Valuation of American Call Options*

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Abstract

The purpose of this paper is to provide an analytical solution for American call options assuming proportional dividends. Proportional dividends are more realistic for long-term options than absolute dividends and the formula does not have the flaws known from absolute dividend formulae.

Introduction

The holder of an American call option has the right but not the obligation to buy the underlying share at a fixed price, the strike, any time until expiry of the option. In the case of no dividends it would not make any sense to execute an American call option prematurely because the option could always be sold at a higher price than the inner value of the option, the difference being the time value. Nevertheless an early execution can be advisable if the share pays dividends. At payment the share price decreases by the dividend. If the dividend is higher than the time value of the option after the payment then the holder should execute the option.

Roll (1977), Geske (1979, 1981), and Whaley (1981) provided an analytical solution for the case of known absolute dividends. As a shortcoming we can only assume the ex-dividend share price (i.e. the share price net of the discounted dividend payment) to follow a geometric Brownian motion. For options with long maturities, and consequently several dividend payments, this corresponds to a considerable reduction of the actual volatility, because only the ex-dividend share price is assumed to fluctuate. Several authors tried to adjust the volatility (Chriss (1997) or Haug and Haug (1998)). In an extreme case, assuming the dividend discount model (i.e. the share price equals the sum of all discounted dividend payments) the share price for a perpetual American call wouldn't even fluctuate anymore. Of course this is an inadmissible exaggeration of the Roll-Geske-Whaley model; its authors only assumed the one dividend payment case as a realistic scenario. Nevertheless the example of a perpetual call exhibits the problem of the ex-dividend share price. As a second drawback we repeat Geske's doubt (1979) that dividends are difficult to forecast in such precise terms.

Dividend Payments: Absolute vs. Proportional

It is easier to forecast dividends as a proportion of the share price, especially for long-term predictions. Dividends are intuitively correlated with the share price. It is therefore unrealistic to assume the same dividend for a high and for a low share price. Dividend estimation as a proportion of the share price takes account of this issue. Furthermore, as comfortable side effect of this formulation we can assume the cum-dividend share price S and the ex-dividend share price $(1-d)S$, d being the dividend proportion, to follow a geometric Brownian motion, i.e. to be log-normally distributed: If $V(S,T)$ is a solution to the Black-Scholes formula[1], then $V((1-d)S,T)$ as well. This corresponds to keeping a binomial tree recombining, if such an algorithm is used.

In what follows we are going to present a formula for American call options on proportional dividend paying shares. Nevertheless, we have to admit that finally the formula is not as canonical as it seems because some parameters have to be found by means of an algorithm.

*The author would like to thank Dr. Liesbeth Thomassen from the University of Antwerp, Belgium, for scrutinising all formulae and many helpful remarks.

Assumptions

We use the same assumptions as Roll, Geske and Whaley with the exception of the formulation of the dividend payment[2]:

1. All individuals can borrow or lend without restriction at the instantaneous riskless rate of interest, r, and that rate is constant through the life of the option, T.
2. At the ex-dividend instant, $t(t < T)$, the stock pays a dividend dS, $0 < d < 1$, which induces a stock price decline of dS.
3. The stock price is described by the stochastic differential equation $\frac{dS}{S} = \mu dt + \sigma dz$, where μ is the instantaneous expected rate of return on the common stock, σ is the instantaneous standard deviation of stock price return (assumed to be constant over the life of the option), and dz is a standard unit normally distributed value.

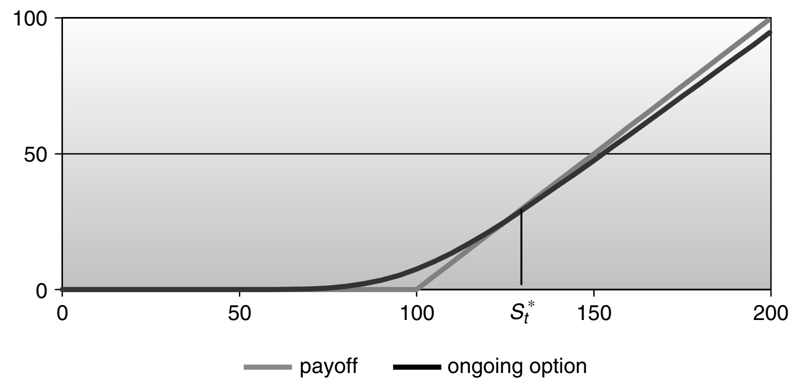
Additionally, an assumption of perfect capital markets is invoked.

Development of the Formula

In a first step we suppose that there will be only one dividend payment at $t > 0$ and the option expires at $T > t$. At t the option holder will exercise the option if the current payoff is higher than the ongoing option value. Cox and Ross (1975) showed that maintaining a hedging portfolio we can express the option value as a riskless discounted expectation:[3]

$$V_1(S, t, T, d) = e^{-rt}\max[S_t - K, V_0((1-d)S_t, T-t)]$$

There is one value S_t^* for which $S_t^* - K = V_0((1-d)S_t^*, T-t)$ holds. Above the threshold S_t^* the holder exercises the option, as shown in the figure.



Hence the option value can be expressed in the following way:

$$V_1(S, t, T, d) = e^{-rt}\left[\int_0^{S_t^*} e^{-r(T-t)}\int_K^{\infty}((1-d)S_T - K)dP(S_t, S_T)dP(S, S_t) + \int_{S_t^*}^{\infty}(S_t - K)dP(S, S_t)\right]$$

$$= e^{-rt}\left[\int_0^{S_t^*} call((1-d)S_t, T-t)dP(S, S_t) + \int_{S_t^*}^{\infty}(S_t - K)dP(S, S_t)\right]$$

$dP(S_t, S_T)$ being the probability distribution of S_T, knowing S_t. This expression becomes (see appendix for precise development):

$$V_1(S, t, T, d) = (1-d)SN_2\left(-d_1\left(\frac{S}{S_t^*}, t\right), d_1\left(\frac{(1-d)S}{K}, T\right), -\sqrt{\frac{t}{T}}\right)$$
$$- Ke^{-rT}N_2\left(-d_2\left(\frac{S}{S_t^*}, t\right), d_2\left(\frac{(1-d)S}{K}, T\right), -\sqrt{\frac{t}{T}}\right)$$
$$+ SN\left(d_1\left(\frac{S}{S_t^*}, t\right)\right) - Ke^{-rt}N\left(d_2\left(\frac{S}{S_t^*}, t\right)\right)$$

with N_2 being the bivariate cumulative normal density function and the two functions $d_1(x, y) = (\ln(x) + (r + (1/2)\sigma^2)y)/\sigma\sqrt{y}$ and $d_2(x, y) = d_1(x, y) - \sigma\sqrt{y}$.

Extension

The above-shown development can be extended to various dividend payments as Geske did (1979). We assume an option V_n on a stock paying n dividends until expiry. The dividend payments amount each time to the proportion d[4] of the stock and take place at the moments $t_1, \ldots, t_n$, with t_{n+1} being the expiry. Like in the above-described case of one dividend it is necessary to calculate the critical values $S_{t_i}^*$, above which the holder exercises the option. The complete calculation of the value then involves $(n+1)$-variate normal distributions.

$$V_n(S, t_1, \ldots, t_{n+1}, d) = \sum_{i=1}^{n+1}(1-d)^{i-1}SN_i\left(-d_1\left(\frac{S}{S_{t_1}^*}, t_1\right);\right.$$
$$\left. -d_1\left(\frac{(1-d)S}{S_{t_2}^*}, t_2\right)\ldots; +d_1\left(\frac{(1-d)^{i-1}S}{S_{t_i}^*}, t_i\right); \Xi^i\right)$$
$$-\sum_{i=1}^{n+1}e^{-rt_i}KN_i\left(-d_2\left(\frac{S}{S_{t_1}^*}, t_1\right); -d_2\left(\frac{(1-d)S}{S_{t_2}^*}, t_2\right)\ldots;\right.$$
$$\left. +d_2\left(\frac{(1-d)^{i-1}S}{S_{t_i}^*}, t_i\right); \Xi^i\right)$$

with $S_{t_i}^*$ such that $V_{n-i}((1-d)S_{t_i}^*, t_{i+1}, \ldots, t_{n+1}, d) = S_{t_i}^* - K$, $S_{t_{n+1}}^* = K$, and Ξ being the correlation matrix with $\Xi_{lk}^i = \sqrt{t_l/t_k}$, $l \leq k < i$, $\Xi_{kl}^i = \Xi_{lk}^i$; $\Xi_{ik}^i = -\sqrt{t_k/t_i}$, $k < i$, $\Xi_{ik}^i = \Xi_{ki}^i$. $N_i(a_1, \ldots, a_i, \Xi)$ being the cumulative distribution function of an i-variate standard normal distribution with mean 0 and correlation matrix Ξ, as defined above.

The calculation of the thresholds $S_{t_i}^*$ requires algorithmic calculation (e.g. bisection algorithm).

For a similar extension of the RGW-formula and n-stage compound options consult Thomassen and Van Wouwe (2004).

Discussion

Accuracy of the formula. Comparing the results with binomial tree values (2,000 time steps) for options with one intermediate dividend payment (e.g. as mentioned by Vorst (2001)) proves the accuracy of the formula:

We can assume that the small differences stem from the polynomial approximation of the bivariate cumulative normal density function and from

Stock	Strike	d	t_1	T	r	σ	Binomial	Formula
100	80	5%	0.5	1	4%	30%	26.088	26.086
100	100	5%	0.5	1	4%	30%	11.498	11.498
100	120	5%	0.5	1	4%	30%	8.607	8.609
100	100	3%	0.5	1	4%	30%	12.164	12.163
100	100	7%	0.5	1	4%	30%	11.015	11.013
100	100	5%	0.2	1	4%	30%	10.900	10.900
100	100	5%	0.8	1	4%	30%	12.633	12.632
100	100	5%	0.5	0.6	4%	30%	9.597	9.597
100	100	5%	0.5	2	4%	30%	17.132	17.132
100	100	5%	0.5	1	3%	30%	11.164	11.164
100	100	5%	0.5	1	5%	30%	11.841	11.840
100	100	5%	0.5	1	4%	20%	7.792	7.791
100	100	5%	0.5	1	4%	40%	15.226	15.227

the inaccuracy of the binomial tree algorithm. The threshold S^* has been calculated with a bisection algorithm of 100 steps, i.e. up to the 28th digit. These threshold values can be calculated to a very high accuracy with relatively small effort. Given these values, the formula provides the exact price, while a tree algorithm can always be improved by increasing the number of steps. Nevertheless, we must add that the complexity of the binomial tree does not depend on the number of dividend payments, while the computation time of the formula increases at least linearly with the number of dividend payments. On the other hand, the formula is independent of the maturity, while the tree needs usually more time-steps for longer maturities.

Dividend estimation. Companies usually announce dividend payments in advance and from this moment the dividend payment is known. In that case the proportional dividend approach blurs actually accurate information. On the other hand, the proportional dividend assumption seems more reasonable for options with long maturities. Proportional dividends are as discussable as absolute dividends regarding the accuracy of prediction. Nevertheless with absolute dividends the valuation must pay attention that the share price does not turn negative. This can be done either by using a mirror at the level of the dividend (i.e. not allowing the share price to fall below the dividend payment, similar to barrier options), or by using only the ex-dividend share price as Roll-Geske-Whaley did. But after having read the next paragraph the reader should definitely be inclined to assuming proportional dividends.

Flaws of the Roll-Geske-Whaley formula. Haug, Haug, and Lewis mention that the Roll-Geske-Whaley formula has some serious flaws allowing arbitrage. They use the following example: The stock price is 100, strike 130, the option matures after exactly one year with a dividend payment of 7 right before maturity, at 0.9999 years. Interest rates are at 6%, volatility at 30%. They compare the RGW price with an option on the same stock maturing after 0,9998 years. The prices should actually be the same, otherwise there is a possibility of arbitrage. And in deed, the respective values are 4.30 for the 1 year option and 4.92 for the 0.9998 years option. This difference is at first sight disturbing. If we dig a bit deeper the worries about the formula fly away, but some annoyance remains: The reason for the price difference stems from the fact that in the RGW formula only the ex-dividend stock is allowed to fluctuate with a volatility of 30% while in the 0.9998 years option the full stock fluctuates. The RGW price corresponds to an 0.9998 years option whose underlying is at 100-exp(-r*0.9999)*7 and strikes at 130-7. For these values we get the expected price of 4.92. The main problem of the RGW formula is therefore that the volatility does not act on the whole stock, but only the ex-dividend stock. The problem exhibited by Haug, Haug, and Lewis is based on the fallacy that the input parameters should be the same. They have shown that this is not the case and that the volatility must be adjusted. Nevertheless, the formula assumes one single volatility for the whole life span of the option and any adjustment is therefore only a compromise. The above-presented proportional dividend method does not suffer from this problem. For the 1 year option, assuming a dividend payment of 4.5% (which is on average exactly 7 if the option is in the money at dividend payment) it yields an option value of 4.92 as required.

The above-presented formula has still some difficulties to cope with a volatility term structure, i.e. with different volatilities for different maturities, as most non-algorithmic valuations. Intuitively it seems promising to use the volatilities according to maturity (t_i) and strike ($S^*_{t_i}$) in the respective terms (d_1, d_2), but this method has no analytical justification and requires some adjustments in the correlation matrix.

Conclusion

We provided an accurate pricing formula for American calls with proportional dividends. The hypothesis of proportional dividends seems more reasonable for options with longer life times. As main advantage of proportional dividends the volatility remains unaffected by the dividend payment. This avoids volatility adjustments.

Appendix

We must develop the term

$$V_1(S,t,T,d) = e^{-rt}\left[\int_0^{S_t^*} call((1-d)S_t, T-t)\,dP(S,S_t) + \int_{S_t^*}^{\infty} (S_t - K)\,dP(S,S_t)\right]$$

$$= \underbrace{e^{-rt}\int_0^{x^*}(1-d)Se^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}x} N\left(\frac{\ln\left(\frac{(1-d)S_t}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)\varphi(x)dx}_{1}$$

$$\underbrace{-\,e^{-rt}\int_0^{x^*} KN\left(\frac{\ln\left(\frac{(1-d)S_t}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)\varphi(x)dx}_{2}$$

$$\underbrace{+\,SN\left(\frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right)}_{3}\underbrace{-\,e^{-rt}KN\left(\frac{\ln\left(\frac{S}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}\right)}_{4}$$

with $x^* = \dfrac{\ln\left(\frac{S^*}{S}\right)-\left(r-\frac{1}{2}\sigma^2\right)t}{\sigma\sqrt{t}}$ and $\varphi(x) = \dfrac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$.

For this we use the identity developed in Geske (1979a)

$$N_2(h,k,\rho)=\int_{-\infty}^{h}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}N_1\left(\frac{k-\rho x}{\sqrt{1-\rho^2}}\right)dx$$

The first expression is the most complex to develop:

$$e^{-rt}\int_0^{x^*}(1-d)Se^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}x}$$

$$\times N\left(\frac{\ln\left(\frac{(1-d)S}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}x+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)\varphi(x)dx$$

$$=\int_0^{x^*}(1-d)Se^{-\frac{1}{2}\sigma^2t+\sigma\sqrt{t}x}$$

$$\times N\left(\frac{\ln\left(\frac{(1-d)S}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{t}x+\sigma^2(T-t)}{\sigma\sqrt{T-t}}\right)\varphi(x)dx$$

$$=\int_0^{x^*}(1-d)Se^{-\frac{1}{2}\sigma^2t+\sigma\sqrt{t}x}N\left(\frac{k}{\sqrt{1-\rho^2}}+\frac{\sqrt{t}(x-\sigma\sqrt{t})}{\sqrt{T-t}}\right)\varphi(x)dx$$

with $k=\dfrac{\ln\left(\frac{(1-d)S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}$ and $\rho=-\sqrt{\frac{t}{T}}$. A change of variable $z=x-\sigma\sqrt{t}$ we get the following:

$$\int_0^{x^*+\sigma\sqrt{t}}(1-d)SN\left(\frac{k-\rho z}{\sqrt{1-\rho^2}}\right)\varphi(z)dz=(1-d)SN_2(x^*+\sigma\sqrt{t};k;\rho)$$

Expression 2 can be solved in a very similar way:

$$e^{-rt}\int_0^{x^*}KN\left(\frac{\ln\left(\frac{(1-d)S}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)t+\sigma\sqrt{t}x+\left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right)\varphi(x)dx$$

$$=e^{-rt}\int_0^{x^*}KN\left(\frac{\ln\left(\frac{(1-d)S}{K}\right)+\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{t}x}{\sigma\sqrt{T-t}}\right)\varphi(x)dx$$

$$=e^{-rt}\int_0^{x^*}KN\left(\frac{k-\sigma\sqrt{T}-\rho x}{\sqrt{1-\rho^2}}\right)\varphi(x)dx=e^{-rt}KN_2(x^*;k-\sigma\sqrt{T};\rho)$$

Expressions 3 and 4 are already solved.

FOOTNOTES & REFERENCES

1. The Black-Scholes formula being $V_t + rSV_s + \frac{1}{2}\sigma^2S^2V_{SS} - rV = 0$.
2. Roll, Geske and Whaley also used a factor $0 < \alpha < 1$, taking account of tax effects. Consequently, shareholders only received αD as dividend, D being the dividend payment. The share price declined by the same amount. We assume without loss of generality an ex-tax dividend.
3. We use the notation V_i for an American call option on a stock paying i dividends until expiry.
4. For ease of notation we assume one dividend proportion d. The formula equally holds for different proportions $d_1,\ldots,d_n$.

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