# A SEMI-ANALYTICAL APPROACH TO CANARY SWAPTIONS IN HJM ONE-FACTOR MODEL

MARC HENRARD

Abstract. Leveraging the explicit formula for European swaptions and coupon-bond options in HJM one-factor model we develop a semi-explicit formula for 2-Bermudan options (also called Canary options). For this we first extend the European swaption formula to future times. Then we are able to reduce the valuation of a 2-Bermudan swaption to a single numerical integration. In that integration the most complex part of the valuation of the embedded European swaptions has been simplified in such a way that it has to be performed only once and not for every point of the numerical integration.

## 1. Introduction

This article is devoted to Bermudan swaptions, more precisely to 2-Bermudan swaptions (swaptions with 2 exercise dates). Those swaptions are also called Canary swaptions because Canary Islands are halfway between Bermuda and Europe.

We leverage the explicit formula for European swaptions and coupon-bonds in the HJM one-factor model. This is done by first extending the European option formula to future times. Such an extension is required as at the first expiry date we need to compare a swap with the remaining European option.

Using the explicit formula we are able to reduce the valuation of a 2-Bermudan option to a single expectation. This is an improvement to respect to a direct or usual tree approach as, even if there is two expiry dates, the numerical process is done only at one date.

By adding a condition on the volatility structure, condition which is satisfied by the Hull-White volatility, we are able to reduce the computation even further. The most time consuming part of the computation, the solution of a non-linear one-dimensional equation, can be done once and then reused for all the other points of the integration.

Finally, for cancelable swaps or options on underlying similar after the second expiry date, we propose a semi-explicit formula. The formula is explicit for the valuation of the part corresponding to the exercise at the first expiry date and still written as an expectation for the rest. The size of the interval on which the expectation has to be computed is reduced by the probability of the exercise at the first expiry date.

The HJM one-factor model and hypothesis are described in the next section. Then we present some preliminary results before presenting the main results in Section 4 and the simplified formulas in Section 5.

## 2. Model and hypothesis

The model and main hypothesis used in this paper are the same that in [2].

We use a model for $P(t,u)$, the price at $t$ of the zero-coupon bond paying 1 in $u$. We will describe this for all $0 \leq t, u \leq T$, where $T$ is some fixed constant.

When the discount curve $P(t,.)$ is absolutely continuous, which is something that is always the case in practice as the curve is constructed by some kind of interpolation, there exists $f(t,u)$ such

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that

$$P(t,u) = \exp\left(-\int_t^u f(t,s)ds\right). \tag{1}$$

The idea of Heath-Jarrow-Morton [1] was to exploit this property by modelling $f$ as

$$df(t,u) = \mu(t,u)dt + \sigma(t,u)dW_t$$

for some suitable (possibly stochastic) $\mu$ and $\sigma$.

Here we use a similar model, but we restrict ourself to non-stochastic coefficients. The exact hypothesis on the volatility term $\sigma$ is described by (H2). We don't need all the technical refinement used in their paper or similar one, like the one described in [5] in the chapter on *dynamical term structure model*. So instead of describing the conditions that lead to such a model, we suppose that the *conclusions* of such a model are true. By this we mean we have a model, that we call a *HJM one-factor model*, with the following properties.

Let $A = \{(s,u) \in \mathbb{R}^2 : u \in [0,T] \text{ and } s \in [0,u]\}$. We work in a filtered probability space $(\Omega, F, \mathbb{P}^{\mathsf{real}}, (\mathcal{F}_t))$. The filtration $\mathcal{F}_t$ is the (augmented) filtration of a one-dimensional standard Brownian motion $(W^{\mathsf{real}})_{0\leq t\leq T}$.

**H1:** There exists $\sigma : [0,T]^2 \to \mathbb{R}^+$ measurable and bounded[1] with $\sigma = 0$ on $[0,T]^2 \setminus A$ such that for some process $(r_s)_{0\leq t\leq T}$, $N_t = \exp(\int_0^t r(s)ds)$ forms, with some measure $\mathbb{N}$, a numeraire pair[2] (with Brownian motion $W_t$),

$$\begin{aligned} df(t,u) &= \sigma(t,u)\int_t^u \sigma(t,s)ds\, dt - \sigma(t,u)dW_t \\ dP^N(t,u) &= P^N(t,u)\int_t^u \sigma(t,s)ds\, dW_t \end{aligned}$$

and $r(t) = f(t,t)$.

The notation $P^N(t,s)$ designates the numeraire rebased value of $P$, i.e. $P^N(t,s) = N_t^{-1}P(t,s)$. To simplify the writing in the rest of the paper, we will use the notation

$$\nu(t,u) = \int_t^u \sigma(t,s)ds.$$

Note that $\nu$ is increasing in $u$, measurable and bounded.

To be able to use the explicit formula for the valuation of the European swaptions, we will also use the following hypothesis.

**H2:** The function $\nu$ satisfies $\nu(s,t_2) - \nu(s,t_1) = f(t_1,t_2)g(s)$ for some positive function $g$.

Moreover in order to simplify further the computation for the 2-Bermudan swaptions, we will use the following hypothesis.

**H3:** The function $\nu$ satisfies $\nu(s,t_2) - \nu(s,t_1) = (f(t_2) - f(t_1))g(s)$ with $f$ decreasing and $g$ positive and increasing.

*Example:* The Ho and Lee volatility model [3] and the Hull and White volatility model [4] satisfy the condition (H2) and (H3). For Ho and Lee one has $\nu(s,t) = \sigma(t-s)$ and $\sigma(s,t) = \sigma$; for Hull and White one has $\nu(s,t) = (1-\exp(-a(t-s)))\sigma/a$ and $\sigma(s,t) = \sigma\exp(-a(t-s))$. The volatility time-dependent versions of the models also satisfy the conditions.

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[1] Bounded is too strong for the proof we use, some $L^1$ and $L^2$ conditions are enough, but as all the examples we present are bounded, we use this condition for simplicity.

[2] See [5] for the definition of a numeraire pair. Note that here we require that the bonds of *all* maturities are martingales for the numeraire pair $(N, \mathbb{N})$.

## 3. Preliminary results

We want to price some option in this model. For this we recall the generic pricing theorem [5, Theorem 7.33-7.34].

**Theorem 1.** *Let $V_T$ be some $\mathcal{F}_T$-measurable random variable. If $V_T$ is attainable, then the time-$t$ value of the derivative is given by $V_t^N = V_0^N + \int_0^t \phi_s dP_s^N$ where $\phi_t$ is the strategy and*

$$V_t = N_t \, \mathrm{E}^{\mathbb{N}}\left[ V_T N_T^{-1} \middle| \mathcal{F}_t \right].$$

We now state two technical lemmas that generalize the lemmas presented in [2].

**Lemma 1.** *Let $0 \leq t \leq u \leq v$. In a HJM one factor model, the price of the zero coupon bond can be written has,*

$$P(u,v) = \frac{P(t,v)}{P(t,u)} \exp\left( -\frac{1}{2} \int_t^u \left(\nu^2(s,v) - \nu^2(s,u)\right) ds + \int_t^u \left(\nu(s,v) - \nu(s,u)\right) dW_s \right).$$

*Proof.* By definition of the forward rate and its equation,

$$\begin{aligned} P(u,v) &= \exp\left( -\int_u^v f(u,\tau) d\tau \right) \\ &= \exp\left( -\int_u^v \left[ f(t,\tau) + \int_t^u \nu(s,\tau) D_2\nu(s,\tau) ds - \int_t^u D_2\nu(s,\tau) dW_s \right] d\tau \right). \end{aligned}$$

Then using again the definition of forward rates and the Fubini theorem on inversion of iterated integrals, we have

$$\begin{aligned} P(u,v) &= \frac{P(t,v)}{P(t,u)} \exp\left( -\int_t^u \int_u^v \nu(s,\tau) D_2\nu(s,\tau) d\tau ds + \int_t^u \int_u^v D_2\nu(s,\tau) d\tau dW_s \right) \\ &= \frac{P(t,v)}{P(t,u)} \exp\left( -\frac{1}{2} \int_t^u \left(\nu^2(s,v) - \nu^2(s,u)\right) ds + \int_t^u \nu(s,v) - \nu(s,u) dW_s \right). \end{aligned}$$

□

**Lemma 2.** *In the HJM one factor model, we have*

$$N_u N_v^{-1} = \exp\left( -\int_u^v r_s ds \right) = P(u,v) \exp\left( \int_u^v \nu(s,v) dW_s - \frac{1}{2} \int_u^v \nu^2(s,v) ds \right).$$

*Proof.* By definition of $r$,

$$\begin{aligned} r_\tau &= f(\tau,\tau) = f(t,\tau) + \int_t^\tau df(s,\tau) ds \\ &= f(t,\tau) + \int_t^\tau \nu(s,\tau) D_2\nu(s,\tau) ds + \int_t^\tau D_2\nu(s,\tau) dW_s. \end{aligned}$$

Then using Fubini, we have

$$\begin{aligned} \int_u^v r(\tau) d\tau &= \int_u^v f(t,\tau) d\tau + \int_u^v \int_s^v \nu(s,\tau) D_2\nu(s,\tau) d\tau ds - \int_u^v \int_s^v D_2\nu(s,\tau) d\tau dW_s \\ &= \int_u^v f(t,\tau) d\tau + \frac{1}{2} \int_u^v \nu^2(s,v) ds + \int_u^v \nu(s,v) dW_s. \end{aligned}$$

□

We give the pricing formula for swaptions for a future time.

**Theorem 2.** *Suppose we work in the HJM one-factor model with a volatility term of the form (H2). Let $\theta \leq t_0 < \cdots < t_n$, $c_0 < 0$ and $c_i \geq 0$ $(1 \leq i \leq n)$. The price of an European receiver swaption, with expiry $\theta$ on a swap with cash-flows $c_i$ and cash-flow dates $t_i$ is given at time $t$ by the $\mathcal{F}_t$-measurable random variable*

$$\sum_{i=0}^n c_i P(t,t_i) N(\kappa + \alpha_i)$$

*where $\kappa$ is the $\mathcal{F}_t$-measurable random variable defined as the (unique) solution of*

$$\sum_{i=0}^{n} c_i P(t,t_i) \exp\left(-\frac{1}{2}{\alpha_i}^2 - \alpha_i \kappa\right) = 0 \tag{2}$$

*and*

$${\alpha_i}^2 = \int_t^\theta (\nu(s,t_i) - \nu(s,\theta))^2 \, ds.$$

*The price of the payer swaption is*

$$-\sum_{i=0}^{n} c_i P(t,t_i) N(-\kappa - \alpha_i)$$

*Proof.* Let $\mu(s,\theta) = \nu(s,\theta)$ if $s \geq t$ and 0 if $s < t$. We define $W^{\#}_\tau = W_\tau - \int_0^\tau \mu(s,\theta) ds$. By Girsanov's theorem ([6, Section 4.2.2, p. 72]), the process $W^{\#}$ is a standard Brownian motion with respect to the probability $\mathbb{P}^{\#}$ of density

$$L_\theta = \exp\left(\int_0^\theta \mu(s,\theta) dW_s - \frac{1}{2}\int_0^\theta \mu^2(s,\theta) ds\right).$$

Using Lemma 1 and rewriting $\nu^2(s,t_i) - \nu^2(s,\theta)$ as $(\nu(s,t_i)-\nu(s,\theta))^2 + 2\nu(s,\theta)(\nu(s,t_i)-\nu(s,\theta))$, we have

$$P(\theta,t_i) = \frac{P(t,t_i)}{P(t,\theta)} \exp\left(-\frac{1}{2}\alpha_i^2 - \alpha_i X\right)$$

where $-\alpha_i X = \int_t^\theta \nu(s,t_i) - \nu(s,\theta) dW^{\#}_s$ and $X$ is a standard normally distributed with respect to $\mathbb{P}^{\#}$. The hypothesis (H2) is used here to prove that the random variable $X$ is the same for all $i$.

Using Lemma 2, we have $N_t N_\theta^{-1} = P(t,\theta) L_\theta$. By the generic pricing theorem 1, the price of the option is

$$V_t = \mathrm{E}^{\#}\left[\max\left(\sum_{i=0}^{n} c_i P(t,t_i) \exp\left(-\frac{1}{2}\alpha_i^2 - \alpha_i X\right), 0\right) \middle| \mathcal{F}_t\right].$$

Note that $P(t,t_i)$ is $\mathcal{F}_t$-measurable and $X$ is independent of $\mathcal{F}_t$. Using a property of the conditional expectation ([6, Proposition A.2.5, p. 166]), we can do this computation in two parts.

Let's fixed $P(t,t_i) = P_i$. Similarly to the proof in the case of the price in 0, we have that $\sum c_i P_i \exp(-\frac{1}{2}\alpha_i^2 - \alpha_i y) > 0$ if and only if $y < \kappa$ where $\kappa$ is the unique solution of $\sum c_i P_i \exp(-\frac{1}{2}\alpha_i^2 - \alpha_i y) = 0$.

So we have $V_t = \phi(P)$ where $\phi(p) = \sum c_i p_i N(\kappa + \alpha_i)$. Or more explicitly

$$V_t = \sum c_i P(t,t_i) N(\kappa + \alpha_i)$$

where $P(t,t_i)$ and $\kappa$ are $\mathcal{F}_t$-measurable and $\kappa$ is implicitly defined by the equation (2) □

## 4. 2-Bermudan swaption

We are now in position to state and prove the main result of this article.

**Theorem 3.** *Let $\theta_1 < \theta_2$, $t_{i,j}$ ($i = 1,2$, $j = 0,\ldots,n_i$) be such that $\theta_i \leq t_{i,0} < t_{i,1} < \cdots < t_{i,n_i}$ and $c_{i,j}$ ($i = 1,2$, $j = 0,\ldots,n_i$) be such that $c_{i,0} < 0$ and $c_{i,j} \geq 0$ ($j > 0$). In the a HJM one-factor model, when the volatility term has the form (H2), the price of an 2-Bermudan receiver swaption with expiries $\theta_i$ and underlying swaps with cash-flow $c_{i,j}$ and cash-flow dates $t_{i,j}$ is given by*

$$\begin{aligned} V_0 \;=\; \mathrm{E}\Bigg(\max\Bigg(&\sum_{j=0}^{n_1} c_{1,j} P(0,t_{1,j}) \exp\left(-\frac{1}{2}\alpha_{1,j}^2(0,\theta_1) - \alpha_{1,j}(0,\theta_1) X\right), \\ &\sum_{j=0}^{n_2} c_{2,j} P(0,t_{2,j}) \exp\left(-\frac{1}{2}\alpha_{2,j}^2(0,\theta_1) - \alpha_{2,j}(0,\theta_1) X\right) N\left(\kappa(X) + \alpha_{2,j}(\theta_1,\theta_2)\right)\Bigg)\Bigg) \end{aligned} \tag{3}$$

*where $\kappa(X)$ is the unique solution of*

$$\sum_{j=0}^{n_2} c_{2,j} P(0,t_{2,j}) \exp\left(-\frac{1}{2}\alpha_{2,j}^2(0,\theta_2) - \alpha_{2,j}(0,\theta_1)X - \alpha_{2,j}(\theta_1,\theta_2)\kappa\right) = 0, \tag{4}$$

*$X$ is a standard normally distributed random variable with respect to* E *and*

$$\alpha_{i,j}^2(u,v) = \int_u^v \left(\nu(s,t_{i,j}) - \nu(s,v)\right)^2 ds.$$

*The price of the payer swaption is*

$$\begin{aligned} V_0 \quad = \quad & \mathrm{E}\left(\max\left(-\sum_{j=0}^{n_1} c_{1,j} P(0,t_{1,j}) \exp\left(-\frac{1}{2}\alpha_{1,j}^2(0,\theta_1) - \alpha_{1,j}(0,\theta_1)X\right)\,,\right.\right. \\ & \left.\left. -\sum_{j=0}^{n_2} c_{2,j} P(0,t_{2,j}) \exp\left(-\frac{1}{2}\alpha_{2,j}^2(0,\theta_1) - \alpha_{2,j}(0,\theta_1)X\right) N\left(-\kappa(X) - \alpha_{2,j}(\theta_1,\theta_2)\right)\right)\right) \end{aligned} \tag{5}$$

*Proof.* In $\theta_1$ the price of the swaption is given by the maximum of the price of the first swap and the price of the European swaption on the second swap.

We define $W_t^\# = W_t - \int_0^t \nu(s,\theta_1) ds$. By Girsanov theorem, the process $W^\#$ is a standard Brownian motion with respect to the probability $P^\#$ of density

$$L_{\theta_1} = \exp\left(\int_0^{\theta_1} \nu(s,\theta_1) dW_s - \frac{1}{2}\int_0^{\theta_1} \nu^2(s,\theta_1) ds\right).$$

Using Lemma 1, we have

$$P(\theta,t_{i,j}) = \frac{P(0,t_{i,j})}{P(0,\theta_1)} \exp\left(-\frac{1}{2}\alpha_{i,j}^2(0,\theta_1) - \alpha_{i,j}(0,\theta_1)X\right) \tag{6}$$

where $-\alpha_{i,j}(0,\theta_1)X = \int_0^{\theta_1} \nu(s,t_{i,j}) - \nu(s,\theta_1) dW_s^\#$ and $X$ is a random variable with standard normal distribution with respect to $\mathbb{P}^\#$. Like in Theorem 2, we use the hypothesis (H2) to prove that the random variable X is the same for all $i$ and $j$.

By Lemma 2, $N_{\theta_1}^{-1} = P(0,\theta_1)L_{\theta_1}$ and so using the generic pricing theorem 1 and the swaption pricing theorem 2, we then have the equation 3 where $\kappa$ is solution of

$$\sum_{j=0}^{n_2} P(\theta_1,t_{2,j}) \exp\left(-\frac{1}{2}\alpha_{2,j}^2(\theta_1,\theta_2) - \alpha_{2,j}(\theta_1,\theta_2)\kappa\right) = 0.$$

By using the equation 6 we obtain the described result for the value of $\kappa$. □

## 5. Simplified formulas

Subject to an extra condition, this result can be written in a form easier to compute.

**Theorem 4.** *Under the conditions of Theorem 3, if the volatility structure satisfies (H3) instead of (H2), then the value of the 2-Bermudan receiver swaption is given by the same formula (3) (payer given by 5) but with $\kappa(X) = (\Lambda - \sqrt{G(\theta_2)-G(\theta_1)}X)/\sqrt{G(\theta_1)-G(0)}$ where $\Lambda$ is the unique solution of*

$$\sum_{j=0}^{n_2} c_{2,j} P(0,t_{2,j}) \exp\left(-\frac{1}{2}\alpha_{2,j}^2(0,\theta_2) + f(t_{2,j})\Lambda\right) = 0. \tag{7}$$

*and $G$ is a primitive of $g^2$.*

*Proof.* Using condition (H3) we can write

$$\alpha_{i,j}^2(u,v) = (f(t_{i,j}) - f(v))^2(G(v) - G(u)).$$

If we inject that description in (4) and simplify some factors, we have as equation for $\kappa$

$$\sum_{j=0}^{n_2} c_{2,j} P(0,t_{2,j}) \exp\left(-\frac{1}{2}\alpha_{2,j}^2(0,\theta_2) - f(t_{2,j})(\sqrt{G(\theta_1)-G(0)}X + \sqrt{G(\theta_2)-G(\theta_1)}\kappa)\right) = 0.$$

If we denote by $\Lambda$ the term $\sqrt{G(\theta_1)-G(0)}X + \sqrt{G(\theta_2)-G(\theta_1)}\kappa$, which is independent of $j$, we have the result. □

In the case where the underlying swaps have the same cash-flows and cash-flow dates after the settlement of the second swap, the formula can be simplified further. The simplification consists in the analytical solution of the expected value of the first swap in case of exercise in $\theta_1$. This is applicable in particular for cancelable swaps and bonds with embedded options.

**Theorem 5.** *Let $\theta_1 \leq t_0 < t_1 < \cdots t_{k-1} < \theta_2 \leq t_k < \cdots < t_n$, $c_j > 0$ ($j = 1,\ldots,n$), $c_0 < 0$ and $d_k < 0$. We consider two receiver swaps which are represented by the cash-flows $(c_0, c_1, \ldots, c_n)$ on dates $(t_0,\ldots,t_n)$ and by the cash-flows $(d_k, d_{k+1},\ldots,d_n) = (d_k, c_{k+1},\ldots,c_n)$ on dates $(t_k,\ldots,t_n)$.*

*In the HJM one-factor model, when the volatility term has the form (H2), the price of a 2-Bermudan receiver swaption with expiries $\theta_i$ and underlying swap described above is given by*

$$\begin{aligned} V_0 &= \sum_{j=0}^{n} c_j P(0,t_j) N\left(\mu + \alpha_j(0,\theta_1)\right) \\ &\quad + \mathrm{E}\left( 1\!\!1(X \geq \mu) \max\left( \sum_{j=0}^{n} c_j P(0,t_j) \exp\left(-\frac{1}{2}\alpha_j^2(0,\theta_1) - \alpha_j(0,\theta_1)X\right), \right.\right. \\ &\quad \left.\left. \sum_{j=k}^{n} d_j P(0,t_j) \exp\left(-\frac{1}{2}\alpha_j^2(0,\theta_1) - \alpha_j(0,\theta_1)X\right) N\left(\kappa(X) + \alpha_j(\theta_1,\theta_2)\right) \right)\right) \end{aligned}$$

*where $\mu$ is the smallest solution of*

$$\begin{aligned} &\sum_{j=0}^{n} c_j P(0,t_j) \exp\left(-\frac{1}{2}\alpha_j^2(0,\theta_1) - \alpha_j(0,\theta_1)\mu\right) \\ &-\sum_{j=k}^{n} c_j P(0,t_j) \exp\left(-\frac{1}{2}\alpha_j^2(0,\theta_1) - \alpha_j(0,\theta_1)\mu\right) N\left(\kappa(\mu) + \alpha_j(\theta_1,\theta_2)\right) = 0 \end{aligned}$$

*where $\kappa$ is the function defined by 4.*

*The price of the 2-Bermudan payer swaption is*

$$\begin{aligned} V_0 &= -\sum_{j=0}^{n} c_j P(0,t_j) N\left(-\mu - \alpha_j(0,\theta_1)\right) \\ &\quad + \mathrm{E}\left( 1\!\!1(X \leq \mu) \max\left( -\sum_{j=0}^{n} c_j P(0,t_j) \exp\left(-\frac{1}{2}\alpha_j^2(0,\theta_1) - \alpha_j(0,\theta_1)X\right), \right.\right. \\ &\quad \left.\left. -\sum_{j=k}^{n} d_j P(0,t_j) \exp\left(-\frac{1}{2}\alpha_j^2(0,\theta_1) - \alpha_j(0,\theta_1)X\right) N\left(-\kappa(X) - \alpha_j(\theta_1,\theta_2)\right) \right)\right) \end{aligned}$$

*Proof.* By the implicit function theorem applied to equation (4), $\kappa$ is continuous (as a function of $X$). Let $Q_j = c_j P(0,t_j)\exp(-\frac{1}{2}\alpha_j^2(0,\theta_1))$. Note that $Q_0 < 0$ and $Q_j > 0$ ($j = 1,\ldots,n$). The difference between the value of the first swap and the swaption can be written as

$$\exp(-\alpha_0 X)\left(Q_0+\sum_{j=1}^{k-1}Q_j\exp((\alpha_o-\alpha_j)X)+\sum_{j=k}^{n}Q_j(1-\frac{d_j}{c_j}N(\kappa(X)+\alpha_j))\exp((\alpha_0-\alpha_j)X)\right).$$

As $\alpha_0-\alpha_j<0$, (see [2] for a proof of it) all the coefficient in the exponentials are negative. Moreover as $d_j=c_j$ for $j>k$, $d_k<0$ and $0<N<1$, all the factors of the exponential are positive. The only negative term is $Q_0$.

Using all those elements, we have that the term within the parenthesis tends to $+\infty$ as $X$ tends to $-\infty$ and converges to $Q_0<0$ when $X$ tends to $+\infty$.

By continuity there exists at least one point for which the difference is 0. Also as it converges to $+\infty$ in $-\infty$, the set of zeros is bounded from below. This proves that the set of solutions, which is closed, has a finite minimum. □

## 6. Numerical implementation

Usually, we compute the expected value through a numerical integration. The expected value is the one of a random variable written as the function of a standard normal random variable. It means that we know quite well the weight of the distribution underlying our expected value. We can use points for the numerical integration with equal *weight* with respect to the underlying normal distribution. By using equally weighted points we concentrate the computation where they have a greater importance and so increase the precision for a given number of points in the numerical integration.

In the Hull-White tree implementation, for a tree with $n$ steps there is at maximum $2n+1$ final points and a total of $(2n+2)n/2$ grid points. In order to obtain a similar precision one needs only $2n+1$ points in the numerical integration.

But even with the same number of points as the points of the numerical integration are chosen to have equal weight, the precision of the numerical integration is better. A different implementation of the numerical integration with points equally distant would give something closer to a tree implementation. A rough testing indicates that a equally distant implementation needs several times more points than an equally weighted implementation to obtain the same precision.

This approach will not work directly in practice for $n$-Bermudan swaption ($n\geq 3$) as $n-1$ integrations would be required for a total of points of the order of $p^{n-1}$ where a Hull-White tree has a number of final point in $pn$ (total of the order of $(pn)^2$).

Nevertheless some extra analytical manipulation and selection of the points for the integration can bring the number of computation for an integration-like formula to $pn^2$. This will be developed in a forthcoming article.

The advantage of Theorem 4 is that $\kappa$ is written explicitly (even affinely) as a function of $X$. The non-linear equation 4 has to be solved only once (to find $\Lambda$) and not for each value of $X$. That improves dramatically the numerical implementation.

The advantages of the semi-explicit presentation of Theorem 5 are also obvious. By solving a one-dimensional equation, one divides by two the interval for the numerical integration. In our implementation this approach reduces the computational time (for at-the-money swaptions) if one uses 50 points or more for the numerical integration. For large number of points, the time is reduced by almost the probability of exercising the swaption at the first exercise date. The analytical part takes almost no time (like 10 or 20 points in our implementation[3]).

**Disclaimer:** The views expressed here are those of the author and not necessarily those of the Bank for International Settlements.

[3]Matlab code available from the author.

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Derivatives group, Bank for International Settlements, CH-4002 Basel (Switzerland)
*E-mail address*: Marc.Henrard@bis.org