## 4-1. Compound Options

Motivating Example: Compound options as a means of contingency hedging

A German construction company is bidding on a contract in the US to be awarded in six months

If the company is successful, it will receive US$30 million 12 months from contract day for the completion of the contract

The company needs to hedge its dollar exposure against the possibility of a lower dollar/DM rate

Consider purchasing a call option on dollar-put option

| Outcome | Dollar-put rises in value | Dollar-put falls in value |
|---|---|---|
| Successful in bid | Exercise the compound option | Buy the dollar-put in the market |
| Not successful in bid | Exercise the compound option | Compound option expires OTM |

In any case, the loss to the company is limited to the premium paid for the compound option

| Options | First payment | Second payment | Total premium |
|---|---|---|---|
| Call on dollar-put (1) | 450 points | 450 points | 900 points |
| Call on dollar-put (2) | 290 points | 800 points | 1090 points |
| ATM vanilla put | 800 points | — | 800 points |

Although more expensive than the vanilla put, the compound option offers greater flexibility and control in their application

The premiums can be adjusted according to the likelihood of the company being awarded the contract

A Second Example: Compound options as a means for hedging debt

A liability manager wishes to hedge a portfolio of dollar floating-rate debt priced off LIBOR

He seeks protection against rises in dollar interest rates over the next three years

## A Second Example (continued)

1. Purchase a standard cap on three-month LIBOR for three years at a strike of 7.50% pa

   Cost: premium equals 2.1% of the face value of the underlying debt being hedged

2. Purchase a compound cap ("caption") for the right to buy a cap on three-month LIBOR with a strike of 7.50% pa

   Cost: agreed total premium of 1.75% (for all 11 caps) and compound cap premium of 0.74%

The compound cap offers a cost saving on the initial premium of a substantial amount

The contingent premium would be triggered only if the cap level is breached at any three-month LIBOR fixing date and the hedger would pay the relevant agreed premium for that particular period

The compound cap hedge involves a lower total cost except where interest rates rise sharply and all or most of the caps are triggered (e.g., when 9 or more caps are triggered)

Payoff: Suppose the compound option has strike $K$ and maturity date $T$ while the underlying option

has strike $K^*$ and maturity date $T^* > T$ (so the underlying option lasts for $T^* - T$ before expiration)

There are four possible variations

1. Call on call: $\text{payoff}_T = \max\{0, C_{\text{std}}(S_T, K^*, T^* - T) - K\}$

2. Put on call: $\text{payoff}_T = \max\{0, K - C_{\text{std}}(S_T, K^*, T^* - T)\}$

3. Call on put: $\text{payoff}_T = \max\{0, P_{\text{std}}(S_T, K^*, T^* - T) - K\}$

4. Put on put: $\text{payoff}_T = \max\{0, K - P_{\text{std}}(S_T, K^*, T^* - T)\}$

$C_{\text{std}}$ and $P_{\text{std}}$ are given by the Black-Scholes formula: for spot $S$, strike $K$ and time to expiration $\tau$
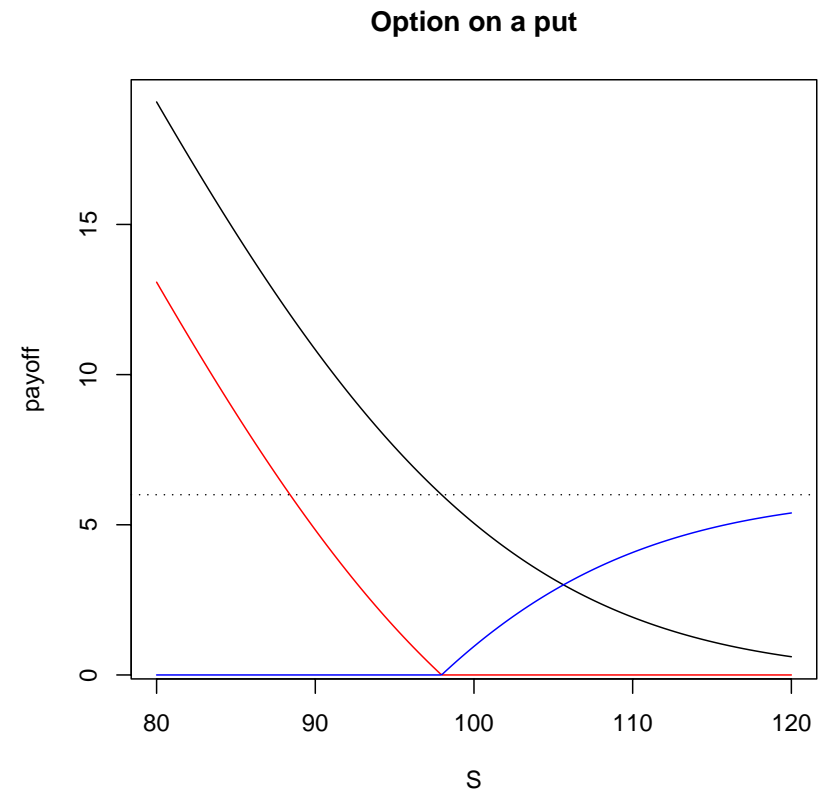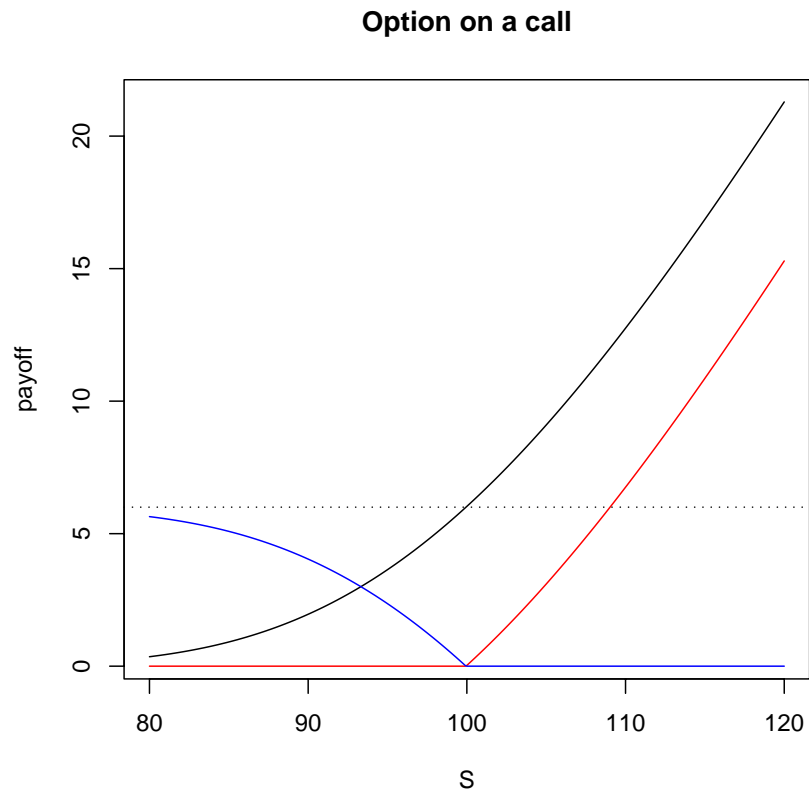
$$C_{\text{std}}(S, K, \tau) = Se^{-q\tau} N(d_1) - Ke^{-r\tau} N(d_2)$$

$$P_{\text{std}}(S, K, \tau) = Ke^{-r\tau} N(-d_2) - Se^{-q\tau} N(-d_1)$$

$$d_1 = d_2 + \sigma\sqrt{\tau} \quad \text{and} \quad d_2 = \frac{\ln(S/K) + (r - q - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

## Payoff (continued)

E.g., $S = 100$, $r = 0.05$, $q = 0.03$, $\sigma = 0.2$, $K = 6$, $T - t = 0$, $K^* = 100$, $T^* - T = 0.5$



**Option on a call**

**Option on a put**

Call on call and call on put inherit characteristics of the underlying options while put on call and put on put have characteristics different from the underlying options

Valuation: By risk-neutral valuation, the value of a call on call is

$$C = e^{-r(T-t)}E\big[\max\{0, C_{\text{std}}(S_T, K^*, T^* - T) - K\}\big]$$

While an analytic expression for $C$ is available, we first consider evaluating the integral (expectation) using a numerical integration routine (e.g., trapezoidal rule)

Let $f(x)$ be the PDF of the normal distribution with mean $(r - q - \sigma^2/2)(T - t)$ and variance $\sigma^2(T - t)$ and apply a change of variable $z = (x - (r - q - \sigma^2/2)(T - t))/\sigma\sqrt{T - t}$

$$C = \int_{-\infty}^{\infty} \max\{0, C_{\text{std}}(Se^x, K^*, T^* - T) - K\}f(x)\,dx$$

$$= \int_{-\infty}^{\infty} \max\{0, C_{\text{std}}(Se^{z\sigma\sqrt{T-t}+(r-q-\sigma^2/2)(T-t)}, K^*, T^* - T) - K\}n(z)\,dz = \int_{-\infty}^{\infty} I(z)\,dz$$

We replace $\int_{-\infty}^{\infty}$ by $\int_{-M}^{M}$ (e.g., $M = 6$ would give a sufficiently accurate approximation), choose a large $n$ and set $h = 2M/n$ to obtain

$$C \approx \frac{h}{2}\left[I(-M) + I(M) + 2\sum_{i=1}^{n-1} I(-M + ih)\right]$$

## Valuation (continued)

Alternatively, let $S^*$ be the asset price for which $C_{\text{std}}(S^*, K^*, T^* - T) = K$ (e.g., $S^*$ can be found using the Newton-Raphson algorithm)

The compound option payoff is nonzero when $S_T > S^*$ (or when $x > \ln(S^*/S)$) so

$$C = e^{-r(T-t)} \int_{\ln(S^*/S)}^{\infty} \left[ C_{\text{std}}(Se^x, K^*, T^* - T) - K \right] f(x) \, dx$$

Each of the three terms in the integral can be evaluated separately

1. $e^{-r(T-t)} \displaystyle\int_{\ln(S^*/S)}^{\infty} Se^x e^{-q(T^*-T)} N(d_1) f(x) \, dx = Se^{-q(T^*-t)} N_2(D_1, D_1^*; \rho)$

2. $e^{-r(T-t)} \displaystyle\int_{\ln(S^*/S)}^{\infty} K^* e^{-r(T^*-T)} N(d_2) f(x) \, dx = K^* e^{-r(T^*-t)} N_2(D_2, D_2^*; \rho)$

3. $e^{-r(T-t)} \displaystyle\int_{\ln(S^*/S)}^{\infty} K f(x) \, dx = Ke^{-r(T-t)} N(D_2^*)$

Here, $\rho = \sqrt{(T-t)/(T^* - t)}$ and $N_2(\cdot, \cdot; \rho)$ is the bivariate standard normal CDF with correlation coefficient $\rho$

## Valuation (continued)

The analytic pricing of the compound option relies on the following result

$$\int_a^\infty f(x) N(bx + c)\, dx = N_2 \left( \frac{\mu - a}{\nu}, \frac{b\mu + c}{\sqrt{1 + b^2 \nu^2}}; \rho \right)$$

where $f(x)$ is the PDF of a $N(\mu, \nu^2)$ distribution and $\rho = b\nu / \sqrt{1 + b^2 \nu^2}$

1. After completing the squares, we have $\mu = (r - q + \sigma^2/2)(T - t)$ and $\nu = \sigma\sqrt{T - t}$

   Futhermore, $a = \ln(S^*/S)$, $b = \dfrac{1}{\sigma\sqrt{T^* - T}}$ and $c = \dfrac{\ln(S/K^*) + (r - q + \sigma^2/2)(T^* - T)}{\sigma\sqrt{T^* - T}}$
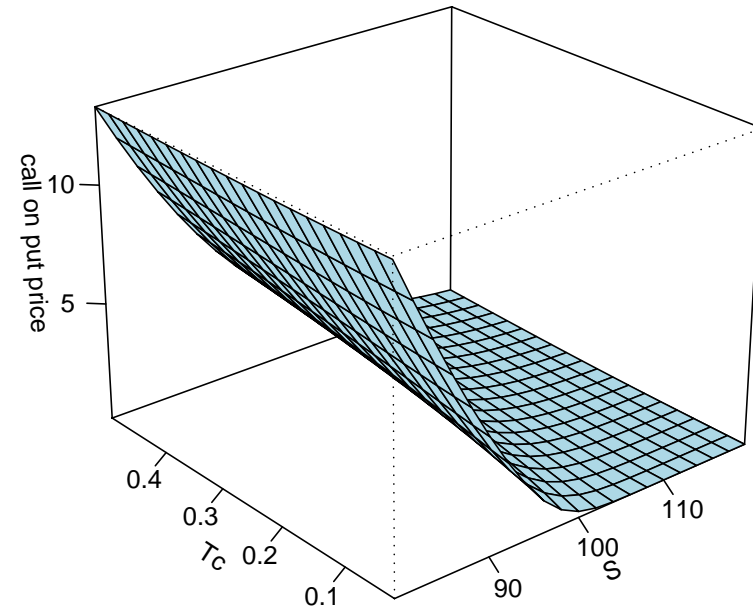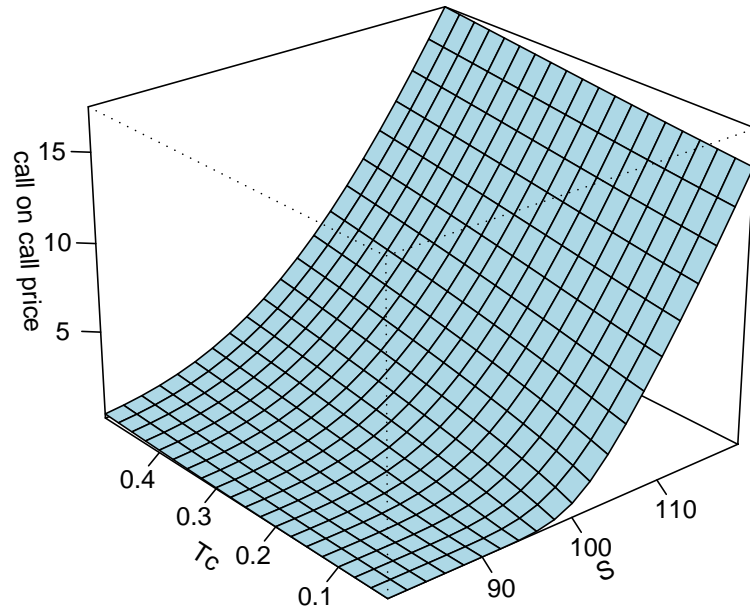
   Therefore, $\dfrac{\mu - a}{\nu} = D_1^*$, $\dfrac{b\mu + c}{\sqrt{1 + b^2 \nu^2}} = D_1$ and $\rho = \sqrt{(T - t)/(T^* - t)}$

2. Now $\mu = (r - q - \sigma^2/2)(T - t)$ and $c = \dfrac{\ln(S/K^*) + (r - q - \sigma^2/2)(T^* - T)}{\sigma\sqrt{T^* - T}}$

   With $a, b, \nu$ as before, we have $\dfrac{\mu - a}{\nu} = D_2^*$ and $\dfrac{b\mu + c}{\sqrt{1 + b^2 \nu^2}} = D_2$
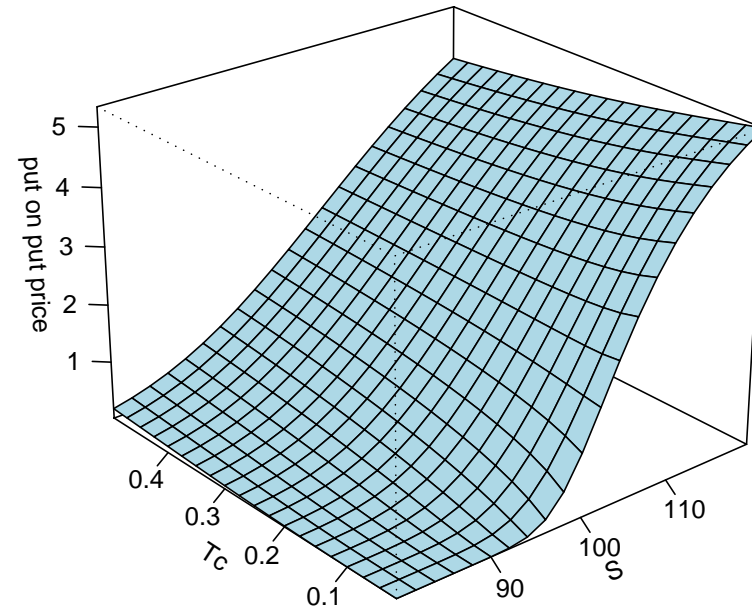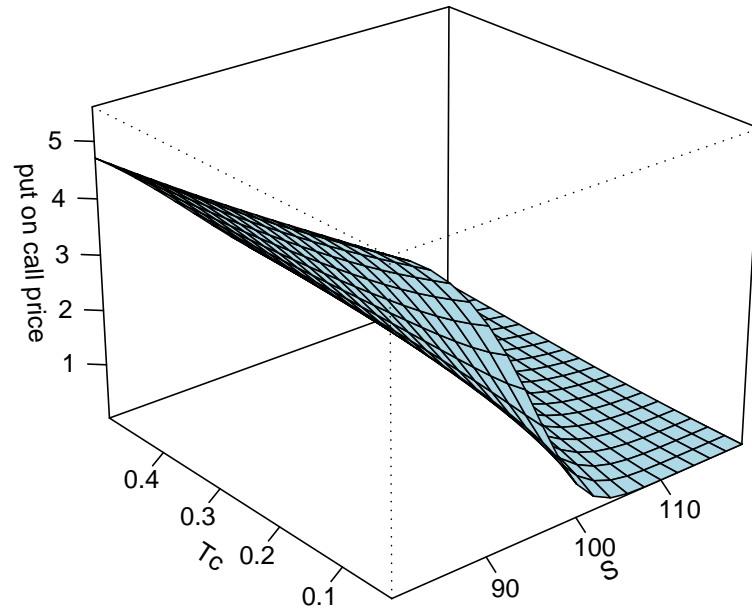
## Valuation (continued)



As expected, compound calls have price characteristics and sensitivities similar to the underlying option

By contrast, as the payoffs on compound puts flatten out (at close to $K$) as the options become further ITM, compound puts have different price, delta, gamma and vega surfaces than the underlying options

## Valuation (continued)



**Hedging:** Hedging compound options using the underlying options can be expensive due to higher transaction costs

Hedging using the asset underlying the underlying options leads us to analyze the price sensitivities of compound options (we focus on compound puts)

Delta: The deltas of compound puts are peaked near the strike price $K^*$ of the underlying option
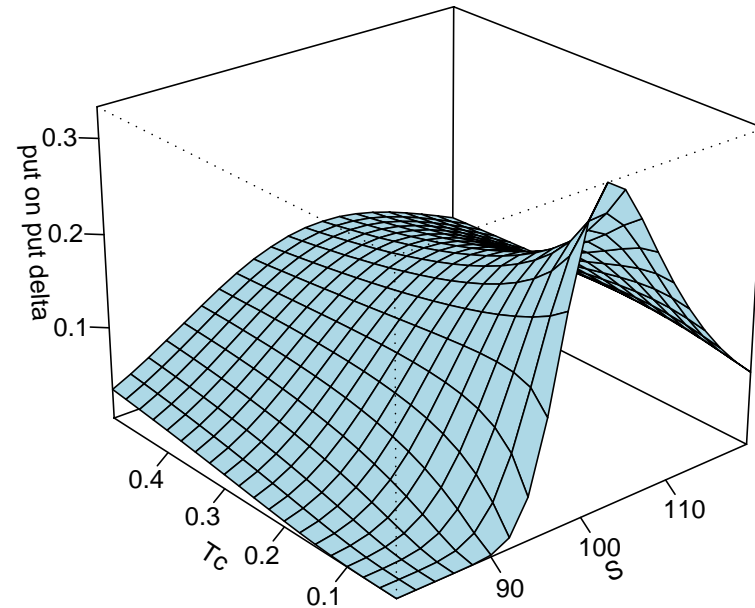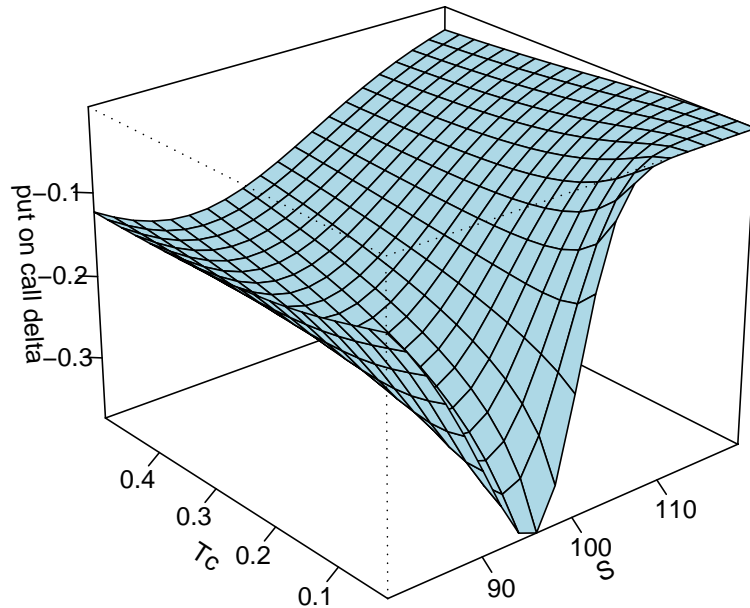


Delta hedging:

- Sale of a put on call is hedged by selling $|\Delta|$ units of the underlying asset

- Sale of a put on put is hedged by buying $\Delta$ units of the underlying asset

Rapid changes in $\Delta$ as $t \to T$ means that hedging is difficult close to compound option expiration

Gamma: The gammas of compound puts can be positive or negative depending on the level of the

underlying asset relative to the peaks
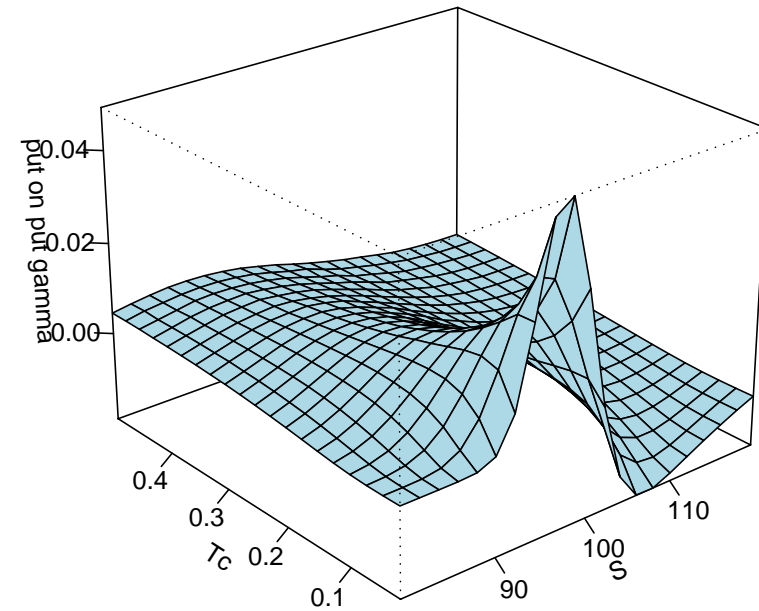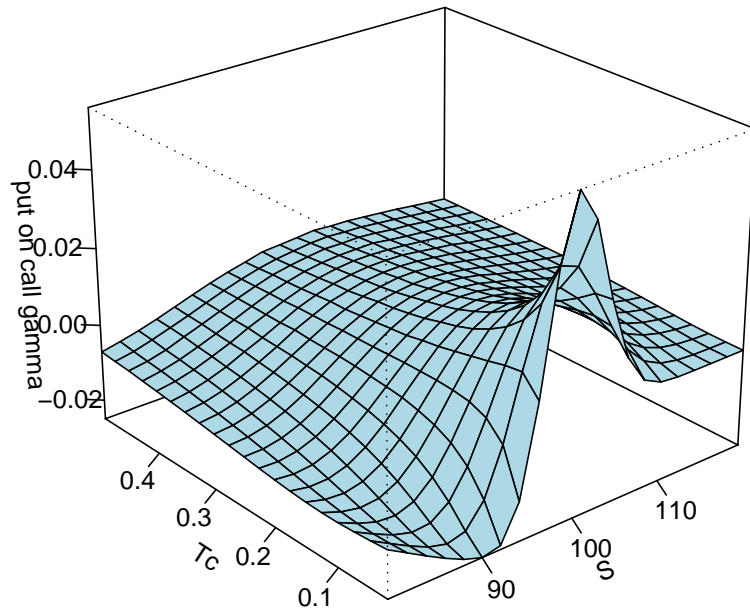


Compound puts are high gamma instruments close to compound option expiration; elsewhere changes

in asset prices do not require drastic rebalancing of the hedging portfolio

Vega: Except when the compound puts are deep OTM, increases in volatility always lead to decreases

in compound option prices (negative vegas)
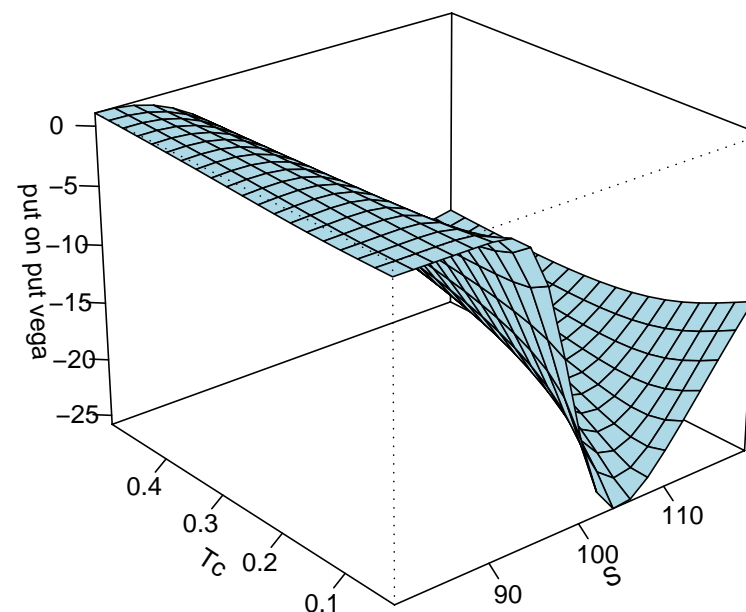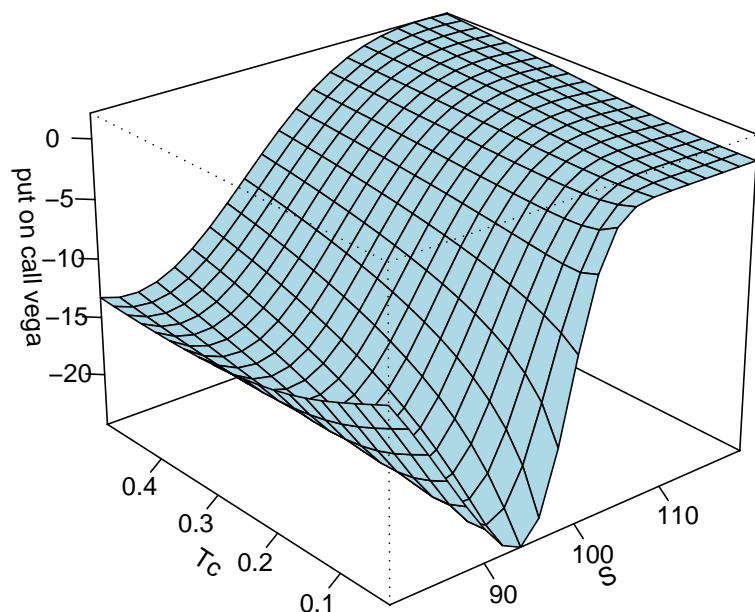


Example: $T - t = 0.025$

- Put on call: For $S > 94$ increases in $\sigma$ leads to increases in $\Delta$ (rapid increases around $S = 100$);

  for $S < 94$ increases in $\sigma$ leads to decreases in $\Delta$

- Put on put: Effect of $\sigma$ on $\Delta$ reversed about $S = 104$

**Variations:** The instalment option allows the puchaser to pay its premium in instalments at regular intervals thereby facilitating a decision to abandon the option and cut the premium cost at each instalment date

Another variation would be to make the underlying option an American option or to use an exotic option in place of a vanilla one

**Summary:** Compound options offer the right to buy or sell an option

They are useful in situations where there is a degree of uncertainly over whether the underlying option will be needed at all (contingency hedging)

The small up-front premium is a known cost that can be budgeted for

## 4-2. Chooser Options

Motivating Example: Chooser options as a means of hedging unknown direction of currency exposure

A corporation has Canadian dollar exposure that is netted from the cross-border flows of two operations

<div align="center">

Forecasts from corporation's sales and manufacturing functions

| Product | Unit sales (,000) | Price | Revenue | Budget rate | Forex hedge | US$ cash |
|---------|-------------------|-------|---------|-------------|-------------|----------|
| A | 1,000 | 38 | 38,000 | 1.2 | −38,000 | 31,667 |
| B | 1,000 | 10 | 10,000 | 1.2 | −10,000 | 8,333 |

| Sourced | Material (,000) | Price | Cost | Budget rate | Forex hedge | US$ cash |
|---------|-----------------|-------|------|-------------|-------------|----------|
| Canada | 2,500 | 24 | −60,000 | 1.2 | 60,000 | −50,000 |

</div>

<div align="center">

Forecast C$ purchases:      12,000

Forecast US$ cashflow:      −10,000

</div>

## Motivating Example (continued)

Two months into the fiscal year, several events are pushing revised forecasts away from the initial budget levels

Results reported by corporation's sales and manufacturing functions

| Product | Unit sales (,000) | Price | Revenue | Budget rate | Forex hedge | US$ cash |
|---------|-------------------|-------|---------|-------------|-------------|----------|
| A | 1,375 | 38 | 52,250 | 1.3 | $-52,250$ | 40,192 |
| B | 500 | 10 | 5,000 | 1.3 | $-$ 5,000 | 3,846 |

| Sourced | Material (,000) | Price | Cost | Budget rate | Forex hedge | US$ cash |
|---------|-----------------|-------|------|-------------|-------------|----------|
| Canada | 2,100 | 21 | $-44,100$ | 1.3 | 44,100 | $-33,923$ |

| | | | | | | |
|---|---|---|---|---|---|---|
| | | | Revised C$ to be sold: | | 13,150 | 10,115 |
| | | | Initial C$ purchases: | | 12,000 | $-$ 769 |
| | | | Post hedge US$ cashflow: | | | 9,346 |

## Motivating Example (continued)

1. Based on the first budget, the purchase of a C$ call would have prevented losses from the forward purchase of C$, given the subsequent direction of the exchange rate exposure

2. C$ call is useless here since the forecast of the underlying exposure has swung from short C$12 million to long C$13.15 million

3. Consider a chooser option struck at C$1.2000 with an expiry of one year and a choice date of three months after the budget's starting point on a face value of C$12 million

Revaluation to account for effects of hedging

| *C$ call hedge* | | |
|---|---|---|
| Pre-option US$ cashflow: | | 10,115 |
| Premium: | — | 160 |
| | | |
| Post-call US$ cashflow: | | 9,955 |

| *Chooser hedge* | | |
|---|---|---|
| Pre-option US$ cashflow: | | 10,115 |
| Premium: | — | 385 |
| Savings: | | 769 |
| Post-chooser US$ cashflow: | | 10,500 |

Motivating Example (continued)

Benefits of chooser option:

1. More effective in protecting US$ cashflow: US$1.154 million more than (incorrect) forward hedge and US$0.545 million more than C$ call hedge

2. Useful when the forecast is inaccurate due to factors well beyond the control of the corporate treasury

A Second Example: Chooser options as a means of yield enhancement

There is an election in three months' time: it is not clear what the result will be and how the markets will react

The investor buys an ATM chooser option on the market index with a choice data three months from now on options which expire six months from now

Payoff: Chooser options allow the holder to choose at some predetermined future date $T$ (choice date) whether the option will be a standard call or put with predetermined strike price and maturity date

1. Simple chooser option: underlying options have identical strike $K$ and maturity date $T^*$

$$\text{payoff}_T = \max\{C_{\text{std}}(S_T, K, T^* - T), P_{\text{std}}(S_T, K, T^* - T)\}$$
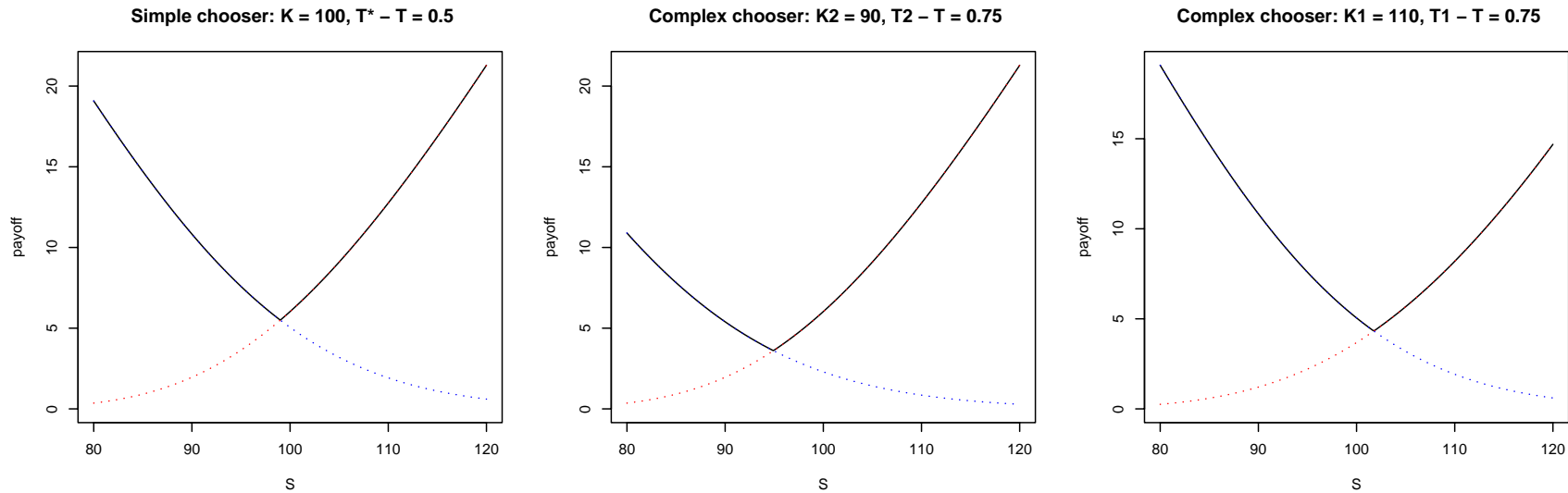


2. Complex chooser option: underlying options have different strikes and maturity dates

$$\text{payoff}_T = \max\{C_{\text{std}}(S_T, K_1, T_1 - T), P_{\text{std}}(S_T, K_2, T_2 - T)\}$$

Valuation: A simple chooser can be priced by appealing to put-call parity

$$P_{\text{std}}(S_T, K, T^* - T) = C_{\text{std}}(S_T, K, T^* - T) + Ke^{-r(T^* - T)} - S_T e^{-q(T^* - T)}$$

Since $\max\{C_{\text{std}}, P_{\text{std}}\} = C_{\text{std}} + \max\left\{0, Ke^{-r(T^* - T)} - S_T e^{-q(T^* - T)}\right\}$, the value of a simple chooser is

$$C = e^{-r(T-t)} E\left[C_{\text{std}}(S_T, K, T^* - T)\right] + e^{-r(T-t)} E\left[\max\left\{0, Ke^{-r(T^* - T)} - S_T e^{-q(T^* - T)}\right\}\right]$$

where the expectations are taken at time $t$

The first term evaluates to the <span style="color:red">current value of a call option</span> with parameters $(S, K, T^* - t)$

The second term is the <span style="color:red">value of a put</span> with underlying asset price $Se^{-q(T^* - T)}$, strike $Ke^{-r(T^* - T)}$, and time to expiration $T - t$

$$C = \left[Se^{-q(T^* - t)} N(d_1) - Ke^{-r(T^* - t)} N(d_2)\right] + \left[Ke^{-r(T^* - t)} N(-d_2^*) - Se^{-q(T^* - t)} N(-d_1^*)\right]$$

where $d_1 = d_2 + \sigma\sqrt{T^* - t}$, $d_1^* = d_2^* + \sigma\sqrt{T^* - T}$ and

$$d_2 = \frac{\ln(S/K) + (r - q - \sigma^2/2)(T^* - t)}{\sigma\sqrt{T^* - t}}, \quad d_2^* = \frac{\ln(S/K) + (r - q)(T^* - t) - \sigma^2(T^* - T)/2}{\sigma\sqrt{T^* - T}}$$

## Valuation (continued)

Example: $S = 100$, $r = 0.08$, $d = 0.03$, $\sigma = 0.2$, $K = 100$, $T^* = 1$, the choice date $T$ is varied

| $T$: | 0 | 1/12 | 1/4 | 1/2 | 3/4 | 1 |
|------|-----|-------|-------|-------|-------|-------|
| $C$: | 8.95 | 10.01 | 11.43 | 12.93 | 14.10 | 15.11 |

The value of the simple chooser for $T = 0$ is equivalent to that of the call, while the value for $T = 1$ is equal to that of a straddle

For the complex chooser, valuation is more complex but the starting point is still the risk-neutral pricing formula

$$C = e^{-r(T-t)} E\left[\max\{C_{\text{std}}(S_T, K_1, T_1 - T), P_{\text{std}}(S_T, K_2, T_2 - T)\}\right]$$

A closed-form pricing formula in terms of the bivariate standard normal CDF is available and can be obtained along the lines of the derviation of the compound option value

Numerical integration using the trapezoidal rule provides a simple alternative for analyzing the price dynamics of complex choosers

Static Hedging: Since the simple chooser decomposes exactly into a portfolio of a call and put option then, if the required strike and maturity are available in the market, these can be used to perfectly hedge the chooser

Dynamic Hedging: We analyze the price sensitivities of chooser options
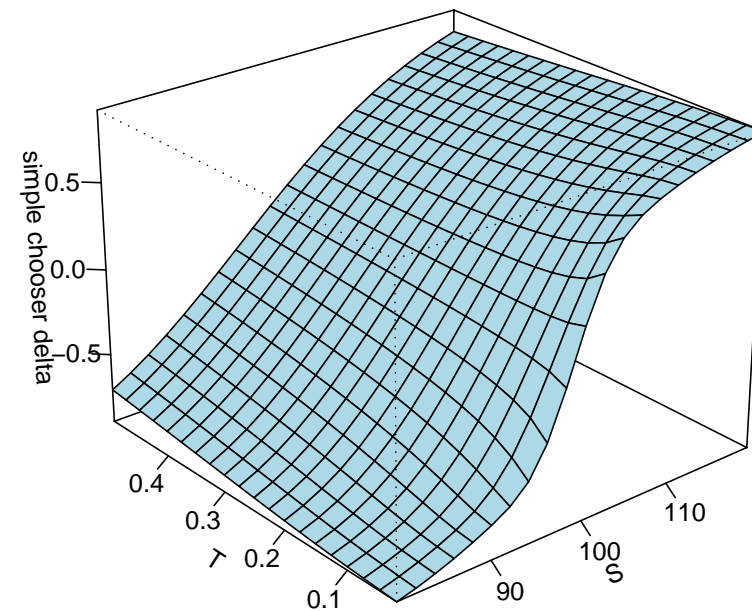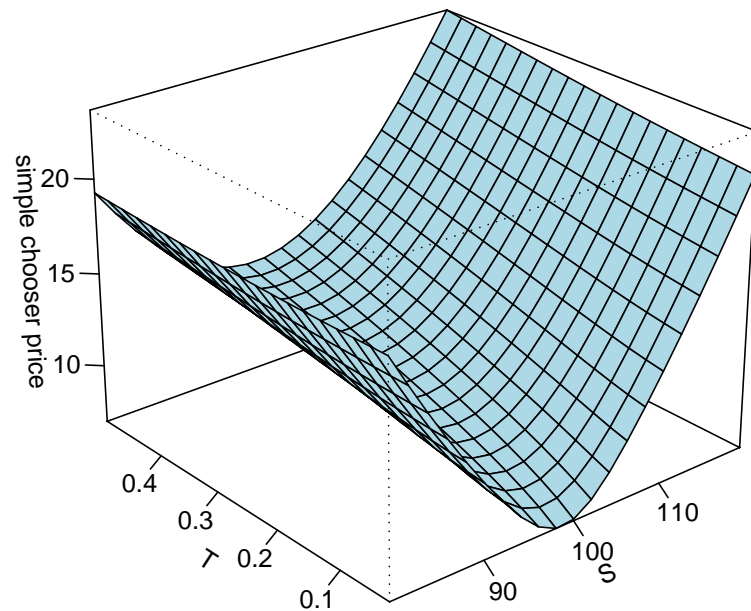


The chooser option exhibits sensitivities very similar to those of a straddle (but at a lower price)
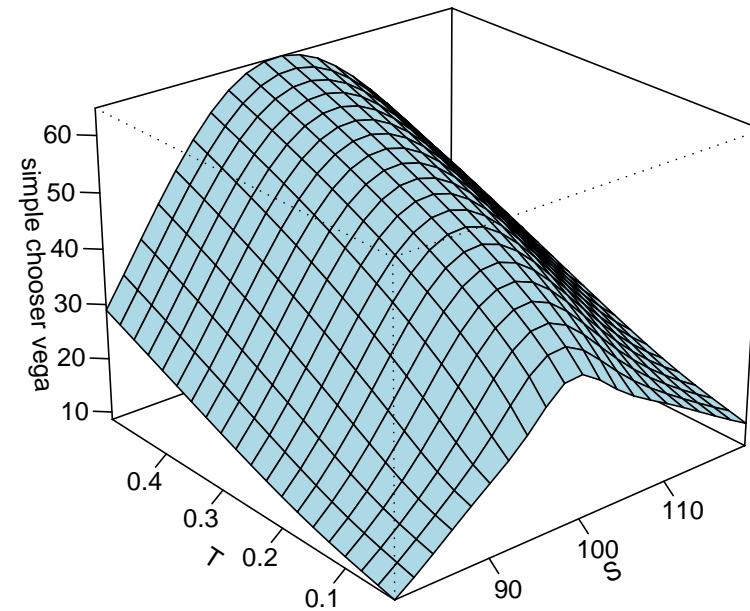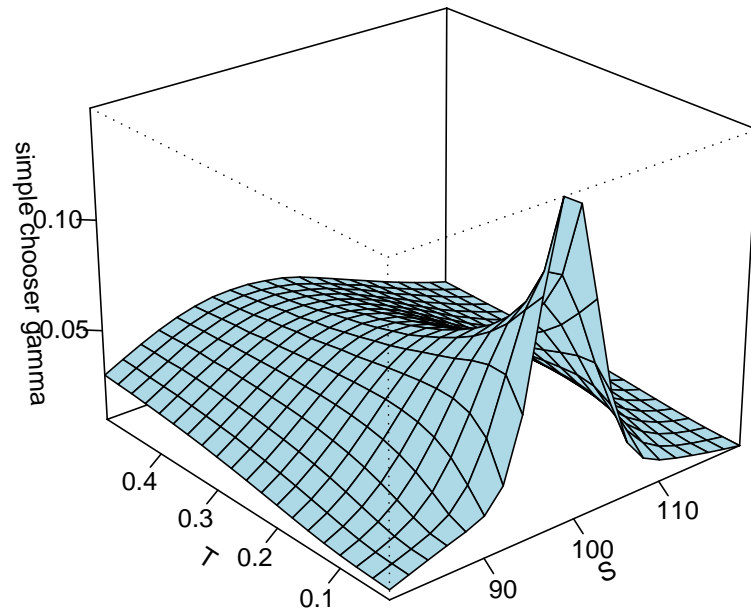
Chooser options can therefore be hedged in the same way as vanilla options

Alternatively, one can use a combination of the underlying options (e.g., chosen to minimized the squared difference in prices) to form a quasi-static hedge and then delta hedge the residual price risk

The positions in the underlying options would not need to be adjusted until quite near the choice date

Variations: By changing the type of underlying option (but perhaps retaining the put-call pair), one can create several natural variations of the basic chooser option

Summary: Chooser options facilitate hedging of asset-price risk under conditions of extreme volatility and/or where the direction of the exposure is not unknown

The hedging can be achieved at an economic cost lower than that of the equivalent straddle

## 4-3. Shout Options

Motivating Example: Shout options as a means of yield enhancement (locking in profits)

A US manufacturer imports raw materials priced in C$ and wishes to limit its exposure to a weaker US$ over a time span of one month

Consider option structures that will allow the importer to lock in the lowest C$ rate during the life of the option

  1. Lookback option: Pays out the difference between strike and lowest rate in the month, if positive

  2. Shout option: Pays out the larger of two quantities

    • Difference between strike and "shouted" rate, if positive

    • Difference between strike and terminal rate, if positive

The lookback option is deemed too expensive and the importer purchases an ATM shout option

Motivating Example (continued)

At the commencement of the transaction the prevailing exchange rate is US\$1 = C\$1.35

One week later, the C\$ rate falls to C\$1.30 and the importer "shouts" at the prevailing exchange rate

One month after:

1. Currency is trading at C\$1.34 (above C\$1.30): the importer gets exercise price − shout rate (or C\$1.35 − C\$1.30)

2. Currency is trading at C\$1.28 (below C\$1.30): the importer gets exercise price − asset price at maturity (or C\$1.35 − C\$1.28)

Payoff: The payoff of a shout call option can be written as

$$\max\{0, S_T - K\} \qquad \text{if not shouted}$$

$$\max\{S_T - K, S_\tau - K\} \qquad \text{if shouted at time } \tau \text{ when } S_\tau > K$$

Payoff (continued)

When the holder has shouted so $S_\tau$ is known, the payoff becomes $\max\{0, S_T - S_\tau\} + (S_\tau - K)$

1. Standard European call option with strike price equal to $S_\tau$

2. PLUS cash amount $S_\tau - K$

Valuation: Consistent with other American-style options, closed-form solutions are not available

We can use a binomial tree to value the shout option: at every node in the tree we set the value equal to the maximum of the discounted expectation and the value if shouted

Since the payoff from shouting can be decomposed exactly into a call and cash, the "shout value" is given by

$$C_{\text{std}}(S_\tau, S_\tau, T - \tau) + (S_\tau - K) = S_\tau \left[ e^{-q(T-\tau)} N(d_1) - e^{-r(T-\tau)} N(d_2) \right] + (S_\tau - K)$$

with $d_2 = \sigma^{-1}(r - q - \sigma^2/2)\sqrt{T - \tau}$ and $d_1 = d_2 + \sigma\sqrt{T - \tau}$

## Valuation (continued)



$n$-step tree

$$\Delta = T/n$$

$$u = e^{\sigma\sqrt{\Delta}}$$

$$d = 1/d$$
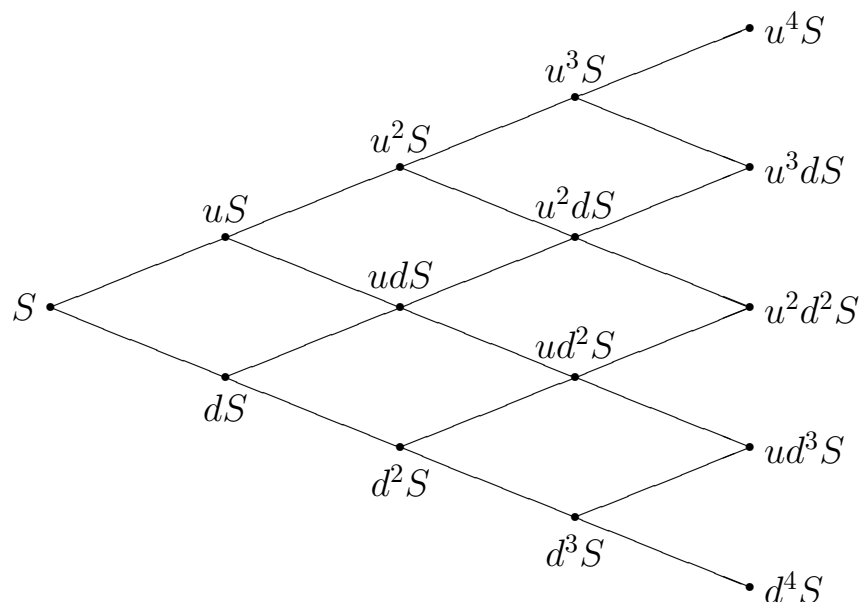
$$p = \frac{e^{r\Delta t} - d}{u - d}$$

1. At the terminal date evaluate the payof $\max\{0, S_T - K\}$ assuming no shouting

2. Work backwards through the tree to get the value at any node as

$$\max\{\text{shout value}, \ e^{-r\Delta}[p \times C_u + (1 - p) \times C_d]\}$$

where $C_u$ is the call value at the next up-node and $C_d$ is the call value at the next down-node

Hedging: We can obtain an approximation for the optimal shout level by locating the level at which it is first optimal to shout within a binomial lattice



**Optimal shout level**
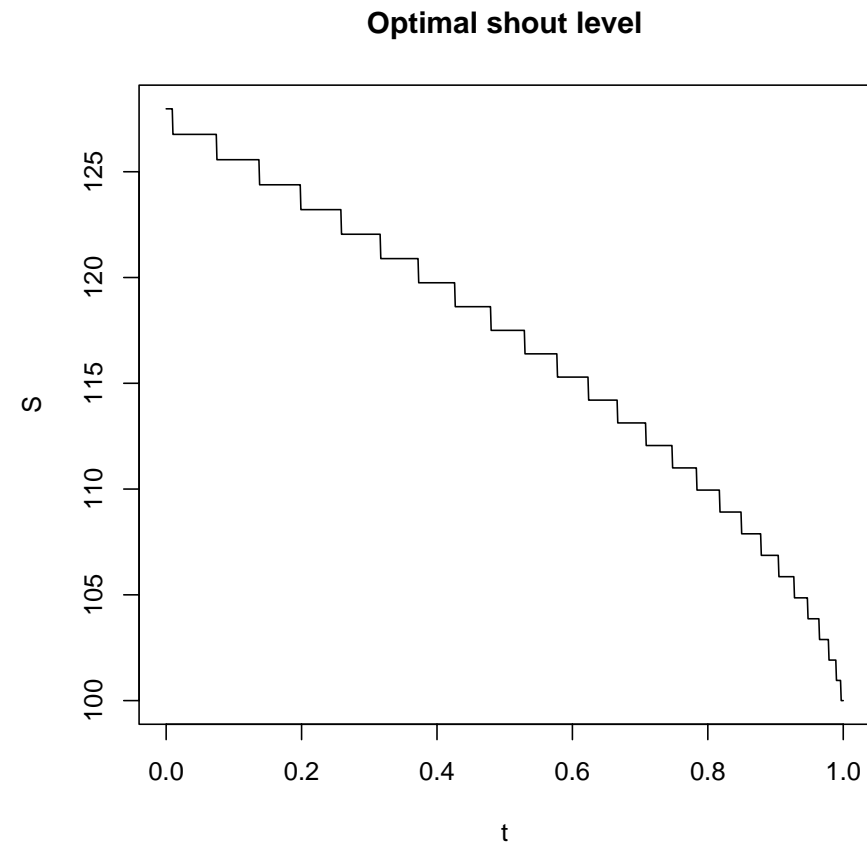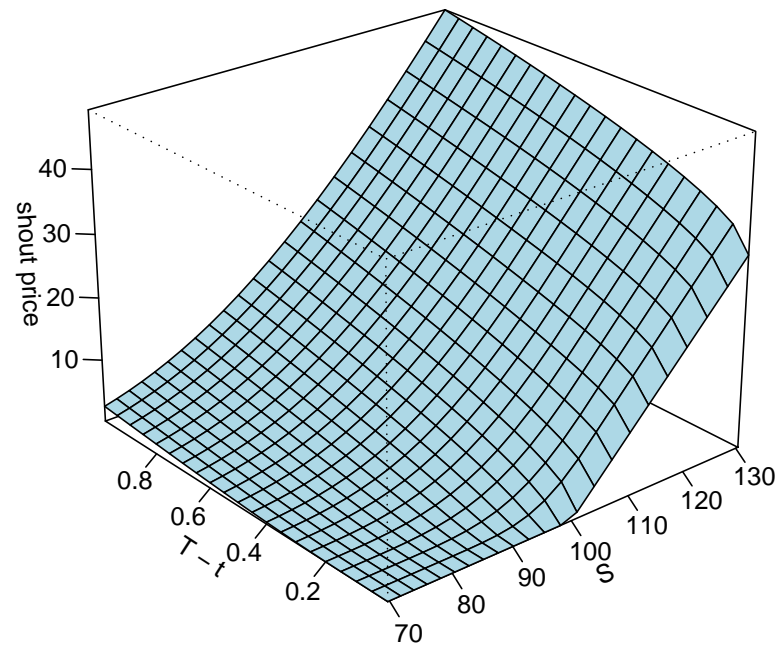
A shout call option is very similar to a European call option but with a higher delta

A related structure is the ladder option

The strike price of a ladder option is periodically reset (automatically) when the price of the underlying asset trades through prespecified price levels

The payoff from a European ladder call option is

$$\text{payoff}_T = \max\{0, S_T - K, L - K\}$$

where $L$ is the highest ladder level achieved over the life of the option

The pricing and hedging of the ladder option can be simplified by replication

1. Long one European call option with strike $K$ and maturity $T$

2. Long one European put spread with strikes $L_1 > K$ and $K$, and maturity $T$

3. Short one European knockout put spread with strikes $L_1$ and $K$, knockout level $L_1$, and maturity $T$

4. For each subsequent rung $L_i$, replace the spread strikes by $L_i$ and $L_{i-1}$, and knockout level by $L_i$

**Summary:** Shout options allow the purchaser to actively lock in the minimum payout of the option, while retaining the right to gain from further favorable movements in the price of the underlying asset

The major demand for shout options derives from its lower premium relative to competing structures, primarily lookback options

Such structures are attractive to users who believe that they are capable of accurately predicting future asset price movements in the market