Static Replication of Barrier Options: Some General Results

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Abstract

This paper presents a number of new theoretical results for replication of barrier options through a static portfolio of European put and call options. Our results are valid for options with completely general knock-out/knock-in sets, and allow for time- and state-dependents volatility as well as discontinuous asset dynamics. We illustrate the theory with numerical examples and discuss the practical implementation.

1. Introduction

The classical approach to the hedging of derivatives involves maintaining an ever-changing position in the underlying assets. The construction of such *dynamic hedges* is a key argument in the seminal paper by Black and Scholes (1973), and is a standard technique for practical hedging of derivative products. A literal interpretation of dynamic hedging strategies, however, requires continuous trading which would generate enormous transaction cost if implemented in practice. Instead, most real-life trading strategies involve time-discrete rebalancing, exposing the hedger to some risk, particularly if the gamma of the option hedged is high.

For some derivatives, it turns out that it is possible to construct a hedge that does not involve continuous rebalancing. Such *static hedges* normally involve setting up a portfolio of simple, European options (typically puts and calls) that is guaranteed to match the payout of the instrument to be hedged. It is fair to say that less is known about static hedging than dynamic hedging, although recent papers have made some progress. Derman *et al* (1995) describe a numerical algorithm for single barrier options in the context of a binomial tree representing the evolution of a stock with time- and level-dependent volatility. Carr and Chou (1997) and Carr *et al* (1998) examine in detail the static replication of barrier options in the Black-Scholes (1973) model. For martingale stock processes, Brown *et al* (1998) demonstrate how to set up model-free over- and underhedges for certain simple classes of single-barrier options.

The approach in this paper differs from previous literature in a number of ways. First, we derive exact, explicit expressions for the composition of the statically replicating portfolio. Second, we are able to derive static hedging portfolios not only for simple, continuously monitored barrier options, but allow for almost arbitrarily complicated knock-out regions and terminal payouts (and can easily handle curved, discrete, partial, and double barrier options). Third, our results are extended to hold for asset dynamics which involve both jumps as well as time- and state-dependent diffusion volatility.

All of our theoretical results are derived under the assumption (or approximation) that European options are traded in inelastic supply for all maturities and strikes. This is not true in practice and we therefore devote a section of the paper to treating some issues that arise in the practical implementation of the static hedging strategies suggested in the first part of the paper.

The rest of the paper is organized as follows: Section 2 derives static hedges for general barrier options written on an asset with a volatility that depends deterministically on time and the asset itself. In Section 3, we extend our results to the case of discontinuous asset dynamics. Section 4 investigates some issues relating to the practical implementation of static hedging strategies and presents numerical results. Finally, Section 5 contains the conclusions of the paper. An Appendix demonstrates how the results in the paper – which are derived using probabilistic techniques – can alternatively be proven by the more traditional tools of differential forms and circulation theorems. Another Appendix briefly considers the case of stochastic volatility and demonstrates that our technique does not lead to a static hedge for this case. A numerical example in Section 4.1. does, however, illustrate how a static *over- and underhedges* are sometimes possible, even when volatility is stochastic.

Finally, let us point out that the study at hand is largely applied in nature. As our main focus are new formulas and the ideas behind them, we have de-emphasized technicalities and set the paper in a relatively loose mathematical frame. In particular, we have put little emphasis in the specification of technical regularity conditions, which we trust that the reader can supply herself.

2. Deterministic Volatility

In this section we derive static hedging portfolios for barrier option written on an underlying stock (or foreign exchange rate) characterized by a local volatility that is only a function of time and stock price level. Such asset price dynamics are discussed in detail in Dupire (1994). For ease of notation (and without loss of generality) we make the simplifying assumption that all interest rates and dividend yields are zero[1]. Next we assume that the underlying stock (or foreign exchange rate) evolves according to

$$\frac{dS(t)}{S(t)} = s\left(t, S(t)\right) dW(t) \ , \tag{1}$$

where s is a continuous, deterministic function, and W is a Brownian motion under the risk-neutral measure. We assume that s is positive and sufficiently regular for (1) to have a unique, non-explosive, positive solution.

We further assume that we can trade European options on the stock with all maturities and strikes. We will let $C(T,K)$ and $P(T,K)$ denote the time 0 prices of

[1] Notice, that if rates and dividends are non-zero but deterministic, one can easily represent the evolution of the underlying as in (1) by simply modeling the forward stock price. In this case barrier levels must be represented in terms of forward stock levels and terminal payments and rebates in terms of their discounted values. As our approach is valid for arbitrary barrier shapes (not just constant barriers) such transformations can easily accomodated in the framework of this paper.

European call and put options, respectively, with maturity T and strike K. We let $C(t;T,K)$ and $P(t;T,K)$ denote the same options' prices at time t. We note that European option prices are linked to the risk-neutral marginal density of the stock price. Specifically, if we let $f(T,S)$ denote the time 0 marginal density of $S(T)$ taken in S, we have that

$$
\begin{aligned}
& f(T,S) \equiv E\left[\delta(S(T)-S)\right] = C_{KK}(T,S) = P_{KK}(T,S), \\
& \int_B^\infty f(T,S)\,dS = E\left[1_{S(T)\geq B}\right] = -C_K(T,B) = 1 - P_K(T,B), \\
& \int_0^B f(T,S)\,dS = E\left[1_{S(T)\leq B}\right] = 1 + C_K(T,B) = P_K(T,B),
\end{aligned}
\tag{2}
$$

where subscripts denote partial derivatives, $\delta(\cdot)$ is Dirac's delta function, $E[\cdot]$ is the time 0 risk-neutral expectations operator, and 1_A denotes the indicator function on the set A.

2.1. Continuous Barriers.

Consider the function $F = F(t,S)$ defined as the solution to

$$
\begin{aligned}
& F_t(t,S) + \tfrac{1}{2}\sigma(t,S)^2 S^2 F_{SS}(t,S) = 0 && ,\, t < T,\, S > B(t) \\
& F(t,S) = R(t) && ,\, t < T,\, S \leq B(t) \\
& F(T,S) = g(S) && ,\, \forall S
\end{aligned}
\tag{3}
$$

where g is function of the stock price only, and B is a continuous function of time on $[0,T]$. We recognize (3) as the PDE formulation of the problem of pricing a down-and-out barrier option with time-dependent rebate $R(t)$ and time-dependent continuously observed barrier level, $B(t)$. Here, and throughout the paper, we assume that R is a continuous, differentiable function[2]. Note that we let g define the terminal value of $F(T,S)$ for all values of S, including the knock-out region $S \leq B(T)$. So, if we for example consider a down-and-out call option then

$$
g(S) = (S-K)^+ 1_{S>B(T)} + R(T) 1_{S\leq B(T)}
$$

It should be stressed that $F(t)$ is the value of a barrier option *initiated* at time t, i.e. if $G(t;s)$ is the time t value of a barrier option originally issued at time $s \leq t$, then $F(t) = G(t;t)$. This means that F, unlike G, is not a martingale under the risk-neutral measure (as will be evident shortly).

Using (3) and the fact that F is continuous, but not generally continuous differentiable, at $S = B$, we get from Tanaka's formula (Karatzas and Shreve 1991) and (3) that

[2] This assumption is made mainly for convenience. In most cases, it is possible to allow for rebate functions with kinks and even discontinuities by interpreting derivatives of R in terms of step- and delta-functions.

$$dF\big(t,S(t)\big) = 1_{S>B(t)} F_S\big(t,S(t)\big)S(t)\, s\big(t,S(t)\big)dW(t) + 1_{S<B(t)} R'(t)dt \\ + \tfrac{1}{2}\, s\big(t,S(t)\big)^2 B(t)^2 F_S\big(t,B(t)+\big)\, d\big(S(t)-B(t)\big)dt \tag{4}$$

where $R' = dR/dt$ is assumed to exist, and $F_S\big(t,B(t)+\big)$ is the limit of $F_S\big(t,B(t)+e\big)$ for $e \downarrow 0$.

Integrating (4) in the time-dimension yields

$$g\big(S(T)\big) - F\big(0,S(0)\big) = \int_0^T 1_{S(t)>B(t)} F_S\big(t,S(t)\big)\, s\big(t,S(t)\big)S(t)dW(t) + \int_0^T 1_{S(t)<B(t)} R'(t)dt \\ + \tfrac{1}{2}\int_0^T F_S\big(t,B(t)+\big)\, s\big(t,B(t)\big)^2 B(t)^2\, d\big(S(t)-B(t)\big)dt\,.$$

Taking expectations and rearranging yields the relation

$$F\big(0,S(0)\big) = E\big[g\big(S(T)\big)\big] - \int_0^T R'(t)E\big[1_{S(t)<B(t)}\big]dt \\ - \tfrac{1}{2}\int_0^T s\big(t,B(t)\big)^2 B(t)^2 f\big(t,B(t)\big)F_S\big(t,B(t)+\big)dt\,.$$

Notice that we have here used the fact that F_S is a deterministic function around the barrier; had the stock volatility been stochastic, this would *not* hold[3]. The formula above relates the barrier option price to the volatility and the (risk-neutral) marginal density. Interestingly, the first passage-time densities and conditional probabilities are not directly involved here. The marginal density can be synthesized using options positions by use of (2). We get

$$F\big(0,S(0)\big) = \int_0^\infty g(S)P_{KK}(T,S)dS - \int_0^T\int_0^{B(t)} R'(t)P_{KK}(t,S)dSdt \\ - \tfrac{1}{2}\int_0^T s\big(t,B(t)\big)^2 B(t)^2 F_S\big(t,B(t)+\big)P_{KK}\big(t,B(t)\big)dt\,, \tag{5}$$

where we have arbitrarily chosen to synthesize the density from put options. We note that (5) expresses the value of a barrier option as a linear combination of puts, specifically:

- long a continuum $\{g(S)\}_{0<S<\infty}$ of *T*-maturity butterfly put spreads $P_{KK}(T,S)$;
- short a double continuum $\{R'(t)\}_{0<t<T}$ of butterfly put spreads $P_{KK}(t,S)$ with strikes in $[0,B(t)]$;
- short a continuum $\{s(t,B(t))^2 B(t)^2 F_S(t,B(t)+)\}_{0<t<T}$ of butterfly spreads with strikes along the barrier.

Consider now using the put portfolio suggested by (5) as a hedge for a barrier option $G(t;0)$ initiated at time 0. Specifically, if $t = \inf\{t: S(t) = B(t)\}$ is the first time the stock touches the barrier, we hold the put portfolio up to $t \vee T$ and, if $t < T$, sell off the

[3] Appendix B takes a closer look at stochastic volatility models.

outstanding portfolio at the time the barrier is breached. As $F(t) = G(t;0)$ up to (and including) the minimum of t and T, such a strategy would clearly generate the correct cashflow at $t \vee T$. For the put portfolio to qualify as a static hedge, we need to verify that the portfolio does not generate any other cashflows at times $t < t \vee T$. But as all the put positions with maturities less than T only involve strikes at or below the barrier, clearly no such cashflows are generated; whence, the put portfolio in (5) *qualifies as a static hedge*.

While (5) is a static hedge, it is not necessarily the most convenient one. In particular, we notice that the second term in (5) can be simplified to

$$\int_0^T \int_0^{B(t)} R'(t) P_{KK}(t,S) dt = \int_0^T R'(t) P_K(t,S) dt \, ,$$

which represents a position of put spreads along the barrier. This position does not generate cash-flows before the option matures or knocks out[4] and the hedge remains static. We can simplify the hedge even further by relating the butterfly spreads to calendar spreads through the forward equations of Dupire (1994):

$$0 = -C_T + \tfrac{1}{2} s(T,K)^2 K^2 C_{KK}; \;\; 0 = -P_T + \tfrac{1}{2} s(T,K)^2 K^2 P_{KK} \; .$$

We can now rewrite (5) as simply

$$F(0,S(0)) = \int_0^\infty g(S) P_{KK}(T,S) dS - \int_0^T F_S(t, B(t)+) P_T(t, B(t)) dt - \int_0^T R'(t) P_K(t, B(t)) dt \quad (6)$$

As calendar put spreads on the barrier do not produce cash-flows as long as the barrier option is “alive”, (6) represents a static hedge, where the barrier option is now replicated by a European option paying g at maturity, minus the (deterministic) continuum $\{F_S(t, B(t)+)\}_{0<t<T}$ of calendar spreads along the barrier, and minus a continuum $\{R'(t)\}_{0<t<T}$ of put spreads along the barrier. As mentioned earlier, the options positions must be unwound when the barrier is hit. If the model is correct, i.e. the delta (F_S) along the barrier of alive options is computed correctly, then the unwind gain equals the rebate.

As written in (6), hedging the European payout paying g is accomplished through butterfly spreads. Alternatively, we assemble the European payout directly from the “hockey-stick” building blocks of puts and calls. Following Carr and Chou (1997), this can be accomplished by writing

$$g(S) = g(k) + g'(k)(S-k) + \int_0^k g''(K)[K-S]^+ dK + \int_k^\infty g''(K)[S-K]^+ dK$$

for some arbitrary positive constant k. Setting $k = B(T)-$, and integrating over the density of S yields

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[4] It is obvious from (2), that a decomposition using call spreads is possible, too. However, this would not constitute a static hedge, as the call positions would generate random cash-flows in the “alive” region of the barrier option.

$$
\begin{aligned}
E\left[g(S(T))\right] = R(T)P_K\left(T,B(T)\right) + \int_{B(T)+}^{\infty} g''(K)C(T,K)dK \\
+g'\left(B(T)+\right)C\left(T,B(T)\right) - C_K\left(T,B(T)\right)g\left(B(T)+\right).
\end{aligned} \tag{7}
$$

(7) represents a static hedge consisting of a continuum of calls with strikes above the barrier, plus a finite number of calls and put/call spreads with strikes at the barrier. Notice that if g has kinks or discontinuities, the derivatives of g in (7) must, of course, be interpreted in the sense of distributions.

It is worth noting that (6)-(7) only requires model-based computation of the delta along the barrier, for instance by a finite difference scheme (see e.g. Andersen and Brotherton-Ratcliffe 1998 for a discussion of the implementation of the dynamics (1) in a finite difference scheme); all other terms in the hedging portfolio can be deduced from the market prices of standard European options.

The technique outlined above is easy to apply to many types of barrier-options, including "in"-style barrier options. Sometimes we can also rely on parity results; for instance, a down-and-in option can be written as a European option minus a down-and-out option (with no rebate), whereby the results derived above can directly be used to statically hedge a down-and-in option. Applications to double barrier options are simple as well, and would merely involve including in (6) an extra integral of call maturity-spreads and an extra integral of call spreads along the second barrier[5]. We will return to more general barrier shapes in a later section.

2.2. Discretely Monitored Barriers

Consider now the case when the down-and-out barrier of the previous section is only monitored at a discrete set of dates:

$$0 \le t_0 < \ldots < t_n < T \quad .$$

The PDE formulation of the pricing problem is

$$
\begin{aligned}
&F_t(t,S) + \tfrac{1}{2}\, s(t,S)^2 S^2 F_{SS}(t,S) = 0 &&, (t,S) \notin \left\{(t,S) | t \in \{t_i\}, S \le B(t_i)\right\} \\
&F(t_i,S) = R(t_i) &&, S \le B\left(t_i\right) \\
&F(T,S) = g(S) &&, \forall S
\end{aligned} \tag{8}
$$

Since the option price is discontinuous in the time-dimension across every barrier time t_i for all $S \le B(t_i)$, Ito expanding the function defined by (8) gives us

[5] For up-style barriers a static hedge representation using calendar spreads (such as (6)) must be based on calls rather than puts, to prevent the hedge from generating cash-flows before the barrier options matures or knocks out. Such considerations are not necessary for the representation (5), which can be based on either puts or calls.

$$dF(t,S(t)) = 1_{t\notin\{t_i\} \text{ or } S(t)>B(t)} F_S(t,S(t))s(t,S(t))S(t)dW(t)$$
$$+\sum_{i=1}^{n} \delta(t-t_i)1_{S(t_i)\le B(t_i)}[F(t_i+,S(t_i))-R(t_i)]dt. \tag{9}$$

Integrating, taking expectations, and rearranging yields

$$F(0,S(0)) = E\left[g(S(T))\right] - \sum_{i=1}^{n} E\left[[F(t_i+,S(t_i))-R(t_i)]1_{S(t_i)\le B(t_i)}\right].$$

The expectations can be substituted with integrals and European option spreads to give us

$$F(0,S(0)) = E\left[g(S(T))\right] - \sum_{i=1}^{n}\int_0^{B(t_i)}[F(t_i+,S)-R(t_i)]C_{KK}(t_i,S)dS. \tag{10}$$

We have now arrived at an equation that explicitly specifies a static hedge for the barrier option. As for the continuous barrier case we need a position that replicates a European payout $g(S)$ (for instance the put-call portfolio (7)) and a number of butterfly spread positions along the barrier[6]. The number of spreads that we need is again dependent on the value of the barrier option along its barrier. As in (6), only barrier option values along the barriers depend directly on the model for stock price evolution.

2.3. A General Result

The results in Sections 2.1 and 2.2 have been proved by Tanaka's formula. As one would expect, it is possible to prove the results by more traditional methods. Appendix A shows how this can be done through the usage of differential forms and circulation theorems. The circulation theorems shown in the Appendix allow for a compact and completely general representation of barrier options with almost arbitrarily complicated knock-out regions. Such extensions can also be accomplished using the Tanaka formula. Specifically, we can summarize the results of the previous two subsections in the following theorem (where we arbitrarily have used European calls as the hedging instruments):

Theorem I

Suppose the underlying stock evolves according to (1) and consider an option that has the value $g(S(T))$ at time T and knocks out on a set $B\subset\Omega$, $\Omega=[0,T]\times(0,\infty)$, with a once differentiable rebate function, R, that only depends on time. Assuming that $\Omega\setminus B$ is an open submanifold in Ω, a static hedge for the option value is defined by

[6] Obviously, we can use the same trick that lead to (7) to rewrite the position in butterfly spreads to a more "direct" position in put and call options.

$$
\begin{aligned}
F(0,S(0)) &= \int_0^\infty g(S) C_{KK}(T,S)dS \\
&\quad -\tfrac{1}{2}\int_0^T \sum_{S\in\left(\overline{\partial B}\right)(t,\cdot)} \left[F_S(t,S+) - F_S(t,S-)\right] s(t,S)^2 S^2 C_{KK}(t,S)dt \\
&\quad -\int_0^\infty \sum_{t\in\left(|\partial B\right)(\cdot,S)} \left[F(t+,S) - F(t,S)\right] C_{KK}(t,S)dS \\
&\quad -\int_{\operatorname{int} B} R'(t) C_{KK}(t,S)dtdS
\end{aligned}
$$

where ∂B and $\operatorname{int} B$ denote respectively the boundary and the interior of B, $R' = dR/dt$, and where we use the convention $F(T+,\cdot) = F(T,\cdot)$. Further, we define

$$
\begin{aligned}
&|\partial B = \left\{(t,S) \in \partial B \middle| \exists e > 0: (t,S+h) \in \partial B, \forall |h| < e\right\}, \\
&\overline{\partial B} = \partial B \setminus |\partial B,
\end{aligned}
$$

and if $A \subset \Omega$, we let

$$
\begin{aligned}
A(t,\cdot) &= \left\{S \in (0,\infty) \middle| (t,S) \in A\right\}, \\
A(\cdot,S) &= \left\{t \in [0,T] \middle| (t,S) \in A\right\}.
\end{aligned}
$$

While Theorem I looks complicated, it is really just simple extension of the previous results. In particular, the barrier price is split into a contribution from the terminal maturity (first term), the non-vertical parts of the barrier (second term), the vertical parts of the barrier (third term), and the rebate (fourth term). Notice that the second term involves both $F_S(t,B+)$ and $F_S(t,B-)$; the former is required for down-and-out portions of the barrier, the latter for up-and-out parts of the barrier. Also, the technical requirement that $\Omega \setminus B$ is a submanifold is simply to ensure an alive-region that is genuinely two-dimensional, ruling out arbitrarily crinkly or even fractal barriers.

As we have seen in the previous two sections, it is often possible to simplify the expressions above by either completing integrals[7] or applying the forward equation of Dupire (1994). However, care must be taken to ensure that the resulting expressions represent static hedges with no cash-flows being generated on the "alive" region of the option.

3. Discontinuous Asset Dynamics

In this section we extend our static hedging results to the case when the process (1) is extended to allow the stock to jump. Specifically, we will assume that the stock evolves according to

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[7] Specifically, as R is only a function of time, it is clear that we can write the plane integral over the interior of B (last integral in the Theorem) as a path integral over the boundary of B.

$$\frac{dS(t)}{S(t-)} = -\,l(t)m(t)dt + s\big(t,S(t)\big)dW(t) + \big(J(t)-1\big)dN(t) \tag{11}$$

where N is a Poisson process with deterministic intensity $l(t)$, and $\{J(t)\}_{t\geq 0}$ is a sequence of independent positive random variables, each with distribution given by the densities $\{z\,(t,\cdot)\}_{t\geq 0}$. We assume that W, N, and J are independent of each other, and let $m(t) = E[J(t)-1]$ denote the mean jump.

Let us now consider the case of a continuous down-and-out barrier option $F\big(t,S(t)\big)$, equivalent to the discussion in Section 2.1. We will need the definition

$$\Delta F(t) = F\big(t,S(t)J(t)\big) - F\big(t,S(t)\big).$$

Ito-Tanaka expansion of F yields

$$\begin{aligned} dF(t) = {} & 1_{S(t)>B(t)}dM(t) + \tfrac{1}{2}\,d\big(S(t)-B(t)\big)F_S\big(t,B(t)+\big)s\big(t,S(t)\big)^2 B(t)^2 dt \\ & +1_{S(t)<B(t)}\big[R'(t)dt + \Delta F(t)dN(t)\big], \end{aligned}$$

where $M(t)$ is a (discontinuous) martingale. Integrating over time and taking expectations yields

$$\begin{aligned} E\big[g\big(S(T)\big)\big] - F(0) = {} & \tfrac{1}{2}\int_0^T F_S\big(t,B(t)+\big)s\big(t,B(t)\big)^2 B(t)^2 f\big(t,B(t)\big)dt \\ & +\int_0^T E\Big[\big(R'(t) + l(t)\Delta F(t)\big)1_{S(t)<B(t)}\Big]dt. \end{aligned}$$

Using (2), we get

$$\begin{aligned} F(0) = {} & \int_0^\infty g(S)P_{KK}(T,S)dS - \tfrac{1}{2}\int_0^T s\big(t,B(t)\big)^2 B(t)^2 F_S\big(t,B(t)+\big)P_{KK}\big(t,B(t)\big)dt \\ & -\int_0^T\int_0^{B(t)}\big[R'(t) + l(t)E[\Delta F|S(t)=S]\big]P_{KK}(t,S)dSdt. \end{aligned} \tag{12}$$

This shows that the our static replication results can be extended to the case of jumps. In this case the static replicating portfolio also includes an extra term from below the barrier. To set up the static hedge, we need a model to compute F_S at the barrier, as well as the quantity $E[\Delta F]$ below the barrier.

Andersen and Andreasen (1999a-b) show that under the model assumptions above,

$$\begin{aligned} 0 &= -C_T + m(T)\,l(T)KC_K + \tfrac{1}{2}\,s(T,K)^2 K^2 C_{KK} + l'(T)E'[\Delta' C], \\ 0 &= -P_T + m(T)\,l(T)KP_K + \tfrac{1}{2}\,s(T,K)^2 K^2 P_{KK} + l'(T)E'[\Delta' P], \end{aligned} \tag{13}$$

where $l'(t) = l(t)\big(1+m(t)\big)$ and

$$E'[\Delta' C](t,K) = \int_0^\infty \frac{J}{1+m(t)} z\,(t,J)C(t,K/J)dJ - C(t,K),$$
$$E'[\Delta' P](t,K) = \int_0^\infty \frac{J}{1+m(t)} z\,(t,J)P(t,K/J)dJ - P(t,K).$$

The quantities $E'[\Delta' C]$ and $E'[\Delta' P]$ can be interpreted as spreads on a continuum of European options around a certain strike. The fact that these spreads contain strikes that lie above the barrier, we *cannot* generally use (13) to eliminate the term $\frac{1}{2} s\,(t,B(t))^2 B(t)^2 C_{KK}$ in (12) without introducing cash-flows on the "alive" region of the barrier and thereby destroying the static hedge. Nevertheless, if z is known, (13) does provide us with a way to compute the volatility function $s\,(t,B(t))$ in (12) from quoted options prices; see Andersen and Andreasen (1999a-b).

We note that the hedging expression for discrete barriers (10) is unaffected by jumps. This together with (12) suggest the following generalization of Theorem I:

Theorem II
Suppose the underlying stock evolves according to (11) and consider a barrier option similar to the one in Theorem I. A static hedge for the option is defined through:

$$\begin{aligned}
F\big(0,S(0)\big) &= \int_0^\infty g(S)C_{KK}(T,S)dS \\
&\quad -\tfrac{1}{2}\int_0^T \sum_{S\in(\partial B)(t,\cdot)} \big[F_S(t,S+) - F_S(t,S-)\big] s\,(t,S)^2 S^2 C_{KK}(t,S)dt \\
&\quad -\int_0^\infty \sum_{t\in(|\partial B)(\cdot,S)} \big[F(t+,S) - F(t,S)\big] C_{KK}(t,S)dS \\
&\quad -\int_{\operatorname{int} B} \Big[R'(t) + \ l(t)E\big[\Delta F(t)\big|S(t)=S\big]\Big] C_{KK}(t,S)dtdS,
\end{aligned}$$

where the notation is the same as in Theorem I.

4. Stability and Reality

The results so far have relied on two key assumptions: a) the model of stock price evolution is a perfect description of reality; and b) put and call options exist in unlimited supply, at all strikes and maturities. In practice, neither assumption is valid, making the construction of perfect hedges impossible. In this section we will deal with these issues, and also consider the problem (which also affects regular dynamic hedging) of certain barrier option contracts having deltas at the barrier that grow infinitely large as the option approaches maturity.

4.1. Model Uncertainty

A theory for model-free hedging of simple barrier options on martingale stock processes is developed in Brown *et al* (1998). We will take another approach in the spirit of Avellaneda *et al* (1995) which involves an assumption of the stock volatility being stochastic but restricted to move within a specified band. For simplicity, we further assume that there are no jumps in the underlying stock. While the technique suggested here is general, let us consider the concrete example of a down-and-out put option with strike *K*, no rebate, and with a discretely monitored constant barrier *B*. Letting $\{t_i\}$ denote the barrier observation dates (which do not include the terminal maturity *T*) and proceeding as in Appendix B, equation (B.2), we get that the price of this barrier option can be written as

$$F(t) = P(t;T,K) - \sum_{i:t_i \geq t} \int_0^B E_t[F(t_i+)|S(t_i) = S)]C_{KK}(t;t_i,S)dS \quad . \tag{14}$$

As discussed in Appendix B, this equation does not represent a static hedge.

Suppose at time 0, we have sold the barrier option and put on a hedge against it using the hedging equation (14), but with the term $E_t[F(t_i+)|S(t_i) = S)]$ approximated by a model with some deterministic volatility function $\underline{s}(t,S)$. Let the corresponding barrier option price function be denoted $\underline{F}(t,S)$.

Equation (14) shows that if we hit the barrier at time t_j then our profit is given by

$$\sum_{i \geq j} \int_0^B \left[E_{t_j}\left[F(t_i+)|S(t_i) = S\right] - \underline{F}(t_i+,S)\right] C_{KK}(t_j;t_i,S)dS$$

Hence, we are guaranteed a profit (and have thus created an *overhedge*) if for all barrier observation times $\underline{F} \leq F$ for $S \leq B$. Now assume that the true volatility is limited to move in a certain band, i.e.

$$s \in [a,b]$$

We are guaranteed to make a profit if we initially calculate the option price at each $t_i +$ from the function defined as the solution to the control problem

$$\underline{F}(t) = \min_{J \in [a,b]} E_t^J \left[g(S(T)) 1_{t \geq T} \right]$$

where t is the larger of *T* and the first (discretely monitored) passage time to the barrier and $E^J[\cdot]$ denotes expectations taken under the probability measure where the instantaneous volatility of the stock is $J(t,S(t))$, J being a deterministic function. The solution of this problem is given as the solution to the Bellman equation

$$0 = \min_{J \in [a,b]} \underline{F}_t + \tfrac{1}{2} J^2 S^2 \underline{F}_{SS}\,, \quad t < T, S > B, \tag{15}$$

subject to boundary conditions

$$\underline{F}(T,S) = (K-S)^+$$
$$\underline{F}(t_i,S) = 0 \quad , \forall i, \;\; S \le B.$$

The optimal solution is the "bang-bang" type control

$$J = \begin{cases} a & , \underline{F}_{SS} > 0 \\ b & , \underline{F}_{SS} < 0 \end{cases} \tag{16}$$

It is important to note that we only use the control problem outlined above to identify the "worst-case" barrier option price along the discrete barriers. Once these are determined, one can use market prices of European option together with the hedging equation (10) to price the option. It is also worth noting that in this particular case, the hedge suggested by the above is the one based on minimizing, rather than maximizing, option prices. This is due to the fact that the static hedge for the option in question involves selling off contingent claims with positive price.

Equations (15) and (16) together define a non-linear PDE which generally is beyond analytical treatment. However, the "bang-bang" nature of the control strategy makes a numerical solution fairly straightforward, see for example the treatment of "discrete" passport options in Andersen, Andreasen, and Brotherton-Ratcliffe (1998). The basic idea is to assume that the control, J, can only be changed at the discrete times:

$$s_n = nT/N \quad , n = 1, \ldots, N$$

Then for large N, the true solution to (15)-(16) is well approximated by the joint solution of the interdependent equations

$$F_t^a + \tfrac{1}{2} a^2 S^2 F_{SS}^a = 0; \quad F_t^b + \tfrac{1}{2} b^2 S^2 F_{SS}^b = 0$$

on $\{0 < t < T, S > B\}$, subject to

$$F^a(T,S) = F^b(T,S) = (K-S)^+$$
$$F^a(t_i,S) = F^b(t_i,S) = 0 \quad , \forall i, S \le B$$
$$F^a(s_n,S) = F^b(s_n,S) = \min\left(F^a(s_n+,S), F^b s_n+,S)\right), \; n = 0, \ldots, N, \;\; S > 0$$

Using the same technique as outlined above, it is possible to also compute an underhedge for the option, i.e. if we let

$$\overline{F}(t) = \max_{J \in [a,b]} E_t^J\left[g(S(T)) 1_{t \ge T}\right]$$

then over- and underhedges ($\overline{V}$ and $\underline{V}$, respectively) are given by

$$\underline{V}(t) = P(t;T,K) - \sum_{i:t_i \geq t} \int_0^B \overline{F}(t_i+,S) C_{KK}(t;t_i,S) dS \ ,$$
$$\overline{V}(t) = P(t;T,K) - \sum_{i:t_i \geq t} \int_0^B \underline{F}(t_i+,S) C_{KK}(t;t_i,S) dS \ .$$

with

$$\overline{V}(t) \geq F(t) \geq \underline{V}(t).$$

Table 1 below reports the prices of static under- and overhedges of a discretely monitored down-and-out put option when the band of volatility is varied, and we assume that we initially can buy and sell European options at the mid volatility. For reference we also report the time 0 values of $\underline{F}$ and $\overline{F}$. The parameters are: $T=1$, $K=100$, $B=90$, $R=0, \{t_i\} = \{0.25,\ 0.5,\ 0.75\}$, $S(0)=100$. The finite difference grid used to generate Table 1 had N = 500 time-steps and 500 steps in the (log-) asset direction.

Table 1: Under- and overhedge portfolios for down-and-out put

$T=1,\ K=100,\ B=90,\ R=0, \{t_i\} = \{0.25,\ 0.5,\ 0.75\},\ S(0)=100$

Volatility band $[a,b]$	Under hedge $\underline{V}(0)$	Over hedge $\overline{V}(0)$	Min price $\underline{F}(0)$	Max price $\overline{F}(0)$
[0.250; 0.250]	1.5281	1.5281	1.5281	1.5281
[0.245; 0.255]	1.4769	1.5789	1.4135	1.6605
[0.240; 0.260]	1.4233	1.6214	1.3015	1.7963
[0.230; 0.270]	1.3210	1.7137	1.0841	2.0684
[0.220; 0.280]	1.2011	1.7919	0.9017	2.3851
[0.210; 0.290]	1.0733	1.8636	0.7406	2.7295
[0.200; 0.300]	0.9375	1.9288	0.5997	3.1016
[0.175; 0.325]	0.5833	2.0677	0.3238	4.1149
[0.150; 0.350]	0.1594	2.1653	0.1534	5.3250

Note: at time 0 all European options are assumed to be trading at mid volatility

While the hedging bands suggested in Table 1 are not particularly tight, the Table demonstrates that our static hedging technique enables us to narrow the bands $[\underline{F}, \overline{F}]$ that we would get from a direct application of the uncertain volatility approach. As the uncertain volatility approach essentially bounds the cost of delta hedging under stochastic volatility this suggests that our hedging technique is useful even in the case of stochastic volatility.

Let us finally point out, that the general technique outlined in this section is fairly easy to apply even in the case where there is more than one uncertain parameter (for example in a jump model with uncertain jump intensity). Designing over- and underhedges, however, will always require a case-by-case study of the properties of the hedging equation.

4.2. Finite Number of European Options

In practical applications we only have a finite and often sparse set of actively traded options. This means that it can be difficult to put together a portfolio that closely enough replicates the barrier option under consideration. As in Section 4.1, it is useful to consider the alternative of setting up static over- or underhedges. As a first example, consider the case of a down-and-out call with strike *K*, no rebate, and a discretely monitored constant barrier *B*. The barrier observation dates are $\{t_i\}$. The hedging equation for this option contract is

$$F(0) = C(T,K) - \sum_i \int_0^B F(t_i+,S)C_{KK}(t_i,S)dS$$

It is clear that in order to overhedge the option at time 0, we need to sell off a profile, $p_i(\cdot)$, that satisfies

$$\begin{aligned} &p_i(S) \le F(t_i,S) \ , S \le B \\ &p_i(S) \le 0 \qquad\quad\ , S > B \end{aligned}$$

for each barrier observation date t_i. If we for maturity t_i can trade European call options with strikes $K_1^\iota,\ldots,K_{m_i}^\iota$, we get that the cheapest overhedge of the option corresponds to the profile

$$p_i(S) = \sum_j a_j^i (S-K_j^i)^+,$$

where the weights $\{a_j^i\}$ are the solution to the linear programming problem:

$$\begin{aligned} &\max_{\{a_j^i\}} \sum_j a_j^i C(t_i,K_j^i) \\ &\text{s.t.} \quad \sum_j a_j^i (S-K_j^i)^+ \le F(t_i,S)1_{S\le B} \ , \ \forall S \end{aligned}$$

After discretizing in the stock price dimension the linear problem can be solved numerically using the simplex algorithm, see for example Press *et al* (1992).

Let us now turn to a slightly more complicated example where the option has a continuous down-and-out barrier at a constant level *B*. Assume that the rebate is constant over time, but allow for a general payout *g*. We assume that we can purchase enough *T*-maturity options at various strikes to allow for an overhedge of *g*. However, we can only transact in *B*-strike European put options with a finite number of maturities $0 = T_0, T_1, T_2, \ldots, T_{n-1}, T_n = T$ Assuming deterministic volatility of the underlying stock, the hedging equation is, from (6),

$$
\begin{aligned}
F\left(t,S(t)\right) &= E_t\left[g(T)\right] - \int_t^T F_S\left(u,B+\right)P_T\left(t;u,B\right)du \\
&= E_t\left[g(T)\right] - F_S\left(T,B+\right)P(t;T,B) + \int_t^T F_{St}(u,B+)P(t;u,B)du \\
&= E_t\left[g(T)\right] - F_S\left(T,B+\right)P(t;T,B) + \sum_{i:T_i>t}^{n}\int_{t\wedge T_{i-1}}^{T_i} F_{St}(u,B+)P(t;u,B)du
\end{aligned}
$$

where the second equation follows from integration by parts and the fact that $P(t;t,B)=0$ when $S(t)>B$. For our process assumption, European option prices are increasing in maturity, whereby we can now write

$$
\underline{V}(t) \le F\left(t,S(t)\right) \le \overline{V}(t), \tag{17}
$$

where

$$
\underline{V}(t) = E_t\left[g(T)\right] - F_S\left(T,B+\right)P(t;T,B) + \sum_{i:T_i>t}^{n}\left[F_S(T_i,B+) - F_S(t\wedge T_{i-1},B+)\right]P(t;\underline{T}_i,B),
$$

$$
\overline{V}(t) = E_t\left[g(T)\right] - F_S\left(T,B+\right)P(t;T,B) + \sum_{i:T_i>t}^{n}\left[F_S(T_i,B+) - F_S(t\wedge T_{i-1},B+)\right]P(t;\overline{T}_i,B),
$$

$$
\underline{T}_i = \begin{cases} T_i, & F_S(T_i,B+) \le F_S(t\wedge T_{i-1},B+) \\ t\wedge T_{i-1}, & F_S(T_i,B+) > F_S(t\wedge T_{i-1},B+); \end{cases}
$$

$$
\overline{T}_i = \begin{cases} t\wedge T_{i-1}, & F_S(T_i,B+) \le F_S(t\wedge T_{i-1},B+) \\ T_i, & F_S(T_i,B+) > F_S(t\wedge T_{i-1},B+). \end{cases}
$$

To test the tightness of the above bounds on F consider the special case of a down-and-out call option with strike $K=100$, maturity $T=1$, spot asset price $S(0)=100$, and barrier $B=90$. We assume that the stock volatility is constant at $s=0.25$. For this case, a closed-form solution exists for the option (e.g. Merton 1973), and all terms in (17) can be computed without resolving to numerical methods. Notice also that here $F_S(T,B+)P(t;T,B)=0$, $E_t[g(T)]=C(t;T,K)$, and $F_{St}\le 0$ at the barrier whereby $\overline{T}_i = t\wedge T_{i-1}$ and $\underline{T}_i = T_i$. Table 2 below shows the bounds in (17) as a function of the number of equally spaced put maturities (n). For reference we also report the value of a hedge based on a simple mid sum approximation to the integral.

4.3. Unbounded Delta

For many barrier options, the terminal payout function g is discontinuous at the barrier, resulting in an unbounded delta at the maturity of the barrier option. Common examples include for instance continuously monitored down-and-out puts with strike above the barrier, and continuously monitored up-and-out call options with strike below the barrier. The unbounded delta of such "in-the-money" barrier options is not only a problem for traditional dynamic delta hedging, but also affect our static hedges which involve option spread positions of size proportional to the delta at the barrier.

Let us focus on the specific example of an up-and-out call option with a flat continuous barrier *B*, no rebate, and a strike $K < B$. Our hedging equation is

$$F(0) = C(T,K) - C(T,B) + (B-K)C_K(T,B) + \int_0^T F_S(t,B-)C_T(t,B)dt\,.$$

Here we use calls in the replicating portfolio to avoid cash flows before the option expires or knocks out. Since $F_S(t,B-) \to -\infty$ for $t \uparrow T$, we would need to short an infinite number of maturity spreads in the hedge portfolio. To circumvent this problem we note that if we move the barrier slightly upwards by $e > 0$ without changing the terminal pay-off[8], we not only get an overhedge but also are able to bound the delta at the barrier. In fact, the delta of this option at *S* = *B* will tend to zero as we approach maturity. The resulting overhedge is

$$\overline{F}(0) = C(T,K) - C(T,B) + (B-K)C_K(T,B) + \int_0^T \overline{F}_S(t,e+B-)C_T\big(t,B+e\big)dt\,.$$

The choice of e is a matter of compromise: the larger e is, the more expensive the hedge becomes; the smaller e is, the larger (in absolute magnitude) the delta can become.

A more scientific approach to the problem of unbounded deltas has been suggested by Wystup (1997), and Schmock *et al* (1999). The authors impose constraints on the delta and shows that the cheapest super-replicating claim that satisfies this constraint can be found as the solution to a stochastic control problem. Interestingly, Wystup (1997) points out that the simple strategy of moving the barrier is a close approximation of the “correct” strategy. Wystup (1997) also gives an approximate link between the size of the barrier shift (e above) and the constraint on delta.

[8] That is, we keep $g(S) = (S-K)^+ 1_{S<B}$.

Table 2: Under- and overhedging of down-and-out call

$T = 1,\ K = 100,\ B = 90,\ R = 0,\ S(0) = 100$

n	Underhedge $\underline{V}(0)$	Mid Sum	Overhedge $\bar{V}(0)$
∞	7.1791	7.1791	7.1791
365	7.1743	7.1790	7.1838
183	7.1695	7.1790	7.1885
122	7.1648	7.1791	7.1933
91	7.1601	7.1791	7.1980
73	7.1553	7.1791	7.2028
52	7.1458	7.1791	7.2123
37	7.1317	7.1820	7.2266
18	7.0850	7.1826	7.2747
12	7.0395	7.1846	7.3241
8	6.9749	7.1912	7.4019
6	6.9151	7.2028	7.4850
4	6.8086	7.2404	7.6665
3	6.7168	7.2948	7.8672
2	6.5660	7.4456	8.3195

5. Conclusion

This paper has discussed the construction of static hedges for generalized barrier-type claims on stocks following a jump-diffusion process with state- and time-dependent volatility. The static hedge takes the form of a linear portfolio of European puts and calls which exactly matches the cash-flow from the option to be hedged. Allowing for time-dependent rebates, we have derived exact expressions for the composition of the hedging portfolio, the form of which depends both on the option to be hedged as well as the stock process. While our theoretical results assumes unlimited supply of European options and perfect knowledge of stock dynamics, we have discussed several practical techniques to relax such idealized assumptions.

Finally, we point out that while this paper has focused on barrier options, many other option types allow for a decomposition in terms of barrier options which again allow our hedging results to be applied. For instance, lookback and "ratchet" options can be synthesized by a "ladder" of continuously monitored barrier options (see e.g. Carr and Chou 1997), and can thus be statically hedged in our framework. Similarly, Bermudan options can, after the determination of the early exercise frontier, be treated as a discretely monitored barrier option (albeit with an asset-dependent rebate) and can be hedged by a static position of puts and calls maturing at each exercise date.

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Appendix A
Derivation of hedging equation using differential forms

Let $f(t,S)$ denote the density of *S* satisfying (1), and let $F(t,S)$ be the value of a knock-out option that knocks out on some set $B \subset \Omega$, $\Omega = [0,T] \times (0,\infty)$, where *B* is closed in Ω. Let $B^c = \Omega \setminus B$ denote the complement of *B*, and define the open set $\hat{B} = B^c \setminus \{(t,S): t = 0 \ or \ t = T\}$. We assume that $\hat{B}$ is a submanifold of Ω. Consider the differential form

$$w = Q(t,S)dS + P(t,S)dt \ ,$$

$$Q(t,S) = -f(t,S)F(t,S) \ ,$$

$$P(t,S) = \tfrac{1}{2}\frac{\partial F(t,S)}{\partial S} s(t,S)^2 S^2 f(t,S) - \tfrac{1}{2} F \frac{\partial\left(s(t,S)^2 S^2 f(t,S)\right)}{\partial S} \ .$$

Lemma A.1.
Let the submanifold $\hat{B}$ be as defined above, and let there be given a submanifold $M \subset \hat{B}$, with boundary curve ∂M lying entirely in $\hat{B}$. Then

$$\int_{\partial M} w = 0$$

Proof:
Given the assumptions about the topology of *M*, proving Lemma A.1 is equivalent to showing that w is closed in *M*, i.e. that

$$\frac{\partial Q}{\partial t} = \frac{\partial P}{\partial S}$$

for all $(t,S) \in M$. Now

$$\begin{aligned} -\frac{\partial Q}{\partial t} + \frac{\partial P}{\partial S} &= \left[f\frac{\partial F}{\partial t} + F\frac{\partial f}{\partial t} \right] + \left[\tfrac{1}{2} s^2 S^2 f \frac{\partial^2 F}{\partial S^2} - \tfrac{1}{2} F \frac{\partial^2 (s^2 Sf)}{\partial S^2} \right] \\ &= f\left[\frac{\partial F}{\partial t} + \tfrac{1}{2} s^2 S^2 \frac{\partial^2 F}{\partial S^2} \right] + F\left[\frac{\partial f}{\partial t} - \tfrac{1}{2} \frac{\partial^2 (s^2 S^2 f)}{\partial S^2} \right] \end{aligned}$$

On the submanifold *M*, *F* satisfies the backward equation (2), whereby the term multiplying *f* is zero. By the Fokker-Planck equation, we also have, for $t > 0$,

$$\frac{\partial f}{\partial t} - \tfrac{1}{2}\frac{\partial^2 (s^2 S^2 f)}{\partial S^2} = 0, \ \text{ s.t. } f(0,S) = d(S - S(0))$$

whence the term on F is also zero.?

As an application of the Lemma, consider now the case of a down-and-out barrier option where $B = \{(t,S): S \le B(t), t \in [0,T]\}$, for some continuous, positive function $B(t)$. Set

$$M = \{(t,S): S \in [B(t)+e, L], t \in [e, T-e)]\}$$

for two parameters $e > 0$ and L, where everywhere $L > B(t)+e$. Integrating around the boundary of *M*, and letting $L \to \infty$ and $e \downarrow 0$ we get from the Lemma:

$$F(0,S(0)) = \int_{B(T)}^{\infty} f(T,S)F(T,S)dS - \tfrac{1}{2}\int_{|0}^{T} F_S(t,B(t)+)s(t,B(t)+)^2 (B(t))^2 f(t,B(t))dt$$
$$+\tfrac{1}{2}\int_0^T R(t)\frac{\partial\left(s(t,S)^2 S^2 f(t,S)\right)}{\partial S}|_{S=B(t)+}dt + \int_0^T f(t,B(t))R(t)B'(t)dt \tag{A.1}$$

where we have used that

$$\lim_{e\downarrow 0}\int_{B+e}^{L} f(0,S)F(0,S)dS = F(0,S(0))$$

and assumed that $f(t,S)$ dies out sufficiently fast when S is increased to make the integral along $S = L$ vanish in the limit. In (A.1), we have introduced the rebate $R(t) = F(t,B(t))$.

To complete the derivation, integration of the Fokker-Planck equation yields

$$\tfrac{1}{2}\frac{\partial}{\partial S}\left[s(t,S)^2 S^2 f(t,S)\right]_{S=B(t)} = \int_0^{B(t)} \frac{\partial f(t,S)}{\partial t}dS = -f(t,B(t))B'(t) + \frac{\partial}{\partial t}\int_0^{B(t)} f(t,S)dS$$

Inserting this into (A.1) and performing integration by parts yields the desired result:

$$F(0,S(0)) = E[F(T,S)] - \tfrac{1}{2}\int_{|0}^{T} F_S(t,B(t)+)s(t,B(t)+)^2 B(t)^2 f(t,B(t))dt$$
$$-\int_0^T E\left[1_{S(t)\le B(t)}\right]R'(t)dt\,.$$

While Lemma A.1 is completely general and can be applied to virtually all types of barrier options, it is slightly inconvenient to work with and requires some rearrangements of the final results to yield a static hedge. Below, we have listed a more convenient form of Lemma A.1 expressed directly in terms involving puts $P(T,K)$[9]:

[9] By (2), Lemma A.2 can also easily be written in terms of call options.

Lemma A.2.
Let everything be as in Lemma A.1, and define

$$\begin{aligned}\overline{w} &= \left[\tfrac{1}{2}F_S(t,S)\,s(t,S)^2 S^2 P_{KK}(t,S) + P_K(t,S)F_t(t,S)\right]dt + P_K(t,S)F_S(t,S)dS \\ &= \tfrac{1}{2}F_S(t,S)\,s(t,S)^2 S^2 P_{KK}(t,S)dt + P_K(t,S)dF\end{aligned}$$

Then

$$\int_{\partial M}\overline{w} = 0.$$

Proof:
Set $Z(t,S) = F(t,S)\int_0^S f(t,s)ds$ and notice that

$$\begin{aligned}dZ(t,S) = F_t(t,S)\int_0^S f(t,s)dsdt + F(t,S)\int_0^S f_t(t,s)dsdt \\ +F_S(t,S)\int_0^S f(t,s)dsdS + F(t,S)\,f(t,S)dS\end{aligned} \tag{A.2}$$

From Lemma 1, on *M*,

$$\begin{aligned}w &= \tfrac{1}{2}F_S(t,S)\,s(t,S)^2 S^2 f(t,S)dt - \tfrac{1}{2}F(t,S)\frac{\partial\left(s(t,S)^2 S^2 f(t,S)\right)}{\partial S}dt - f(t,S)F(t,S)dS \\ &= \tfrac{1}{2}F_S(t,S)\,s(t,S)^2 S^2 f(t,S)dt - F(t,S)\int_0^S f_t(t,s)dsdt - f(t,S)F(t,S)dS \\ &= \tfrac{1}{2}F_S(t,S)\,s(t,S)^2 S^2 f(t,S)dt - dZ(t,S) + dF(t,S)\int_0^S f(t,s)ds.\end{aligned}$$

Here the first equation follows from the Fokker-Planck equation, and the second from (A.2). As dZ is an exact differential, the lemma follows by application of (2). ?

As a simple example, consider applying Lemma A.2. to a down-and-out option with a single step-down discontinuity at $t = T^*$. Specifically, we set

$$B(t) = \begin{cases} B_1(t),\ 0 \le t \le T^* \\ B_2(t),\ T^* < t \le T \end{cases}$$

where B_1 and B_2 are smooth functions, with $B_1(T^*) > B_2(T^*+)$. Using the same type of integration contour as in our previous example, we now get

Time 0+ vertical piece:

$$\int_{\infty}^{B(0)+} P_K(0,S)F_S(0,S)dS = -F(0,\infty) + F\big(0,S(0)\big)$$

Piece along $B_1(t)+$*, for* $t \in (0,T^*)$ *(where* $dF = R'(t)dt$*):*

$$\tfrac{1}{2}\int_0^{T^*-} F_S\big(t,B_1(t)+\big)s\big(t,B_1(t)\big)^2 B_1(t)^2 P_{KK}\big(t,B_1(t)\big)dt + \int_0^{T^*-} P_K\big(t,B_1(t)\big)R'(t)dt$$

Horizontal Piece from $(t,S) = \big(T^*-,B_1(T^*)+\big)$ *to* $(t,S) = \big(T^*+,B_1(T^*)+\big)$*:*

$$P_K\big(T^*,B_1(T^*)\big)\Big(F\big(T^*+,B_1(T^*)\big) - R(T^*)\Big)$$

Time T^*+ *vertical piece:*

$$\int_{B_1(T^*)}^{B_2(T^*+)} P_K(T^*,S)F_S(T^*+,S)dS = P_K\big(T^*,B_2(T^*+)\big)R(T^*) - P_K\big(T^*,B_1(T^*)\big)F\big(T^*+,B_1(T^*)\big)$$
$$-\int_{B_1(T^*)}^{B_2(T^*+)} F\big(T^*+,S\big)P_{KK}(T^*,S)dS$$

Piece along $B_2(t)+$*, for* $t \in (T^*,T)$*:*

$$\tfrac{1}{2}\int_{T^*+}^{T} F_S\big(t,B_2(t)+\big)s\big(t,B_2(t)\big)^2 B_2(t)^2 P_{KK}\big(t,B_2(t)\big)dt + \int_{T^*}^{T} P_K\big(t,B_2(t)\big)R'(t)dt$$

Time T- vertical piece:

$$\int_{B_2(T)+}^{\infty} P_K(T,S)F_S(T,S)dS = F(T,\infty) - P_K\big(T,B_2(T)\big)R(T) - \int_{B_2(T)+}^{\infty} g(S)P_{KK}(T,S)$$
$$= F(T,\infty) - \int_0^{\infty} g(S)P_{KK}(T,S)$$

Horizontal Piece at $S = L,\ L \to \infty$*:*

$$\int_T^0 P_K(t,\infty)F_t(t,\infty)dt = F(0,\infty) - F(T,\infty)$$

Adding all pieces, setting the sum to zero, and rearranging yields the desired static hedge decomposition:

$$F\big(0,S(0)\big) = \int_0^{\infty} g(S)P_{KK}(T,S) - \tfrac{1}{2}\int_0^T F_S\big(t,B(t)+\big)s\big(t,B(t)\big)^2 B(t)^2 P_{KK}\big(t,B(t)\big)dt$$
$$-\int_0^T P_K\big(t,B(t)\big)R'(t)dt - \int_{B_1(T^*)}^{B_2(T^*+)} \Big(F\big(T^*+,S\big) - R(T^*)\Big)P_{KK}(T^*,S)dS.$$

Finally, we wish to demonstrate that it is possible to formulate Theorem I in terms of circulation integrals. Consider the following:

Theorem A.1

Let everything be as in Lemmas A.1 and A.2. Let the connected components of the knock-out set B be denoted B_i. Then

$$F(0,S(0)) = E\left[F(S(T))\right] - \sum_i \int_{\partial B_i +} \overline{w} \tag{A.3}$$

where $\partial A+$ for a set A indicates a contour infinitesimally close to ∂A but just outside the set A wherever $\partial A \subset \operatorname{int}\Omega$, and which coincides with ∂A otherwise. The circulation integral in (A.3) should be performed counterclockwise.

Proof:

We define w and $\overline{w}$ as in Lemmas A.1 and A.2, but, using the rebate function $R(t)$, we extend their domains of definition from $\hat{B}$ to all of $\overline{\Omega}$, the closure of Ω in $\Re^2$. $\overline{\Omega}$ is a compact space, whose boundary includes the points at $S = \infty$. (We may alternatively obtain this type of boundary by a standard limiting procedure, as demonstrated earlier). The forms so defined contain singularities (at ∂B, and at $t = 0$ and $t = T$); however these are all integrable singularities, as their component functions are products of derivatives of piecewise-smooth functions, bounded on compact subsets. These forms are closed everywhere in B^c, and so we may choose a contour $\partial B^c - \subset B^c$ that is infinitesimally close to ∂B^c. By Lemmas A.1 and A.2, we find that

$$\int_{\partial B^c -} w = \int_{\partial B^c -} \overline{w} = 0 .$$

Now, B^c satisfies $B^c = \overline{\Omega} \setminus \overline{B}$, so that $\partial B^c = \partial\overline{\Omega} \setminus \partial\overline{B}$ in $\Re^2$, and also, that $\partial B^c - = \partial\overline{\Omega} \setminus \partial\overline{B} +$, so that

$$\int_{\partial B^c -} w = \int_{\partial \Omega} w - \sum_i \int_{\partial B_i +} w = 0 ; \quad \int_{\partial B^c -} \overline{w} = \int_{\partial \Omega} \overline{w} - \sum_i \int_{\partial B_i +} \overline{w} = 0 .$$

Note that the integrals on the right may run over much larger regions than those on the left, but these additional integrals cancel out. The extra pieces are in $\partial\overline{\Omega} \cap \partial\overline{B}$ (closure in $\Re^2$), and represent integrals along the lines $\{S = 0, t \in [0,T]\}$ and $\{S = \infty, t \in \lfloor 0,T \rfloor\}$ or the other parts of $\partial\overline{\Omega}$. These extra pieces make use of the extension of the forms w and $\overline{w}$ to $\overline{\Omega}$, because they are not in ∂B^c, and so w and $\overline{w}$ on these contours cannot be obtained as a limit of values in B^c.

Finally, we note that $w = \overline{w} + dZ$, with Z defined in the proof of Lemma A.2. The function Z is well defined everywhere in $\overline{\Omega}$, and, as dZ is an exact form on all of $\overline{\Omega}$,

$$\int_{\partial\overline{\Omega}} \overline{w} = \int_{\partial\overline{\Omega}} w \; .$$

Using the same technique as used in the example after Lemma A.1, it is easy to verify that, integrating counterclockwise,

$$\int_{\partial\overline{\Omega}} w = -F\left(0, S(0)\right) + E\left[F\left(T, S(T)\right)\right].$$

Thus, we have the final result. ?

Appendix B
Stochastic Volatility

Consider now the case where S follows the process

$$dS(t)/S(t) = s(t)dW(t),$$

where $s(t)$ is a stochastic process. As in Section 2.1, let F denote the price of a down-and-out option with a continuous barrier. Since the volatility is allowed to be stochastic, F will in general depend on other variables than time and stock price level, i.e.

$$F(t) = F\big(t, S(t), x(t)\big),$$

where x is a vector of state variables additional to time and current stock price. So Ito-Tanaka expansion of F yields

$$\begin{aligned} dF(t) &= 1_{S>B}dM(t) + 1_{S<B}R'(t)dt \\ &\quad + \tfrac{1}{2}\,d\big(S(t)-B(t)\big)F_S\big(t, B(t)+, x(t)\big)s(t)^2 B(t)^2 dt + \\ &\quad + 1_{S=B}\tfrac{1}{2}\Big[\sum_i F_{x_i S}dx_i dS + \sum_i\sum_j F_{x_i x_j}dx_i dx_j\Big]. \end{aligned}$$

However, if the limit F_{x_i} for $S \downarrow B$ exists and is finite almost everywhere, then continuity on $\{S > B\}$ and the fact that $F|_{S=B} = R$ imply that $F_{x_i} \to 0$ (a.e.) and hence we can ignore the terms in the sums. So, integrating over time and taking expectations yields

$$\begin{aligned} F(t) &= E_t\big[g(S(T))\big] - \int_t^T R'(u)E_t\big[1_{S(u)<B(u)}\big]du \\ &\quad - \tfrac{1}{2}\int_t^T E_t\big[F_S\big(u, B(u)+, x(u)\big)s^2(u)\big|S(u)=B(u)\big]B(u)^2 C_{KK}\big(t; u, S(u)\big)du. \end{aligned} \tag{B.1}$$

where M is a martingale. While equation (B.1) is a perfectly valid expression for the price of the barrier option, it does not constitute a static hedge. The reason is, of course, that the terms $E_t[F_S\big(u, B(u)+, x(u)\big)s^2(u)|S(u)=B(u)]$ are stochastic and move around as calendar time passes. As a consequence, any butterfly hedge set up to replicate the last integral in (B.1) would need re-balancing over time. We note, that (B.1) may in some circumstances lead to static *over*- and *underhedges*, if one can find a robust way to bound $E_t[F_S\big(u, B(u)+, x(u)\big)s^2(u)|S(u)=B(u)]$.

If the barrier is discretely monitored, as in Section 2.2, we find the expression

$$F(t) = E_t\big[g(S(T))\big] - \sum_{i:t_i \geq t}\int_t^{B(t_i)}\Big(E_t\big[F\big(t_i+, S, x(t_i)\big)\big] - R(t_i)\Big)C_{KK}\big(t; t_i, S(t_i)\big)dS\ . \tag{B.2}$$

Again, this expression does not represent a static hedge.