2-1. Digital Options

Motivating Example: Digital options as a cost-effective means of taking proprietary positions

Overnight rate: 3%

One-month forward on overnight rate: 3.50%

Position: overnight rate will stay down at 3%

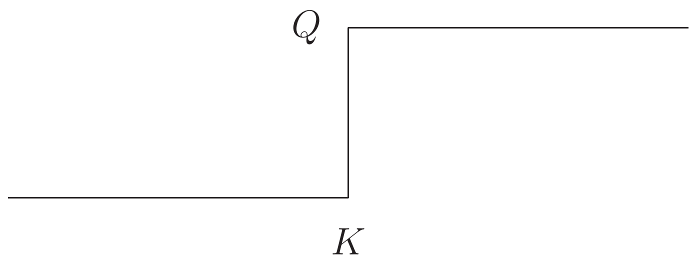
Strategy: low premium, high payout

1. Engage in a short forward transaction: zero cost but potentially unlimited loss if position is wrong
2. Buy an at-the-money put: pay a premium of 14.20 basis points to potentially make 50
3. Buy an out-of-the-money put (e.g., struck at 3.05%): pay a premium of 1.35 basis points to potentially make 5
4. Buy a digital put (e.g., struck at 3.05%): pay a premium of 4.85 to potentially make 50

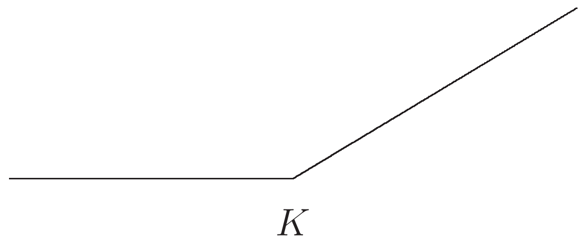
Why Go Digital? For the writer, the option means a known and limited loss in the event of exercise

For the buyer, the option payoff can be related to a fundamental amount of the underlying hedging transaction

Payoff: The simplest digital call (resp. put) option pays nothing when $S_T \le K$ (resp. $S_T \ge K$) or pays a predetermined constant amount Q when $S_T > K$ (resp. $S_T < K$)



cf.



Mathematically it is convenient to express the payoff as an indicator function

Let $\mathbb{I}_A = 1$ if the event A occurs, 0 otherwise

For the call, payoff $= Q\,\mathbb{I}_{\{S_T > K\}}$; for the put, payoff $= Q\,\mathbb{I}_{\{S_T < K\}}$

Valuation: By risk-neutral valuation, value of digital call $= e^{-r\tau}E(\text{payoff}) = Qe^{-r\tau}P\{S_T > K\}$

This is essentially the present value of area under the lognormal density to the right of the strike price

$\ln(S_T/S)$ is normally distributed with mean $(r - q - \sigma^2/2)\tau$ and standard deviation $\sigma\sqrt{\tau}$ so

$$Z = \frac{\ln(S_T/S) - (r - q - \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \sim N(0,1)$$

Thus, $P\{S_T > K\} = P\left\{Z > \dfrac{\ln(K/S) - (r - q - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right\}$

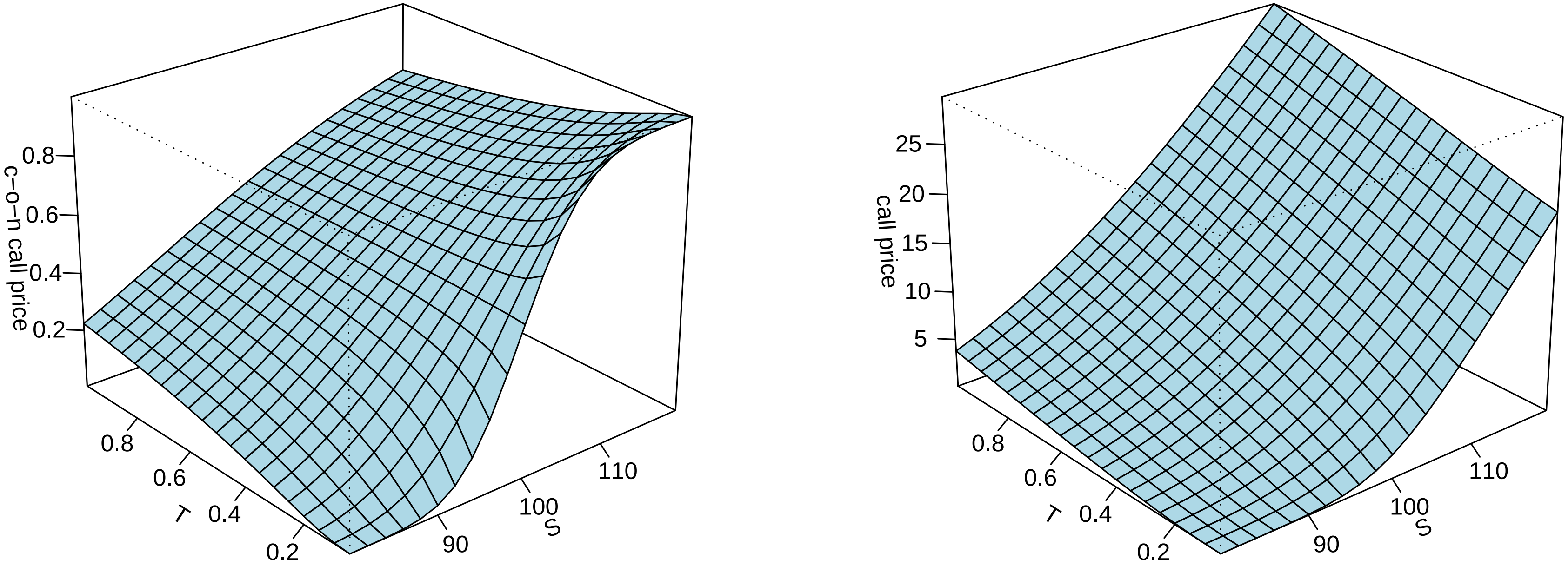
Finally, observe that $P\{Z > z\} = P\{Z < -z\} = N(-z)$ to get

$$C = Qe^{-r\tau}N(d_2) \qquad \text{where} \qquad d_2 = \frac{\ln(S/K) + (r - q - \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

Example: Digital call with $Q = \$100$, $S = \$480$, $K = \$500$, $\tau = 0.5$, $r = 0.08$, $q = 0.03$, $\sigma = 0.2$

$$d_2 = -0.1826 \quad \Rightarrow \quad N(d_2) = 0.4276 \quad \Rightarrow \quad C = 100 \times 0.4108 = \$41.08$$

Valuation (continued)



Moneyness: As S increases, C increases for digital call [same as standard call]

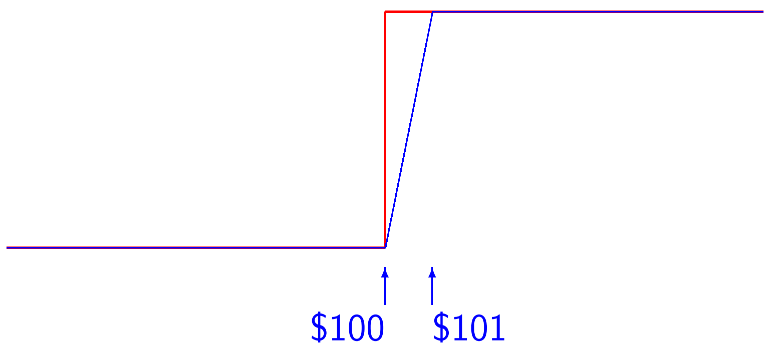
Time to expiration: As T decreases, C decreases/remains constant/increases for OTM/ATM/ITM digital call [cf. standard call: $C_{\text{std}} \downarrow$ as $T \downarrow$]

Static Hedging: The idea underlying this hedging strategy is the bull spread

For example, compare the following structures

1. A digital call struck at $100 that pays $1

2. A portfolio that is long a call struck at $100 and short a call struck at $101 (bull spread)



The payoff of the bull spread can be made arbitrarily close to the digital option payoff by moving the strike of the short call closer to the strike of the long call (at the same time increasing the quantity of options constituting the bull spread)

Static Hedging (continued)

In general, the static hedge for a digital call option struck at K that pays \$1 consists of a bull spread long $1/\varepsilon$ calls struck at K and short $1/\varepsilon$ calls struck at $K+\varepsilon$

The static hedge is not perfect: we can find ε so that the hedge is correct within a certain probability

Choose ε so that $P(K < S < K+\varepsilon) < 1-\alpha$, where α is the desired probability of being hedged

The following directive for exotic options traders sums it up:

> *"Digital calls are hedged with a bull spread with a 20-tick difference, unless volatility is lower than 10%, in which case we move to a 10-tick difference."*

Dynamic Hedging: Digital options can also be hedged dynamically by trading in the underlying asset and cash

The delta and gamma of a digital call option are (for $Q = 1$):

$$\Delta = \frac{e^{-r\tau} n(d_2)}{S\sigma\sqrt{\tau}} \quad \text{and} \quad \Gamma = -\frac{e^{-r\tau} d_1 n(d_2)}{S^2\sigma^2\tau} \quad \text{with} \quad d_1 = d_2 + \sigma\sqrt{\tau}$$

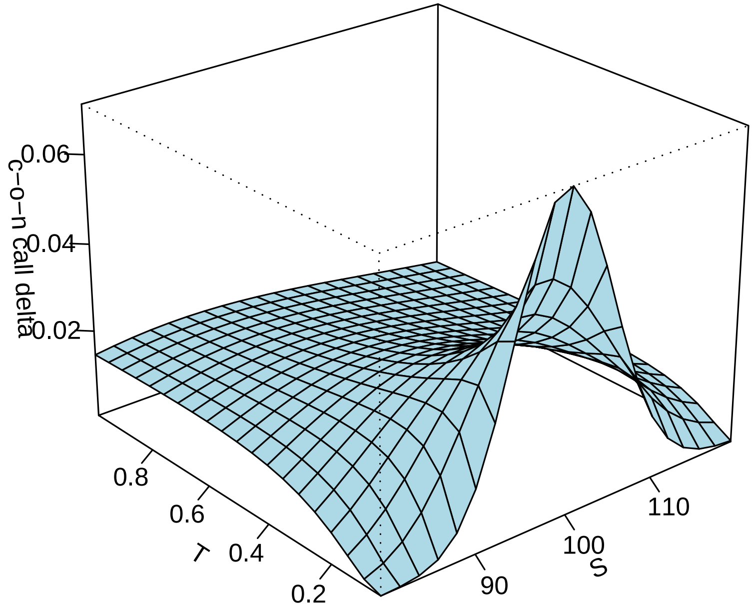
A delta-neutral portfolio would consist of a long position in one unit digital option and a short position in Δ units underlying asset, or vice versa

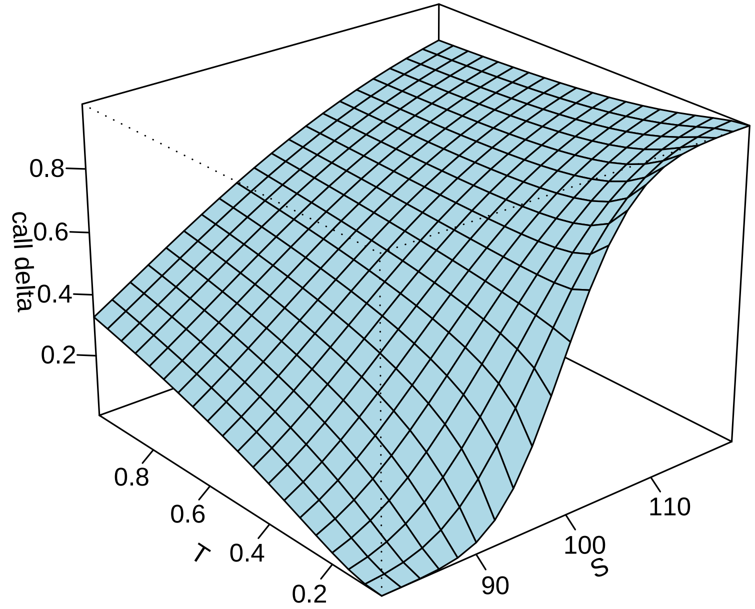
Because the digital option payoff is discontinuous at the strike, digital options are difficult to hedge near expiration

Around the strike, small moves in the underlying asset price can have very large effects on the value of the option: the absolute value of delta can be large close to maturity

The delta may exhibit violent changes as the underlying price changes when the option is close to maturity: the digital option is a high gamma instrument

Delta of Cash-or-Nothing Option:

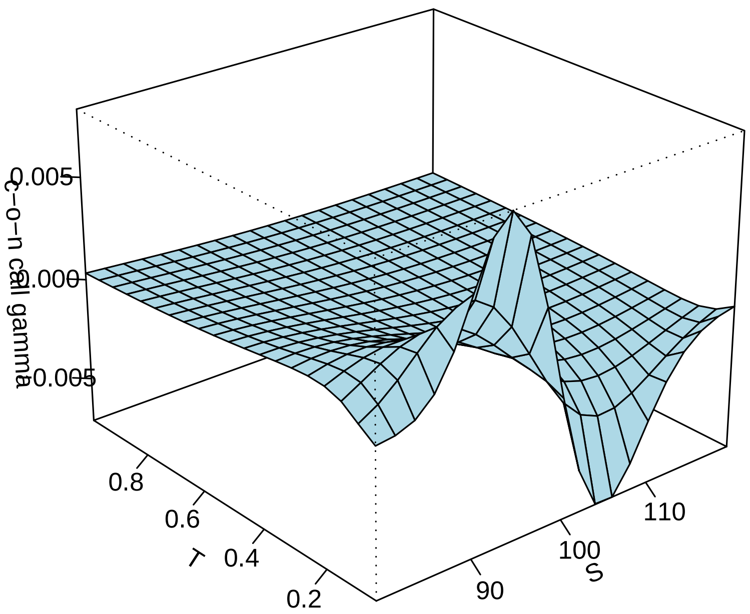


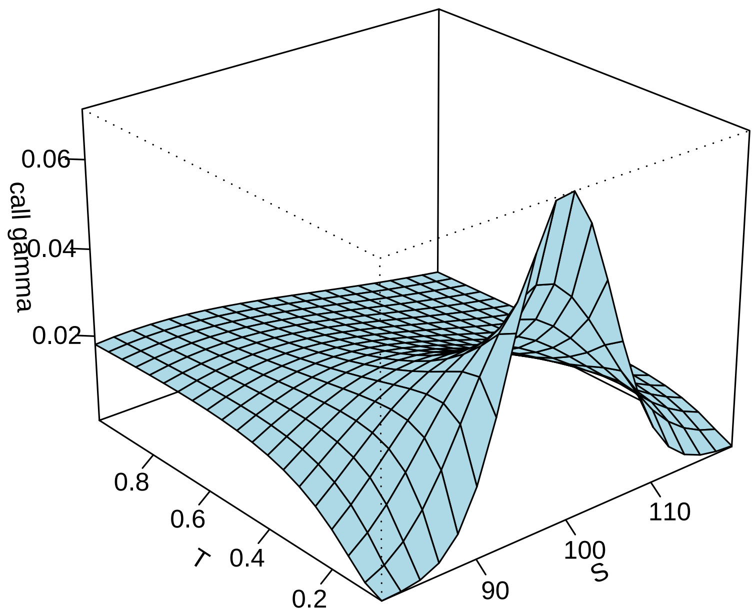


Δ for digital call is unbounded [cf. standard call: $0 \leq \Delta_{\text{std}} \leq e^{-q\tau}$]

Δ for digital call is fairly constant except near expiration: Δ increases rapidly as T decreases for around-the-money digital call but drops to zero for OTM/ITM digital call [cf. standard call: $\Delta_{\text{std}} \downarrow / \uparrow$ for OTM/ITM option]

Gamma of Cash-or-Nothing Option:



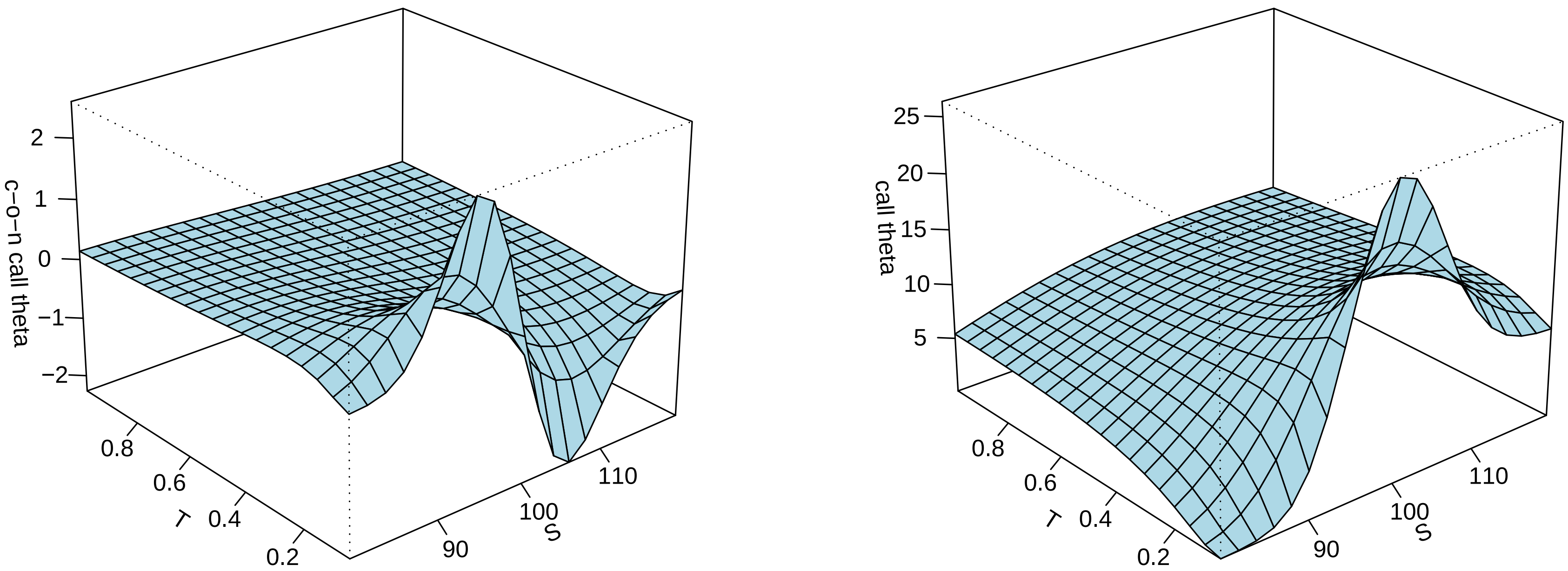


Γ for digital call can be negative or positive [cf. standard call: $\Gamma_{\text{std}} > 0$]

Γ for digial call is essentially zero except near expiration [cf. standard call: Γ_{std} constant but nonzero]

Digital options are [even more] difficult to hedge [than standard options] near expiration around the money due to large $|\Gamma|$ and Γ changing signs as S passes through strike K

Theta of Cash-or-Nothing Option:

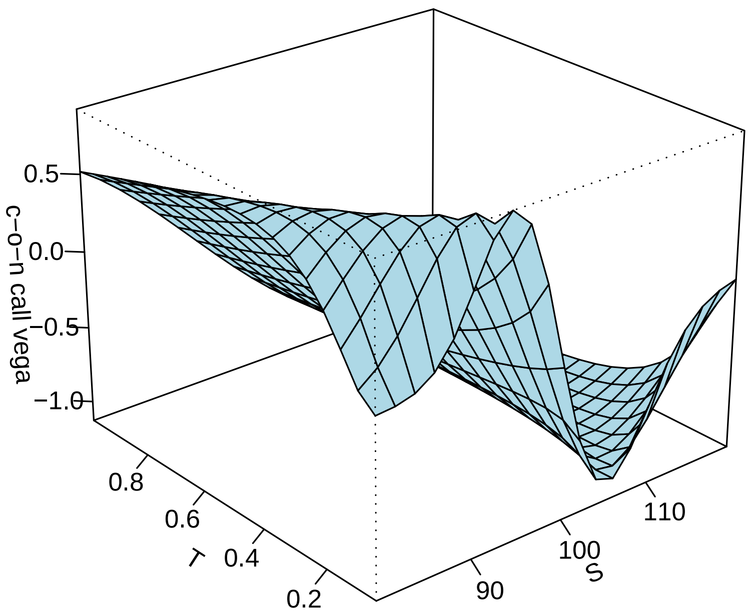


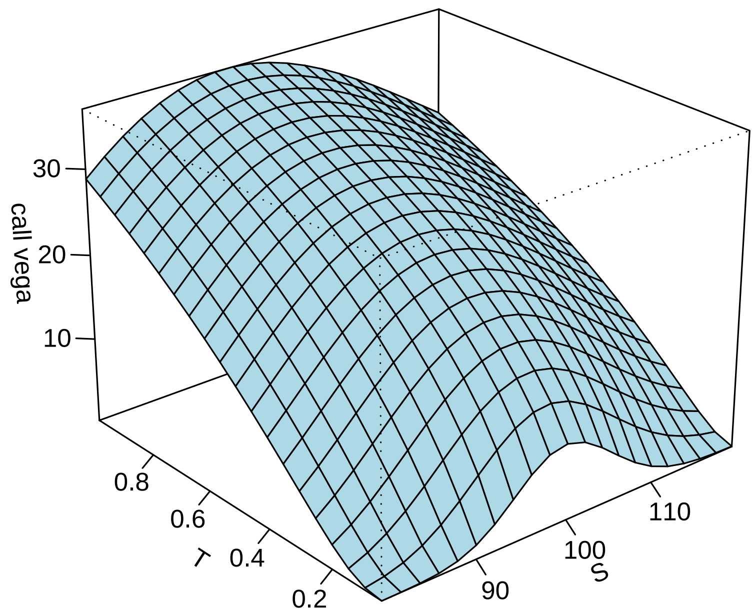
Θ for digital call can be negative or positive [cf. standard call: $\Theta_{\text{std}} > 0$]

Θ for digial call is essentially zero except near expiration [cf. standard call: Θ_{std} constant but nonzero]

Consistent with observation on C: OTM/ITM digital call loses/earns value over time [cf. standard call: always loses value]

Vega of Cash-or-Nothing Option:





V for digital call can be negative or positive [cf. standard call: $V_{\text{std}} > 0$]

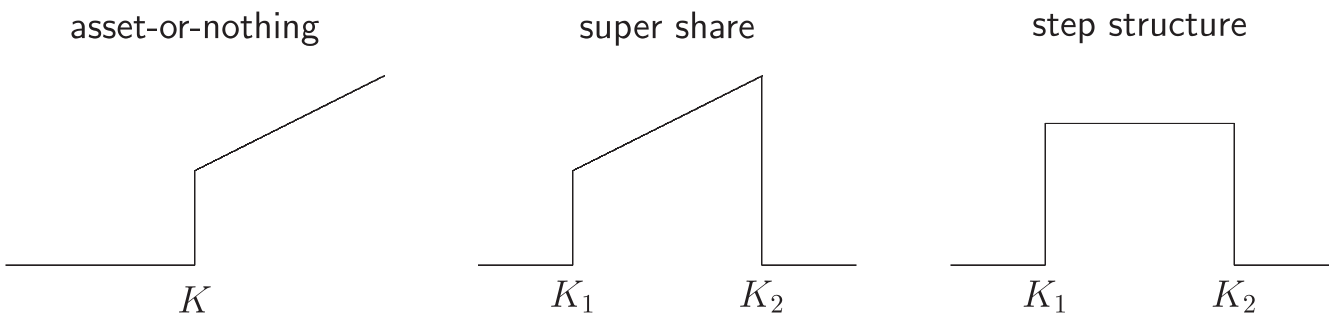
Long-dated options: increase in volatility leads to increase/decrease in value for OTM/ITM digital call [cf. standard call: always increases value]

As expiration is approached, the effect of volatility is more marked around the money

Variations: Besides the cash-or-nothing (CON) or binary option, several other digital structures exist

(1) asset-or-nothing (AON) options (2) digital-gap options (3) super shares (4) step structures

(5) one-touch options (American-style) (6) path-dependent digitals



Asset-or-Nothing Options: An asset-or-nothing call option is worth the amount $C_{\text{aon}} = Se^{-q\tau}N(d_1)$

This value can be obtained by direct computation taking the risk-neutral valuation approach

Alternatively (and more easily), we can note that the payoff of a standard call option can be exactly replicated by a portfolio which is long one AON call and short one CON call paying out K on expiration

Super Shares: A super share can be thought of as a portfolio that is long an AON call struck at K_1 and short an AON call struck at K_2, both written on the same underlying asset

The value of a super share is therefore $C_{\text{aon}}(K_1) - C_{\text{aon}}(K_2)$

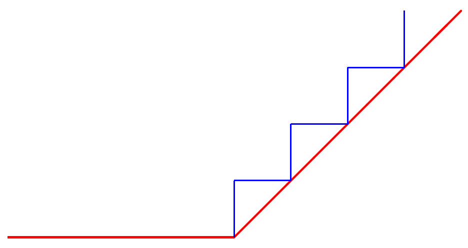
Step Structures: A step structure can be thought of as a portfolio that is long a CON call struck at K_1 and short a CON call struck at K_2, both having the same cash payout on expiration

The value of a step structure is therefore $C(K_1) - C(K_2)$

Applications: Digital options have proved to be among the most enduring and popular types of exotic options and have been incorporated in a variety of securities and derivatives transactions

- Valuation and hedging of more complex options
- Structured off-balance sheet products
- Contingent premium options

Hedging European Options: A European option can be approximately replicated with a series of digital options with linearly increasing strikes (the smaller the increases, the more precise the replication)



The Spel: Digital options are usually integrated into structured off-balance sheet products to improve yields or outperform the market under certain interest rate and volatility scenarios

The Spel (*s*wa*p* with *e*mbedded *l*everage) combines a short-term swap and a series of digital options (e.g., binary strangles)

The most common type of product is the one-year swap in which the corporate pays a reference index and expects to receive an enhanced fixed or floating rate at the end of the period, depending on the number of fixings for which this index remain within a given range

Summary: Digital options belong to the class of payoff-modified options

This alteration of the payoff pattern creates unique transactional hedging opportunities

- Allows writers a known cost in the event that the option expires in the money
- Allows option purchasers to nominate a particular payoff related to underlying hedging requirement

It is not too difficult to price and hedge digital options unless they are very near expiration and very near the strike

2-2. Contingent Premium Options

Motivating Example: Contingent premium options (CPOs) as a means of deferring premium payment

Riskless rate: 10%; dividend yield: 5%; volatility: 20%

1. Buy a three-month at-the-money European call option struck at $100: the premium is $4.55
2. Add a short position in a digital option, also struck at $100: the payout is $8.80
3. This long call/short digital position incurs no up-front premium

On expiration, if the underlying asset has a price

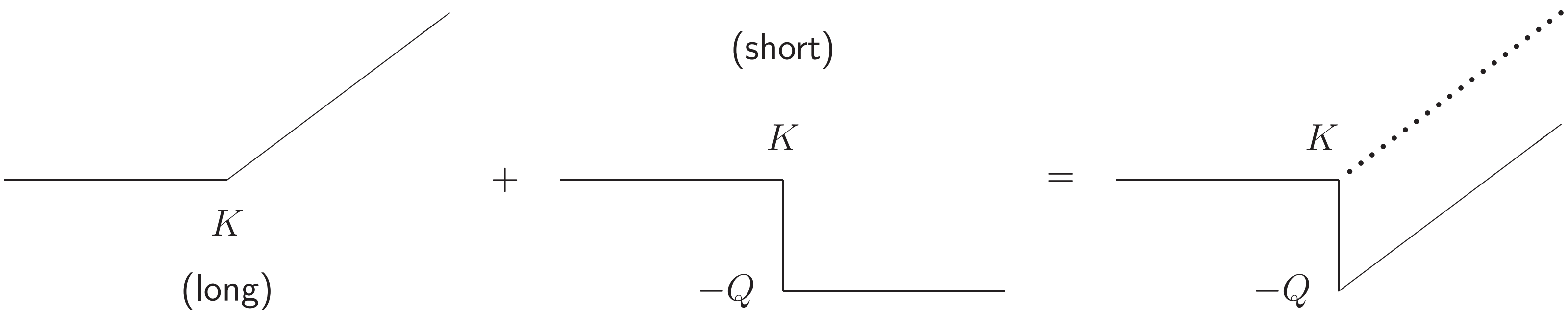
- Under $100, both options expire out of the money: no payments
- Between $100 and $108.80 (inclusive), the client has to pay $8.80 for the short digital, part of which can be recovered from the call option: net loss
- Above $108.80, the call option payout exceeds $8.80: net gain

Where's the Benefit? CPO is more expensive than standard call?

1. Standard call premium of \$4.55 is payable up-front in any case

2. Contingent premium of \$8.80, though higher, is payable on expiration and only if the underlying is over \$100

Payoff: The simplest CPO is a combination of a standard option and a digital option with a payout Q [the "contingent premium"] determined to give the combination zero initial premium



1. If the option expires out of the money then the option is not exercised and no premium is payable

2. If the option expires in the money then the option is exercised and the premium is payable

Valuation: Through the decomposition of a CPO into its constituent elements, it is easy to see that the CPO has the value

$$C = Se^{-q\tau}N(d_1) - Ke^{-r\tau}N(d_2) - Qe^{-r\tau}N(d_2)$$

Setting $C = 0$ leads to the following:

$$\text{contingent premium} = Q = Se^{(r-q)\tau}\frac{N(d_1)}{N(d_2)} - K$$

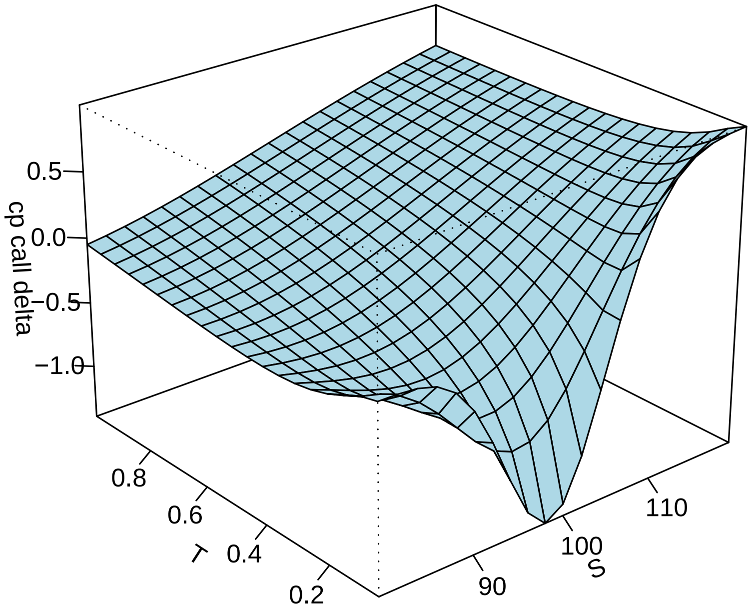
Example: CPO (call) with $S = 1/97\,\$/¥$, $K = 1/100\,\$/¥$, $\tau = 0.25$, $r = 0.059$ (domestic rate), $q = 0.032$ (foreign rate), $\sigma = 0.2$

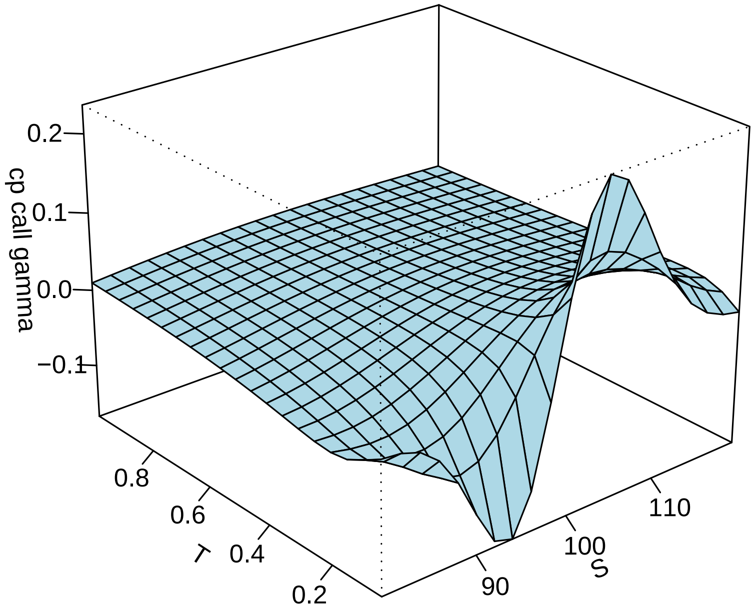
$$\left.\begin{array}{lcl} d_2 = 0.3221 & \Rightarrow & N(d_2) = 0.6263 \\ d_1 = 0.4471 & \Rightarrow & N(d_1) = 0.6726 \end{array}\right\} \quad \Rightarrow \quad Q = 0.001146$$

The premium payable on a ¥1 million notional in three months is $10^6 \times 0.001146 = \$1146$, only if the dollar-yen rate stays below ¥100 per dollar

Static Hedging: Since the CPO is constituted by an ordinary option minus a digital option, the market makers can hedge the sale of a CPO by buying a standard option and shorting a digital

Dynamic hedging: Where the components are not available, the CPO can be hedged dynamically by trading in the underlying asset and cash using $\Delta = \Delta_{\text{std}} - \Delta_{\text{con}}$ and $\Gamma = \Gamma_{\text{std}} - \Gamma_{\text{con}}$

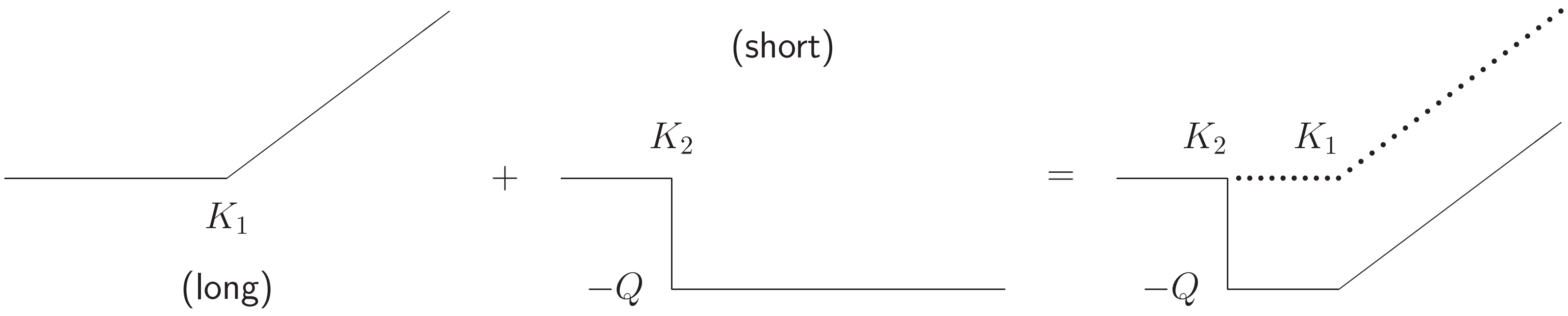




As expected, we could encounter hedging difficulties around the strike close to expiration

Variations: The digital and the European options can have different strikes

By moving the strike of the digital to the left while keeping the strike of the call option unchanged, we create a low-strike digital



Another variation is offered by the money-back option, which pays back the initially paid premium under certain conditions on the underlying price at expiration

Furthermore, we can introduce path-dependency into the CPO structure (e.g., in the form of barriers)

Barrier-driven CPOs have a payout conditional on whether the underlying price has hit one or more preset barrier levels during the life of the option

Applications: CPOs appeal to corporate managers and asset managers as a form of disaster "insurance" against an unfavorable move in the underlying

For example, a corporate borrower needs to hedge currency exposure under a loan agreement

If the corporate liability manager feels strongly that a currency hedge is unnecessary because exchange rates are likely to move in his favor rather than against him, the corporation can — without immediate cash outlay — enter into a CPO agreement

The corporation will pay nothing for the hedge if the manager's expectations are correct

If these expectations are not realized, however, the corporation will be faced with a higher option premium than it would otherwise have to pay

Summary: CPOs are attractive to purchasers as they effectively enable deferring payment of a premium and make the premium payment contingent upon the option expiring in the money

This structure enables the purchaser to insure itself against large one-way movements at no initial cost

The primary risk for the purchaser of such an option is when the option expires slightly in the money and the intrinsic value is insufficient to recover the higher premium payable on such a structure

2-3. Power Options

Motivating Example: Consider one of the three DM-denominated warrant issues undertaken by Trinkhaus & Burkhardt, the German merchant banking arm of HSBC Markets, in May 1995

The DM currency tranche was structured as follows:

1. Two tranches with strike prices of dollar/DM at 1.45 and 1.55
2. European-style exercise on 17 June 1996
3. Pays 100 $\times$ $($positive difference between spot above strike$)^2$ capped at DM25 per warrant

The transaction was motivated by the investor's expectations of a small move in the underlying price

The maximum payoff of DM25 would require an appreciation in the dollar to DM1.70 (17.24%)

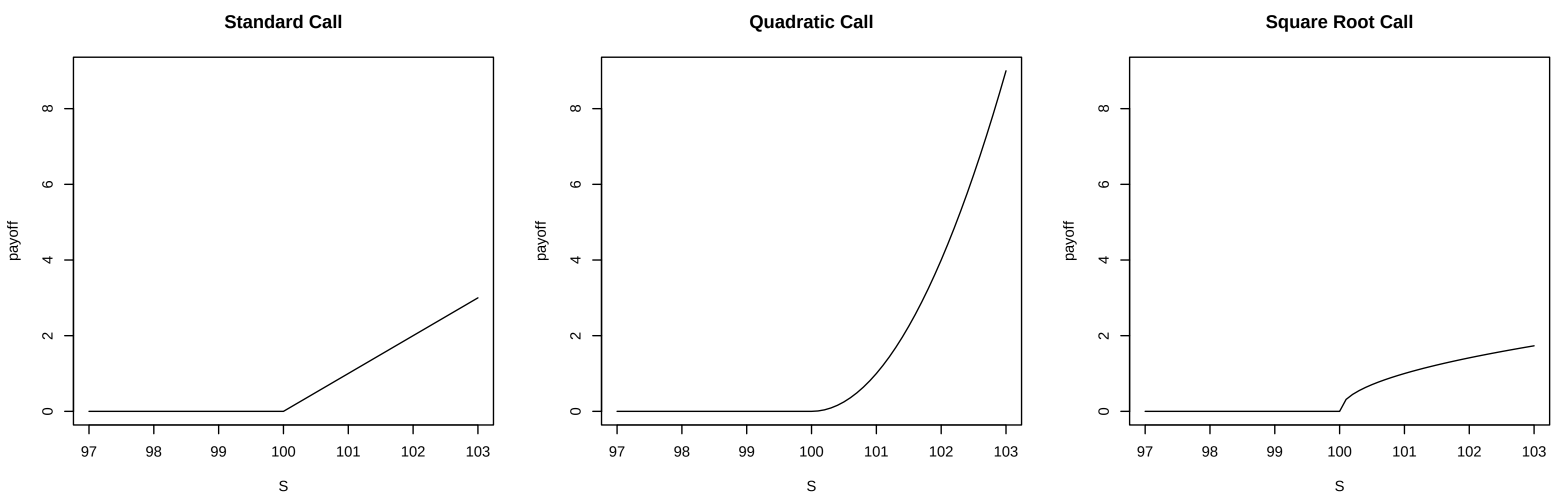
In contrast, the power warrant would provide a similar return to the option purchaser for an increase in the dollar to DM1.50 (3.45%).

Going Exponential: The amplification of the return provided by the power feature allows an investor to maximize the expected return of its expectations in a low risk manner

The structure also offers a relatively defensive investment strategy allowing the warrant buyer to benefit from modest market movements

Payoff: The option payoff is a defined function of the difference between the underlying asset price at maturity and a strike price, usually raised to a power $\alpha \neq 1$

1. Power call option: $|S_T - K|^{\alpha}\, \mathbb{I}_{\{S_T > K\}}$
2. Power put option: $|K - S_T|^{\alpha}\, \mathbb{I}_{\{S_T < K\}}$



Valuation: To fix ideas, consider a power call with $\alpha = 2$ (quadratic call)

By risk-neutral valuation, value of quadratic call $= e^{-r\tau} E(\text{payoff})$

$$= e^{-r\tau} E\left[(S_T^2 - 2KS_T + K^2)\, \mathbb{I}_{\{S_T > K\}} \right]$$

We can evaluate the three terms separately:

1. $E\left[K^2\, \mathbb{I}_{\{S_T > K\}}\right] = K^2 P\{S_T > K\} = K^2 N(d_2)$

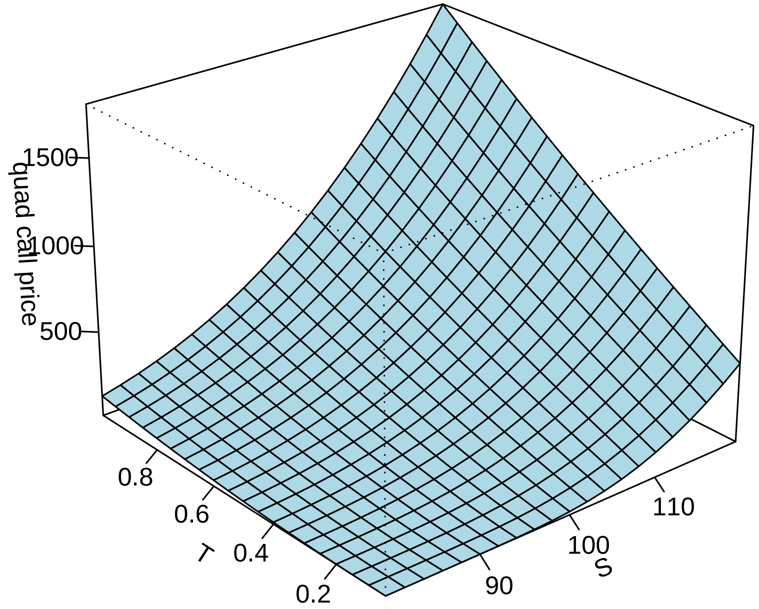
2. $E\left[2KS_T\, \mathbb{I}_{\{S_T > K\}}\right] = 2K \int_{\ln(K/S)}^{\infty} Se^x f(x)\, dx = 2KSe^{(r-q)\tau} N(d_2 + \sigma\sqrt{\tau})$

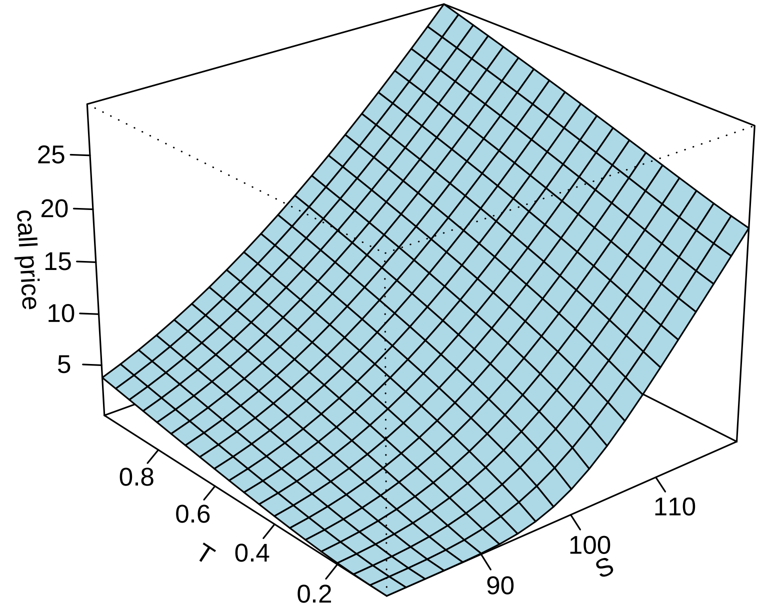
3. $E\left[S_T^2\, \mathbb{I}_{\{S_T > K\}}\right] = \int_{\ln(K/S)}^{\infty} S^2 e^{2x} f(x)\, dx = S^2 e^{(2r-2q+\sigma^2)\tau} N(d_2 + 2\sigma\sqrt{\tau})$

Here, $f(x)$ is the normal density function with mean $(r - q - \sigma^2/2)\tau$ and standard deviatiom $\sigma\sqrt{\tau}$

Therefore, $C = S^2 e^{(r-2q+\sigma^2)\tau} N(d_2 + 2\sigma\sqrt{\tau}) - 2KSe^{-q\tau} N(d_2 + \sigma\sqrt{\tau}) + K^2 e^{-r\tau} N(d_2)$

Valuation (continued)





Quadratic call is substantially more expensive than standard call (so structures utilizing the power payoff are usually capped)

Δ and Γ for quadratic call are also larger compared to those for standard call

Quadratic call also loses value in time much faster than standard call

Valuation (continued)

We carry through the evaluation of $I = \int_{\ln(K/S)}^{\infty} e^x f(x)\,dx$ as illustration of the technique

First, we complete the squares in the exponent:

$$\exp\left\{x - \frac{1}{2\sigma^2\tau}\left[x - (r - q - \frac{\sigma^2}{2})\tau\right]^2\right\} = e^{(r-q)\tau} \times \exp\left\{-\frac{1}{2\sigma^2\tau}\left[x - (r - q + \frac{\sigma^2}{2})\tau\right]^2\right\}$$

The integral can be rewritten as

$$I = e^{(r-q)\tau} \int_{\ln(K/S)}^{\infty} f^*(x)\,dx$$

where $f^*(x)$ is the normal density function with mean $(r - q + \sigma^2/2)\tau$ and standard deviation $\sigma\sqrt{\tau}$

It follows then that

$$I = e^{(r-q)\tau} P\{X > \ln(K/S)\} = P\left\{Z > \frac{\ln(K/S) - (r - q + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}\right\} = N(d_2 + \sigma\sqrt{\tau})$$

Valuation (continued)

In general, for any positive integer α, we can rely on the following results to price the power option

1. For any positive integer α:

$$(x-y)^\alpha = \sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} x^{\alpha-i} y^i \qquad \text{where} \qquad \binom{\alpha}{i} = \frac{\alpha!}{(\alpha-i)!i!}$$

2. For lognormally distributed asset prices:

$$E\left[S_T^{\alpha-i}\mathbb{I}_{\{S_T>K\}}\right] = S^{\alpha-i}\exp\left\{\frac{(\alpha-i)\tau}{2}\left[2r-2q+\sigma^2(\alpha-i-1)\right]\right\} N\left[d_2+(\alpha-i)\sigma\sqrt{\tau}\right]$$

The value of such a power call option is

$$C = \sum_{i=0}^{\alpha} (-1)^i \binom{\alpha}{i} S^{\alpha-i} K^i$$

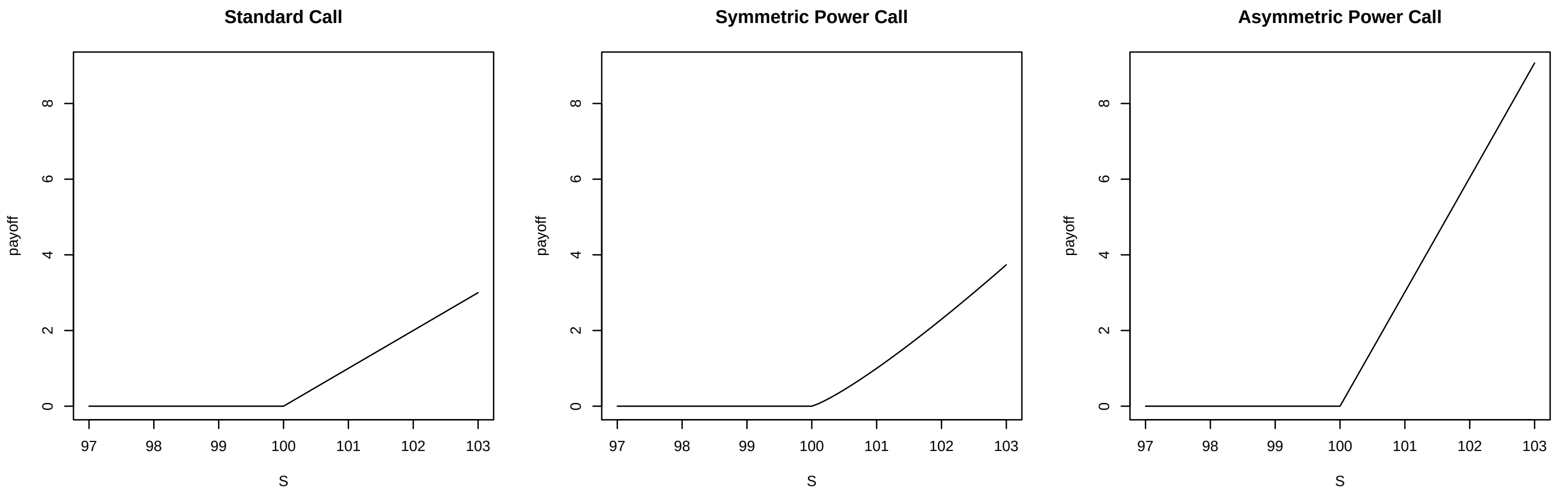
$$\times \exp\left\{\left[(\alpha-i-1)r-(\alpha-i)q+\frac{\sigma^2(\alpha-1)(\alpha-i-1)}{2}\right]\tau\right\} N\left[d_2+(\alpha-i)\sigma\sqrt{\tau}\right]$$

Hedging: While it is possible to hedge power options statically by using a series of standard options with differing strikes, where the number of options for each strike prices increases the further out of the money the strike, it is perhaps easier to perform dynamic hedging by borrowing cash and trading in the underlying asset

Variations: One may choose to deal with asymmetric power options, which have the following payoffs:

1. Asymmetric-power call option: $(S_T^\alpha - K^\alpha)\,\mathbb{I}_{\{S_T>K\}}$

2. Asymmetric-power put option: $(K^\alpha - S_T^\alpha)\,\mathbb{I}_{\{S_T<K\}}$



Variations (continued)

Whether a user chooses to use an asymmetric or a symmetric power option depends on his particular need: for example, the compound exposure of an importer's revenue to the exchange rate can be hedge using symmetric quadratic options

It turns out that asymmetric power options are substantially easier to value than symmetric ones

Summary: By modifying its payoff, the power option serves to enhance the return by a defined exponent

The power option in this regard is the opposite of the digital-option structure, which is predominantly concerned with the limitation of the payoff to a known amount

The power option generates a nonlinear payoff profile which has a range of applications, including the hedging of volatility risk and the transfer of financial risk in markets where the risk profile itself is nonlinear

2-4. Forward Start Options

Motivating Example: Forward start options (FSOs) as a means of issuing bonus and reducing liability

A softwar firm offers an employee, if he stays employed until the end of the year, a large number of one-year call options — the strike price of which will be determined to be equal to the price of the firm's shares as of January 1

The employee cannot use the Black-Scholes formula to determine the value of such an option because as of today, the strike price is still unknown (it has not yet been set)

Current accounting practices in the U.S. permit firms to value many types of employee options at cost and not at fair value

As most share prices have increased substantially, such options are evaluated at a large discount to their fair value so the liability to the issuer is valued at less than its actual value

Payoff: FSOs are options whose strike will be determined at some later date

The payoff of an FSO with expiration at time T and such that the strike is determined at the money [to be S^*] at time t^* is as follows:

1. Forward start call option: $\max\{0, S_T - S^*\}$
2. Forward start put option: $\max\{0, S^* - S_T\}$

Valuation: At time t^*, the strike has been set, and the call option then turns into a standard option whose value is

$$C^* = S^* \left[e^{-q\tau^*} N(d_1) - e^{-r\tau^*} N(d_2) \right]$$

with

$$d_2 = \frac{(r - q - \sigma^2/2)\tau^*}{\sigma\sqrt{\tau^*}}, \quad d_1 = d_2 + \sigma\sqrt{\tau^*} \quad \text{and} \quad \tau^* = T - t^*$$

Valuation (continued)

At time t (prior to t^*) when the FSO is valued, S^* is not known so we talk about the expectation of C^*:

$$E(C^*) = Se^{(r-q)(t^*-t)}\left[e^{-q\tau^*}N(d_1) - e^{-r\tau^*}N(d_2)\right]$$

Discount this expectation (by multiplying with $e^{-r(t^*-t)}$) to give

$$C = e^{-q(t^*-t)} \times S\left[e^{-q\tau^*}N(d_1) - e^{-r\tau^*}N(d_2)\right]$$

Example: FSO (call) with $S = \$50$, $t^* - t = 0.5$, $\tau^* = 0.5$, $r = 0.1$, $q = 0.05$, $\sigma = 0.15$

$$\left.\begin{array}{lll} d_2 = 0.1827 & \Rightarrow & N(d_2) = 0.5725 \\ d_1 = 0.2887 & \Rightarrow & N(d_1) = 0.6136 \end{array}\right\} \Rightarrow \quad C = \$2.63$$

Hedging: Dynamic hedging works well in this instance because the FSO has a zero gamma and

$$\Delta = e^{-q(t^*-t)}\left[e^{-q\tau^*}N(d_1) - e^{-r\tau^*}N(d_2)\right]$$

Variations: The ratchet option starts out as a normal call option, and then, on every fixing date, the owner gets the increase of the underlying (typically an equity index)

The ratchet option is extremely useful for fund managers who wish to profit if the index keeps rising, but who do not want to lose if the index declines

Increases in the index from one fixing date to the next are locked in

In case of a drop in the equity markets, the investor does not lose the gains he has already made

The ratchet can be considered to be a collection of forward start options except for one detail

Each forward start option pays out when it expires while the ratchet would pay out only at the final fixing date

Therefore, to price the ratchet we sum the prices of the individual forward start options, making sure to adjust for the time value of money

Summary: Forward start options are used in situations where it is impossible to determine the strike price in advance

In practice, they are most often used to create long-term option structures in the equity and interest-rate markets

In some cases, fixing the strike price in advance would cause the option to be very expensive

Letting the strike be determined by the level of the underlying may make for a cheaper option