{d}\,\zeta_{i-1}(\mathrm{t}) = \zeta_{i-1}(\mathrm{t})\sigma_{I,i-1}[-\frac{\tau_i\sigma_{n,i}F_n(t;T_{i-1},T_i)}{1+\tau_iF_n(t;T_{i-1},T_i)}\rho_{I,n,i}dt + dW_{i-1}^I(t)],$$

$$E_n^{T_i}[\frac{\zeta_i(T_{i-1})}{\zeta_{i-1}(T_{i-1})} - 1|\mathcal{F}_t] = \frac{\zeta_i(\mathrm{t})}{\zeta_{i-1}(\mathrm{t})} e^{D_i(t)},$$

where the correlation and volatility terms are defined similarly to the case in the first market model and particularly

$$D_i(t) = \sigma_{I,i-1}[\frac{\tau_i \sigma_{n,i} F_n(t;T_{i-1},T_i)}{1+\tau_i F_n(t;T_{i-1},T_i)} \rho_{I,n,i} - \rho_{I,i}\sigma_{I,i} + \sigma_{I,i-1}](T_{i-1} - t).$$

The main weakness of the second market model is that the correlations between the nominal rates and inflation rates are different from zero in longer maturity bonds. Lastly, he compared his two models with the JY model and found that the three models produced similar results except in the case of out of the money option.

4.5 Belgrade-Benhamou-Koehler Market Model (2004)

Belgrade, Benhamou and Koehler (2004) developed a no arbitrage model analog to the LIBOR market model by adjusting information from the zero-coupon and year-on-year swaps market. The model uses fewer parameters than the three inflation models discussed above. Furthermore, the simple model is robust enough to replicate some inflation derivatives in the market. Instead of modeling the forward inflation rate directly, they modelled the return of the forward inflation rate and assume the diffusion to follow a deterministic volatility structure. They considered three distinct volatility structures: constant, exponentially decaying, and exponentially adjusting.

They also gave a procedure of making convexity adjustments when martingale measures in the numerator and the dominator are different in the swap pricing formula. The limitations of the model are that there are not enough market observations for the zero-coupon and year-on-year swaps and the fact that the model is more computationally intensive than the three previous models.

4.6 Information-Based Pricing Framework

This model was developed by Macrina (2006). Using the incomplete market setting, he challenged the fundamental assumption of filtration being fully adapted to the pricing process. Under the realistic assumption of partial information, he used a certain number of economic variables to first generate the associated filtration field $\mathcal{F}_t$. Then he took the conditional expectation with an appropriate risk neutral measure to generate price.

In pricing inflation-related derivatives, Macrina (2006) first started from the real benefit from the money supply at a particular time t_i:

$$l_i = \frac{\lambda_i M_i}{C_i},$$

where λ_i is the nominal liquidity, C_i is the CPI at t_i,and M_i is the money supply per capita. He used a separable power-utility function

$$\mathrm{U(x,y)} = \frac{A}{p}x^p + \frac{B}{q}y^q,$$

and then ran a Lagrange optimization scheme, he obtained a pricing formula for a contingent claim with payoff H_j:

$$\mathrm{H}_0 = \frac{\lambda_0 M_0}{k_0^{q(1-p)/(1-q)}} e^{-\gamma t_j} \mathrm{E}\left[\frac{\mathrm{H}_j k_j^{\frac{q(1-p)}{1-q}}}{\lambda_j M_j}\right],$$

where k_i represents the aggregate consumption level of the economy at t_i.

The formula tells us that the price of an inflation-indexed derivative depends on a number of economic variables: money supply, liquidity, utility function parameters, short rate and aggregate consumption. The drawback of this approach is that it is difficult to assume reasonable stochastic processes for the money supply, liquidity and aggregate consumption.

Chapter 5

Inflation-Indexed Swap Products

After discussing the pros and cons of the major inflation models, we now focus on using only the JY model in pricing inflation-indexed derivatives. This will give us model consistency and will aid us in evaluating the price of an inflation-indexed American cap when we run the Q-Factor simulation procedure in chapter 6.

In this chapter, we introduce and derive pricing formulae for the vanilla swap products in the inflation market. These products lay a strong foundation for us to understand the inflation-indexed cap pricing in the coming chapter. The notations and formulae in this chapter are similar to Brigo and Mercurio (2006, p. 643-663).

Given a tenor structure $0 \le T_0 < T_1 < \ldots < T_M$. In an inflation-indexed swap (IIS), a party A pay the other party B the inflation rate while party B pays a fixed rate to party A on each payment date in the tenor structure. An IIS is similar to a vanilla interest rate swap except that IIS uses an inflation index rate as the reference rate instead of the LIBOR rate. In the actual inflation market, there are two actively traded IIS: zero-coupon and year-on-year. To avoid confusion in notations, let party A be the party paying a float rate in exchange for a fixed rate. Denote N as the nominal value ,K as the pre-defined fixed rate, and I(t) the inflation index (i.e. CPI) function for some time t≥0.

5.1 Zero-Coupon Inflation-Indexed Swap (ZCIIS)

In a ZCIIS, payment occurs at the maturity of the swap, namely, T_M. At T_M, party A will pay party B the floating payment

$$\mathrm{N}\left[\frac{I(T_M)}{I_0} - 1\right] \tag{5.1}$$

The quantity inside the bracket is the percentage change of the inflation index from the beginning to the end of the swap contract. We call this floating leg the inflation-indexed leg. At T_M, the counterparty B will pay a known amount

$$N[(1+K)^M - 1] \tag{5.2}$$

Using the no-arbitrage theory developed in Chapter 3, we could determine the value of the inflation-indexed leg and thereby determine the fixed rate K by balancing both fixed and floating leg in the swap contract. For $0 \le t \le T_M$, denote the function ZCIIS(t, T_M, T_0,N) as the value of the inflation-indexed leg at time t, for a contract with nominal value N, beginning at T_0 and ending at the contract maturity T_M. Discounting the floating payment at time T_M in (5.1) back to time t, we have

$$\text{ZCIIS}(t,\ T_M, T_0, N) = N E_n\left\{e^{-\int_t^{T_M} n(u)du}\left[\frac{I(T_M)}{I_0} - 1\right] | \mathcal{F}_t\right\}, \tag{5.3}$$

where $\mathcal{F}_t$ is the associated filtration up to time t and E_n is the expectation operator with respect to the nominal interest rate n.

Using the notation of a real bond price $P_r(t,T)$, we know by no-arbitrage assumption

$$P_r(t,T) = E_r\left\{e^{-\int_t^T r(u)du} | \mathcal{F}_t\right\}. \tag{5.4}$$

Furthermore, we get the relations

$$I(t)P_r(t,T) = P_n(t,T)I(T) \tag{5.5}$$

Extending (5.5) using definition of bond price under no-arbitrage, we obtain

$$I(t)E_r\left\{e^{-\int_t^T r(u)du} | \mathcal{F}_t\right\} = E_n\left\{e^{-\int_t^T n(u)du} I(T) | \mathcal{F}_t\right\}. \tag{5.6}$$

Substituting (5.5) and (5.6) into (5.3), we have a simpler form of ZCIIS equation,

$$\text{ZCIIS}(t,\ T_M, T_0, N) = N\left[\frac{I(t)}{I_0} P_r(t,T_M) - P_n(t,T_M)\right]. \tag{5.7}$$

To simplify (5.7), we set t=0 and obtain

$$\text{ZCIIS}(t,\ T_M, T_0, N) = N[P_r(t,T_M) - P_n(t,T_M)] \tag{5.8}$$

Notice that (5.8) implies that the price of a ZCIIS ultimately depends on three quantities, nominal value N, the real zero-coupon bond price and the corresponding nominal bond price. (5.8) also implies ZCIIS is model independent, a rare and desirable feature in derivatives pricing. The assumption of the underlying interest rate model, whether realistic or unrealistic, plays no role in ZCIIS. Interestingly, we can use (5.8) to deduce a no-arbitrage price for a real zero-coupon bond with the same maturity as the ZCIIS. By observing various ZCIIS prices with different maturity dates in the market, a term structure for inflation-protected bond, namely the real bond, can be easily constructed.

5.2 Year-on-Year Inflation-Indexed Swap (YYIIS)

In contrast to a ZCIIS, where payment occurs only at maturity, a YYIIS has payment exchanged on each date on the tenor structure $0 \leq T_0 < T_1 < \ldots < T_M$. Party A pays the floating amount

$$N\varphi_i \left[\frac{I(T_i)}{I(T_{i-1})} - 1\right], \qquad , \tag{5.9}$$

where φ_i ,for i=0,..., M, is the floating-leg year fraction for the interval $[T_{i-1}, T_i]$ in the swap contract. The counterparty pays an amount,

$$N\alpha_i K, \tag{5.10}$$

where α_i, for i=0,..., M, is the fixed-leg year fraction for the interval $[T_{i-1}, T_i]$ in the swap contract. Due to market conventions, α_i may not be equal to φ_i. Hence two distinct variables are used to denote the time length between two consecutive payments.

Given a tenor structure, $0 \leq T_0 < T_1 < \ldots < T_M$. For i=0,…,M, denote YYIIS(t, T_{i-1}, T_i, φ_i, N) as the price of a year-on-year inflation-indexed swap at time t. The swap has nominal value N. It starts at T_{i-1}, ends at T_i, and has floating leg year fraction of φ_i. Using no-arbitrage argument and the payoff of a YYIIS in (5.9), we obtain

$$\text{YYIIS}(t,\ T_{i-1}, T_i, \varphi_i, N) = N\varphi_i E_n\left\{e^{-\int_t^{T_i} n(u)du}\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right] | \mathcal{F}_t\right\}, \quad (5.11)$$

with t< T_{i-1}.

To simplify (5.11) even more, we use the law of iterated expectation. We have,

$$N\varphi_i E_n\left\{e^{-\int_t^{T_{i-1}} n(u)du} E_n\left[e^{-\int_{T_{i-1}}^{T_i} n(u)du}\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right] | \mathcal{F}_{T_{i-1}}\right] | \mathcal{F}_t\right\} \quad (5.12)$$

The inner expectation of (5.12) is very similar to (5.3). In fact we can write it as

$$E_n\left[e^{-\int_{T_{i-1}}^{T_i} n(u)du}\left[\frac{I(T_i)}{I(T_{i-1})} - 1\right] | \mathcal{F}_{T_{i-1}}\right] = \text{ZCIIS}(T_{i-1},\ T_i, T_{i-1}, 1).$$

Combining the relation above with (5.8) and (5.12), we subsequently turn (5.12) into

$$N\varphi_i E_n\left\{e^{-\int_t^{T_{i-1}} n(u)du}[P_r(T_{i-1}, T_i) - P_n(T_{i-1}, T_i)] | \mathcal{F}_t\right\}$$

$$= N\varphi_i E_n\left\{e^{-\int_t^{T_{i-1}} n(u)du}[P_r(T_{i-1}, T_i)] | \mathcal{F}_t\right\} - N\varphi_i P_n(t, T_i). \quad (5.13)$$

Notice that if the real rates were deterministic, (5.13) can be reduced to

$$N\varphi_i P_r(T_{i-1}, T_i) P_n(t, T_{i-1}) - N\varphi_i P_n(t, T_i) \quad (5.14)$$

The simple formula (5.14) in practice does not hold because real rates are stochastic and has its own dynamics and volatilities. So, contrast to ZCIIS, YYIIS is model dependent in general. We denote Q_n^T as the nominal T-forward measure and use it to rewrite (5.14) as

$$N\varphi_i P_n(t, T_{i-1}) E_n^{T_{i-1}}\{[P_r(T_{i-1}, T_i)] | \mathcal{F}_t\} - N\varphi_i P_n(t, T_i) \quad (5.15)$$

To get the forward real bond price $P_r(T_{i-1}, T_i)$ in the expectation in (5.15), we first use the Hull and White (1994b) model to derive $P_r(t, T)$. We have

$P_r(t, T) = A_r(t, T) e^{-B_r(t,T) r(t)}$,

$B_r(t, T) = \frac{1}{a_r}[1 - e^{-a_r(T-t)}]$,

$$A_r(t, T) = \frac{P_r^M(0, T)}{P_r^M(0, t)} \exp\left\{B_r(t, T) f_r^M(0, t) - \frac{\sigma_r^2}{4a_r}(1 - e^{-2a_r t}) B_r(t, T)^2\right\}, \quad (5.16)$$

Recall that the JY model described in the previous chapter contains three SDEs.

Since P(t,T)= exp(-$\int_t^T n(u)du$), we rewrite n=f(P(t,T)) for some function f(·). Then apply Ito's lemma to obtain the SDE dn(t). Now the underlying stochastic system is under the risk neutral measure Q. We then utilize the multi-dimensional version of Girsanov theorem and the change of numeraire technique to obtain the real rate dynamics under $Q_r^{T_{i-1}}$. Mathematically,

$$dW_r^{T_{i-1}}(t) = dW_r(t)+p_{n,r}\sigma_n B_n(t,T_{i-1})dt, \tag{5.17}$$

$$\mathrm{dr(t)} = \left[-p_{n,r}\sigma_n B_n(t,T_{i-1}) + \vartheta_r(t) - \sigma_r\sigma_I\rho_{r,I} - a_r r(t)\right]dt + \sigma_r dW_r^{T_{i-1}}(t), \tag{5.18}$$

With the setup above and some long algebra, we obtain a simplified equation for the expectation in (5.15) under the JY model

$$E_n^{T_{i-1}}\{[P_r(T_{i-1},T_i)]|\mathcal{F}_t\}=\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{C(t,T_{i-1},T_i)}, \tag{5.19}$$

where

$$C(t,T_{i-1},T_i)=\sigma_r B_r(T_{i-1},T_i)[\,B_r(\mathrm{t},T_{i-1})(\rho_{r,I}\sigma_I - \frac{1}{2}\sigma_r B_r(\mathrm{t},T_{i-1}) + \frac{\rho_{r,n}\sigma_n}{a_n+a_r}(1 + a_r B_n(\mathrm{t},T_{i-1})))\text{-}\frac{\rho_{r,n}\sigma_n}{a_n+a_r}B_n(\mathrm{t},T_{i-1})].$$

Substituting (5.19) into (5.15), we have the value of a YYIIS on $[T_{i-1},T_i]$ at time t is given by the formula:

YYIIS(t, $T_{i-1},T_i,\ \varphi_i,N$)

$$= \mathrm{N}\varphi_i P_n\big(\mathrm{t},T_{i-1}\ \big)\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}e^{C(t,T_{i-1},T_i)}\text{- }\mathrm{N}\varphi_i P_n(\mathrm{t},T_i) \tag{5.20}$$

If we allow $\complement\coloneqq \{T_1,\dots,T_M\}$ *and* $\varphi \coloneqq \{\varphi_i,\dots,\varphi_M\}$ be the time index set and year fraction index set respectively, then the formula for the whole floating leg at time 0 will be

YYIIS(0, $\complement$, φ,N)=N$\varphi_1[P_r(0,T_1) - P_n(0,T_1)]$

+N$\sum_{i=2}^{M}\varphi_i[\,P_n\big(0,T_{i-1}\ \big)\frac{P_r(0,T_i)}{P_r(0,T_{i-1})}e^{C(0,T_{i-1},T_i)}$- $P_n(0,T_i)]$

$$= \mathrm{N}\sum_{i=1}^{M} \varphi_i P_n(0,T_i)\left[\frac{1+\tau_i F_n(0,T_{i-1},T_i)}{1+\tau_i F_r(0,T_{i-1},T_i)} e^{C(0,T_{i-1},T_i)} - 1\right] \qquad (5.21)$$

where $C(0,T_{i-1},T_i)$ is given in (5.19).

Several things can be observed from (5.21). Firstly, the first payment starts from 0 and ends at T_i. So we can price it as a ZCIIS(0, T_1,0, N) given by (5.8). At the last equation in (5.21), we observe that we need the nominal and real forward rate $F_n(\cdot)$ and $F_r(\cdot)$. If we assume both of the functions are normal, then YYIIS becomes analytically tractable. But this advantage is offset by the disadvantages that the real and nominal rates are possibly negative and the difficulties in obtaining the real rate parameters and the real forward curve solely from historical data. To overcome such major drawbacks in the JY model, the first and second market were proposed. Both the nominal and real forward rates are assumed to follow lognormal distributions. This feature guarantees rates strictly positive rates.

The YYIIS formula in the first market model can be expressed as an integral in terms of the forward rates and some other parameters such as volatility of real rate and correlation. The pricing relies on the numerical integration technique. But it's hard to estimate the real rate volatility in the integrand during calibration. Fortunately, this problem is resolved in the later proposed second market model. The derivations of the YYIIS pricing formula for the two inflation market models above can be found in Brigo and Mercurio (2006, p.654-659).

5.3 Vanilla Caplet

An inflation-indexed caplet (IIC) is a call option on the inflation rate, usually, the CPI Index in the domestic country. Likewise, an inflation-indexed put is the corresponding put option. Mathematically, we write the IICF payoff at time T_i as

$$\mathrm{N}\varphi_i\left[w\left(\frac{I(T_i)}{I(T_{i-1})} - 1 - k\right)\right]^+, \qquad (5.22)$$

where φ_i is the year fraction for the interval $[T_{i-1}, T_i]$, N is the contract principal value, I(·) is the inflation function, k is the strike, and w= 1 implies a caplet contract, and w=-1 implies a floorlet contract. Set K=1+k to simplify (5.22). At time t<T_i, the pricing of an IICF in (5.22) becomes

IICplt(t, T_{i-1}, T_i, φ_i, K, N, w)

$= \mathrm{N}\varphi_i \mathrm{E}_n \left\{ e^{-\int_t^{T_i} n(u)du} [w\left(\frac{I(T_i)}{I(T_{i-1})} - 1 - k\right)]^+ | \mathcal{F}_t \right\}$

$= \mathrm{N}\varphi_i P_n(\mathrm{t}, T_i) \mathrm{E}_n^{T_i} \left\{ [w\left(\frac{I(T_i)}{I(T_{i-1})} - K\right)]^+ | \mathcal{F}_t \right\}$ (5.23)

The last equation in (5.23) involve a change of measure from physical measure to the T_i-forward measure.

The model assumes the nominal and real rates follow the Gaussian processes. Then the inflation ratio in (5.23) is lognormal under Q_n. After a change of measure, the ratio $\frac{I(T_i)}{I(T_{i-1})}$, conditional on filtration $\mathcal{F}_t$, still follows the lognormal distribution. So if we know the mean of the ratio and the variance of its logarithm, then we can write the expectation in (5.23) as

$$\mathrm{E}_n^{T_i} \left\{ [w\left(\frac{I(T_i)}{I(T_{i-1})} - K\right)]^+ | \mathcal{F}_t \right\} = wm\mathrm{N}\left(w\frac{ln\frac{m}{K} + \frac{v^2}{2}}{v}\right) - wK\mathrm{N}\left(w\frac{ln\frac{m}{K} - \frac{v^2}{2}}{v}\right) \quad (5.24)$$

where m=E($\frac{I(T_i)}{I(T_{i-1})} | \mathcal{F}_t$) , v^2=var[ln$\frac{I(T_i)}{I(T_{i-1})} | \mathcal{F}_t$], and N(·) is the normal cumulative density function.

The derivation of (5.24) is analogous to the derivation of an European call option in the Black Scholes model. Now the task is to find the quantities m and v. Then we can express the whole pricing formula for the caplet.

m

$= \mathrm{E}_n^{T_i} \left\{ \frac{I(T_i)}{I(T_{i-1})} | \mathcal{F}_t \right\}$

$=\mathrm{E}_n^{T_i}\left\{(\frac{P_r(t,T_i)}{P_n(t,T_i)})/(\frac{P_r(t,T_{i-1})}{P_n(t,T_{i-1})})|\mathcal{F}_t\right\}$

$=\mathrm{E}_n^{T_i}\left\{(\frac{P_n(t,T_{i-1})}{P_n(t,T_i)})(P_r(T_{i-1},T_i)|\mathcal{F}_t\right\}$

$=(\frac{P_n(t,T_{i-1})}{P_n(t,T_i)})\mathrm{E}_n^{T_i}\{(P_r(T_{i-1},T_i)|\mathcal{F}_t\}$

$=(\frac{P_n(t,T_{i-1})}{P_n(t,T_i)})\ (\frac{P_r(t,T_i)}{P_r(t,T_{i-1})})e^{C(t,T_{i-1},T_i)}$ (5.25)

where the function $C(t,T_{i-1},T_i)$ is given in (5.18).

v^2=var[ln$\frac{I(T_i)}{I(T_{i-1})}$ $|\mathcal{F}_t$]

=$V^2(t,T_{i-1},T_i)$

$=\frac{\sigma_n^2}{2a_n^3}(1-e^{-a_n(T_i-T_{i-1})})^2\left[1-e^{-2a_n(T_{i-1}-t)}\right]+\sigma_I^2(T_i-T_{i-1})+\frac{\sigma_r^2}{2a_r^3}(1-$

$e^{-a_r(T_i-T_{i-1})})^2\left[1-e^{-2a_r(T_{i-1}-t)}\right]-$

$2\rho_{n,r}\frac{\sigma_n\sigma_r}{a_na_r(a_n+a_r)}\left(1-e^{-a_n(T_i-T_{i-1})}\right)\left(1-e^{-a_r(T_i-T_{i-1})}\right)\left[1-e^{-(a_r+a_n)(T_{i-1}-t)}\right]+$

$\frac{\sigma_n^2}{a_n^2}\left[T_i-T_{i-1}+\frac{2}{a_n}e^{-a_n(T_i-T_{i-1})}-\frac{1}{2a_n}e^{-2a_n(T_i-T_{i-1})}-\frac{3}{2a_n}\right]+$

$\frac{\sigma_r^2}{a_r^2}\left[T_i-T_{i-1}+\frac{2}{a_r}e^{-a_r(T_i-T_{i-1})}-\frac{1}{2a_r}e^{-2a_r(T_i-T_{i-1})}-\frac{3}{2a_r}\right]-2\rho_{n,r}\frac{\sigma_n\sigma_r}{a_na_r}\Big(T_i-T_{i-1}-$

$\frac{1}{a_n}\left(1-e^{-a_n(T_i-T_{i-1})}\right)-\frac{1}{a_r}\left(1-e^{-a_r(T_i-T_{i-1})}\right)+\frac{1}{a_n+a_r}(1-e^{-(a_r+a_n)(T_i-T_{i-1})}\Big)$

$+2\rho_{n,I}\frac{\sigma_n\sigma_I}{a_n}[T_i-T_{i-1}-\frac{1}{a_n}\left(1-e^{-a_n(T_i-T_{i-1})}\right)]-2\rho_{r,I}\frac{\sigma_r\sigma_I}{a_r}[T_i-T_{i-1}-$

$\frac{1}{a_r}\left(1-e^{-a_r(T_i-T_{i-1})}\right)]$. (5.26)

Eliminating the expectation term in (5.23), we obtain the pricing formula for the inflation-indexed caplet

IICplt(t, T_{i-1}, T_i, φ_i, K, N, w)

=w N$\varphi_i P_n$(t, T_i)[mN(w$\frac{ln\frac{m}{K}+\frac{v^2}{2}}{v}$)- KN(w$\frac{ln\frac{m}{K}-\frac{v^2}{2}}{v}$)]

=wN$\varphi_i P_n(\text{t}, T_i)$[

$\frac{P_n(t,T_{i-1})}{P_n(t,T_i)}\ \frac{P_r(t,T_i)}{P_r(t,T_{i-1})}\ e^{C(t,T_{i-1},T_i)}$N(w$\frac{ln(\frac{P_n(t,T_{i-1})}{KP_n(t,T_i)})\ (\frac{P_r(t,T_i)}{P_r(t,T_{i-1})})+C(t,T_{i-1},T_i)+\frac{V^2(t,T_{i-1},T_i)}{2}}{V(t,T_{i-1},T_i)}$)-

KN(w$\frac{ln\left(\frac{P_n(t,T_{i-1})}{KP_n(t,T_i)}\right)\left(\frac{P_r(t,T_i)}{P_r(t,T_{i-1})}\right)+C(t,T_{i-1},T_i)-\frac{V^2(t,T_{i-1},T_i)}{2}}{V(t,T_{i-1},T_i)}$)] (5.27)

where $C(t,T_{i-1},T_i)$ is given in (5.19).

5.4 Inflation-Indexed Caps and Floors

An inflation-indexed cap (floor) is just a stream of inflation-indexed caplets (floorlets). Given the starting date T_0=0 and the tenor structure T_0, T_1, …, T_M. At T_i, i=1,…, M, an IICapFloor pays

$$\text{N}\varphi_i[w\left(\frac{I(T_i)}{I(T_{i-1})}-1-k\right)]^+,$$

where k is the strike stated in the IICapFloor contract.

Following the derivation of the caplet above, the pricing formula for an IICapFloor at time 0 is,

IICapFloor(0, ζ, ψ, K, N, w)

=N$\sum_{i=1}^{M} P_n(0,T_i)\psi_i \text{E}_n^{T_i}\left\{[w\left(\frac{I(T_i)}{I(T_{i-1})}-K\right)]^+|\mathcal{F}_t\right\}$

=N$\sum_{i=1}^{M} P_n(0,T_i)\psi_i w$[

$\frac{P_n(0,T_{i-1})}{P_n(0,T_i)}\ \frac{P_r(0,T_i)}{P_r(0,T_{i-1})}\ e^{C(0,T_{i-1},T_i)}$N(w$\frac{ln(\frac{P_n(0,T_{i-1})}{KP_n(0,T_i)})\ (\frac{P_r(0,T_i)}{P_r(0,T_{i-1})})+C(0,T_{i-1},T_i)+\frac{V^2(0,T_{i-1},T_i)}{2}}{V(0,T_{i-1},T_i)}$)-

KN(w$\frac{ln\left(\frac{P_n(0,T_{i-1})}{KP_n(0,T_i)}\right)\left(\frac{P_r(0,T_i)}{P_r(0,T_{i-1})}\right)+C(0,T_{i-1},T_i)-\frac{V^2(0,T_{i-1},T_i)}{2}}{V(0,T_{i-1},T_i)}$)] (5.28)

where the payment date set ζ:={T_1, …, T_M} and the year fraction setψ:={ψ_1, …, ψ_M}. The last equation in (5.28) follows from (5.25)-(5.27) and note that the values of m and v depend on the index i.

Chapter 6

Inflation-Indexed American Cap

6.1 American Caplet

Using the same setup in the previous chapters, introduce stopping time

$T_{\tau_i}\epsilon\Gamma_i$={$T_i, T_i + h_i, T_i + 2h_i, \ldots,\ T_i + n_i h_i$} and $h_i = \frac{T_{i+1}-T_i}{n_i}$, i=0,…, M-1. In other words, on the interval $[T_i, T_{i+1}]$, the step size is equal to a constant h_i and n_i is the number of time steps. When the step size is small and the time step is large on $[T_i, T_{i+1}]$, the price of an American option is very close to the price of its corresponding Bermudan option. Hence, we will price the American option as if it is a Bermudan option. This fact is widely used in the mathematical finance literature.

The associated caplet payoff at T_{τ_i} is,

$$Z_{\tau_i}=\{N\psi_{\tau_i}[w\left(\frac{I(T_{\tau_i})}{I(T_i)}-K\right)]^+\}, \tag{6.1}$$

where ψ_{τ_i} is the day year fraction on the interval $[T_i,\ T_{\tau_i}]$. Let the value function of the inflation-indexed American caplet at time to be IIACplt(·). Denote sup as the supremum (aka the least upper bound) operator. Given a risk neutral measure with the bank money market account B_t, 0≤t≤τ_i, as numeraire, we write

$$\text{IIACplt}(t,\ T_i,\ T_{\tau_i},\ \varphi_{\tau_i}, K, N, w)= \sup_{\tau_i>t} E_n^Q[\frac{B_t}{B_{\tau_i}} Z_{\tau_i}|\mathcal{F}_t]\, B_t, \tag{6.2}$$

6.2 Analytical Approach

Using the risk neutral formula and forward risk measure $Q^{T_{\tau_i}}$ and zero coupon bond price with maturity T_{τ_i} as numeraire, we write

$$\sup_{\tau_i>t} E_n^Q[\frac{B_t}{B_{\tau_i}} Z_{\tau_i}|\mathcal{F}_t]=\sup_{\tau_i>t}P(t,\tau_i)\, E_n^{Q^{T_{\tau_i}}}[Z_{\tau_i}|\mathcal{F}_t]$$

$$=\sup_{\tau_i>t}P(t,\tau_i)\{N\varphi_{\tau_i}E_n^{Q^{T_{\tau_i}}}[[w\left(\frac{I(T_{\tau_i})}{I(T_i)}-K\right)]^+|\mathcal{F}_t]\}$$

$$= N\sup_{\tau_i > t} P(t,\tau_i)\ \varphi_{\tau_i}\{E_n^{Q^{T\tau_i}}[w\left(\frac{I(T_{\tau_i})}{I(T_i)} - K\right)]^+ |\mathcal{F}_t\} \tag{6.3}$$

Combine (5.25), (5.26) and (6.2) into (6.3) and adjusting the time indices accordingly, we have the full formula for the American caplet

IIACplt(t, T_i, T_{τ_i}, φ_i, K, N, w)

$$= N\sup_{\tau_i > t} P(t,\tau_i)\ \varphi_{\tau_i}\{E_n^{Q^{T\tau_i}}[w\left(\frac{I(T_{\tau_i})}{I(T_i)} - K\right)]^+ |\mathcal{F}_t\}$$

$$= wN\sup_{\tau_i > t} P(t,\tau_i)\ \varphi_{\tau_i}\ [\frac{P_n(t,T_i)}{P_n(t,T_{\tau_i})}\ \frac{P_r(t,T_{\tau_i})}{P_r(t,T_i)}\ e^{C(t,T_i,T_{\tau_i})} \mathrm{N}(\ w\frac{ln(\frac{P_n(t,T_i)}{KP_n(t,T_{\tau_i})})\ (\frac{P_r(t,T_{\tau_i})}{P_r(t,T_i)}) + C(t,T_i,T_{\tau_i}) + \frac{V^2(t,T_i,T_{\tau_i})}{2}}{V(t,T_i,T_{\tau_i})})$$

$$\text{-KN}(w\frac{ln\left(\frac{P_n(t,T_i)}{KP_n(t,T_{\tau_i})}\right)\left(\frac{P_r(t,T_{\tau_i})}{P_r(t,T_i)}\right) + C(t,T_i,T_{\tau_i}) - \frac{V^2(t,T_i,T_{\tau_i})}{2}}{V(t,T_i,T_{\tau_i})})]. \tag{6.4}$$

Since cap is just a stream of caplets. So now it is simple to express the pricing formula for an American cap as

$$\text{IICapFloor(t, T, }\Phi,\ \varphi\text{ , K, N, w)}=\sum_{i=0}^{M-1} IIACplt(\text{t, } T_i,\ T_{\tau_i},\ \varphi_i\text{, K, N, w}) \tag{6.5}$$

where the function IIACplt(·) is given by (6.4) , the set φ :={ φ_0,…, φ_M}, the set T:={T_0, T_1,…, T_M} represents the tenor structure of the cap, and the set Φ:={T_{τ_0}, T_{τ_1}, …, T_{τ_M}} represents the set of stopping time.

We now have the analytical formulae for American caplet and American cap given in (6.4) and (6.5), respectively. But in practice, it is very difficult to calculate the caplet price in (6.4). This is because the analytical formula involves the least upper bound operator sup. This means we must run a dynamic programming procedure to find the optimality in the complex objective function, which consists of real and nominal bond prices, real rate volatilities and worse, the normal cumulative distribution function. From our discussion in previous chapter, we know the real rate volatilities across different maturities are very hard to estimate. Given these two main disadvantages, we focus on other computational intensive technique to deduce us the price of an IIACap.

6.3 Approximate Dynamic Programming

One main technique to find the price of an American option in the mathematical finance literature is the approximate dynamic programming method (ADP). Haugh and Kogan (2008) represented how to price American option using the ADP method and duality theory. And we now apply the ADP method on the IIACap and leave the application of duality theory in the next section. The corresponding floors can be found as well, simply by setting w=-1. Instead of starting from (6.4), we keep the term with expectation operator and backtrack to the simple risk neutral formula for an American caplet. Then, we will start value iteration when it is numerically convenient to do so. At time 0, the pricing of an inflation indexed caplet on $[T_i,\ T_{i+1}]$, i=0,…M-1,

$$V_{0,i} = \sup_{\tau_i > t} E_{n,0}^{Q^B}[\frac{Z_{\tau_i}}{B_{\tau_i}}]. \tag{6.6}$$

where the expectation operator is taken with respect to the nominal rate, conditional on filtration at time 0, and under the money market account risk neutral measure Q^B.

We can find $V_{0,i}$ by using value iteration. We rewrite the option payoff in (6.1). For i=1,…,M-1,

$$Z_{t,i}(X_t\ ) = Z_{t,i} = \begin{cases} N\varphi_{\rm i} w[\left(\frac{I(T_{\rm i})}{I(T_{i-1})} - K\right)]^+ \ if\ t = T_i \\ N\varphi_{\tau_i} w[\left(\frac{I\left(T_{\tau_i}\right)}{I(T_i)} - K\right)]^+ \ if\ t = \tau_i \end{cases}, \tag{6.7}$$

where X_t is the state vector of the payoff at time t,

$$V_{t,i} = \max(Z_{t,i}, E_{n,t}^{Q^{\rm B}}\left[\frac{B_t}{B_{t+1}} V_{t+1,i}\right]) \tag{6.8}$$

$$Q_t = E_{n,t}^{Q^{\rm B}}\left[\frac{B_t}{B_{t+1}} V_{t+1,i}\right] \tag{6.9}$$

The value Q_t is called the Q-value function and is said to be the value of the caplet conditional on it not being exercised today. The next iteration of $V_{t,i}$ is then given by

$$V_{t+1,i} = \max(Z_{t+1,i}, Q_{t+1}). \quad (6.10)$$

Substituting (6.10) into (6.9), we obtain the Q-value iteration formula

$$Q_t = E_{n,t}^{Q^B}\left[\frac{B_t}{B_{t+1}}\max(Z_{t+1,i}, Q_{t+1})\right]. \quad (6.11)$$

Notice that the Q-value function takes in an expectation respect to the nominal rates conditional on time t. If we were to use the real rates, then we must have used the real money market account R_t instead of B_t. The risk neutral measure will be $Q_{n,t}^R$.

The caplet payoff in (6.1) can be further expressed in terms of real and nominal zero-coupon bond prices. The detail was discussed in the previous chapter. So there are three state variables in determining the Q value in (6.11). Because the state vector is high dimensional (greater than or equal to three), the exact version of Q-value iteration in (6.11) is not feasible in practice. Fortunately, there are great improvements in applying the ADP algorithms in the mathematical finance literature (see Longstaff & Schwartz 2001 and Tsitsiklis & Van Roy 2001) Now we illustrate how to apply the approximate Q-value iteration in pricing American caplet.

Let $\widetilde{Q_t}$ be the approximate value of Q_t. Set, $\widetilde{Q_t}=\sum_{i=1}^{m} r_t^i \phi_i$, where $\phi_1,\ldots,\phi_m$ is a set of basis functions and r_t :=($r_t^1,\ldots,$ r_t^m) is a vector of parameters at time t. We now choose some candidates for the basis functions. The rule of thumb is to pick instruments that have features closely resembling to the American caplet. These can be market price of the other inflation caplets or even swaptions. So far we have derived some formulae for caplets, floorlets and swaps under the JY model. To enforce model consistency, it is reasonable to pick all instruments with their prices strictly derived from the JY model. We summarized this in the table 2.

Instrument	Basis Function	Pricing Formula

IICaplet	ϕ_1	(5.27)
IIFloorlet	ϕ_2	(5.27)
ZCIIS	ϕ_3	(5.8)
YYIIS	ϕ_4	(5.20)

Table 2

Next we determine the vector r_t using the so called Least Square Monte Carlo Method (LSMC).

Steps:

1) Generate N paths of state vector X conditional on the initial state X_0
2) Set $\widetilde{Q_T}(X_T^i)$=0 for all i=1 to N
3) For t=T-1 to 1,

   Regress $B_t\widetilde{V_{t+1}}(X_{t+1}^i)/B_{t+1}$ on $(\phi_1(X_t^i), \dots, \phi_4(X_t^i))$,

   where $\widetilde{V_{t+1}}(X_{t+1}^i) := \max(Z_{t+1}(X_{t+1}^i), \widetilde{Q_{t+1}}(X_{t+1}^i))$.

   Set $\widetilde{Q_t}(X_t^i)$=$\sum_{k=1}^{4} r_t^k\, \phi_k(X_t^i)$

   where r_t^ks are the estimated coefficient after the regression.

End for loop

4) Generate another M samples of the state vector, X_1 , conditional on the initial state, X_0 .
5) Set $\widetilde{V_0}(X_0\ )$= ($\sum_{j=1}^{M} \max(Z(X_1^j),\ \widetilde{Q_1}(X_1^j))$)/M$B_1$.

In step 3, we are required to choose a regression method. Typically, we choose the standard least square regression because it runs very fast in the loop when compiling the entire program. This helps us to avoid the problem of having to wait too long for results every time we debug the codes. The number of path N usually ranges from 10,000 to 50,000. Others regression methods can be used and their levels of efficiency remain to be investigated. $\widetilde{V_0}$ is the approximate price of the American caplet $V_{0,i}$ on $[T_i,\ T_{i+1}]$ given in (6.4). So we need to run the Q-value iteration M times to find each American caplet price within the tenor structure of the American cap. Finally, the inflation-indexed American cap is then given by,

$$\text{IICapFloor}(0, T, \Phi, \varphi, K, N, w) = \sum_{i=0}^{M-1} IIACplt(0,\ T_i,\ T_{\tau_i},\ \varphi_i, K, N, w)$$

$$= \sum_{i=0}^{M-1} V_{0,i}$$

$$\approx \sum_{k=0}^{M-1} \widetilde{V_{0,k}} \quad (6.12)$$

Comparing (6.12) with (6.4) and (6.5), the pricing formula of the cap is greatly simplified, at least computationally. The dynamic programming technique shown above belongs to the rollout algorithm class in control theory. For future researchers who want to extend the Q-iteration algorithm with multiple steps ahead and examine various solution stability issues, see Bertsekas (2005). Besides the ADP method, one can also find the cap price using the duality principle used in American option pricing.

6.4 The Duality Approach

We start from (6.6) and then obtain an upper bound $U_{0,i}$,i=0,…,M-1. Then we examine the relationship between $U_{0,i}$ and $V_{0,i}$. Let s_t be an arbitrary supermartingale.

$$V_{0,i} = \sup_{\tau_i > t} E_{n,0}^{Q^B}[\frac{Z_{\tau_i}}{B_{\tau_i}}] = \sup_{T_{\tau_i} \in \Gamma_i} E_{n,0}^{Q^B}[\frac{Z_{\tau_i}}{B_{\tau_i}}] = \sup_{T_{\tau_i} \in \Gamma_i} E_{n,0}^{Q^B}[\frac{Z_{\tau_i}}{B_{\tau_i}} - s_{T_{\tau_i}} + s_{T_{\tau_i}}]$$

$$\leq \sup_{T_{\tau_i} \in \Gamma_i} E_{n,0}^{Q^B}[\frac{Z_{\tau_i}}{B_{\tau_i}} - s_{T_{\tau_i}}] + s_0$$

$$\leq E_{n,0}^{Q^B}[\max_{t \in \Gamma_i}(\frac{Z_{t,i}}{B_t} - s_t)] + s_0 \quad (6.13)$$

The optional sampling theorem for supermartingale is applied in the first inequality. If we take the infimum over all s_t on the right hand side of (6.13), we get

$$V_{0,i} \leq \; U_{0,i} \coloneqq \inf_{s} E_{n,0}^{Q^B} \left[\max_{t \in \Gamma_i} \left(\frac{Z_{t,i}}{B_t} - s_t \right) \right] + s_0. \tag{6.14}$$

It is well known that the deflated process $\frac{V_t}{B_t}$ is a supermartingale (see Duffie 1996).

Using this fact on (6.14), the new upper bound becomes

$$U_{0,i} \coloneqq \inf_{s} E_{n,0}^{Q^B} \left[\max_{t \in \Gamma_i} \left(\frac{Z_{t,i}}{B_t} - s_t \right) + s_0 = E_{n,0}^{Q^B} \left[\max_{t \in \Gamma_i} \left(\frac{Z_{t,i}}{B_t} - \frac{V_{t,i}}{B_t} \right) \right] + V_{0,i} \right.$$

(6.15)

Furthermore, we know that $V_{t,i} \geq Z_{t,i}$ holds true for all t and I because $V_{t,i}$ is the price of an American caplet while $Z_{t,i}$ is just its exercise value. Then the first term in the last expression in (6.15) is non-positive. Define a monotonically decreasing process X_t and use the inequality$V_{t,i} \geq Z_{t,i}$, we obtain

$$U_{0,i} = V_{0,i} + E_{n,0}^{Q^B} [\max_{t \in \Gamma_i} X_t] \Leftrightarrow U_{0,i} \leq V_{0,i} \tag{6.16}$$

Combining (6.14) and (6.16) we can conclude $V_{0,i}$=$U_{0,i}$ where $U_{0,i}$ is defined in (6.15). Therefore, the price of an inflation-indexed American cap would be

IICapFloor(0, T, Φ, φ , K, N, w)=$\sum_{i=0}^{M-1} IIACplt$(0, T_i, T_{τ_i}, φ_i, K, N, w)

=$\sum_{i=0}^{M-1} V_{0,i}$

=$\sum_{k=0}^{M-1} U_{0,i}$ (6.17)

There are also variants of the duality approach such as multiplicative dual approach and optimal exercise frontier approximation (see Andersen & Broadie 2004, Chen & Glasserman 2007, Jamshidan 2003, and Meinshausen & Hambly 2004).

Chapter 7

Conclusion

To price inflation-indexed derivatives, one would need to first assume dynamics on nominal interest rate, real interest rate, inflation rate and forward inflation rates. Furthermore, knowledge of risk neutral measures would help deducing simplified pricing formulae for inflation derivatives. So far our main inflation model has been the JY model. In practice, it is quite difficult to accurately estimate all the parameters in the SDE system under the JY model using historical data. Therefore, one usually prices the inflation derivatives using the first and second market models and then compares the results with the prices given by the JY model.

When pricing an inflation-indexed American cap, we must first obtain the price of American caplets with different maturities. The caplet prices could be deduced in three ways: analytical method, approximate dynamic programming, and duality approach. In analytical method, we need to compute an expression with real and nominal bond prices, volatilities of real and nominal rates, and variance of inflation ratio. To ease the estimation issue in the analytical method, we could run a LSMC routine to approximate caplet prices. This approach requires reasonable basis functions when approximate the Q-Factor. And we used the prices of ZCIIS,YYIIS, IICaplet, and IIFloorlet in forming such basis functions. The Duality approach requires the deflated process $\frac{V_{\mathrm{t}}}{B_{\mathrm{t}}}$ to be a supermartingale. Otherwise, there will be distinct bounds for the American caplet price.

Chapter 8

Critical Comments and Personal Reflections

The information-based model requires techniques in filtration generation and relevant factor identification. The model provides interesting future research directions in inflation modeling. Particularly, it was shown that given reasonable assumptions of the stochastic processes of aggregate consumption, money supply and money liquidity, prices of many inflation derivatives can be derived using appropriate filtrations. Future researches on inflation modeling should involve intense collaborations of researchers in mathematical finance, information theory, macroeconomics economics and political economics.

The markets prices of YYIIS, ZCIIS, caplet and floors are very accessible. This should help to form reliable basis functions when pricing corresponding American caplets. The procedure to price inflation-indexed Bermudan (American) swaptions is very similar to the one in pricing the American caplet. The approximate dynamic programming method discussed in the previous chapter can be used if we price swaptions under the JY model and the relevant swap risk neutral measures. Further extensions of the ADP algorithms and the least square Monte Carlo method can be applied to American caps and swaptions too. The efficiency and price convergence of these algorithms in the inflation derivatives are yet to be investigated. Other than these efficient methods, one could also attempt to use the latest Bermudan swaption pricing methodologies (see Carr & Yang 2001 and Andersen & Andreasen 2001) in the interest rate modeling literature to obtain another estimates for the inflation-indexed American caps and swaptions.

So far we have been avoiding the smile effect commonly observed in options.

When pricing an American caplet, it is expected that the analytical approach with a stochastic volatility model will give a more complicated expression than (6.4). So we should rather employ the Q-value iteration. In the approximate dynamic programming method, we can use Black's formulae for caplet and floorlet with stochastic volatility extensions to form two of the basis functions. Brigo and Mercurio (2006, p.673-687) have demonstrated how to derive the price of an inflation-indexed European caplet when the volatility of forward inflation rate is stochastic and follows the Heston (1993) model. The formula of such caplet is rather involved. But by doing so, a more accurate price for American cap can be deduced. However, volatility smile effects in inflation derivatives can be better captured if the SABR stochastic volatility model is used instead of the Heston model.

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