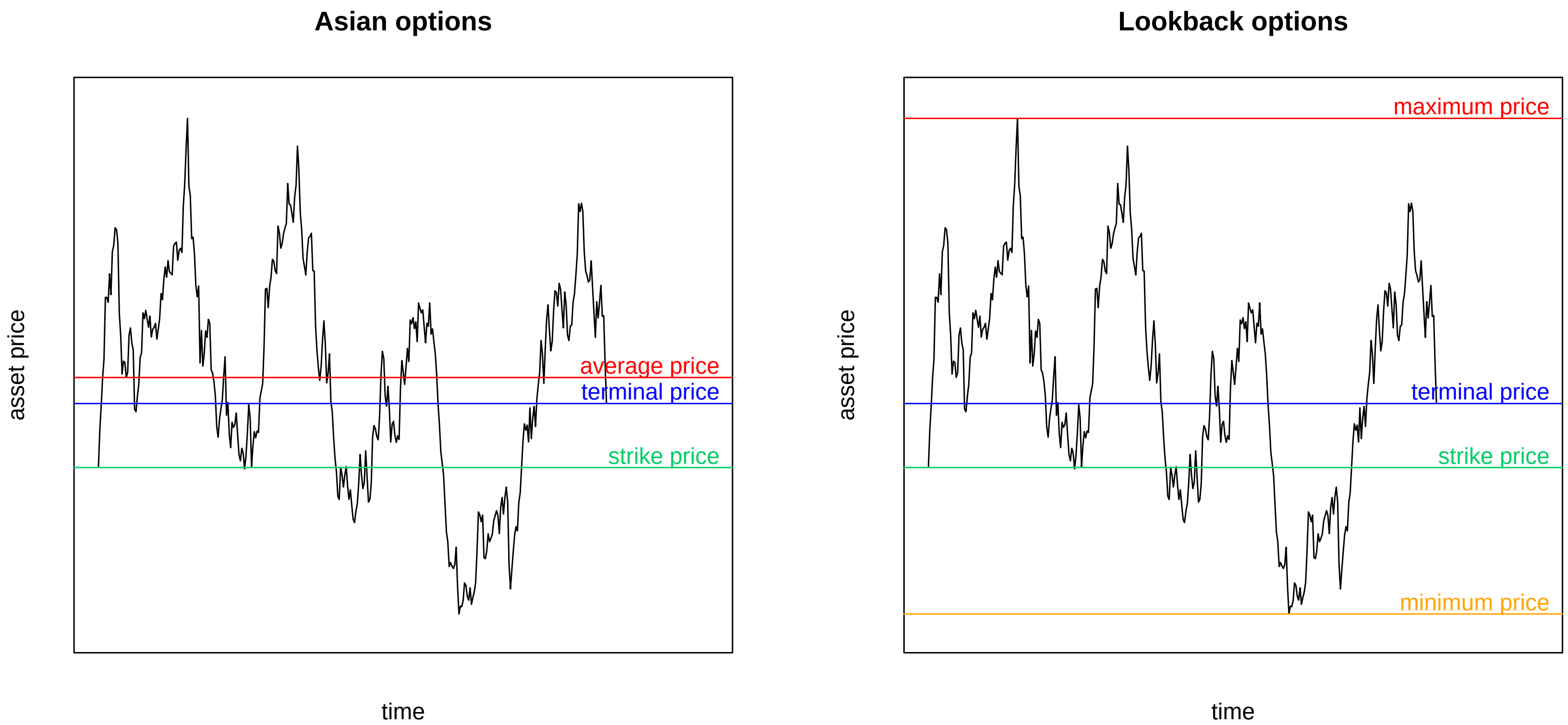
5. Path-Dependent Options

What Are They? Special-purpose derivatives whose payouts depend not only on the final price reached on expiration, but also on some aspect of the path the price follows prior to expiration



Common Types: Asian options, lookback options, barrier options

Common Types (continued)

1. Asian options are options whose final payoff is based on the average level of the underlying asset during some or all of the life of the option

   Two basic styles: the average rate option (average price vs. strike) and the average strike option (average price vs. spot)

2. Lookback options have the feature that the payout is some function of the highest or lowest price at which the asset trades during the lifetime of the option

   Two main variants: the fixed strike option (extremum price vs. strike) and the floating strike option (extremum price vs. spot)

3. Barrier options are essentially identical to vanilla calls or puts except they can be activated or extinguished when one or more barriers are breached

   Four basic forms: down-and-out, up-and-out, down-and-in, and up-and-in options

Examples of Applications:

1. Knock-in option: popular in equity and currency markets

   Appeals to option buyers who hope to buy in a market reversal for a minimum option premium

2. Knock-out option: "bargain-price" substitute for a standard option for buyers who are highly confident about a small chance of breaching the outstrike

   Provides low-cost insurance for investors who feel that the insurance feature is unlikely to be needed in the market environment after a movement through the outstrike

3. Lookback option: opportunity to pay an additional premium for the right to a more attractive strike

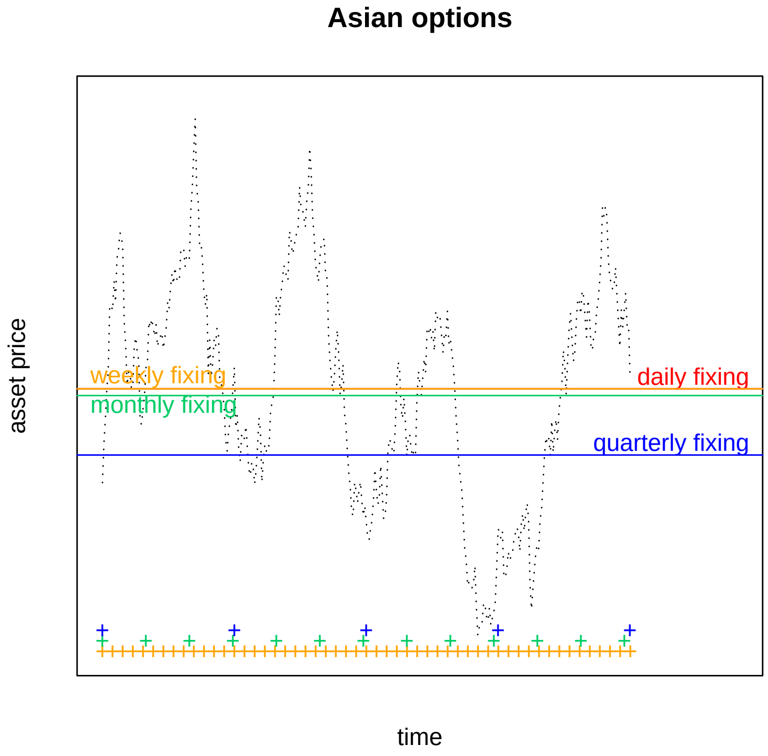
   Buyers anticipate that a near-term development might cause a temporary setback to their long-term expectations

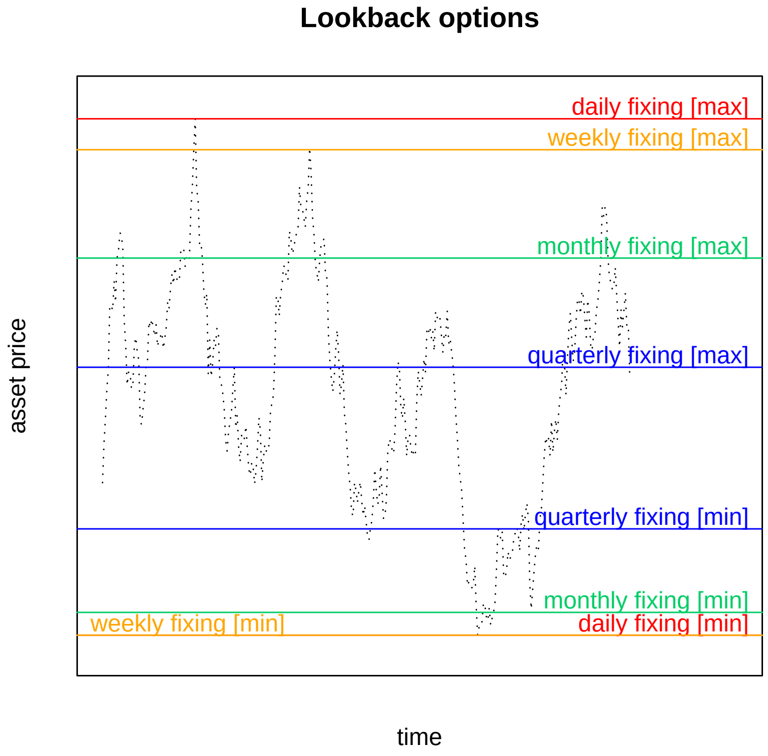
4. Asian option: popular in the management of foreign currency exposures

   Assures an opportunity to buy or sell an asset at its average price for portfolio managers who pays or receives periodic cash flows more frequently than a portfolio is rebalanced

Contract Details: Additional information needs to be supplied in a path-dependent option contract

1. Frequency of fixings: specifies the dates on which the asset price is recorded for calculating the path-dependent quantity





2. Currency of settlement: such contracts are always settled in cash

3. An independent source of observations for the asset prices to be supplied at the fixing dates

Pricing Techniques:

1. Closed-form formulas: these exist for most European-style lookback and barrier options, but not for arithmetic average options and American-style options

2. Monte Carlo simulation: generally applicable to all European-style options, but some form of variance reduction is crucial to ensure precision of price estimates

3. Tree-based methods: standard binomial or trinomial tree can be "easily" extended to incorporate the path-dependent quantity, but it is difficult to keep the methods computationally efficient

   "Continuity corrections" may be necessary to speed up convergence of price estimates

4. Numerical integration: consists of numerically evaluating the (joint) density function of the path-dependent quantity when it is fixed discretely

5. Analytical approximations: whole host of literature on approximating the distribution of the arithmetic average of asset prices (e.g., with another lognormal distribution)

5-1. Asian Options

Motivating Example: Average rate option (ARO) as an instrument for hedging a series of known future foreign exchange cashflows

A company is due to receive every month for the next six months, USD which will be sold for GBP as they become available: it needs protection against the average rate

Without AROs the company had a number of alternatives when confronted with the prospect of a regular future forex exposure

1. Leave the exposure uncovered [obviously not a good idea]
2. Cover all the positions with a series of forward forex deals

   Cannot participate in beneficial moves in the forex rate
3. Purchase a series of standard options mirroring the exposures

   Leads to overhedging (and would therefore be more expensive than necessary)

Motivating Example (continued)

Compare the benefits of a series of USD put/GBP call options and an ARO, all struck at 1.4800

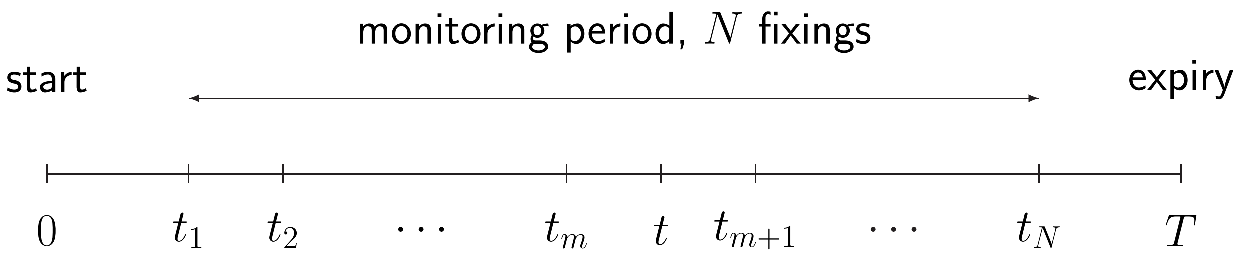
Period	Spot Rate	Benefit of (3)	Benefit of ARO
1 month	1.4850	0.50	
2 month	1.4800	—	
3 month	1.4950	1.50	
4 month	1.4700	—	
5 month	1.4850	0.50	
6 month	1.4800	—	
Total	1.4825	$2.50/6 = 0.41$	0.25

Advantage of ARO: no waste from non-exercise of unnecessary options

This benefit can be passed on to the client through the pricing mechanism, which makes the ARO cheaper than its standard option equivalent

Payoff: The ARO and the average strike option (ASO) are widely offered using the arithmetic average

The average is defined over the interval $[t_1, t_N]$ and at times $t_1, t_2, \ldots, t_N$ (not necessarily equidistant)



The "running" arithmetic average to date $A(t)$ is defined for any time point $t \in [t_m, t_{m+1})$ by

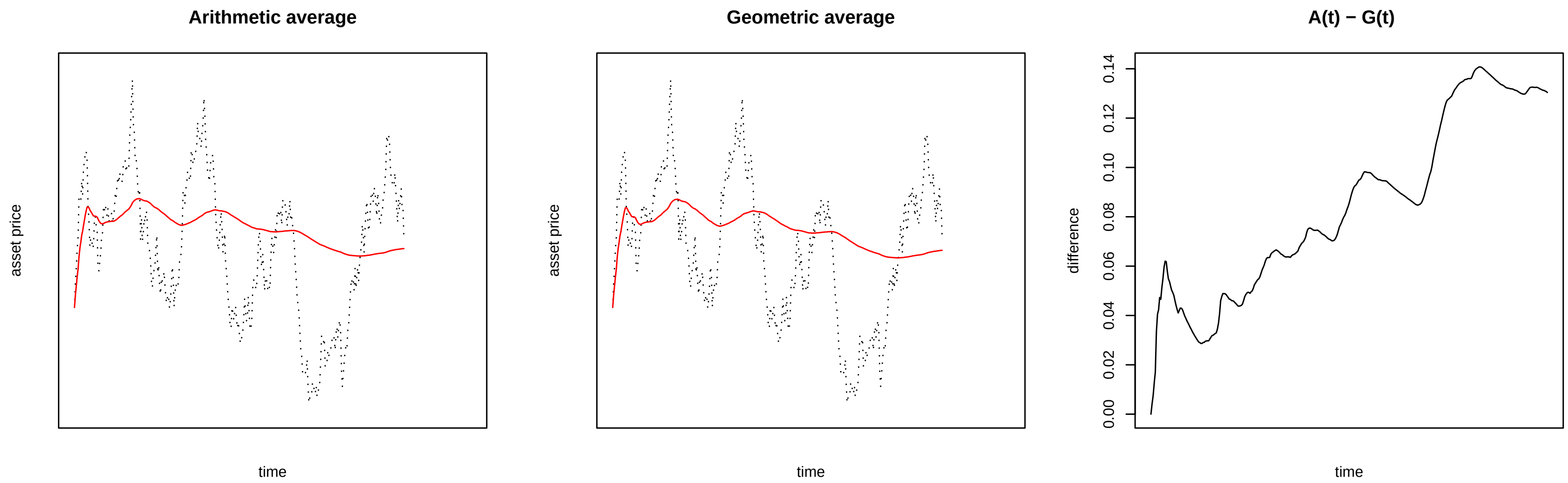
$$A(t) = \frac{1}{m} \sum_{i=1}^{m} S(t_i) \quad \text{with} \quad A(t) = 0 \quad \text{for } t < t_1$$

Thus $A(t_N)$ represents the arithmetic average of all N prices

Very occasionally the geometric average can be preferred

$$G(t) = \left[S(t_1) S(t_2) \ldots S(t_m) \right]^{1/m}$$

Payoff (continued)



1. The ARO pays off at maturity the difference (if positive) between the average of prices and a pre-specified strike price
   - For the call option, payoff$_T = \max\{0, A(t_N) - K\}$
   - For the put option, payoff$_T = \max\{0, K - A(t_N)\}$

Payoff (continued)

2. The ASO pays the difference (if positive) between the asset price on the expiry date and the average of prices

   - For the call option, $\text{payoff}_T = \max\{0, S(T) - A(t_N)\}$
   - For the put option, $\text{payoff}_T = \max\{0, A(t_N) - S(T)\}$

Put-Call Parity: At time t_N, we would already know the values of the ARO call and put, so

$$C_{\text{ARO}}(t_N) - P_{\text{ARO}}(t_N) = A(t_N) - K \tag{1}$$

The cost at time $t \in [t_m, t_{m+1})$ of replicating the payoff $A(t_N)$ at time t_N is given by

$$V_R(t) = \frac{m}{N} A(t) e^{-r(t_N - t)} + \frac{1}{N} S(t) \sum_{i=m+1}^{N} e^{-q(t_i - t)} e^{-r(t_N - t_i)}$$

This yields the put-call parity relation at time t

$$C_{\text{ARO}}(t) - P_{\text{ARO}}(t) = V_R(t) - K e^{-r(t_N - t)}$$

Put-Call Parity (continued)

To evaluate the cost of replicating the payoff $A(t_N)$, we use the identity

$$A(t_N) = \frac{m}{N}A(t) + \frac{1}{N}\big[S(t_{m+1}) + \dots S(t_N)\big]$$

To replicate a payment of $S(t_i)$ ($t_i > t$) at time t_N:

1. We purchase at time t, $e^{-q(t_i - t)}e^{-r(t_N - t_i)}$ units of the asset and hold till time t_i when we have $e^{-r(t_N - t_i)}$ units of the asset

2. Sell these asset at time t_i for $S(t_i)e^{-r(t_N - t_i)}$ cash to be invested for a period of $t_N - t_i$ to give $S(t_i)$ at time t_N

Thus, the cost of replicating a payment of $S(t_i)$ ($t_i > t$) at time t_N is $S(t)e^{-q(t_i - t)}e^{-r(t_N - t_i)}$

The put-call parity relation can also be established by directly evaluating the expectations of eq. (1) and discounting to time t

Valuation Formulas: By risk-neutral valuation, we have

$$C_{\mathsf{ARO}}(t) = e^{-r\tau} E\big[\max\{0, A(t_N) - K\}\big] \quad \text{and} \quad C_{\mathsf{ASO}}(t) = e^{-r\tau} E\big[\max\{0, S(T) - A(t_N)\}\big]$$

Both of these expressions are integrals w.r.t. the appropriate (joint) densities for which no closed-form solutions are known

If instead, the geometric average is used, then since $(\ln S(T), \ln G(t_N))$ has a bivariate normal distribution, the expectation with $A(t_N)$ replaced by $G(t_N)$ can be evaluated in closed form

For example, the value of a geometric ARO call is given by

$$C_{\mathsf{GARO}}(t) = e^{-r\tau} E\big[\max\{0, G(t_N) - K\}\big] = e^{\mu_G + \sigma_G^2/2 - r\tau} N(x_1) - e^{-r\tau} K N(x_2)$$

where $x_1 = (\mu_G - \ln K + \sigma_G^2)/\sigma_G$, $x_2 = x_1 - \sigma_G$ and

$$\mu_G = \frac{m}{N} \ln G(t) + \frac{N-m}{N} \ln S(t) + \frac{r - q - \sigma^2/2}{N} \sum_{i=m+1}^{N} (t_i - t)$$

$$\sigma_G^2 = \frac{\sigma^2}{N^2} \left[\sum_{i=m+1}^{N} (t_i - t) + 2 \sum_{i=m+1}^{N-1} (N - i)(t_i - t) \right]$$

Monte Carlo Simulation: Note that $S(t_i)$ and $S(t_{i-1})$ are related by

$$S(t_i) = S(t_{i-1})e^{(r-q-\sigma^2/2)(t_i-t_{i-1})} \exp\left[Z_i\,\sigma\sqrt{t_i - t_{i-1}}\right]$$

where $Z_i \sim N(0,1)$

1. Draw [independent] outcomes of $Z_1, Z_2, \ldots, Z_N$ from the standard normal distribution to construct a sequence of $S(t_1), S(t_2), \ldots, S(t_N)$

2. (a) For the ARO call, calculate the payoff $\max\{0, A(t_N) - K\}$

   (b) For the ASO call, the payoff is $\max\{0, S(T) - A(t_N)\}$, where $Z_T \sim N(0,1)$ and

$$S(T) = S(t_N)e^{(r-q-\sigma^2/2)(T-t_n)} \exp\left[Z_T\,\sigma\sqrt{T - t_n}\right]$$

3. Repeat Steps 1 and 2 many times and average all the calculated payoffs

4. Taking the present value of this average produces an option price estimate ($\hat{C}_{\text{ARO}}$ or $\hat{C}_{\text{ASO}}$)

Variance Reduction in MC: A high degree of correlation exists between the arithmetic average and the geometric average

The accuracy of the MC price estimates can be improved through the use of control variates

Idea: We calculate $G(t_N)$ every time $A(t_N)$ is calculated

At the end of the simulation we have two MC estimates: one for the arithmetic average option (say $\hat{C}_{\mathsf{A}}$) and one for the geometric average option (say $\hat{C}_{\mathsf{G}}$)

A better price estimate for the arithmetic average option is

$$\hat{C}_{\mathsf{A}}^* = \hat{C}_{\mathsf{A}} + C_{\mathsf{G}} - \hat{C}_{\mathsf{G}}$$

where C_{G} is the exact pricing formula for the geometric average option

Lognormal Approximation: For certain choices of parameters (e.g., σ and τ not too large) the distribution of the arithmetic average can be reasonably approximated by a lognormal distribution

Specifically, if $\ln A(t_N) \sim N(\alpha, \nu^2)$, then

$$C_{\mathsf{ARO}} = e^{-r\tau} E\big[\max\{0, A(t_N) - K\}\big] = e^{-r\tau}\left[e^{\alpha+\nu^2/2} N(x_1) - K N(x_2)\right]$$

where $x_1 = (\alpha - \ln K + \nu^2)/\nu$ and $x_2 = x_1 - \nu$

To obtain expressions for α and ν^2, we match the first and second moments under the exact distribution $A(t_N)$ to those under the assumed lognormal distribution

$$\alpha = 2 \ln E\big[A(t_N)\big] - \frac{1}{2} \ln E\left[A(t_N)^2\right] \quad \text{and} \quad \nu^2 = \ln E\left[A(t_N)^2\right] - 2 \ln E\big[A(t_N)\big]$$

where $g = r - q$ and

$$E\big[A(t_N)\big] = \frac{S(t)}{N} \sum_{i=1}^{N} e^{(r-q)t_i}, \quad E\left[A(t_N)^2\right] = \frac{S(t)^2}{N^2}\left[\sum_{i=1}^{N} e^{(2g+\sigma^2)t_i} + 2 \sum_{i=1}^{N-1} e^{(g+\sigma^2)t_i} \sum_{j=i+1}^{N} e^{g t_j}\right]$$

Lognormal Approximation (continued)

To adjust for higher moment effects one can employ an Edgeworth series expansion of the true distribution of $A(t_N)$ about an assumed lognormal distribution and use this expansion to evaluate the expectation in the pricing formulas

Numerical Integration: We can write the density of the arithmetic average as a convolution of densities from the normal distribution using the decomposition

$$A(t_N) = \frac{S(t_1) + \cdots + S(t_{N-1}) + S(t_N)}{N} = \frac{S(t_1) + \cdots + S(t_{N-1})[1 + e^{Y_N}]}{N}$$

$$= \frac{S(t_1) + \cdots + S(t_{N-2})\left\{1 + e^{Y_{N-1}}[1 + e^{Y_N}]\right\}}{N} = \cdots$$

where $Y_i = (r - q - \sigma^2/2)(t_i - t_{i-1}) + Y_i\,\sigma\sqrt{t_i - t_{i-1}}$ are independently and normally distributed

To get the density of $A(t_N)$, we unroll using the above sequence of definitions: first obtain the density of $X_N = 1 + e^{Y_N}$, then obtain the density of $X_{N-1} = 1 + e^{Y_{N-1}}X_N$ by independence, and so on

The implementation of this technique is demonstrated in Carverhill and Clewlow (1990)

Extended Lattice Approach: The methods we have described so far only allows us to price European-style Asian options

Lattice methods such as binomial trees are typically used to investigate the value of early exercise

These methods are computationally efficient so long as the number of outcomes possible at each node is proportional to the number of steps required to get to that node

This is not the case for arithmetic Asian options: in an N-step binomial tree with a fixing for each time step, there are a total of $2^{N+1} - 1$ distinct values of $A(t)$

To maintain a proportionality of i for monitoring the average, at each node the Asian option is priced for a predetermined set of representative values for $A(t)$

Interpolation on this set at each node is used to calculate the price as required

Extended Lattice Approach (continued)

Example: $S = K = 100$, $N = 3$, $t_1 = 1/3$, $t_N - t_1 = 2/3$, $r = 0.1$, $q = 0$, $\sigma = 0.2$

Binomial tree parameters are $u = 1.122040$, $d = 1/u$, $e^{-r\Delta} = 0.967216$, $p = 0.617609$

$i=0$		$i=1$			$i=2$				$i=3$					
$A(0,0)$	0.00	$A(1,k)$	89.09	112.24	$A(2,k)$	84.24	100.17	119.11	$A(3,k)$	79.73	93.00	108.48	126.54
$S(0,0)$	$C(0,0,0)$	$S(1,j)$	$C(1,j,0)$	$C(1,j,1)$	$S(2,j)$	$C(2,j,0)$	$C(2,j,1)$	$C(2,j,2)$	$S(3,j)$	$C(3,j,0)$	$C(3,j,1)$	$C(3,j,2)$	$C(3,j,3)$
									141.40	0.00	0.00	8.48	26.54
					125.98	3.48	10.58	22.07					
		112.24	10.06	15.85					112.24	0.00	0.00	8.48	26.54
100.00	10.11				100.00	0.19	4.37	13.41					
		89.09	1.73	4.61					89.09	0.00	0.00	8.48	26.54
					79.38	0.00	1.14	7.46					
									70.72	0.00	0.00	8.48	26.54

For American-style AROs, we set $C(2,1,2) = C(2,2,2) = 19.11$ (intrinsic value) so that upon recalculation $C(1,0,1) = 16.51$, $C(1,1,1) = 12.24$ (intrinsic value) and $C(0,0,0) = 10.50$

The early exercise nodes are $(2,1,2)$, $(2,2,2)$ and $(1,1,1)$

Comparison of Valuation Methods: A comparison of the various valuation methods performed by Levy and Turnbull (1992)

Hedging: Practically speaking, the valuation of an ARO can be viewed equal to that of a standard option with the same market parameters except with $\sigma' = \sigma/\sqrt{3}$ or with $T' = T/3$

A rule of thumb is then choose a European option with similar contract details as the ARO but with expiration $(t_N - t_1)/3$ (i.e., one-third the averaging period of the ARO)

Otherwise, delta hedging can be performed with cash and the underlying asset

Summary: Asian options have proven to be the most popular of the exotic family, since it addresses the everyday hedging problems facing treasurers

They are flexible instruments that allow more accurate hedging of regular forex exposures and translation exposures based on an average rate

5-2. Lookback Options

Motivating Example: Lookback option as an instrument for overcoming incorrect timing of market entry and/or exit

1. Market entry: An investor anticipates a substantial rise of the market over the next six months and buys an ATM call option on the index, which is currently at 100
   - Immediately after this purchase, the index drops unexpectedly to 92, then it gets in a strong upward trend which lasts until expiration, ending at 110
   - The option pays off 10, substantially less than if it had been bought a few weeks late
2. Market exit: An investor buys an ATM call option on the index, which is currently at 100
   - As expected, the index rises to a level of 117
   - In the last two weeks before expiration, the index shows an unexpected decline ending at a level of 108 so the option pays off only 8, substantially less than if it had been sold two weeks earlier

Motivating Example (continued)

Usefulness of lookback options:

1. The problem of market entry can be solved with a floating strike lookback option which effectively sets the strike at the lowest index value realized

   The floating strike lookback option would pay off 18 (cf. 10)

2. The problem of market exit can be solved with a fixed strike call option which is effectively a call on the highest price

   An ATM fixed strike lookback option would pay off 17 (cf. 8)

Payoff: We can distinguish between a fixed strike and a floating strike lookback option

We deal first with continuously monitored lookback options

Payoff (continued)

1. Fixed strike lookback options: The lookback period starts at an arbitrary date t_1 before the expiration date T and $t = 0$ denotes the present $(0 \leq t_1 \leq T)$

   - For the call option, set $M_{t_1}^T = \max\{S_t : t_1 \leq t \leq T\}$ and payoff$_T = \max\{0, M_{t_1}^T - K\}$
   - For the put option, set $m_{t_1}^T = \min\{S_t : t_1 \leq t \leq T\}$ and payoff$_T = \max\{0, K - m_{t_1}^T\}$

   If $t_1 = 0$, we have a "full" (cf. partial) lookback option

2. Floating strike lookback options: The lookback period starts at the option's initial date T_0 and ends at an arbitrary date t_1 before T and $t = 0$ denotes the present $(T_0 \leq 0 \leq t_1 \leq T)$

   - Call option: $m_{T_0}^{t_1} = \min\{S_t : T_0 \leq t \leq t_1\}$ and payoff$_T = \max\{0, S_T - \lambda m_{T_0}^{t_1}\}$ with $\lambda \geq 1$
   - Put option: $M_{T_0}^{t_1} = \max\{S_t : T_0 \leq t \leq t_1\}$ and payoff$_T = \max\{0, \lambda M_{T_0}^{t_1} - S_T\}$ with $0 < \lambda \leq 1$

   If $\lambda \neq 1$, we have a "fractional" lookback option (cheaper than if $\lambda = 1$)

   If $t_1 = T$, we have a "full" (cf. partial) lookback option

Valuation: By risk-neutral valuation, we have at $t = 0$

$$C_{\text{fix}} = e^{-rT} E\big[\max\{0, M_{t_1}^T - K\}\big] \quad (P_{\text{fix}} : \text{use } m_{t_1}^T)$$

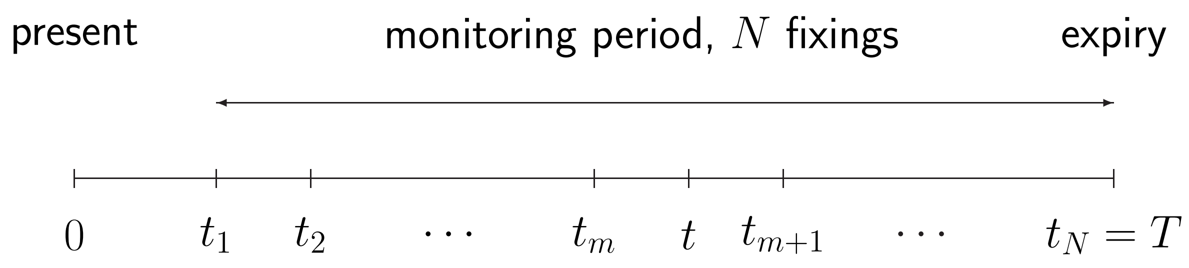
$$C_{\text{float}} = e^{-rT} E\big[\max\{0, S_T - \lambda m_{T_0}^{t_1}\}\big] \quad (P_{\text{float}} : \text{use } M_{T_0}^{t_1})$$

Because we know the (joint) distributions of $M_{t_1}^T$, $m_{t_1}^T$, $(S_T, m_{T_0}^{t_1})$ and $(S_T, M_{T_0}^{t_1})$, all these pricing formulas can be evaluated in closed form assuming continuous monitoring

1. Fixed strike lookback options become more expensive as the lookback period increases, more rapidly nearer the expiry date than further away from the expiry date
2. Floating strike lookback options become more expensive as the lookback period increases, more rapidly in the beginning than nearer the expiry date

   Raising λ causes the option price to fall very rapidly

Discrete Monitoring: To fix ideas, we concentrate on the fixed strike lookback call option, for which we are required to monitor the "running" maximum over the period $[t_1, T]$ and at times $t_1, t_2, \ldots, t_N = T$ (not necessarily equidistant)



The "running" maximum to date $M(t)$, for $t \in [t_m, t_{m+1})$, is given by

$$M(t) = \max\{S(t_1), S(t_2), \ldots, S(t_m)\} \quad \text{with} \quad M(t) = 0 \quad \text{for } t < t_1$$

The payoff of this option is $\max\{0, M(T) - K\}$ so the option has a value at $t = 0$ given by

$$C_{\text{dfix}} = e^{-rT} E\big[\max\{0, M(T) - K\}\big]$$

Numerical methods are needed to evaluate this expectation

Monte Carlo Simulation:

1. Draw outcomes of $Z_i \sim N(0,1)$ $(i = 1, \ldots, N)$ to construct a sequence of $S(t_1), \ldots, S(t_N)$ via

$$S(t_i) = S(t_{i-1}) e^{(r-q-\sigma^2/2)(t_i - t_{i-1})} \exp\left[Z_i \, \sigma \sqrt{t_i - t_{i-1}} \right]$$

   and calculate the payoff $\max\{0, M(T) - K\}$

2. Repeat Step 1 many times and average the calculated payoffs to get an option price estimate $(\hat{C}_{\text{dfix}})$ after discounting

Numerical Integration: We apply recursive numerical integration (cf. Asian options) to compute the density of the maximum using results from sequential analysis [details omitted] and use this density to evaluate the expectation in the pricing formula

Binomial Tree Approach: Given a tree for the underlying price, for each node in the tree one determines all the different paths the underlying price can take to reach that node

Binomial Tree Approach (continued)

For each path one determines the relevant maximum to date and prices the option for all alternative values of the maximum $M(t)$ (cf. Asian options, where representative values of $A(t)$ are used)

The method is memory consuming because a tree with N steps requires the calculation of some $N^3/12 + 5N^2/8 + 17N/12$ prices (cf. $N^2/2 + 3N/2$ calculations for a standard binomial tree)

Overshoot Correction: Based on the probabilistic relationship between a continuously monitored maximum and a discretely monitored maximum (on a set of equally spaced dates)

Broadie, Glasserman, and Kou (1999) came up with an approximation to the option price for "large" N at time $t = 0$

$$C_{\mathsf{dfix}}(K) = e^{-\beta_1 \sigma \sqrt{T/N}} C_{\mathsf{fix}}(K e^{\beta_1 \sigma \sqrt{T/N}}) + o(1/\sqrt{N})$$

where $\beta_1 = -\zeta(1/2)/\sqrt{2\pi} \approx 0.5826$

Hedging: Delta hedging can be performed with cash and the underlying asset

- Closed-form formulas for the Greeks can be obtained under continuous monitoring
- With discrete monitoring, the Greeks will have to be computed numerically

Static hedging using down-and-in bonds (digital options) is demonstrated in Bowie and Carr (1994)

A down-and-in bond with barrier H pays off $1 if the underlying gets as low as H during the lifetime of the bond, zero otherwise

To hedge the sale of a floating strike lookback call option, an investor should go long one (zero cost) forward contract and buy ε units of each of the $N-1$ different down-and-in bonds which have positive barriers below the current spot

For example, if the current spot is at 2 and $\varepsilon = \frac{1}{8}$ (tick size), then $N-1=15$

The investor buys $\frac{1}{8}$ of a DIB with barrier $1\frac{7}{8}$, $\frac{1}{8}$ of a DIB with barrier $1\frac{6}{8} = 1\frac{3}{4}$, and so on, with the last $\frac{1}{8}$ of a DIB having a barrier at $\frac{1}{8}$

Summary: By keeping track of past events, lookback options allow holders to take advantage of anticipated market movements without knowing the exact date of their occurrences, and may provide psychological comfort by minimizing regrets

Full lookback options are exceedingly expensive

If we look at options with relatively short lookback periods, it is clear that partial lookback options offer a good solution to the problem of timing market entry and/or exit at a reasonable price

5-3. Barrier Options

Motivating Example: Knock out options vs. synthetic knock-outs

A common criticism against the use of knock-out options goes as follows: "A company would be better off purchasing a regular option, and then selling it when the spot reaches the out-strike, rather than buying a knock-out"

Consider a UK company that has USD receivables in six months' time

1. To hedge the exposure the company wishes to buy a US$1.4900 GBP call/USD put
2. The spot and the six-month forward at the time are US$1.5110 and US$1.4930 respectively
3. The company obtains two quotations:
    - Regular option: 554 USD points
    - Knock-out at US$1.4500: 394 USD points

Motivating Example (continued)

Break-even analysis:

Days to knock-out	Forward points	Value of new option	Net cost of synthetic KO	Synthetic less actual KO
50	131	241	313	−81
100	82	171	383	−11
150	32	75	479	+85

1. If the out-strike is reached before 105 days, the regular option is cheaper than the knock-out

2. Otherwise the knock-out is cheaper

The question now should be: "Is it likely to be knocked out in the next three and a half months?"

If the answer is no, then the premium saving using the knock-out is a genuine cost-saving

Motivating Example (continued)

In case of a knock-out:

1. The company can reinstate the hedge by using another option
2. It also has the opportunity to re-evaluate the hedge in the light of new information available since the original hedge was put on

Payoff: Suppose the barrier is set at H and the monitoring period coincides with $[0, T]$ (the lifetime of the option)

1. A knock-out option cancels immediately when the underlying price hits or crosses the predetermined barrier level H
   - Down-and-out call option: $\text{payoff}_T = (S_T - K)^+ \mathbb{I}_{\{m_0^T > H\}}$
   - Up-and-out call option: $\text{payoff}_T = (S_T - K)^+ \mathbb{I}_{\{M_0^T < H\}}$

Payoff (continued)

1. A knock-in option does not come into existence (and therefore expires worthless) unless the underlying price hits or crosses the barrier

   - Down-and-in call option: $\text{payoff}_T = (S_T - K)^+ \mathbb{I}_{\{m_0^T \le H\}}$
   - Up-and-in call option: $\text{payoff}_T = (S_T - K)^+ \mathbb{I}_{\{M_0^T \ge H\}}$

Note that DO + DI = UO + UI = $(S_T - K)^+$, the payoff of a standard call option

Therefore it suffices to consider only knock-out options in the sequel

Valuation: By risk-neutral valuation, we have at $t = 0$

$$C_{\text{do}} = e^{-rT} E\left[(S_T - K)^+ \mathbb{I}_{\{m_0^T > H\}}\right]$$

$$C_{\text{uo}} = e^{-rT} E\left[(S_T - K)^+ \mathbb{I}_{\{M_0^T < H\}}\right]$$

As in the case of lookback options, we know the joint distributions of (S_T, m_0^T) and (S_T, M_0^T), so all these pricing formulas can be evaluated in closed form under continuous monitoring

Valuation (continued)

Discrete monitoring causes knock-out options to become more expensive and knock-in options to become less expensive relative to options with continuous monitoring

With discrete monitoring (and also discrete dividend payouts), numerical methods are required to value barrier options

Monte Carlo Simulation: Analogous to the procedure for lookback options

1. Draw outcomes of $Z_i \sim N(0,1)$ $(i = 1, \ldots, N)$ to construct a sequence of $S(t_1), \ldots, S(t_N)$ via

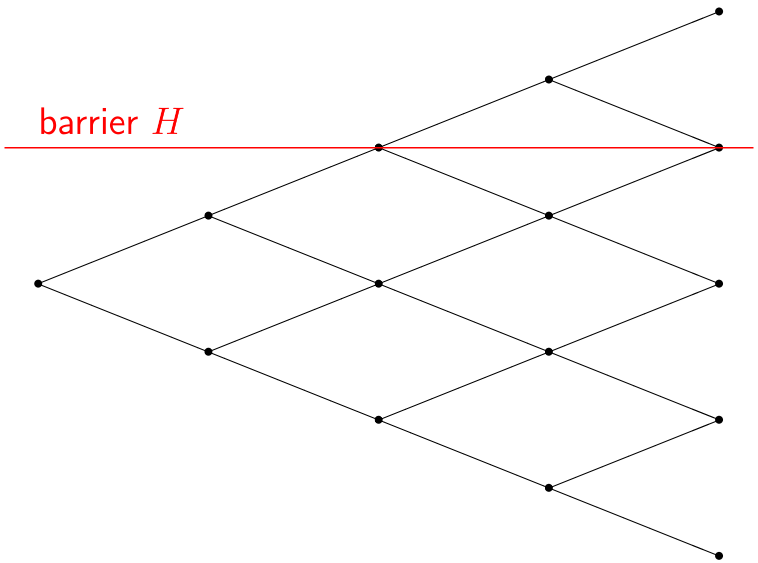
$$S(t_i) = S(t_{i-1}) e^{(r-q-\sigma^2/2)(t_i - t_{i-1})} \exp\left[Z_i \, \sigma \sqrt{t_i - t_{i-1}} \right]$$

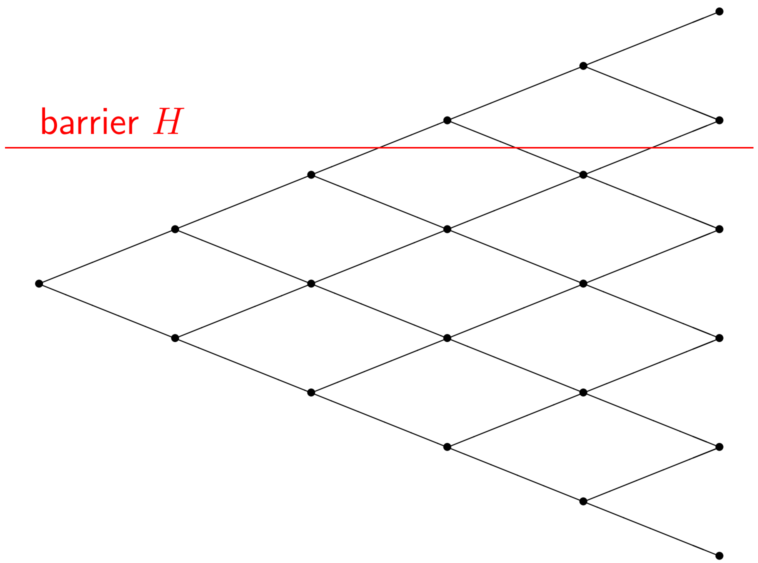
   and calculate the payoff: $(S_T - K)^+ \mathbb{I}_{\{m_0^T > H\}}$ for the DO call option or $(S_T - K)^+ \mathbb{I}_{\{M_0^T < H\}}$ for the UO call option

2. Repeat Step 1 many times and average the calculated payoffs to get an option price estimate after discounting

Numerical Integration: Recursive numerical integration techniques (cf. lookback options) can be used to compute the joint densities needed to evaluate the expectations in the pricing formulas

Binomial Tree Approach: Given a tree for the underlying price, the most obvious way is to simply assign zero values to all nodes of the tree which lie in the knock-out region and calculate the desired option price with the usual backward induction procedure





Unfortunately, this yields prices which heavily depend on the number of time steps used to construct the tree: saw-tooth pattern around the “true” price

Binomial Tree Approach (continued)

Since the jumps in the binomial prices occur every time a line of nodes changes place relative to the barrier, we first identify the number of time steps where this happens

The number of time steps required for the underlying price to reach the barrier H in exactly m upward or downward jumps is given by

$$N(m) = \frac{m^2\sigma^2 T}{\big[\ln(S/H)\big]^2} \qquad (m = 1, 2, \dots)$$

We therefore price the option using a tree with $\lfloor N(m) \rfloor$ time steps, then using a tree with $\lfloor N(m) \rfloor + 1$ time steps and the average of these two values provides an improved binomial price

Hedging: Delta hedging can be performed with cash and the underlying asset

However, since adding a barrier typically seriously distorts the payoff profile of the option it is added to, the delta of a barrier option changes very rapidly as the underlying approaches the barrier

Hedging (continued)

As a consequence, more static ways to hedge barrier options are desired: for example, Bowie and Carr (1994)

We illustrate the idea with a DI call option when $r = q$ (no carrying cost) so that the usual put-call parity can be expressed as $C_{\text{std}} - P_{\text{std}} = (S - K)e^{-rT}$

1. Barrier at strike ($H = K$): the sale of a DI call is hedged by going long a standard put with the same underlying, maturity and strike as the knock-in

    - Underlying stays above H: both options expire worthless
    - Underlying hits H: at the barrier, $P_{\text{std}} = C_{\text{std}}$

      The first time the underlying touches H, the hedger can sell off his standard put and buy a standard call (no out-of-pocket expense)

    This hedge works because put price equals call price at the barrier

    We will capitalize on this condition when $H < K$

Hedging (continued)

2. Barrier below strike ($H < K$): the sale of a DI call with strike K and barrier $H < K$ is hedged by going long K/H puts struck at H^2/K

   Why does this work? At the barrier H, the value of K/H puts struck at H^2/K is

$$\frac{K}{H}e^{-r\tau}\left[\frac{H^2}{K}N(-d_{2P}) - HN(-d_{1P})\right] = e^{-r\tau}\left[HN(-d_{2P}) - KN(-d_{1P})\right]$$

   where

$$-d_{2P} = -\frac{\ln(K/H) - \sigma^2\tau/2}{\sigma\sqrt{\tau}} = \frac{\ln(H/K) + \sigma^2\tau/2}{\sigma\sqrt{\tau}} = d_{1C} \quad \text{and} \quad -d_{1P} = d_{2C}$$

   This value equals $e^{-r\tau}\left[HN(d_{1C}) - KN(d_{2C})\right]$, which is exactly the amount needed to finance the purchase of a call struck at K

Double Barrier Options: The option could have two barrier levels, one above and one below the current asset price, either of which needs to be triggered for the option to be activated or extinguished

Outside Barrier Options: With the use of an "outside" barrier, one distinguishes between the "payoff variable," which determines the actual option payoff, and the "barrier variable," which determines whether the option knocks in or out

With an outside barrier one can tailor the option not only to the view which an investor has on the payoff variable, but also on the barrier variable

A key determinant of the option price is then the correlation between the two variables

Summary: Barrier options provide seemingly endless opportunities to create highly structured risk-reward or payoff profiles consistent with asset price expectations as well as asset price path patterns

From a valuation and hedging perspective, barrier options present unique problems due to the appearing and/or disappearing nature of the option which requires instant adjustment of hedge portfolios to replicate barrier options