3. Correlation Options

Emergence: Cross-market integration and globalization have increased the need to hedge cross-market and global positions

What Are They? Correlation options are options with payoffs affected by at least two underlying instruments

Complexity: Exchange options $\longrightarrow$ Outperformance options $\longrightarrow$ Spread options $\longrightarrow$ Basket options

Statistical Concepts: Let X and Y be random variables with (marginal) PDFs $f_X(x)$ and $f_Y(y)$, and joint PDF $f(x, y)$

1. Expectation of X: $\mu_X = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

2. Variance X: $\sigma_X^2 = \mathrm{Var}(X) = E\left\{\left[X - E(X)\right]^2\right\} = \int_{-\infty}^{\infty} \left[x - \mu_X\right]^2 f(x)\, dx$

Statistical Concepts (continued)

3. Covariance between X and Y:

$$\sigma_{XY} = \mathrm{Cov}(X, Y) = E\left\{\left[X - E(X)\right]\left[Y - E(Y)\right]\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[x - \mu_X\right]\left[y - \mu_Y\right] f(x, y)\, dx\, dy$$

4. Correlation coefficient between X and Y: $\rho = \dfrac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} = \dfrac{\sigma_{XY}}{\sigma_X \sigma_Y}$

In general, $\mathrm{Var}(aX + bY) = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2\rho\sigma_X\sigma_Y$ so

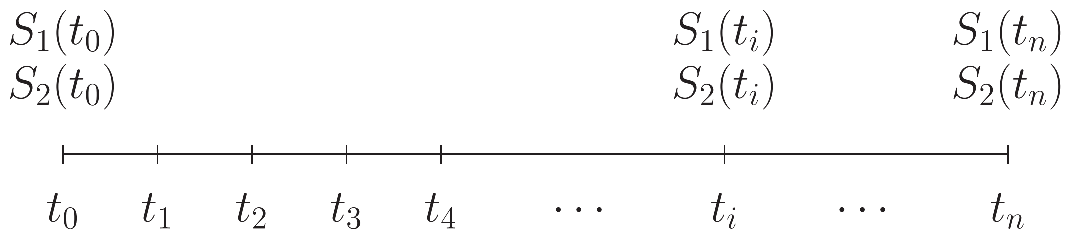
$$\mathrm{Var}(aX + bY) < a^2\sigma_X^2 + b^2\sigma_Y^2 \qquad \text{if } \rho < 0$$

Estimation of ρ: (1) Historical data (2) Implied ρ (3) Time-series analysis (e.g., GARCH model)

Caution: Treating ρ as constant may be misleading and can create serious problems in risk-taking, pricing, and hedging

Historical Data: Asset price is observed at fixed intervals of time (e.g., daily, weekly, monthly)

Define: $n+1$ = number of observations, $S_j(t_i)$ = jth asset price at end of ith interval, τ = length of time interval in years



Let $X(t_i) = \ln\left[S_1(t_i)/S_1(t_{i-1})\right]$ and $Y(t_i) = \ln\left[S_2(t_i)/S_2(t_{i-1})\right]$

Assume $X(t_i)$ and $Y(t_i)$ have zero mean

We estimate the standard deviation of X and of Y by

$$\hat{v}_X = \sqrt{\frac{1}{n}\sum_{i=1}^{n} X(t_i)^2} \qquad \text{and} \qquad \hat{v}_Y = \sqrt{\frac{1}{n}\sum_{i=1}^{n} Y(t_i)^2}$$

Historical Data (continued)

We estimate the volatility of S_1 and of S_2 by

$$\hat{\sigma}_1 = \frac{\hat{v}_X}{\sqrt{\tau}} \qquad \text{and} \qquad \hat{\sigma}_2^* = \frac{\hat{v}_Y}{\sqrt{\tau}}$$

We estimate the covariance between X and Y by

$$\hat{v}_{XY} = \frac{1}{n} \sum_{i=1}^{n} X(t_i) Y(t_i)$$

We estimate the correlation coefficient between X and Y (or between S_1 and S_2) by

$$\hat{\rho} = \frac{\hat{v}_{XY}}{\hat{v}_X \hat{v}_Y} = \frac{\hat{v}_{XY}}{\hat{\sigma}_1 \hat{\sigma}_2 \tau}$$

Generally, time should be measured by trading days so days when the exchange is closed should be ignored for volatility/correlation calculation

Price Processes: The two asset prices are assumed to be bivariate lognormally distributed

Specifically, the price processes S_1 and S_2 both follow geometric Brownian motion:

$$dS_i(t) = (\mu_i - q_i)S_i(t)\,dt + \sigma_i S_i(t)\,dB_i(t), \qquad i = 1, 2$$

where $B_1(t)$ and $B_2(t)$ are standard Brownian motions with correlation coefficient ρ

In a risk-neutral world, we set $\mu_1 = \mu_2 = r$:

$$S_i(t) = S_i \exp\left[\left(r - q_i - \frac{\sigma_i^2}{2}\right) t + \sigma_i B_i(t)\right], \qquad i = 1, 2$$

1. $X(t) = \ln[S_1(t)/S_1] \sim N(\mu_X, \sigma_X^2)$ with

$$\mu_X = (r - q_1 - \sigma_1^2/2)t \qquad \text{and} \qquad \sigma_X^2 = \sigma_1^2 t$$

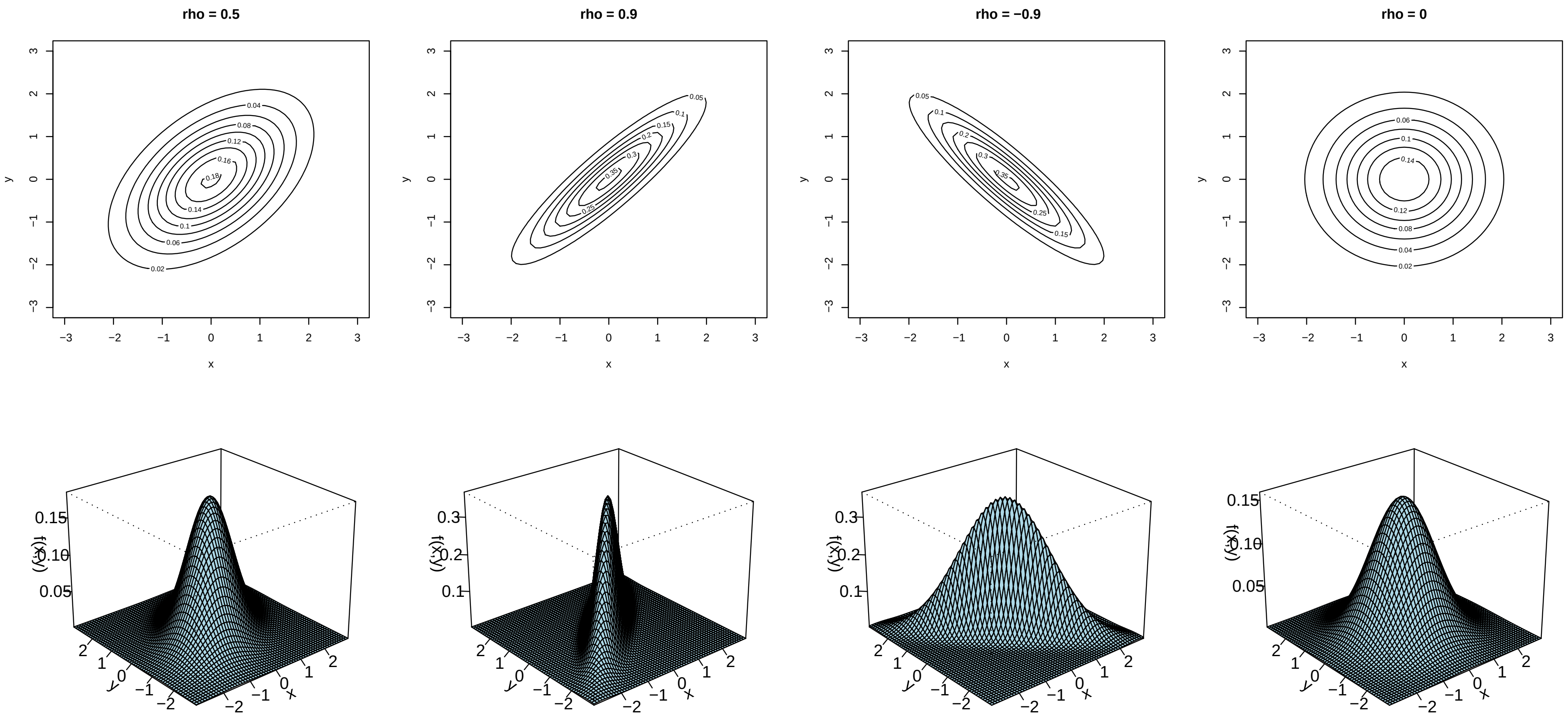
2. $Y(t) = \ln[S_2(t)/S_2] \sim N(\mu_Y, \sigma_Y^2)$ with

$$\mu_Y = (r - q_2 - \sigma_2^2/2)t \qquad \text{and} \qquad \sigma_Y^2 = \sigma_2^2 t$$

3. X and Y jointly normally distributed with correlation coefficient ρ

Bivariate Normal Distribution: The joint PDF is

$$f(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left[-\frac{u^2 - 2\rho uv + v^2}{2(1-\rho^2)}\right] \quad \text{where} \quad u = \frac{x-\mu_X}{\sigma_X} \quad \text{and} \quad v = \frac{y-\mu_Y}{\sigma_Y}$$



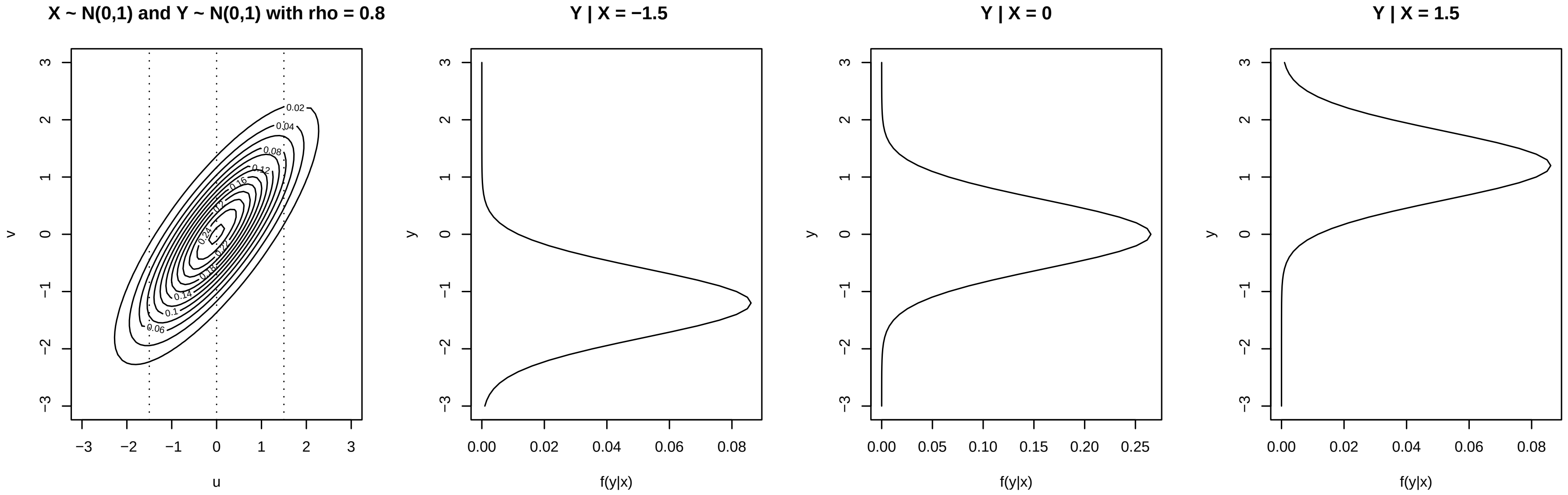
Contours of constant density are ellipses (not invariant under rotation about its center)

Conditional Distributions: Given that $Y = y$, the conditional distribution of X is again normal

$$X \mid Y = y \sim N(\mu_{X \mid y}, \sigma^2_{X \mid y}) \quad \text{where} \quad \mu_{X \mid y} = \mu_X + \frac{\rho \sigma_X}{\sigma_Y}(y - \mu_Y) \quad \text{and} \quad \sigma_{X \mid y} = \sigma_X \sqrt{1 - \rho^2}$$

Similarly

$$Y \mid X = x \sim N(\mu_{Y \mid x}, \sigma^2_{Y \mid x}) \quad \text{where} \quad \mu_{Y \mid x} = \mu_Y + \frac{\rho \sigma_Y}{\sigma_X}(x - \mu_X) \quad \text{and} \quad \sigma_{Y \mid x} = \sigma_Y \sqrt{1 - \rho^2}$$



Standardization: X and Y are jointly normally distributed with parameters $(\mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho)$

1a. $U = \dfrac{X - \mu_X}{\sigma_X} \sim N(0,1)$ and $V = \dfrac{Y - \mu_Y}{\sigma_Y} \sim N(0,1)$

1b. U and V are correlated since $\mathrm{Cov}(U, V) = \rho$

2a. $Z_1 = \dfrac{U + V}{\sqrt{2(1+\rho)}} \sim N(0,1)$ and $Z_2 = \dfrac{U - V}{\sqrt{2(1-\rho)}} \sim N(0,1)$

2b. Z_1 and Z_2 are uncorrelated since $\mathrm{Cov}(Z_1, Z_2) = 0$

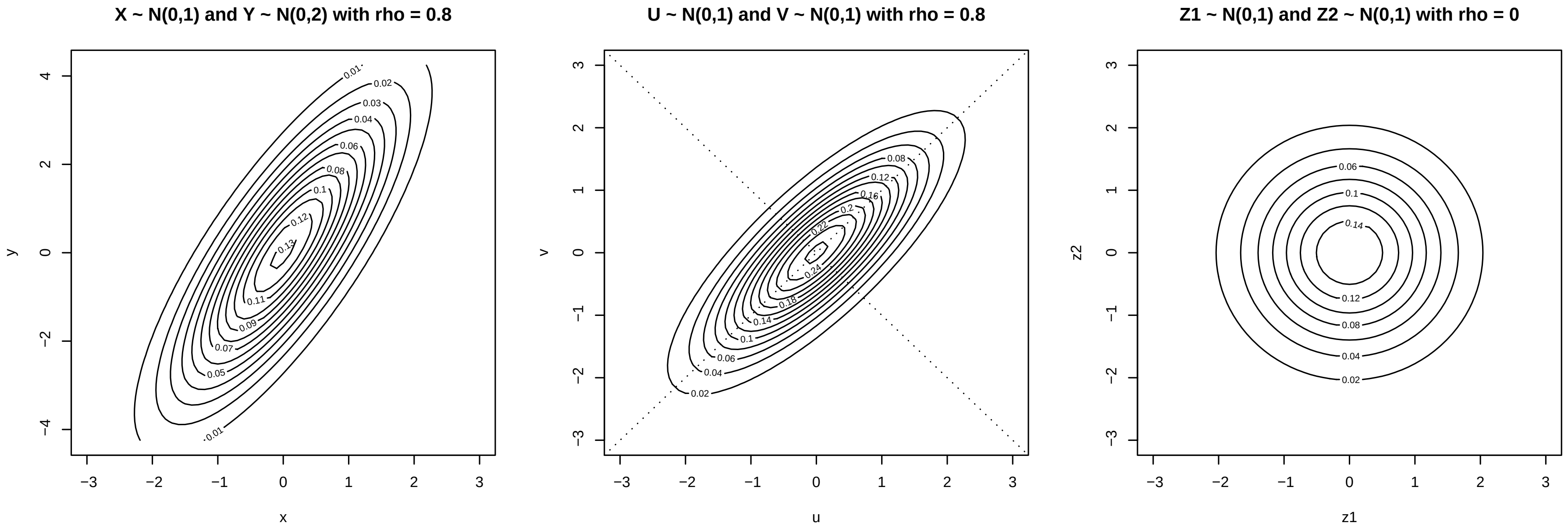
2c. Z_1 and Z_2 follow the bivariate standard normal distribution

The bivariate standard normal density is

$$n_2(z_1, z_2) = \frac{1}{2\pi} \exp\left[-\frac{z_1^2 + z_2^2}{2}\right] = n(z_1)\, n(z_2)$$

Contours of constant density are circles (invariant under rotation about its center at the origin)

Standardization (continued)



Geometrically:

1. Step 1 involved rescaling along the x and y axes so that marginal distributions of U and V are standard normal (no correction to the "tilt")

2. Step 2 involved adopting the principle axes of the ellipse as new coordinate system (thus correcting the "tilt") and another rescaling so that Z_1 and Z_2 are standard normal

Simulation of Pairs of Correlated Normal Variables: A recipe based on our previous discussion

1. Generate independent standard normal variables Z_1 and Z_2

2. Set $U = \frac{1}{2}\left[Z_1\sqrt{2(1+\rho)} + Z_2\sqrt{2(1-\rho)}\right]$ and $V = \frac{1}{2}\left[Z_1\sqrt{2(1+\rho)} - Z_2\sqrt{2(1-\rho)}\right]$

   $\Rightarrow$ U and V are standard normal variables with correlation ρ

3. Set $X = \mu_X + \sigma_X U$ and $Y = \mu_Y + \sigma_X V$

   $\Rightarrow$ $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$ with correlation ρ

Multivariate Normal Distribution: Concepts developed for the bivariate normal distrbution generalized naturally to the n-variate normal distribution

1. n means [mean vector]

2. n variances and $n(n-1)/2$ pairwise covariances/correlations [covariance/correlation matrix]

3. Contours of constant density are hyperellipsoids

Useful Property: Application of rotational invariance of the bivariate standard normal distribution

If Z_1 and Z_2 follow a bivariate standard normal distribution, then

$$P\{Z_2 < a + bZ_1\} = \int_{-\infty}^{\infty} \int_{-\infty}^{a+bz_1} n_2(z_1, z_2)\, dz_2\, dz_1 = N\left(\frac{a}{\sqrt{1+b^2}}\right)$$

3-1. Exchange Options

Motivating Example: Exchange options embedded in DIB notes

Redemption value $=$ gearing $\times \min\{\text{index}_1, \text{index}_2\} =$ gearing $\times \left[\text{index}_1 - \max\{0, \text{index}_1 - \text{index}_2\}\right]$

Embedded in the DIB note is the sale by the investor of an option to exchange index_1 for index_2

The option premium is used to increase the gearing of the note so the DIB note will pay in excess of 100% of the upside of the minimum of the two indices

Example: An investor with a bullish view on both oil and gold can purchase a note with a redemption value of 107% ("gearing") of the minimum value of 44,444 oz gold and 1,075,350 barrels oil

For an investor who is bullish on both indices, this structure offers a higher expected return than a portfolio in the two indices

Payoff: An exchange option offers its purchaser the right to exchange one asset for another

In the case of an option to exchange the first asset for the second

$$\text{payoff}_T = \max\{0, S_2(T) - S_1(T)\}$$

The payoff is nonzero whenever $S_2 e^{Y(T)} > S_1 e^{X(T)}$, or equivalently, $Y(T) > X(T) - \ln(S_2/S_1)$

Valuation: By risk-neutral valuation, we can show that

$$C = e^{-r\tau} E[\text{payoff}_T] = S_2 e^{-q_2\tau} N(d_1) - S_1 e^{-q_1\tau} N(d_2)$$

$$\text{with} \;\; d_1 = \frac{\ln(S_2/S_1) + (q_1 - q_2 + \sigma_a^2/2)\tau}{\sigma_a\sqrt{\tau}}, \quad d_2 = d_1 - \sigma_a\sqrt{\tau} \;\; \text{and} \;\; \sigma_a^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2$$

Alernatively, one can derive a PDE satisfied by the option value at time t and solve it subject to the initial condition $C(T) = \max\{0, S_2(T) - S_1(T)\}$

For $q_1 = q_2 = 0$, the PDE is $\dfrac{\partial C}{\partial t} + \dfrac{1}{2}\left[\sigma_1^2 S_1^2 \dfrac{\partial^2 C}{\partial S_1^2} + 2\rho\sigma_1\sigma_2 S_1 S_2 \dfrac{\partial^2 C}{\partial S_1 \partial S_2} + \sigma_2^2 S_2^2 \dfrac{\partial^2 C}{\partial S_2^2}\right] = 0$

Steps in Risk-Neutral Valuation: Based on the bivariate normal distribution

1. Express $X = X(T)$ and $Y = Y(T)$ in terms of uncorrelated standard normal variables Z_1 and Z_2

$$X = \mu_X + Z_1 \sigma_X \sqrt{\frac{1+\rho}{2}} + Z_2 \sigma_X \sqrt{\frac{1-\rho}{2}}$$

$$Y = \mu_Y + Z_1 \sigma_Y \sqrt{\frac{1+\rho}{2}} - Z_2 \sigma_Y \sqrt{\frac{1-\rho}{2}}$$

2. Express the integration domain $Y(t) > X(t) - \ln(S_2/S_1)$ in the form $Z_2 < A + BZ_1$

3. Complete the squares in the integrand (on the exponentials)

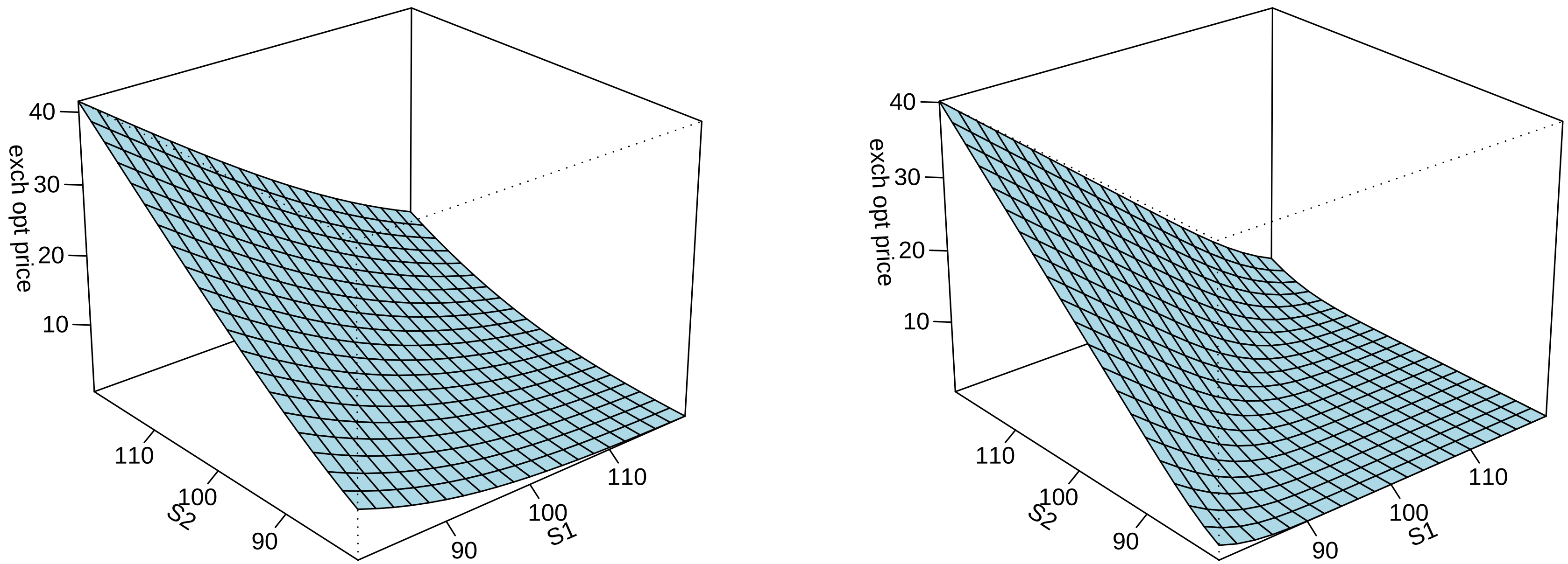
4. Express the integrand in terms of standard normal densities through a change of variable

5. Use rotational invariance of the bivariate standard normal distribution to evaluate the integral

Change of Numeraire: If prices are quoted in units of asset 1, the price of an exchange option is given by the Black-Scholes formula with $S = S_2/S_1$, $K = 1$, $r = q_1$, $q = q_2$, and $\sigma = \sigma_a$

$$C/S_1 = (S_2/S_1)e^{-q_2\tau}N(d_1) - e^{-q_1\tau}N(d_2)$$

An exchange option is a call option on asset 2 with a strike price equal to the future value of asset 1



Example: $q_1 = 0.08$, $q_2 = 0.04$, $\sigma_1 = 0.25$, $\sigma_2 = 0.1$, $\rho = 0.8$ and $T = 1$ [left panel] or 0.1 [right panel]

Static Hedging: The payoff of an exchange option can be rewritten in one of two ways

1. $\max\{0, S_2(T) - S_1(T)\} = \max\{S_1(T), S_2(T)\} - S_1(T)$ suggests a static portfolio which is long a better-of-two-assets option and short asset 1

2. $\max\{0, S_2(T) - S_1(T)\} = S_2(T) - \min\{S_1(T), S_2(T)\}$ suggest a static portfolio which is long asset 2 and short a worse-of-two-assets option

Dynamic hedging: Where the components are not available, we can hedge a short exchange option position with the purchase of Δ_1 units of asset 1 and Δ_2 units of asset 2

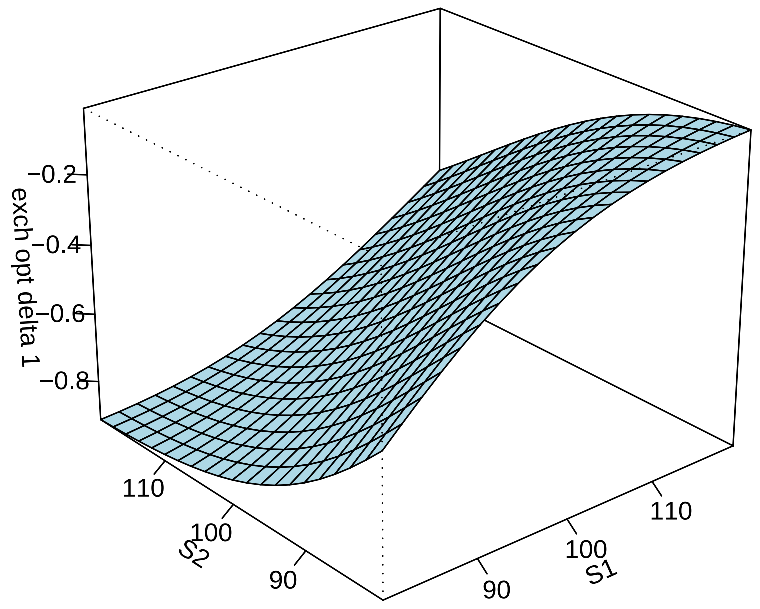
This portfolio is delta-neutral with respect to both asset 1 and asset 2

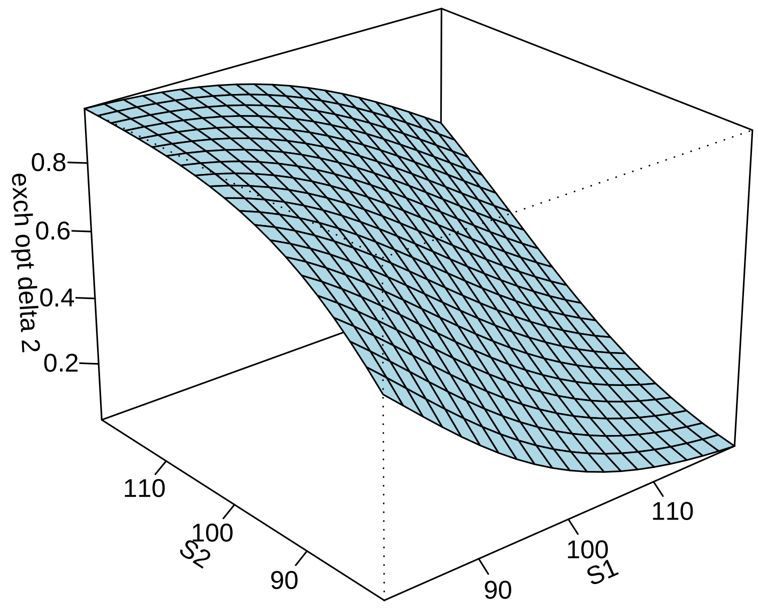
The partial deltas are

$$\Delta_1 = \frac{\partial C}{\partial S_1} = -e^{-q_1\tau}N(d_2) \qquad \text{and} \qquad \Delta_2 = \frac{\partial C}{\partial S_2} = e^{-q_2\tau}N(d_1)$$

Note that $-e^{-q_1\tau} < \Delta_1 < 0$ and $0 < \Delta_2 < e^{-q_2\tau}$ (cf. vanilla options)

Dynamic Hedging (continued)





We use the gammas to access the sensitivity of the partial deltas with respect to S_1 and S_2

1. $\Gamma_{11} = \dfrac{\partial \Delta_1}{\partial S_1} = \dfrac{e^{-q_1\tau} n(d_2)}{S_1 \sigma_a \sqrt{\tau}}$ and $\Gamma_{22} = \dfrac{\partial \Delta_2}{\partial S_2} = \dfrac{e^{-q_2\tau} n(d_1)}{S_2 \sigma_a \sqrt{\tau}}$ [both positive]

2. $\Gamma_{12} = \dfrac{\partial \Delta_1}{\partial S_2} = -\dfrac{e^{-q_1\tau} n(d_2)}{S_2 \sigma_a \sqrt{\tau}} = -\dfrac{e^{-q_2\tau} n(d_1)}{S_1 \sigma_a \sqrt{\tau}} < 0$ [particularly useful]

Price Sensitivity to Correlation: A higher ρ reduces the aggregate volatility σ_a and tends to lower the option premium

$$\frac{\partial C}{\partial \rho} = -S_2 e^{-q_2\tau} n(d_1) \frac{\sigma_1 \sigma_2 \sqrt{\tau}}{\sigma_a} < 0$$

Variations: The payoff profile of the basic exchange option can be modified

1. Different weights on the assets: pricing formula "easy" to obtain by discounting the expected payoff

$$\text{payoff}_T = \max\{0, a_2 S_2(T) - a_1 S_1(T)\} \qquad \text{with} a_1, a2 > 0$$

2. Introduce a strike: gives rise to spread options

$$\text{payoff}_T = \max\left\{0, \left[a_2 S_2(T) - a_1 S_1(T)\right] - K\right\}$$

3. Allow the exchange of multiassets: e.g., swap one of two assets A or B for one for two assets C or D

$$\text{payoff}_T = \max\left\{0, \max\{C, D\} - \min\{A, B\}\right\}$$

Applications: Some examples from Margrabe (1978)

(1) Performance incentive fee (2) Margin account (3) Exchange offer (4) Standby commitment

Exmaple: An adviser receives a performance incentive fee $R_m - R_s$ multiplied by a fixed percentage of the total managed portfolio, where $R_m =$ return of managed portfolio and $R_s =$ return of standard portfolio against which performance is measured

If the adviser has the protection of limited liability in case the fee became negative, the portfolio management fee equates to the value of an exchange option

Summary: Exchange options are the basic correlation options which can be used to analyze many other correlation options

Their simple payoff patterns allow the option prices to be expressed in terms of univariate normal CDFs

To a certain degree, exchange options have been superceded by more complex multifactor option structures such as spread options, which have similar characteristics

3-2. Outperformance Options

Motivating Example: Outperformance options as a means of maintaining competitive position

A sterling based manufacturer with dollar based and DM based competitors would be influenced by these currency movements

1. An appreciation in the pound relative to the dollar
2. A depreciation of the DM relative to the dollar

The manufacturer's exposure, while connected to both rates, is only with reference to the currency that depreciates by the higher amount against the pound

An outperformance option is a relatively cheap strategy for matching the underlying exposure

Gain from the option will offset potential losses in revenue resulting from loss of competitiveness

Motivating Example (continued)

Compare

1. A pound call/dollar put plus a pound call/DM put: Premium is 5.58% of the pound amount (£/\$: 4.08% plus £/DM: 1.50%)

2. An outperformance option to buy pounds and sell dollar or DM, whichever is cheaper: Premium is 4.61% of the pound amount

Payoff: Simplest outperformance option pays better of two assets

$$\text{payoff}_T = \max\{S_1(T), S_2(T)\}$$

We can decompose this payoff into two parts

1. Payoff of asset 1: $S_1(T)$

2. Payoff of an option to exchange asset 2 for asset 1: $\max\{0, S_2(T) - S_1(T)\}$

Valuation: From the decomposition of the option payoff, it is easy to price a better-of-two-assets option

$$C = S_1 e^{-q_1 \tau} + C_{\text{exch}} = S_2 e^{-q_2 \tau} N(d_1) + S_1 e^{-q_1 \tau} N(-d_2)$$

Static Hedging: The sale of a better-of-two-assets option is to hold one of the following portfolios

1. A unit of asset 1 and an option to exchange asset 2 for asset 1

2. A unit of asset 2 and an option to exchange asset 1 for asset 2

Dynamic Hedging: Alternatively, we form a dynamic delta-neutral portfolio consisting of Δ_1 units of asset 1 and Δ_2 units of asset 2

$$\Delta_1 = e^{-q_1 \tau} N(-d_2) \quad \text{and} \quad \Delta_2 = e^{-q_2 \tau} N(d_1)$$

In this instance $0 < \Delta_1 < 1$ and $0 < \Delta_2 < 1$

Price Sensitivity to ρ: Similar to exchange option

Variations: The payoff profile of the basic outperformance option can be modified

1. Worse-of-two-assets (underperformance) options: can also be priced analytically

$$\text{payoff}_T = \min\{S_1(T), S_2(T)\}$$

2. Compare against cash: e.g., options that pay best or worst of two assets and cash

$$\text{payoff}_T = \max\{S_1(T), S_2(T), K\} \quad \text{or} \quad \min\{S_1(T), S_2(T), K\}$$

3. Extend to more than two assets

4. Introduce call or put features: e.g., call options on the maximum of several assets

$$\text{payoff}_T = \max\left\{0, \max\{S_1(T), \ldots, S_n(T)\} - K\right\}$$

With the exception of underperformance options, such additional features make pricing more complex

We rely on numerical techniques: binomial pyramid, Monte Carlo simulation, numerical integration

Summary: By having payouts that depend on the best or worst performance of two or more assets, outperformance options can be used to take advantage of investors' perception of the relative performance of two or more underlying assets

3-3. Spread Options

Motivating Example: A put option on the yield spread between the two-year Treasury note and the 30-year Treasury bond has these terms

1. Notional amount: US$1 million
2. Strike spread level: 150 basis points
3. Type of exercise right: European
4. Expiration date: one month (15 September 1991)

At the end of the month, the put option will expire

1. In the money where the yield spread is < 150 basis points
2. Out of the money where the yield spread is > 150 basis points

Example: If the option premium is 12 basis points and the yield curve spread expires at 130 (< 150) basis points, the return to the investor would be 8 basis points (or $80,000 at $10,000/bp)

Motivating Example (continued)

In general, spread options can be utilized to capture the price differential between commodities that are closely related

1. Demand substitution: potentially related bonds or indices
2. Transformation potential: refined/unrefined energy products

Payoff: For positive constants a_1 and a_2, the payoffs of spread options are

1. Spread call option: $\max\left\{0, \left[a_2 S_2(T) - a_1 S_1(T)\right] - K\right\}$
2. Spread put option: $\max\left\{0, K - \left[a_2 S_2(T) - a_1 S_1(T)\right]\right\}$

Valuation: Traditionally, one would model the spread itself as an asset, lognormally distributed, and apply the usual Black-Scholes formula

This one-factor approach, due to its many deficiencies, has been abandoned in favor of two-factor approaches

Valuation (continued)

Risk-neutral valuation does not lead to analytic pricing formulas

$$C = e^{-r\tau} E\left[\max\left\{0, \left[a_2 S_2(T) - a_1 S_1(T)\right] - K\right\}\right]$$

A natural alternative is Monte Carlo simulation

Monte Carlo simulation: We will estimate the expectation in the risk-neutral pricing formula by averaging the payoff over numerous simulation runs

1. Generate a realization (x, y) from the bivariate normal distribution with parameters $(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$
2. Compute the payoff: $\max\left\{0, \left[a_2 S_2 e^y - a_1 S_1 e^x\right] - K\right\}$
3. Repeat Steps 1–2 to obtain a sequence of simulated payoffs: $\text{payoff}_1, \ldots, \text{payoff}_N$
4. Estimate C by $\hat{C} = e^{-r\tau}(\text{payoff}_1 + \cdots + \text{payoff}_N)/N$

Numerical Integration: Note that the risk-neutral valuation formula can be written in the form

$$C = e^{-r\tau} E_2 \left\{ E_1 \left[\max\left\{0, \left[a_2 S_2(T) - a_1 S_1(T)\right] - K\right\} \Big| S_2(T) \right] \right\}$$

Here E_i means taking expections with respect to the distribution of $S_i(T)$ ($i = 1, 2$)

Since the conditional distribution of $X \,|\, Y$ is normal for $X = \ln(S_1(T)/S_1)$ and $Y = \ln(S_2(T)/S_2)$, the conditional expectation can be evaluated exactly [see Ravindran (1993) eq. (6)]

The remaining expectation $E_2\{$closed-form formula$\}$ can be approximately calculated by using a numerical integration scheme

Hedging: The spread option can be hedged dynamically with the partial deltas calculated numerically

$$\hat{\Delta}_1 = \frac{\hat{C}(S_1 + dS_1, S_2) - \hat{C}(S_1, S_2)}{dS_1} \quad \text{and} \quad \hat{\Delta}_2 = \frac{\hat{C}(S_1, S_2 + dS_2) - \hat{C}(S_1, S_2)}{dS_2}$$

Here $\hat{C}(S_1, S_2)$ denotes the Monte Carlo estimate of the spread option price for spot prices (S_1, S_2)

Note: Spread options may have negative vegas so lower volatilities can produce higher option prices

Variations: Spread option structures can be set up to have multiasset payoffs

Summary: Spread option products represent an innovative means for capturing the price or rate differential between two markets or financial market variables

Interest rate spread options represent a significant component of this market

Asset and liability managers can use spread options to

1. Hedge underlying spread exposures
2. Take positions on anticipated changes in forward yield spreads relative to those already implied in the yield curve

3-4. Basket Options

Motivating Example: Basket option as an effective means of hedging currency positions

A dollar-based treasury has the following currency positions

Position		Spot	Forward
DM	50 million	1.6900	1.7054
¥	3 billion	101.00	100.92
FFr	120 million	5.9500	6.0194
SFr	45 million	1.5000	1.5053
£	25 million	US$1.4900	US$1.4805
Lit	48 trillion	1600.00	1624.54
NLG	60 million	1.9000	1.9154
A$	30 million	US$0.6775	US$0.6728

Portfolio value = US$226.91 million at current forward rates

Motivating Example (continued)

1. Basket option: premium 1.94% = US$4.402 million

   Guarantee: Minimum value of the portfolio after three months will be US$226.91 million less the premium, or US$222.508 million

2. Individual options: total premium US$5.574 million

The major benefit of basket options relates to its lower premium

When low-correlation currencies are taken out of the portfolio, the price differential between the basket option and a series of normal options diminishes

Now it is less likely that one or more currency components will move out of step with other positions

Payoff: For positive constants a_1 and a_2 with $a_1 + a_2 = 1$, we define a basket on two underlying assets by $S_b(t) = a_1 S_1(t) + a_2 S_2(t)$

The payoffs of basket options are

1. Basket call option: $\max\{0, S_b(T) - K\}$
2. Basket put option: $\max\{0, K - S_b(T)\}$

Valuation: Risk-neutral valuation does not lead to analytic pricing formulas

$$C = e^{-r\tau} E\left[\max\{0, S_b(T) - K\}\right]$$

We can again proceed by way of Monte Carlo simulation

As an alternative, the binomial pyramid (i.e., three-dimensional binomial tree) approach can be taken

Binomial Pyramid: From a node (S_1, S_2), we have a 0.25 chance of moving to each of the following

$$(S_1u, S_2A) \quad (S_1u, S_2B) \quad (S_1d, S_2C) \quad (S_1d, S_2D)$$

We use these values of $(u, d; A, B, C, D)$

$$u = \exp\left[(r - q_1 - \sigma_1^2/2)\Delta t + \sigma_1\sqrt{\Delta t}\right]$$

$$d = \exp\left[(r - q_1 - \sigma_1^2/2)\Delta t - \sigma_1\sqrt{\Delta t}\right]$$

$$A = \exp\left[(r - q_2 - \sigma_2^2/2)\Delta t + \sigma_2(\rho + \sqrt{1-\rho^2})\sqrt{\Delta t}\right]$$

$$B = \exp\left[(r - q_2 - \sigma_2^2/2)\Delta t + \sigma_2(\rho - \sqrt{1-\rho^2})\sqrt{\Delta t}\right]$$

$$C = \exp\left[(r - q_2 - \sigma_2^2/2)\Delta t + \sigma_2(\rho - \sqrt{1-\rho^2})\sqrt{\Delta t}\right]$$

$$D = \exp\left[(r - q_2 - \sigma_2^2/2)\Delta t + \sigma_2(\rho + \sqrt{1-\rho^2})\sqrt{\Delta t}\right]$$

Note that when $\rho = 0$, this method is equivalent to constructing separate trees for S_1 and S_2

Approximate Distribution Techniques: The main difficulty with basket options is that the distribution of a sum of correlated lognormal variables is not lognormal

One approach is to approximate this "true" distribution with a lognormal distribution whose first two moments (i.e., mean and variance) matches those of the unknown distribution

Under this approximation, the Black-Scholes formula can essentially be used

A more careful matching of the lognormal distribution to the unknown distribution is available via the Edgeworth-series expansion

Hedging: The spread option can be hedged dynamically with the partial deltas calculated numerically

Variations: In general, a basket is defined by

$$S_b(t) = \sum_{i=1}^{n} a_i S_i(t) \qquad \text{where} \quad a_i > 0 \quad \text{with} \quad \sum_{i=1}^{n} a_i = 1$$

Variations (continued)

The n underlying asset price processes S_i are assumed to follow the geometric Brownian motion

1. $X_i(t) = \ln[S_i(t)/S_i] \sim N(\mu_{X_i}, \sigma_{X_i}^2)$ with $\mu_{X_i} = (r - q_i - \sigma_i^2/2)t$ and $\sigma_{X_i}^2 = \sigma_i^2 t$
2. X_i and X_j are jointly normally distributed with correlation coefficient ρ_{ij} $(i, j = 1, \ldots, n)$

To value this multi-color basket option, we can use Monte Carlo simulation (replacing Step 1–2 before)

1'. Generate a realization vector $(x_1, \ldots, x_n)$ from the n-variate normal distribution with parameters $(\mu_1, \ldots, \mu_n)$, $(\sigma_1^2, \ldots, \sigma_n^2)$, $(\rho_{ij})_{i,j=1,\ldots,n}$

2'. Compute payoff: $\max\left\{0, \left[a_1 S_1 e^{x_1} + \cdots + a_n S_n e^{x_n}\right] - K\right\}$

Summary: The principal rationale of basket options is the use of correlation among basket components to reduce the option premium

The lower cost of the option premium allows portfolio managers (both asset and liability) to utilize these products for the management of exposures