Valuing Moving Barrier Options

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Abstract. We show how to compute prices of options knocked out when the underlying price crosses smoothly-moving barriers. The method is to reduce the problem to fixed barriers, by transformation of the statespace, and then to change time so as to make the underlying diffusion into a Brownian motion with time-dependent drift.

1 Introduction

The problem of pricing barrier options has been extensively discussed in the literature; see, for example, the papers of Boyle & Lau [BL94], Goldman, Sosin & Gatto [GSG79], Kunitomo & Ikeda [KI92], Geman & Yor [GY96], ... , Broadie, Glasserman & Kou [BGK95] and [BGK96], Rogers & Stapleton [RS97]. We will restrict attention here to the case of a two-sided knock-out option, so that there is a fixed expiry $T > 0$ of the option, and two functions $f : [0,T] \to \mathbf{R}$ and $g : [0,T] \to \mathbf{R}$, $g < f$ such that the option pays off $(S_T - K)^+$ at time T if $g(t) < \log S_t < f(t)$ for all $t \in [0,T]$, and otherwise pays zero. Here, $S_t \equiv S_0 \exp(\sigma W_t + (r - \sigma^2/2)t)$ is the price of the share at time t, with volatility σ and interest rate r both assumed constant. Prices of knock-in options, and all other variants, can be computed in a manner analogous to that described here; we leave details to the interested reader.

We emphasise that the barrier functions f and g will be assumed to be *once continuously differentiable*. This excludes important cases of interest, such as step barrier options, where the barrier takes a constant value for part of the life of the option, then moves to another constant value for the remainder. We remark that any such option must inevitably be priced by considering the two parts of the interval $[0,T]$ separately, so if we had moving barriers with steps, we could break the interval into subintervals in which the barriers were C^1, and solve in each subinterval using the method to be described, working backward from the last subinterval.

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If $X_t \equiv \log S_0 + \sigma W_t + (r - \sigma^2/2)t \equiv \log S_0 + \sigma W_t + \mu t$ denotes the log-price process, the idea is firstly to introduce the process

$$Y_t \equiv \frac{X_t - g_t}{f_t - g_t}. \tag{1.1}$$

The payoff of the option is then simply expressed as

$$(\exp((f_T - g_T)Y_T + g_T) - K)^+ I_{\{0 < Y_u < 1 \ \forall \ u \in [0,T]\}} \tag{1.2}$$

which transforms the moving barriers for X into constant barriers for Y. We next do a deterministic transformation of time to produce a process $\tilde{Y}$ which has unit volatility and time-and-space-dependent drift (the details are given in Section 2). The numerical evaluation is done by dividing the time interval into N equal pieces, and selecting a spatial grid size to match, so that 0 and 1 become grid points. We then do a trinomial tree evaluation, taking care to vary the up and down probabilities at the different nodes of the grid so as to match the time-and-space-dependent drift.

One of the problems identified early on with pricing (fixed) barrier options was the fact that if the barriers did not lie on the spatial grid, then the knock-out event was poorly approximated by the binomial method. Boyle & Lau [BL94] chose to vary the number of time steps so that the barriers *did* lie approximately on the grid, and other ways of getting around this have been proposed since. Among these, we mention the work of Broadie, Glasserman and Kou [BGK95], allowing one to correct a discretely-sampled knock-out to get a better price for a continuously-sampled knock-out, and the work of Rogers & Stapleton [RS97], providing a novel interpretation of the standard binomial method which is ideally suited to pricing barrier options. It is clear that the method described above will not in general result in the start price lying on the grid, and we get round this by a suitable interpolation procedure.

For fixed barriers, closed-form solutions exist. These involve integrating with respect to the transition density of a Brownian motion killed when it leaves an interval; this can only be expressed as an infinite series, but in practice it converges so rapidly that only a few terms are required. Kunitomo & Ikeda [KI92] give a closed-form solution expressed in similar terms for the case where the functions f and g are linear, and the exact formulae are clearly superior for speed and accuracy than a tree method *in these special cases*. See the numerical results reported in [KI92]. However, for general barriers no closed-form solution is known, and some tree method is a natural way to proceed. We give the details of the methodology in Section 2, report on the numerical results in Section 3, and conclude in Section 4.

2 Methodology

The transformation of X to Y given in (1) leads to the dynamics

$$dY_t = \frac{\sigma}{f_t - g_t}\, dW_t \;+\; \frac{dt}{f_t - g_t}\,\{\mu - g_t' - (f_t' - g_t')Y_t\} \tag{2.3}$$

for Y. If we now introduce the quadratic-variation-to-go, $q(t)$, defined by

$$q(t) = \int_t^T \left(\frac{\sigma}{f_s - g_s}\right)^2 ds, \tag{2.4}$$

the effective time $\tilde{T}$, defined by

$$\tilde{T} = q(0), \tag{2.5}$$

and the time-transformed process $\tilde{Y}$ defined for $0 \le t \le \tilde{T}$ by

$$\tilde{Y}_t \equiv Y(\tau_t), \tag{2.6}$$

where the time-transformation τ is defined by

$$q(\tau_t) = \tilde{T} - t, \tag{2.7}$$

we end up with a diffusion $\tilde{Y}$ which has unit volatility, and variable drift, satisfying the stochastic differential equation

$$d\tilde{Y}_t = d\tilde{W}_t \;+\; \frac{f(\tau_t) - g(\tau_t)}{\sigma^2}\,\{\mu - g'(\tau_t) - (f'(\tau_t) - g'(\tau_t))\tilde{Y}_t\}dt \tag{2.8}$$

$$\equiv d\tilde{W}_t \;+\; b(t, \tilde{Y}_t)dt.$$

The price of the option is simply expressed in terms of the diffusion $\tilde{Y}$:

$$\text{Price} = e^{-rT}\; E\left[(\exp((f_T - g_T)\tilde{Y}(\tilde{T}) + g_T) - K)^+ : 0 < \tilde{Y}_u < 1 \;\forall\, u \in [0, \tilde{T}]\right]. \tag{2.9}$$

The goal is therefore to evaluate the right-hand side, which we do using a trinomial tree method. In more detail, firstly fix the number N of time steps, each of length $\Delta t \equiv \tilde{T}/N$, and then choose the spatial grid step Δx (in a way to be explained shortly) so that $\Delta x = 1/m$ for some integer m. Now define the value function for the discretised problem via the dynamic-programming equations:

$$V(0, j) = (\exp((f_T - g_T)j\Delta x + g_T) - K)^+,$$

$$V(n+1,j) = p(n+1,j)V(n,j+1) + \theta(n+1,j)V(n,j) + q(n+1,j)V(n,j-1)$$

for $1 \leq j \leq m-1$ and $0 \leq n < N$, and set $V(n,0) = V(n,m) = 0$ for all n. The non-negative parameters $p(n+1,j)$, $\theta(n+1,j)$, and $q(n+1,j)$ add to one, and are chosen to match the first and second moments of the increments of $\tilde{Y}$:

$$(p-q)\,\Delta x = b\,\Delta t (p+q)\,\Delta x^2 = (b\,\Delta t)^2 + \Delta t,$$

where b denotes the drift of $\tilde{Y}$ evaluated at time $(N-n-1)\Delta t$ and position $j\Delta x$. Hence

$$p = \frac{1}{2}\Big[\frac{\Delta t}{\Delta x^2} + \frac{b\Delta t}{\Delta x}\,(\frac{b\Delta t}{\Delta x} + 1)\Big]$$

$$q = \frac{1}{2}\Big[\frac{\Delta t}{\Delta x^2} + \frac{b\Delta t}{\Delta x}\,(\frac{b\Delta t}{\Delta x} - 1)\Big]$$

and $\theta = 1 - p - q$. In order that all of these should be positive, we need to ensure that $\Delta t/\Delta x^2$ should be less than one, so fix some $\lambda \in (0,1)$ (we used $\lambda = 0.7$) and then take m to be the integer part of $\sqrt{\lambda/\Delta t}$. This does not *guarantee* that p, q and θ are all positive, but as the second part of the expression for p is $O(\sqrt{\Delta t})$, we can confidently expect that except in cases of extreme drift they will be. In fact, we never had any problems with this.

If $Y_0 = j\Delta x$ for some integer j, then the price of the option would just be $e^{-rt}V(N,j)$. However, if the initial value of Y does not lie exactly on the spatial lattice, we simply do a cubic-spline interpolation of the values at the two grid points above and the two grid points below Y_0. In fact, we used one further refinement; we continued the dynamic-programming recursion to time step $N+1$, and replaced the values $V(N,j)$ by $(V(N-1,j) + 2V(N,j) + V(N+1,j))/4$. This removes any 'odd-even' effect in the time-parameter (a known snag with the binomial method), and does so in a way that also corrects for linear dependence on time.

3 Results

We computed prices of various options; constant barriers, barriers varying linearly in the log scale (as in Kunitomo & Ikeda [KI92]), and barriers varying linearly in the actual price. For the first two examples, there are known formulae and we are able to compare our results for accuracy with those obtained in [GY96], [KI92] and [RS97]. For the last example, no closed-form expressions exist, and no answer in terms of an integral is known, so we are forced to rely on numerical methods.
In all cases, we computed the prices using N = 25, 50, 100, 200, 400, 800,

1600, 3200. The prices have been computed using a Sun SparcStation 5 and the CPU times that are quoted throuhgout the paper have been measured using the x05baf subroutine from the NAG library.

3.1 Constant barriers

3.1.1 Geman-Yor

These are the results contained in [GY96] where they are compared to the results in [KI92] and to the Monte Carlo price.
In the table L is the lower barrier and U is the upper barrier for the price.

σ	.2	.5	.5
r	.02	.05	.05
T	1	1	1
S_0	2	2	2
K	2	2	1.75
L	1.5	1.5	1
U	2.5	3	3
GY	.0411	.0178	.07615
KI	.041089	.017856	.076172
MC	.0425	.0191	.0772
N=800	.041041(.31)	.017791(.16)	.076018(.26)
N=1600	.041061(.87)	.017819(.47)	.076111(.76)
N=3200	.041079(2.46)	.017837(1.33)	.076147(2.09)

3.1.2 Rogers-Stapleton

The results are now compared to the example in [RS97] (page 11).

σ	r	T	S_0	K	L	U
.2	.02	1	100	100	75	125

These are the results:

N	RS price	RS time	RZ price	RZ time
25	2.1472	0.007	1.9886	0.003
50	2.0518	0.0095	2.0244	0.005
100	2.0984	0.012	2.0339	0.015
200	2.0591	0.025	2.0430	0.04
400	2.0642	0.047	2.0503	0.11
800	2.0578	0.11	2.0520	0.31
1600	2.0558	0.30	2.0530	0.87
3200	2.0558	0.82	2.0539	2.46

Numerical integration price (see [RS97]) 2.0544 (time: 0.03).

3.2 Linear barriers for the log-price

3.2.1 Kunitomo-Ikeda

Let us compare the prices with the results in [KI92] (page 284). We have linear bounds for the log-price. The bounds for the price are given by the functions $U\ \exp(\delta_1\, t)$ and $L\ \exp(\delta_2\, t)$ (so $f(t) = \log(U) + \delta_1\, t$ and $g(t) = \log(L) + \delta_2\, t$). The following parameters are fixed:

σ	r	T	S_0	K
.2	.05	.5	1000	1000

Let $\delta_1 = -\delta_2 = 0.1$, we have

L	U	KI price	RZ price	RZ time
500	1500	67.78	67.7834	2.37
600	1400	64.63	64.6401	1.87
700	1300	55.20	55.1992	1.41
800	1200	34.58	34.5713	0.98

Let $\delta_1 = -\delta_2 = -0.1$, we have

L	U	KI price	RZ price	RZ time
500	1500	62.75	62.7532	2.14
600	1400	52.50	52.5021	1.66
700	1300	33.45	33.4429	1.19
800	1200	10.86	10.8217	0.77

3.2.2 Rogers-Stapleton

The results are now compared to the example in [RS97] (page 11).

σ	r	T	S_0	K	δ_1	U	δ_2	L
.25	.1	1	95	100	0.1	160	-0.1	90

These are the results:

N	RS price	RS time	RZ price	RZ time
25	5.2504	0.01	5.3577	0.002
50	5.3107	0.015	5.3567	0.005
100	5.3270	0.023	5.3675	0.01
200	5.3598	0.043	5.3680	0.03
400	5.3602	0.093	5.3684	0.10
800	5.3660	0.21	5.3682	0.30
1600	5.3668	0.53	5.3681	0.85
3200	5.3672	1.31	5.3680	2.42

The price computed using the Kunitomo-Ikeda approach is 5.3679 (time 0.05)(see [RS97]).

3.3 Linear barriers for the price

Finally we give the results for the price of a double-barrier option in the case in which the barriers for the price process are linear in time, i.e. $f(t) = \log(\alpha_1 + \alpha_2 t)$ and $g(t) = \log(\beta_1 + \beta_2 t)$.
Let us consider the following parameter values

σ	r	T	S_0	K	α_1	β_1
.25	.1	1	95	100	160	90

If $\alpha_2 = -\beta_2 = 5$, we have:

N	price	time
25	4.2622	0.003
50	4.3342	0.005
100	4.3392	0.01
200	4.3422	0.03
400	4.3431	0.09
800	4.3435	0.27
1600	4.3435	0.75
3200	4.3438	2.09

If $\alpha_2 = -\beta_2 = -5$, we have:

N	price	time
25	2.3634	0.003
50	2.5197	0.005
100	2.5314	0.01
200	2.5387	0.03
400	2.5414	0.09
800	2.5426	0.24
1600	2.5436	0.67
3200	2.5438	1.84

To compare the results, we see that this method gets the constant barrier price accurate to 1 part in 1000 in about 1 second. The accuracy is much better than Monte Carlo, and comparable with the transform method of Geman& Yor, but it is impossible to compare speeds as these were not reported in [GY96]. (We assume that the quasi-analytic method of Kunitomo & Ikeda [KI92] gives high accuracy for this problem).
Comparing with the approach of Rogers & Stapleton [RS97], both methods give accuracy of 1 part in 1000 in about 0.3 seconds.
For linearly-moving barriers for *log-price*, except in one case we have agreements with Kunitomo & Ikeda [KI92] to much better that 1 part in 1000 in times of the order of 1-2 seconds.
On the example used by Rogers & Stapleton, this method is accurate to 1 part in 10,000 in a time of 0.01 seconds, whereas the method of [RS97] achieves this accuracy in 1.31 seconds.
The computations for the barriers which are linear in *price* show stability and rapid convergence: for the first example of a growing region, we may be reasonably confident of the answer 4.344 in 2 seconds, and in the second example we may be reasonably confident of 2.544 in about the same time. There is no quasi-analytic method now available to confirm these results, but for the same problem with *static* barriers at 90 and 160, we get the price 3.4607 by numerical integration; this is less than 4.344 and more than2.544, as it should be.

4 Conclusions

We have presented here a simple method to transform smoothly-moving barrier option pricing problems to fixed barrier problems, and have demonstrated that typically accuracy of the order of 1 in 1000 can be achieved in times of about

1 second and often much better than that. This appears to be an effective solution to the problem.

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