Equity derivatives, with a particular emphasis on the South African market

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Financial Modelling Agency is the name of the consultancy of Graeme and Lydia West. They are available for training on, consulting on and building of derivative and financial models.

This is a course concerning equity and equity derivatives, with a particular emphasis on South African market conditions. Thus, we begin off topic: by bootstrapping a South African yield curve. This we have achieved by Chapter 3. Next we look at equity, equity futures and equity futures option trading on the JSE and its subsidiary SAFEX, in Chapter 4 to Chapter 8. In Chapter 9 we briefly consider how structures of options are packaged together.

We prepare for exotic equity derivative pricing: we have prepared ourself with a good model for interest rates, and we discuss volatility in Chapter 6 and Chapter 8. We are similarly concerned with dividends in Chapter 4 and Chapter 10.

In Chapter 11 we consider that unique South African instrument: the BEE transaction.

We then consider various exotic equity derivatives: variance swaps in Chapter 12, compound options in Chapter 13, asian options in Chapter 14, barrier options (including rebates) in Chapter 15, and forward starting options in Chapter 16. This list is probably pretty close to the complete set of options that have recently or are currently traded in the South African market.

The model we consider is typically geometric Brownian motion. This is not always the best model to use, but typically it is used, with various skew modifications. The most sophisticated market participants will use a stochastic volatility framework, or Lévy models possibly with jumps, for pricing and hedging. This will be mentioned here, but will not be examined in any detail.

Contents

Chapter 1

The time value of money

1.1 Discount and capitalisation functions

Much of what is said here is a reprise of the excellent introduction in [Rebonato, 1998, §1.2].

Time is measured in years, with the current time typically being denoted t.

We have two basic functions: the capitalisation function and the discount function. $C(t,T)$ and called the capitalisation factor: it is the redemption amount earned at time T from an investment at time t of 1 unit of currency. Of course $C(t,T) > 1$ for $T > t$ as the owner of funds charges a fee, known as interest, for the usage of their funds by the counterparty.

The price of an instrument which pays 1 unit of currency at time T - this is called a discount or zero coupon bond - is denoted $Z(t,T)$. This is the present value function. A fundamental result in Mathematics of Finance is (intuitively) that the value of any instrument is the present value of its expected cash flows. So the present value function is important.

We say that 1 grows to $C(t,T)$ and $Z(t,T)$ grows to 1. If $Z(t,T)$ grows to 1, then $Z(t,T)C(t,T)$ grows to $C(t,T)$, and so

$$Z(t,T)C(t,T) = 1. \tag{1.1}$$

Note also that $Z(t,t) = 1 = C(t,t)$. The next most obvious fact is that $Z(t,T)$ is decreasing in T (equivalently, $C(t,T)$ is increasing).

Suppose $Z(t,T_1) < Z(t,T_2)$ for some $T_1 < T_2$. Then the arbitrageur will buy a zero coupon bond for time T_1, and sell one for time T_2, for an immediate income of $Z(t,T_2) - Z(t,T_1) > 0$. At time T_1 they will receive 1 unit of currency from the bond they have bought, which they could keep under their bed for all we care until time T_2, when they deliver 1 in the bond they have sold.

What we have said so far assumes that such bonds do trade, with sufficient liquidity, and as a continuum i.e. a zero coupon bond exists for every redemption date T. In fact, such bonds rarely trade in the market. Rather what we need to do is impute such a continuum via a process known as bootstrapping.

Figure 1.1: The arbitrage argument that shows that $Z(t,T)$ must be decreasing.

1.2 Continuous compounding

The term structure of interest rates is defined as the relationship between the yield-to-maturity on a zero coupon bond and the bond's maturity. If we are going to price derivatives which have been modelled in continuous-time off of the curve, it makes sense to commit ourselves to using continuously-compounded rates from the outset. The time t continuously compounded risk free rate for maturity T, denoted $r(t,T)$, is given by the relationships

$$C(t,T) = \exp(r(t,T)(T-t)) \tag{1.2}$$

$$r(t,T) = \frac{1}{T-t}\ln C(t,T) \tag{1.3}$$

$$Z(t,T) = \exp(-r(t,T)(T-t)) \tag{1.4}$$

$$r(t,T) = -\frac{1}{T-t}\ln Z(t,T) \tag{1.5}$$

The rates will be known, or derived from a gentle model, for a set of times $t_1,\ t_2,\ \ldots,\ t_n$; let us abbreviate these rates as $r_i = r(t_i)$ for $1 \le i \le n$. Suppose that the rates $r_1,\ r_2,\ \ldots,\ r_n$ are known at the ordered times $t_1,\ t_2,\ \ldots,\ t_n$. Any interpolation method of the yield curve function $r(t)$ will construct a continuous function $r(t)$ satisfying $r(t_i) = r_i$ for $i = 1,\ 2,\ \ldots,\ n$. Various interpolation methods are reviewed, and a couple of new ones are introduced, in Hagan and West [2006], Hagan and West [2008]. In so-called normal markets, yield curves are upwardly sloping, with longer term interest rates being higher than short term. A yield curve which is downward sloping is called inverted. A yield curve with one or more turning points is called mixed. It is often stated that such mixed yield curves are signs of market illiquidity or instability. This is not the case. Supply and demand for the instruments that are used to bootstrap the curve may simply imply such shapes. One can, in a stable market with reasonable liquidity, observe a consistent mixed shape over long periods of time.

1.3 Annual compounding frequencies

Suppose we are using an actual/365 day count basis.

Example 1.3.1. I can receive 12.5% at the end of the year, or 3% at the end of each 3 months for a year. Which option is preferable?

Option 1: receive 112.5% at end of year. So $C(0,1) = 1.125$. We say: we earn 12.5% NACA (Nominal Annual Compounded Annually)

Option 2: receive 3% at end of 3, 6 and 9 months, 103% at end of year. We say: we earn 12% NACQ (Nominal Annual Compounded Quarterly). Then $C(0,1) = 1.03^4 = 1.1255...$, so this option is preferable.

This introduces the idea of compounding frequency. We have NAC* rates (* = A, S, Q, M, W, D) as well as simple rates, where

A	annual	standard
S	semi-annual	bonds
Q	quarterly	LIBOR, JIBAR (not really!)
M	monthly	call rate, credit cards
W	weekly	carry market
D	daily	overnight rates

Example 1.3.2. I earn 10% NACA for 5 years. The money earned by time 5 is $(1+10\%)^5 = 1.610510$.

Example 1.3.3. I earn 10% NACQ for 5 years. The money earned by time 5 is $\left(1+\frac{10\%}{4}\right)^{4\cdot 5} = 1.638616$.

Example 1.3.4. I earn 10% NACD for 5 years. The money earned by time 5 is $\left(1+\frac{10\%}{365}\right)^{365\cdot 5} = 1.648608$.

There is no such calculation for weeks. There are NOT 52 weeks in a year.
If r is the rate of interest NACn, then at the end of one year we have $\left(1+\frac{r}{n}\right)^n$, in general

$$C(t,T) = \left(1+\frac{r}{n}\right)^{n(T-t)} \tag{1.6}$$

We always assume that the cash is being reinvested as it is earned. Now, the sequence $\left(1+\frac{r}{n}\right)^n$ (for r fixed) is increasing in n, so if we are to earn a certain numerical rate of interest, we would prefer compounding to occur as often as possible. The continuous rate NACC is actually the limit of discrete rates:

$$e^r = \lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n \tag{1.7}$$

Thus the best possible option is, for some fixed rate of interest, to earn it NACC.
The assumption will always be that, in determining equivalent rates, repayments are reinvested. This implies that for the purpose of these equivalency calculations,

$$C(t,T) = C(t,t+1)^{T-t} \tag{1.8}$$

Example 1.3.5. The annual rate of interest is 12% NACA. Then the six month capitalisation factor (in the absence of any other information) is $\sqrt{1.12}$.

If $r^{(n)}$ denotes a NACn rate, then the following gives the equation of equivalence:

$$\left(1+\frac{r^{(n)}}{n}\right)^n = C(t,t+1) = \left(1+\frac{r^{(m)}}{m}\right)^m \tag{1.9}$$

and for a NACC rate,

$$C(t,T) = e^{r(T-t)} \tag{1.10}$$

where time is measured in exact parts of a year, using the relevant day count basis.

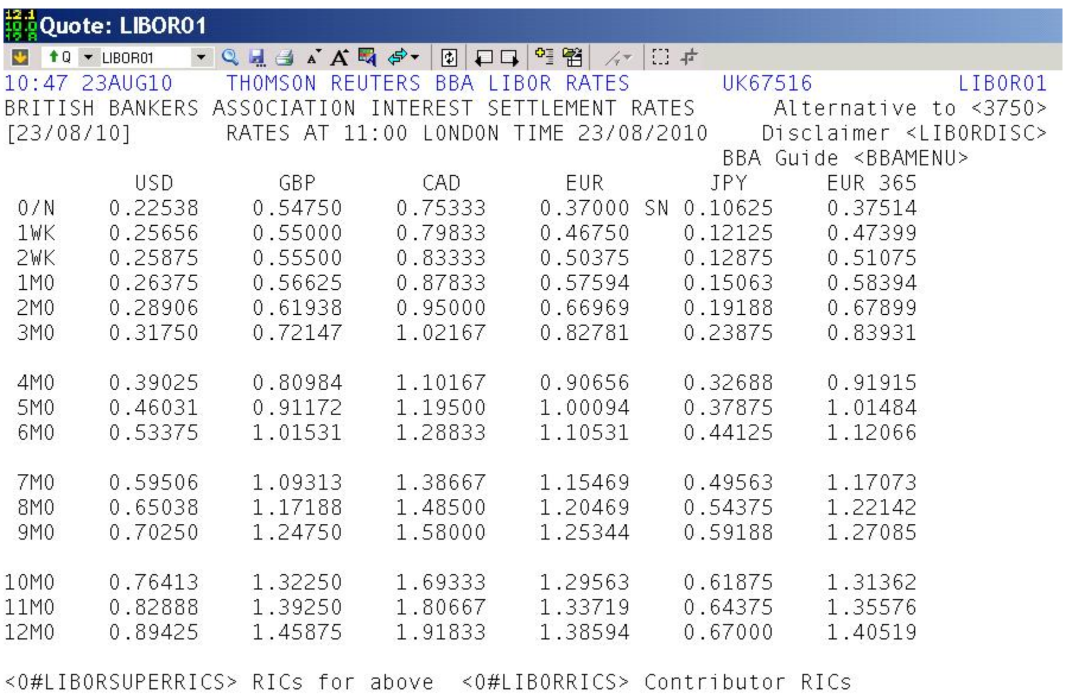
1.4 Simple compounding

In fact, money market interest rates always take into account the EXACT amount of time to payment, on an actual/365 basis, which means that the time in years between two points is the number of days between the points divided by 365.[1] Thus, whenever we hear of a quarterly rate, we must ask: how much exact time is there in the quarter? It could be $\frac{89}{365}$, $\frac{90}{365}$, or some other such value. It is never $\frac{1}{4}$! Thus, in reality, NACM, NACQ etc don't really exist in the money market. Simple rates are defined as follows: the period of compounding must be specified explicitly, and then

$$C(t,T) = 1 + y(T-t) \tag{1.11}$$

$$y = \frac{1}{T-t}\left[C(t,T) - 1\right] \tag{1.12}$$

where y is the simple yield rate, and where in all cases time is measured in years using the relevant day count convention. All IBOR rates are simple, such as LIBOR (London Inter Bank Offered Rate) or JIBAR (Johannesburg Inter Bank Agreed Rate).


Quote: LIBOR01

10:47 23AUG10 THOMSON REUTERS BBA LIBOR RATES UK67516 LIBOR01
BRITISH BANKERS ASSOCIATION INTEREST SETTLEMENT RATES Alternative to <3750>
[23/08/10] RATES AT 11:00 LONDON TIME 23/08/2010 Disclaimer <LIBORDISC>
BBA Guide <BBAMENU>

	USD	GBP	CAD	EUR	JPY	EUR 365
O/N	0.22538	0.54750	0.75333	0.37000	SN 0.10625	0.37514
1WK	0.25656	0.55000	0.79833	0.46750	0.12125	0.47399
2WK	0.25875	0.55500	0.83333	0.50375	0.12875	0.51075
1MO	0.26375	0.56625	0.87833	0.57594	0.15063	0.58394
2MO	0.28906	0.61938	0.95000	0.66969	0.19188	0.67899
3MO	0.31750	0.72147	1.02167	0.82781	0.23875	0.83931
4MO	0.39025	0.80984	1.10167	0.90656	0.32688	0.91915
5MO	0.46031	0.91172	1.19500	1.00094	0.37875	1.01484
6MO	0.53375	1.01531	1.28833	1.10531	0.44125	1.12066
7MO	0.59506	1.09313	1.38667	1.15469	0.49563	1.17073
8MO	0.65038	1.17188	1.48500	1.20469	0.54375	1.22142
9MO	0.70250	1.24750	1.58000	1.25344	0.59188	1.27085
10MO	0.76413	1.32250	1.69333	1.29563	0.61875	1.31362
11MO	0.82888	1.39250	1.80667	1.33719	0.64375	1.35576
12MO	0.89425	1.45875	1.91833	1.38594	0.67000	1.40519

<0#LIBORSUPERRICS> RICs for above <0#LIBORRICS> Contributor RICs

Figure 1.2: Reuter's page for LIBOR rates in various countries and tenors.

Simple discount rates are also quoted, although, with changes in some of the market mechanisms in the last few years, this is becoming less frequent. The instrument is called a discount instrument. Bankers Acceptances were like this; currently it needs to be specified if a BA is a discount or a yield instrument. Here

$$Z(t,T) = 1 - d(T-t) \tag{1.13}$$

[1] Actual/actual also occurs internationally, which is a method for taking into account leap years. 30/360 occurs in the USA: every month has 30 days and (hence) there are 360 days in the year. This is certainly the most convenient system for doing calculations. Actual/360 also occurs in the USA, which seems a bit silly.

where d is the discount rate, and where in all cases time is measured in years.

Example 1.4.1. Lend 1 at 20% simple for 2 weeks. Then, after 2 weeks, amount owing is $1+0.20\frac{14}{365}$ ie. $C\left(t, t+\frac{14}{365}\right)=1+0.20\frac{14}{365}$.

Example 1.4.2. Lend 1 at 12% simple for 91 days. The amount owing at the end of 91 days is $1+0.12\left(\frac{91}{365}\right)$.

Example 1.4.3. Buy a discount instrument of face value 1,000,000. The discount rate is 10% and the maturity is in 31 days. I pay $1,000,000\left(1-0.10\frac{31}{365}\right)$ for the instrument.

Another issue that arises is the 'Modified Following' Rule. This rule answers the question - when, exactly, is n months time from today? To answer that question, we apply the following criteria:

- It has to be in the month which is n months from the current month.
- It should be the first business day on or after the date with the same day number as the current. But if this contradicts the above rule, we find the last business day of the correct month.

For the following examples, refer to a calender with the public holidays marked. Once we are familiar with the concept, we can use a suitable excel function (which will be provided).

Example 1.4.4. What is 12-Feb-09 plus 6 months? Answer: 12-Aug-09. This is $16+31+30+31+30+31+12=181$ days, so the term is $\frac{181}{365}$.

Example 1.4.5. What is 11-May-09 plus 3 months? Answer: 11-Aug-09. This is $20+30+31+11=92$ days, so the term is $\frac{92}{365}$.

Example 1.4.6. What is 31-Jul-09 plus 2 months? Answer: 30-Sep-09. This is $31+30=61$ days, so the term is $\frac{61}{365}$.

Example 1.4.7. What is 30-Jan-09 plus 1 month? Answer: 27-Feb-09. This is $1+27=28$ days, so the term is $\frac{28}{365}$.

Always assume that if a rate of interest is given, and the compounding frequency is not otherwise given, that the quotation is a simple rate, and the day count is actual/365, and that Modified Following must be applied.

1.5 Yield instruments

JIBAR deposits, Negotiable Certificates of Deposit, Forward Rate Agreements, Swaps. The later two are yield based derivatives. The first two is a vanilla yield instrument: it trades at some round amount, that amount plus interest is repaid later. They trade on a simple basis, and generally carry no coupon; there can be exceptions to this if the term is long, but in general the term will be three months. Thus the repayment is calculated using (1.12).
The most common NCDs are 3 month and 1 months. The yield rate that they earn is (or is a function of) the JIBAR rate.

1.6 Discount instruments

Bankers Acceptances, Treasury Bills, Debentures, Commercial Paper, Zero Coupon Bonds. These are a promise to pay a certain round amount, and it trades prior to payment date at a discount, on a simple discount basis. The price a discount instrument trades at is calculated using (1.13). Treasury Bills and Debentures are traded between banks and the Reserve Bank in frequent Dutch auctions.

Promissory notes trade both yield and discount, about equally so at time of writing.

The default is that an instrument/rate is a yield instrument/rate. This wasn't always the case.

If d is the discount rate of a discount instrument and y is the yield rate of an equivalent yield instrument then

$$(1 + y(T - t))\,(1 - d(T - t)) = 1 \tag{1.14}$$

It follows by straightforward arithmetic manipulation that

$$\frac{1}{d} - \frac{1}{y} = T - t \tag{1.15}$$

1.7 Forward Rate Agreements

These are the simplest derivatives: a FRA is an OTC contract to fix the yield interest rate for some period starting in the future. If we can borrow at a known rate at time t to date t_1, and we can borrow from t_1 to t_2 at a rate known and fixed at t, then effectively we can borrow at a known rate at t until t_2. Clearly

$$C(t, t_1)C(t; t_1, t_2) = C(t, t_2) \tag{1.16}$$

is the no arbitrage equation: $C(t; t_1, t_2)$ is the forward capitalisation factor for the period from t_1 to t_2 - it has to be this value at time t with the information available at that time, to ensure no arbitrage.

In a FRA the buyer or borrower (the long party) agrees to pay a fixed yield rate over the forward period and to receive a floating yield rate, namely the 3 month JIBAR rate. At the beginning of the forward period, the product is net settled by discounting the cash flow that should occur at the end of the forward period to the beginning of the forward period at the (then current) JIBAR rate. This feature does not have any effect on the pricing, because the settlement received at the beginning of the forward period could be invested at that exact JIBAR rate and then only redeemed at the end of the forward period.

Figure 1.3: A long position in a FRA.

In South Africa FRA rates are always quoted for 3 month forward periods eg 3v6, 6v9, ..., 21v24 or even further. The quoted rates are simple rates, so if f is the rate, then

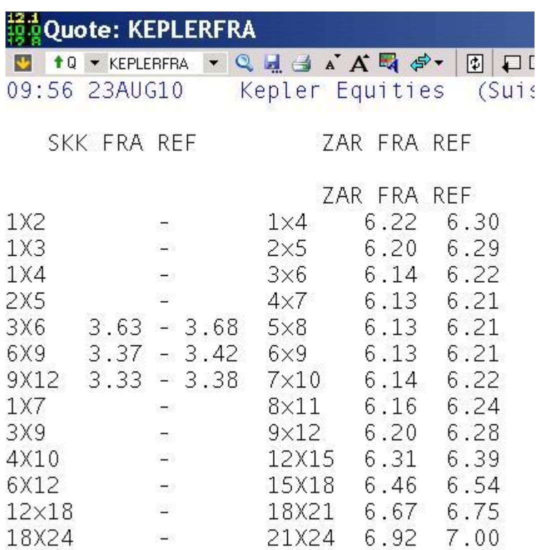
$$1 + f(t_2 - t_1) = C(t; t_1, t_2) = \frac{C(t, t_2)}{C(t, t_1)} \tag{1.17}$$

Thus

$$f = \frac{1}{t_2 - t_1}\left(\frac{C(t, t_2)}{C(t, t_1)} - 1\right) \tag{1.18}$$

One has to be careful to note the period to which a FRA applies. In an $n \times m$ FRA the forward period starts in n months modified following from the deal date. The forward period ends $m - n$ months modified following from that forward date, and NOT m months modified following from the deal date. Here, typically $m - n = 3$ - this is always the case in South Africa - or 6 or even 12. We will call this strip of dates the FRA schedule. In a swap, the i^{th} payment is $3i$ months modified following from the deal date. We will call this strip of dates the swap schedule. One must be careful not to apply the wrong schedule.

1.8 FRA quotes


Quote: KEPLERFRA
KEPLERFRA
09:56 23AUG10 Kepler Equities (Sui

SKK FRA REF				ZAR FRA REF	
1X2	-		1x4	6.22	6.30
1X3	-		2x5	6.20	6.29
1X4	-		3x6	6.14	6.22
2X5	-		4x7	6.13	6.21
3X6	3.63 -	3.68	5x8	6.13	6.21
6X9	3.37 -	3.42	6x9	6.13	6.21
9X12	3.33 -	3.38	7x10	6.14	6.22
1X7	-		8x11	6.16	6.24
3X9	-		9x12	6.20	6.28
4X10	-		12X15	6.31	6.39
6X12	-		15X18	6.46	6.54
12x18	-		18X21	6.67	6.75
18X24	-		21X24	6.92	7.00

Figure 1.4: Slovak koruna FRAs and South African ZAR FRAs quoted by a broker. Note that all the ZAR FRAs are 3 month, the SKK FRAs could have other lengths too (although it would appear there is very little liquidity). The rates are 'under reference' which also indicates poor liquidity.

1.9 Valuation

At the deal date, a fair FRA has value 0. But at some later date s before the setting date, $t < s < t_1$, the FRA will not have 0 value. How do we value the FRA?

Quote: GBPFRA=ICAP

GBPFRA=ICAP INTERCAPITAL LON DEALING

GBP	FRA's		
1x4	0.700	0.750	13:15
2x5	0.720	0.770	13:02
3x6	0.730	0.780	17:15
4x7	0.740	0.790	17:04
5x8	0.755	0.805	17:15
6x9	0.775	0.825	18:25
9x12	0.855	0.905	17:26
12x15	0.960	1.010	17:13
1x7	0.970	1.020	17:15
2x8	0.980	1.030	18:25
3x9	0.980	1.030	17:15
4x10	1.005	1.055	17:15
5x11	1.020	1.070	17:15
6x12	1.025	1.075	18:25
9X15	1.125	1.175	17:26
12X18	1.225	1.275	17:15
1X13	1.450	1.550	17:15
2X14	1.435	1.535	17:15
3X15	1.430	1.530	17:15
4X16	1.425	1.525	17:15
5X17	1.430	1.530	17:20
6X18	1.435	1.535	17:15
9X21	1.535	1.635	18:31
12X24	1.710	1.810	18:14

Figure 1.5: GBP FRAs for lengths 3, 6 and 12 months. The market is very liquid, as you can see from the quote time indication in the right column.

The value of the fixed payment is

$$V(s) = Z(s,t_2)r_K(t_2 - t_1) \tag{1.19}$$

What about the value of the floating payment? Consider Figure 1.6.
The payment we wish to value is the wavy line a (its size is not known up front). We add in b, c, d and the value of this is 0, because it is a fair transaction (we simply wait till time t_1 and borrow at the then ruling JIBAR rate). Now a and b are equal and opposite. Thus

$$V_a = V_a + V_b + V_c + V_d = V_c + V_d = Z(s,t_1) - Z(s,t_2) \tag{1.20}$$

Hence, the value of the FRA is given by

$$V(s) \quad = \quad Z(s,t_1) - Z(s,t_2)[1 + r_K(t_2 - t_1)] \tag{1.21}$$

Example 1.9.1. On 5-Aug-09 the JIBAR rate is 7.675% and the 5-Feb-10 rate is 7.4470% NACC (as read from my yield curve, for example). What is the 3v6 forward rate?
t = 5-Aug-09, t_1 = 5-Nov-09, t_2 = 5-Feb-10.
First we have

$$(1 + 7.675\% t_1)\, C(t;t_1,t_2) = e^{7.4470\%\, t_2}$$

where $t_1 = \frac{92}{365}$ and $t_2 = \frac{184}{365}$. So $C(t;t_1,t_2) = 1.018551$ and $f = 7.360\%$.

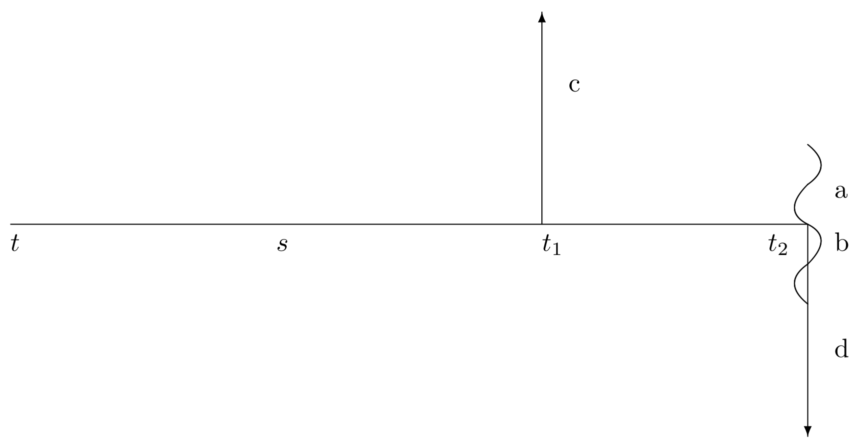


Figure 1.6: Valuing the floating payment.

For the next example, see the sheet MoneyMarketCurves.xls

Example 1.9.2. On 4-May-07 the JIBAR rate is 9.208%, the 3v6 FRA rate is 9.270%, and the 6v9 FRA rate is 9.200%. What are the 3, 6 and 9 month rates NACC?
Note that

$$\begin{aligned}
C(\text{4-May-07}, \text{6-Aug-07}) &= \left(1 + 9.208\%\frac{94}{365}\right) = 1.023714 \\
C(\text{4-May-07;6-Aug-07}, \text{6-Nov-07}) &= \left(1 + 9.270\%\frac{92}{365}\right) = 1.023365 \\
C(\text{4-May-07}; \text{5-Nov-07}, \text{5-Feb-08}) &= \left(1 + 9.200\%\frac{92}{365}\right) = 1.023189
\end{aligned}$$

Straight away we have $r(\text{6-Aug-07}) = 9.101\%$.
Note that we would really like to have $C(\text{4-May-07}; \text{6-Aug-07}, \text{5-Nov-07})$, but we don't. Also, we would like the sequence to end on 4-Feb-08, but it doesn't. So already, some modelling is required.
We have

$$C(\text{4-May-07}, \text{6-Nov-07}) = 1.023714 \cdot 1.023365 = 1.047633$$

We now want to find $C(\text{4-May-07}, \text{5-Nov-07})$.
To model this we interpolate between $C(\text{4-May-07}, \text{6-Aug-07})$ and $C(\text{4-May-07}, \text{6-Nov-07})$. For reasons that are discussed elsewhere, the most desirable simple interpolation scheme is to perform linear interpolation on the logarithm of capitalisation factors - so called raw interpolation. See Hagan and West [2008].
Thus we model that $C(\text{4-May-07}, \text{5-Nov-07}) = 1.047370$.
Thus $r(\text{5-Nov-07}) = 9.131\%$.
Now

$$C(\text{4-May-07}, \text{5-Feb-08}) = 1.047370 \cdot 1.023189 = 1.071658$$

Now interpolate between $C(\text{4-May-07}, \text{6-Nov-07})$ and $C(\text{4-May-07}, \text{5-Feb-08})$ to get

$$C(\text{4-May-07}, \text{4-Feb-08}) = 1.071391$$

Thus $r(\text{4-Feb-08}) = 9.119\%$.

Even within this model, the solution has not been unique. For example, we could have used $C(\text{4-May-07}, \text{6-Aug-07})$ and $C(\text{4-May-07;6-Aug-07}, \text{6-Nov-07})$ unchanged, and interpolated within $C(\text{4-May-07}; \text{5-Nov-07}, \text{5-Feb-08})$ to obtain $C(\text{4-May-07}; \text{6-Nov-07}, \text{4-Feb-08})$. However, this is quite an ad hoc approach. It makes sense to decree that one strictly works from left to right.

1.10 Exercises

1. You have a choice of 2 investments:

   (i) R100 000 invested at 12.60% NACS for 1 year

   (ii) R100 000 invested at 12.50% NACQ for 1 year.

   Which one do you take and why? (Explain Fully)

2. On 17-Jun-08 you sell a 1 Million, 3 month JIBAR instrument. At maturity you receive R1,030,246.58. What was the JIBAR rate?

3. (a) If I have an x% NACA rate, what is the general formula for y, the equivalent NACD rate in i.t.o x?

   (b) Now generalise this to convert from **any** given rate, r_1 (compounding frequency d_1) to another rate r_2 (compounding frequency d_2).

   (c) Now construct an efficient algorithm that converts from or to simple, NAC* or NACC. (Hint: given the input rate, find what the capitalisation factor is (for a year or for the period specified), if the rate is simple. Then, using this capitalisation factor, find the output rate.)

4. On 17-Jun-08 you are given that the 3-month JIBAR rate is 12.00% and the 3x6-FRA rate is 12.10%. What is the 6 month JIBAR rate?

5. If the 4 year rate is 9.00% NACM and the 6 year rate is 9.20% NACA, what is the 2 year forward rate (NACA) for 4 years time?

6. Suppose on 22-Jan-08 the 3 month JIBAR rate is 10.20% and the 6 month JIBAR rate is 10.25%. Show that the fair 3x6 FRA rate is 10.0446%.

7. Suppose on 3-Jan-08 a 6x9 FRA was traded at a rate of 12.00%. It is now 3-Apr-08. The NACC yield curve has the following functional form: $r(t,T) = 11.50\% + 0.01(T-t) - 0.0001(T-t)^2$ where time as usual is measured in years. Find the MtM of the receive fixed, pay floating position in the FRA.

   Note that the 6x9 FRA has now become a 3x6 FRA. However, the rate 12.00% is not the rate that the market would agree on for a new 3x6 FRA. Using the yield curve, what would be the fair rate for a newly agreed 3x6 FRA? (There are two ways of doing this: from scratch, or by using the information already calculated.)

Chapter 2

Swaps

2.1 Introduction

A plain vanilla swap is a swap of domestic cash flows, which are related to interest rates. We have a fixed payer and a floating payer; the fixed payer receives the floating payments and the floating payer receives the fixed payments.

The fixed payments are based on a fixed interest rate paid on a simple basis using the relevant day count convention, on some unit notional, which we will make 1. These interest rates are made in arrears (at the end of fixed periods), and these periods are always three or six monthly periods. The i^{th} payment is on date $\text{ModFol}(t, 3i)$ or $\text{ModFol}(t, 6i)$ where t is the initiation date of the product, and the period to which the interest rate applies is the period from $\text{ModFol}(t, 3(i-1))$ to $\text{ModFol}(t, 3i)$ (replace 3 with 6 when appropriate). Let this period be of length α_i. Then, if the fixed rate is R, the fixed payment made at the end of the period is $R\alpha_i$, the floating payment will be denoted $J_i\alpha_i$. J_i, the JIBAR rate, is observed at time t_{i-1}.

The floating payments are on the relevant JIBAR rate, on the same notional, paid in arrears every 3 months. The day count schedule thus could be quite different. As we do not know what the JIBAR rate will be in the future, we do not know what these floating payments will be, except for the next one (because that is already set - remember that the rate is set at the beginning of the 3 month period and the interest is paid at the end of the period). The date where the rate is set is known as the reset date.

Thus, a swap is not merely a strip of FRAs: not only are payments in arrears and not in advance, the day count schedule is slightly different.

The notionals are not exchanged here. In some more exotic swaps, such as currency swaps, the notional is exchanged at the beginning and at the termination of the product. Of course the fixed and floating payments, occurring on the same day, are net settled.

The diagram is that of the long party to the swap - fixed payer, floating receiver. He is making fixed payments, hence they are in the negative direction. He is receiving floating payments, so these are in the positive direction. We don't know what the floating payment is going to be until the fixing date (3 months prior to payment).

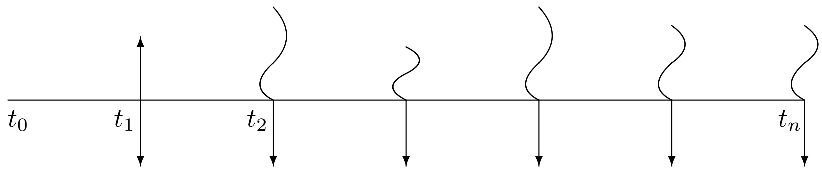


Figure 2.1: Fixed payments (straight lines) and floating payments (wavy lines) for a swap with fixed payments every three months.

2.2 Rationale for entering into swaps

Why would somebody wish to enter into a swap? This is dealt with in great detail in [Hull, 2005, Chapter 7]. The fundamental reason is to transform assets or liabilities from the one type into the other. If a company has assets of the one type and liabilities of the other, they are severely exposed to possible changes in the yield curve. Another reason dealt with there is the Comparative Advantage argument, but this is quite a theoretical concept.

Example 2.2.1. A service provider who charges a monthly premium (for example, DSTV, newspaper delivery, etc.) has undertaken with their clients not to increase the premium for the next year. Thus, their revenues are more or less fixed. However, the payments they make (salaries, interest on loans, purchase of equipment) are floating and/or related to the rate of inflation, which is cointegrated[1] with the floating interest rate. Thus, they would like to enter a swap where they pay fixed and receive floating. They ask their merchant bank to take the other side of the swap. This removes the risk of mismatches in their income and expenditure.

Example 2.2.2. A company that leases out cars on a long term basis receives income that is linked to the prime interest rate, again, this is cointegrated with the JIBAR rate. In order to raise capital for a significant purchase, adding to their fleet, they have issued a fixed coupon bond in the bond market. Thus, they would like to enter a swap where they pay floating and receive fixed. They ask their merchant bank to take the other side of the swap. This removes the risk of mismatches in their income and expenditure.

Of course, if the service provider and the car lease company know about each other's needs, they could arrange the transaction between themselves directly. Instead, they each go to their merchant bank, because

- they don't know about each other: they leave their finance arrangements to specialists.
- they don't wish to take on the credit risk of an unrelated counterparty, rather, a bank, where credit riskiness is supposedly fairly transparent.
- they wish to specify the nominal and tenor, the merchant bank will accommodate this; the other counterparty will have the wrong nominal and tenor.
- they don't have the resources, sophistication or administration to price or manage the deal.

[1] A statistical term, for two time series, meaning that they more or less move together over time. It is a different concept to correlation.

Sometimes the above arguments are not too convincing. Thus in some instances major service organisations form their own banks or treasuries - for example, Imperial Bank, Eskom Treasury, SAA Treasury, etc.

2.3 Valuation of the fixed leg of a swap

How do we value such a swap? Given a yield curve compatible with the swap, the fixed payments are clearly worth

$$V_{\text{fix}} = R \sum_{i=1}^{n} \alpha_i Z(t, t_i) \tag{2.1}$$

where R is the agreed fixed rate (known as the swap rate), n is the number of payments outstanding, and α_i is the length of the i^{th} 3 month period using the day count basis. This valuation formula holds whether or not today t is a reset date.

2.4 Valuation of the unknown flows in the floating leg of a swap

The difficulty in the valuation of the floating-rate side of the swap is that the cashflows $t_2, \ldots, t_n$ are not yet known. Remarkably, it is easy enough to value these flows anyway. What we do is to add to the schedule of actual swap cashflows another set of (imaginary) cashflows with zero total value, in such a way that the total set of cash flows can actually be calculated.

Our argument work for both a new and a seasoned swap. The imaginary cashflows to be added is a par floating rate note, with the commencement date being at t_1 and redemption at t_n. As we see in Figure 2.2, the result has easily determined value, it is

$$V_{\text{unknown}} = Z(t, t_1) - Z(t, t_n)$$

2.5 Valuation of the entire swap

The only flow that we have yet to deal with is the known flow on the floating side which occurs at time t_1. J_1 has already been fixed, at the previous reset date. The size of that payment is then $\alpha_1 J_1$ and its value is $Z(t, t_1)\alpha_1 J_1$.

Thus

$$\begin{aligned} V_{\text{float}} &= Z(t, t_1)\alpha_1 J_1 + Z(t, t_1) - Z(t, t_n) \\ &= Z(t, t_1)[1 + \alpha_1 J_1] - Z(t, t_n) \end{aligned} \tag{2.2}$$

Note that if today is the reset date, in other words $t = t_0$, then $1 + \alpha_1 J_1 = C(t, t_1)$, and so

$$V_{\text{float}} = 1 - Z(t, t_n)$$

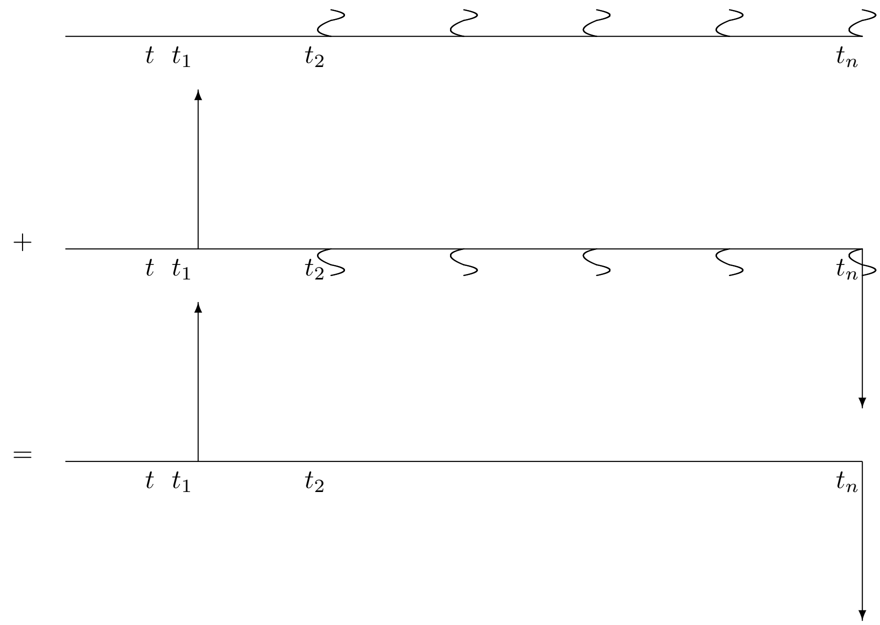


Figure 2.2: The unknown floating payments of the swap, combined with borrowing starting at time t_1 in a floating rate note, gives two fixed cash flows. Fixed payments and known floating payments of the swap are not shown.

2.6 New swaps

At inception, a swap is dealt at a rate $R = R_n$ which makes the value of the swap 0, in other words, the fixed payments equal in value to the floating payments. (Of course, when a bank does such a deal with a corporate client, they will weight the actual R_n traded in their favour.) The rates quoted in the market are, by virtue of the forces of supply and demand, deemed to be the fair rates, and R_n is then called the fair swap rate. Thus

$$R_n \sum_{i=1}^{n} \alpha_i Z(t, t_i) = 1 - Z(t, t_n)$$

$$R_n = \frac{1 - Z(t, t_n)}{\sum_{i=1}^{n} \alpha_i Z(t, t_i)} \tag{2.3}$$

This view has R_n as a function of the Z function. In fact the functional status is exactly the other way round when we do bootstrapping. A naïve attempt is as follows: we can inductively suppose that $Z(t, t_i)$ is known for $i = 1,\ 2,\ \ldots,\ n-1$, and R_n is known, to get

$$Z(t, t_n) + R_n \alpha_n Z(t, t_n) = 1 - R_n \sum_{i=1}^{n-1} \alpha_i Z(t, t_i)$$

$$Z(t, t_n) = \frac{1 - R_n \sum_{i=1}^{n-1} \alpha_i Z(t, t_i)}{1 + R_n \alpha_n} \tag{2.4}$$

Now $Z(t, t_i)$ is known for small i from the money market. Swap rates are quoted up to 30 years. The above formula could be used to bootstrap the curve out for that term, if there was no holes in the data - but there always are. A recommendation on how to resolve this will feature in Chapter 3.

Chapter 3

Bootstrap of a South African yield curve

There is a need to value all instruments consistently within a single valuation framework. For this we need a risk free yield curve which will be a NACC zero curve (because this is the standard format, for all option pricing formulae).
Thus, a yield curve is a function $r = r(\tau)$, where a single payment investment made at time t for maturity T will earn a rate $r = r(\tau)$ where $\tau = T - t$. We create the curve using a process known as bootstrapping.
We present and clarify here the algorithm for the bootstrap procedure discussed in [Hagan and West, 2006, p.93] and [Hagan and West, 2008, §2].

3.1 Algorithm for a swap curve bootstrap

The instruments available are JIBAR-type instruments, FRAs, and swaps. The assumption is that the LIBOR instruments and FRAs expire before the swaps.

(i) Arrange the instruments in order of expiry term in years. Let these terms be $\tau_1,\ \tau_2,\ \ldots,\ \tau_N$.

(ii) Find the NACC rates corresponding to the JIBAR instruments: suppose for the i^{th} instrument the rate quoted is J_i. Then

$$r_i = \frac{1}{\tau_i}\ln(1 + J_i\tau_i) \tag{3.1}$$

(iii) Now create a first estimate curve. For example, for each of the FRA and swap instruments let r_i be the market rate of the instrument.

(iv) (*) We now have a first estimate of our curve: terms $\tau_1,\ \tau_2,\ \ldots,\ \tau_M$ and rates $r_1,\ r_2,\ \ldots,\ r_M$. These values are passed to the interpolator algorithm.

(v) Update FRA estimates: when a FRA is dealt, for it to have zero value, we must have

$$C(t, t_1)(1 + \alpha f) = C(t, t_2) \tag{3.2}$$

Here t_1 is the settlement date of the FRA, t_2 the expiry date (thus, it is a $t_1 \times t_2$ FRA), f the FRA rate, and $\alpha = t_2 - t_1$ the period of the FRA, taking into account the relevant day count convention. We can rewrite (3.2) as

$$r(t_2) = \frac{1}{\tau_2}\left[C(t,t_1) + \ln(1+\alpha f)\right] \tag{3.3}$$

and this gives us the required iterative formula for bootstrap: $C(t,t_1)$ on the right is found by reading off (interpolating!) off the estimated curve, the term that emerges on the left is noted for the next iteration.

(vi) When a swap is dealt, for it to have zero value, we have

$$R_n \sum_{i=1}^{n} \alpha_i Z(t,t_i) = 1 - Z(t,t_n)$$

and hence

$$Z(t,t_n) = \frac{1 - R_n \sum_{i=1}^{n-1} \alpha_i Z(t,t_i)}{1 + R_n \alpha_n} \tag{3.4}$$

We can rewrite (3.4) as

$$r(t_n) = \frac{-1}{\tau_n} \ln\left[\frac{1 - R_n \sum_{j=1}^{n-1} \alpha_j Z(t,t_j)}{1 + R_n \alpha_n}\right] \tag{3.5}$$

and this gives us the required iterative formula for bootstrap: again all terms on the right are found by reading off (interpolating!) on the estimated curve, the term that emerges on the left is noted for the next iteration.

(vii) We do this for all instruments. We now return to (*) and iterate until convergence. Convergence to double precision is recommended; this will occur in about 10-20 iterations for favoured methods (monotone convex, raw).

3.1.1 Example of a swap curve bootstrap

We will consider one bootstrap of the money market part of the yield curve via an example. Suppose on 26-Apr-11, we have the inputs in Table 3.1.

For the JIBAR and FRA rates we take into account the Modified Following rule, thus, the JIBAR and FRA dates are: 26-May-11, 26-Jul-11, 26-Oct-11, 26-Jan-12, 26-Apr-12, 26-Jul-12, and 26-Oct-12.

The terms (in days) are: 30, 91, 183, 275, 366, 457, and 549.

Conveniently (26-Apr-11 has been well chosen!) all the FRAs are synchronous, and so the FRA date schedule and the swap date schedule coincide).

- $C(\text{26-Apr-11}, 30d) = \left(1 + 5.500\% \frac{30}{365}\right) = 1.0045205$
- $C(\text{26-Apr-11, 26-Jul-11}) = \left(1 + 5.575\% \frac{91}{365}\right) = 1.0138993$

From the above listed observation dates, the fra periods are of length 92, 92, 91, 91, 92 days. Thus

Curve Inputs			
Data	Type	Rate type	Input rate
1 Month	JIBAR	simple yield	5.500%
3 Month	JIBAR	simple yield	5.575%
3x6	fra	simple yield	5.640%
6x9	fra	simple yield	6.000%
9x12	fra	simple yield	6.370%
12x15	fra	simple yield	6.790%
15x18	fra	simple yield	7.250%

Table 3.1: Money market curve data for 26-Apr-11

- $C(\text{26-Apr-11},\ \text{26-Oct-11}) = C(\text{26-Apr-11},\ \text{26-Jul-11}) \cdot \left[1 + 5.640\% \frac{92}{365}\right] = 1.0283128.$
- $C(\text{26-Apr-11},\ \text{26-Jan-12}) = C(\text{26-Apr-11},\ \text{26-Oct-11}) \cdot \left[1 + 6.000\% \frac{92}{365}\right] = 1.0438643.$
- $C(\text{26-Apr-11},\ \text{26-Apr-12}) = C(\text{26-Apr-11}, \text{26-Jan-12}) \cdot \left[1 + 6.370\% \frac{91}{365}\right] = 1.0604423.$
- $C(\text{26-Apr-11},\ \text{26-Jul-12}) = C(\text{26-Apr-11}, \text{26-Apr-12}) \cdot \left[1 + 6.790\% \frac{91}{365}\right] = 1.0783940.$
- $C(\text{26-Apr-11},\ \text{26-Oct-12}) = C(\text{26-Apr-11},\ \text{26-Jul-12}) \cdot \left[1 + 7.250\% \frac{92}{365}\right] = 1.0981005.$

We can now obtain rates for any tenor, as in Table 3.2.

date	term	capfactor	NACC rate
26-May-11	30	1.0045205	5.5366%
26-Jul-11	91	1.0138993	5.5686%
26-Oct-11	183	1.0283128	5.6979%
26-Jan-12	275	1.0438643	5.8526%
26-Apr-12	366	1.0604423	6.0279%
26-Jul-12	457	1.0783940	6.2217%
26-Oct-12	549	1.0981005	6.4295%

Table 3.2: Results of the bootstrap of the money market portion of the curve.

Suppose we also have the input data in Table 3.3 (of course, inputs will go to 25y or 30y, but we only extend the curve to the 4y point.)

Data	Type	Input rate
swap	2y	6.630%
swap	3y	7.150%
swap	4y	7.500%

Table 3.3: Swap curve data for 26-Apr-11

First, we guess the rates at the 2y = 24m, 3y = 36m, and 4y = 48m nodes as the NACC equivalent of the input rate, which is approximately a NACQ rate. Then we use raw interpolation (or whatever interpolation method we choose) for the 21m, 27m, 30m, 33m, 39m, 42m and 45m rates. We obtain column ${}_0r$ in Table 3.4.

Next we find the column ${}_1r$. We find the values for 24m, 36m and 48m by applying (3.5) and using the curve ${}_0r$. For example,

$$ {}_1r(\tau_{24m}) = \frac{-1}{\tau_{24m}} \ln \left[\frac{1 - R_{24m} \sum_{j=1}^{7} \alpha_{3jm} \exp\left(-\tau_{3jm}\, {}_0r(\tau_{3jm})\right)}{1 + R_{24m}\alpha_{24m}} \right] $$

We again interpolate for the other rates.

Then we repeat the whole process, iterating as long as we need to. As we can see in Table 3.3, convergence is very fast.

	Term	α	${}_0r$	${}_1r$	${}_2r$	${}_3r$
1m			5.5366%	5.5366%	5.5366%	5.5366%
3m	0.249	0.249	5.5686%	5.5686%	5.5686%	5.5686%
6m	0.501	0.252	5.6979%	5.6979%	5.6979%	5.6979%
9m	0.753	0.252	5.8526%	5.8526%	5.8526%	5.8526%
12m	1.003	0.249	6.0279%	6.0279%	6.0279%	6.0279%
15m	1.252	0.249	6.2217%	6.2217%	6.2217%	6.2217%
18m	1.504	0.252	6.4295%	6.4295%	6.4295%	6.4295%
21m	1.762	0.258	6.4295%	6.4482%	6.4480%	6.4480%
24m	2.003	0.241	6.5757%	6.6073%	6.6071%	6.6071%
27m	2.252	0.249	6.7447%	6.7871%	6.7859%	6.7860%
30m	2.510	0.258	6.8840%	6.9353%	6.9334%	6.9334%
33m	2.759	0.249	6.9941%	7.0524%	7.0499%	7.0500%
36m	3.011	0.252	7.0868%	7.1510%	7.1481%	7.1482%
39m	3.258	0.247	7.1914%	7.2660%	7.2616%	7.2618%
42m	3.507	0.249	7.2822%	7.3659%	7.3602%	7.3605%
45m	3.756	0.249	7.3609%	7.4525%	7.4457%	7.4461%
48m	4.008	0.252	7.4306%	7.5291%	7.5214%	7.5218%

Table 3.4: Iteration of a swap curve bootstrap

3.2 Exercises

1. Suppose on 3-Mar-08 I have the JIBAR and FRA data given in the table.

jibar	fra	fra	fra	fra	fra	fra	fra
3m	3v6	6v9	9v12	12v15	15v18	18v21	21v24
11.33%	11.43%	11.41%	11.29%	11.04%	10.74%	10.74%	10.74%

   (a) Bootstrap the rates for every 3 month interval out to 24 months.

   (b) What is the fair swap rate for a 24 month swap?

(c) Suppose we decide to use raw interpolation to find rates at dates which are not are not node points on my yield curve. Write a vba function to find the NACC rate at any non-node point.

(d) Suppose on 12-Feb-08 the 3m JIBAR rate was 11.20% and the 6v9 FRA rate was 11.40%. I entered into a pay fixed 6v9 FRA. What is the MtM of the FRA now (on 3-Mar-08)?

(e) Suppose on 12-Feb-08 I entered into a 1 year swap, paying a fixed rate of 11.30%. What is the MtM of the swap now (on 3-Mar-08)?

2. Suppose on 3-Apr-08 my yield curve is 12.00% NACC for a term of zero, and increases by one-tenth of a basis point every calendar day into the future (relevant for this question).

   - Find the fair rate for vanilla swaps with expiry at 6m, 9m, 12m, etc. to 20y.
   - What can you observe about the trend in the fair swap rates? What is the reason for this?
   - If I am paying fixed, receiving floating, in what periods do I expect to be receiving payments and in what periods do I expect to make payments?

3. (UCT exam 2004) Suppose we are given the inputs to the swap curve as follows:

Date	31-Mar-05	
Swap Curve Inputs		
Data	Type	Rate
1 day	on	7.6400%
1 month	jibar	7.2210%
3 month	jibar	7.4040%
3x6	fra	7.2500%
6x9	fra	7.2900%
9x12	fra	7.4000%
12x15	fra	7.8500%
15x18	fra	8.1900%
2y	swap	7.8300%
3y	swap	8.3200%
4y	swap	8.6200%
5y	swap	8.8700%

Bootstrap the yield curve out to 5 years using the method discussed in class. Report the rates for every 3 month point following the valuation date out for 5 years.

To determine these three month points, use the modfol function provided. Use raw interpolation on the NACC rates, for which it is recommended that you write your own linear interpolation function in vba.

4. (UCT exam 2008) Suppose a $T_1 \times T_2$ FRA is traded today date t at rate F.

Suppose I have a complete yield curve, that is, I can borrow or lend zero coupon bonds for any maturity. The mark to market borrowing price for date T is denoted $Z(t,T)$. There are bid and offer curves: $Z^b(t,T)$ and $Z^o(t,T)$. For avoidance of doubt:

   - $Z^b(t,T)$ is the bid curve: there are participants in the market willing to pay $Z^b(t,T)$ at date t and receive 1 at date T.

- $Z(t,T)$ is the MtM price on date t of a payment of 1 on date T.
- $Z^o(t,T)$ is the offer curve: there are participants in the market willing to receive $Z^o(t,T)$ at date t and pay 1 at date T.

These quantities exist for every date T.

(a) What is a FRA?

(b) Write down a formula for today's MtM value V of the pay fixed, receive floating side of the above FRA.

(c) What is the no-arbitrage range for the FRA rate F? (It will be a function of $Z^b(t,T_1)$, $Z^o(t,T_1)$, $Z^b(t,T_2)$ and $Z^o(t,T_2)$.) Prove your assertions.

You may assume that FRAs are settled in arrears, and that at time T_1 one can either pay or receive the then-ruling JIBAR rate J (that is, there is no spread on J).

Chapter 4

The Johannesburg Stock Exchange equities market

The JSE has various trading and settlement platforms:

- equities, which we consider in this chapter, along with some general equity issues.
- financial futures and ...
- ... agricultural products, run by SAFEX, which since 2001 is a wholly owned subsidiary of the JSE. We look at SAFEX in Chapter 7.
- Yield-X, an interest rate futures market.
- BESA has been owned by the JSE since 2009.

The JSE is the world's 16th largest exchange (on a value traded basis).

4.1 What is the purpose of a stock exchange?

An exchange allows for sale and purchase of shares. There is transparency about ruling prices, price queues, liquidity, etc. As trading is nowadays always electronic, there is a very small chance of the trade not being completed.

4.2 Some commonly occurring acronyms at the JSE

- STRATE: Share TRAnsactions Totally Electronic. An electronic settlement system for the South African equities market.
- UST: Uncertificated Securities Tax, a tax paid on the purchase of shares by all entities, currently 0.25% of market price. This tax replaced MST: Marketable Securities Tax. However, all members of the exchange are exempt from UST.

  Foreign banks can't obtain memberships except thorough local offices, however, they usually run their South African equity derivatives desk offshore. So Foreign banks with local branches avoid UST by having the local branch buy the stock, and then taking the foreign bank taking a long position in a single stock future with the local branch taking a short position.

- SETS: the trading system of the partnership of the JSE with the London Stock Exchange. SETS may offer a number of potential strategic benefits, including remote membership and primary dual listing opportunities, the possibility of which should decrease the exodus of blue chip South African companies to foreign exchanges. The LSE and the JSE have also put in place a separate business agreement covering dual listing for issuers, remote access for JSE and Exchange member firms, and the marketing and sale of each other's market information.

  The government shut the door after five of the JSE's largest companies moved to London as a prerequisite for inclusion in the FTSE 100 index. Dual primary listings could reopen this door without exposing the country to capital flight.

- SENS: The JSE Securities Exchange South Africa News Service was established with the aim of facilitating early, equal and wide dissemination of relevant company information, and improving communication between companies and the market. It is a real time news service for the dissemination of company announcements and price sensitive information. The company must submit all relevant company and price sensitive information to SENS as soon as possible after authorisation. An announcement must be sent to SENS before it's published in the press.

4.3 What instruments trade on the JSE equities board?

4.3.1 Ordinary Shares

These are the typical shares that trade. Ownership of shares entitles one to vote on the management structure. Dividends are received as declared periodically by management. Shares are bought at the bid price and sold at the offer price if they are screen traded. Larger blocks of shares might be traded OTC in which case they may very well be traded at an agreed discount known as a haircut.

4.3.2 Other Shares

N stands for non-voting. These are just like ordinary shares except that the owner has no voting rights. Because they have fewer rights these shares should trade at a discount to the corresponding ordinary shares, and be less liquid.
However, in SA the exact opposite can occur. Issuing N-shares was in the 1990's a popular black empowerment vehicle in South Africa - the ordinary shares are closely held by black management and the majority of shares are N-shares trading in the open market.

4.3.3 The Main Board and Alt-X

AltX is a division of the JSE where small companies list. These are companies who are too small to access enough capital to be listed on the main board. There will be other simple reasons to list: having a mark facilitates the workings of an employee share ownership scheme, for example.
The current trend for AltX is not favourable. Any company that succeeds on AltX is likely to start wishing for the main board, and will migrate if they can. The reason for this is that institutional buyers might then be interested in buying the stock, thus increasing the value of equity. Institutional buyers will typically not buy AltX shares, and may very well be constrained from doing so by the rules of any one particular scheme.

4.3.4 SATRIX and ETFs

This piece is distilled from Johannesburg Stock Exchange [2000].

Satrix (SA TRacking IndeX) securities are JSE listed contracts that replicate the dividend and price performance of a particular index. They provide the same returns as would be received had the investor directly purchased shares in each company in the relevant JSE index in the appropriate ratios. Satrix securities are issued by a wholly owned subsidiary of the JSE (known as IndexCo). They are listed on the JSE and traded like any other JSE listed share. The underlying shares of the constituent companies in the relevant index are held by a Trust under a contractual relationship with the issuing company. Holding the underlying basket of shares at all times enables the Satrix Trust to replicate the index performance (price and dividends). The value of a Satrix security will rise and fall in line with the relevant Index. The price of the Satrix security on the JSE will approximately reflect the index level divided by 1000. Any changes to the index will trigger a change to the underlying assets of the Satrix security in order to ensure continual alignment with the index composition. This ensures exact tracking of the index for Satrix security holders.

Quarterly dividend distributions are made to holders of Satrix securities. The amount used for distribution will be all the dividends and interest which have accrued within the Trust (which holds the underlying shares) less the costs incurred in managing the Trust's assets. Because dividends are only paid quarterly from the Trust, the dividend yield will not always match that of the underlying index (which adjusts its Dividend Yield daily based on dividends paid by its composite counters).

Satrix was initiated by Corpcapital Bank and Gensec Bank in November 2000.

Satrix products are examples of ETFs (exchange traded funds). Nowadays there are several exchange traded funds (9 as of 2010) on the JSE. An example is Bettabeta, a Nedbank ETF which has equal weightings of all of the TOP40 shares.

Besides the above, there will be ETFs that track international indices. Trading in such a fund means taking a quanto position - the foreign index is continually denominated in Rands, so one is exposed to the exchange rate. All of the Zshares listed by Investec are such.

4.3.5 Options issued by the underlying

These are options issued by the underlying and which trade like shares on the JSE. The pricing of these instruments is affected by the dilution effect eg. if a call is exercised, the underlying can simply issue more shares (and thereby dilute the price) to honour the call. In textbooks such as Hull [2005] such instruments are known as warrants.

At any time there are very few such instruments listed; as of June 2010 none could be determined.

Until 2001 only calls were possible but now puts are also legal. Companies could only issue calls since it was illegal for a company to buy its own shares because of fears of market manipulation. This has now been legalised because of better market surveillance. However, the company cannot actively trade the options - their issue must be strategic.

4.3.6 Warrants

On a stock exchange options issued by a third party are known as warrants.

The third party is always a merchant bank or financial institution, such as Standard Bank, Sanlam Capital Markets, Investec and Deutsche Bank. The issuer makes a market in these vanilla options ie. will buy/sell the option at the bid/offer. Pricing is a simple matter for the issuer via standard pricing formulae. This price will not necessarily lie between the bid and the offer, but the issuer intends to influence and hedge the position with the theoretical model.

Warrants typically have conversion ratios eg. need 10 warrants to buy one share. Then the value is just 1/10 of the value given by a formula.

It is not allowed to short warrants because the gearing makes this too credit-risky.

Compound warrants (a warrant which entitles you to another warrant on exercise) were issued by Gensec Bank. Down and out barrier warrants (a warrant which disappears if the stock price goes below a certain level) have been issued by Standard Bank, and out puts and calls were issued by Deutsche Bank on 13 September 2002 (they called them WAVEs). The later was a public relations disaster as nearly all of the out puts went out within a couple of months - the market dropped significantly.

4.3.7 Debentures

This is a loan to the company; the loan being raised from small 'lenders' i.e. shareholders, but who are paid a fixed level of interest each year. The debentures are redeemable at a future date; in other words they must be paid back to the lenders. Debentures may also be convertible into ordinary shares.

4.3.8 Nil Paid Letters

A security which is temporarily listed on the stock exchange and which represents the right to take up the shares of a certain company at a certain price and on a certain date, in other words, a call option. Nil paid letters are the result of a rights issue to the existing shareholders (or debenture holders) of a company. A rights issue is one way of raising additional capital by offering existing shareholders the opportunity to take up more shares in the company - usually at a price well below the market price of the shares. These rights are represented by the 'nil paid letter, and are renounceable - this means that they may be bought and sold on the stock exchange.

4.3.9 Preference shares

These are shares which receive a fixed dividend which must be paid before ordinary shareholders receive a dividend; and in the event of liquidation, preference shareholders will be paid out before ordinary shareholders (but only after all prior claims on the company have been met).

Dividends are semi-annual. Here by 'fixed dividend' one means a fixed percentage of the prime rate of interest. Currently dividends range between 63% and 80% of prime. The big banks have the highest credit-worthiness and thus pay the lowest percentages; industrial companies are lower rated and pay higher percentages.

Preference shares are usually cumulative, so that if the company is unable to pay a dividend, the arrears accumulates and must be paid out when the company is in a position to do so. In the non-cumulative case, the dividend is simply missed. Typically preference shares issued by the banks are non-cumulative, the others are cumulative.

They might be convertible into ordinary shares at some date, or they might be redeemable for capital. So, really, a preference share is typically a corporate bond or a convertible bond.

In the past, dividends in South Africa have been tax free (whereas normal interest i.e. coupons on bonds was not) so there was an advantage to holding preference shares. With the normalisation of the tax treatment of financial instruments in South Africa (final implementation 1 April 2012) this distinction is being removed.

4.4 Matching trades on the stock exchange - the central order book

How does screen trading work? It is entirely automated, and works using the laws of supply and demand. Consider the following example:

- Suppose the buyers and sellers are initially lined up as follows:

Trader	Amount	Price
Seller A	1000	333
Seller B	10000	310
Buyer C	1000	300
Buyer D	10000	290
Buyer E	5000	285

  Nothing will happen since there isn't agreement on a price. All of the above information is in the JET system, but the screen will only show 300 (bid) and 310 (offer).

- Suppose a new person enters at 310 and wants 4000 shares. We have:

Trader	Amount	Price
Seller A	1000	333
Seller B	10000	310
Buyer F	4000	310
Buyer C	1000	300
Buyer D	10000	290
Buyer E	5000	285

  On the screen: 310 will flash indicating a sale. A sale of 4000 shares is done automatically, and now:

Trader	Amount	Price
Seller A	1000	333
Seller B	6000	310
Buyer C	1000	300
Buyer D	10000	290
Buyer E	5000	285

  The screen bid is 300 and the offer is 310.

- B decides to encourage the purchase of the remainder of his shares and lowers from 310 to 305. On screen the bid is 300 and the offer 305.

  A new person enters, wanting 7 000 shares at R305.

Trader	Amount	Price
Seller A	1000	333
Seller B	6000	305
Buyer G	7000	305
Buyer C	1000	300
Buyer D	10000	290
Buyer E	5000	285

On the screen: 305 will flash indicating a sale. A sale of 6000 shares is done automatically.

Trader	Amount	Price
Seller A	1000	333
Buyer G	1000	305
Buyer C	1000	300
Buyer D	10000	290
Buyer E	5000	285

305 is the bid (G still wanting 1000 shares) and the offer is 333.

4.5 Indices on the JSE

Indices are needed as an indication of the level of the market. The JSE All-Share index comprises 100% of JSE shares, and has existed since 1995. Prior to that the indices comprised the top 80% by market capitalisation and it was these indices that the futures market was based. This index comprised about 140 of the 620 shares listed on the JSE.

In 1995 there was the realisation that indices for futures need to comprise a small number of shares which have high market capitalisation and are highly traded. This permits hedging and arbitrage operations through trades in baskets of shares. Nevertheless, arbitrage trades will rarely include all the relevant shares and so there will be some residual correlation risk (tracking error).

The JSE, in collaboration with the SA Actuarial Society, determine a number of indices which are used as barometers of the market (or sectors thereof). The most popular of these are:

- ALSI (All share index). This consists of all shares on the JSE bar about 100, these being, for example, pyramids or debentures.
- TOPI (Top 40 listed companies index). Until June 2002 called the ALSI40.
- INDI25 (Top 25 listed industrial companies index)
- FINI15 (Top 15 listed financial companies index)
- RESI20 (Top 20 listed resources companies index)

The primary derivatives indices are the TOPI, INDI25. There was/is also the GLDI10, RESI20, FINI15, FINDI30. There are no futures contracts based on the ALSI.

The indices are revised quarterly to synchronise with the futures closeouts, being on the third Thursday of March, June, September and December. If this is a public holiday we go to the previous business day. The criteria for inclusion/exclusion are market capitalisation (measured in rands) and the daily average value traded (measured in rands). These two criteria are equally important. Shares are listed according to their dual rank, which is the larger of the two ranks, and then selected in order. Ties are broken by higher market capitalisation.

How is the index constructed? The index has some nominated but arbitrary starting date and value I_0. For example, the ALSI40 started on 19 June 1995 at 2000, the INDI25 on the same date at 3000. There is an initial k-factor given by

$$k_0 = I_0 \frac{\sum_{i=1}^{n} W_i}{\sum_{i=1}^{n} P_i W_i} \tag{4.1}$$

where the index consists of the n shares, the prices are $P_1,\ P_2,\ \ldots,\ P_n$ and the number of eligible shares[1] are $W_1,\ W_2,\ \ldots,\ W_n$. Now

$$I_t = k_t \frac{\sum_{i=1}^{n} P_{t,i} W_{t,i}}{\sum_{i=1}^{n} W_{t,i}} \tag{4.2}$$

The ratio $\frac{\sum_{i=1}^{n} P_{t,i} W_{t,i}}{\sum_{i=1}^{n} W_{t,i}}$ is the average price of the shares making up the index. We adjust by the k-factor to make the index level manageable.

Until now, k_t for $t > 0$ is undefined. However, such a factor cannot be omitted. Occasionally certain changes occur to certain shares which should not affect the index. For example, when there is a share split, constituent reselection, or if a share is delisted or suspended the index should evolve continuously. Essentially,

$$k_{t^-} \frac{\sum_{i \in I_{t^-}} P_{t^-,i} W_{t^-,i}}{\sum_{i \in I_{t^-}} W_{t^-,i}} = k_{t^+} \frac{\sum_{i \in I_{t^+}} P_{t^+,i} W_{t^+,i}}{\sum_{i \in I_{t^+}} W_{t^+,i}}. \tag{4.3}$$

This ensures that at the instant of the event, the level of the index is unchanged.

[1]At the June 2002 closeout, this number of eligible shares changed from shares in issue, to free float shares in issue. These are the shares which trade freely in the market, and are not held in family ownership, etc. And that closeout was made a day late, on the Friday.

	Day 1		Day 2		Between Days 2 and 3		Day 3 closing		Between Days 3 and 4		Day 4 closing	
	k value	51.9286834	k value	51.9286834	k value	57.3144941	k value	57.3144941	k value	51.7378892	k value	51.7378892
	Price	ffSISS	Price	ffSISS	Price	ffSISS	Price	ffSISS	Price	ffSISS	Price	ffSISS
AAA	23.00	4,000,000	24.00	4,000,000	24.00	4,000,000	24.00	4,000,000	24.00	4,000,000	24.00	4,000,000
BBB	1.20	3,542,564	1.20	3,542,564	1.20	3,542,564	1.20	3,542,564	1.20	3,542,564	1.20	3,542,564
CCC	0.55	10,000,500	0.55	10,000,500	0.55	10,000,500	0.55	10,000,500	0.55	10,000,500	0.55	10,000,500
DDD	300.00	555,666	310.00	555,666	31.00	5,556,660	34.00	5,556,660	34.00	5,556,660	34.00	5,556,660
EEE	120.00	4,040,400	120.00	4,040,400	120.00	4,040,400	118.00	4,040,400	118.00	4,040,400	118.00	4,040,400
FFF	12.55	10,000,255	13.00	10,000,255	13.00	10,000,255	13.00	10,000,255	13.00	10,000,255	13.00	10,000,255
GGG	4.40	1,078,989	4.50	1,078,989	4.50	1,078,989	4.50	1,078,989	4.50	1,078,989	4.50	1,078,989
HHH	3.00	15,000,000	2.50	15,000,000	2.50	15,000,000	2.50	15,000,000				
III									4.10	10,000,000	4.20	10,000,000
den		48,218,374		48,218,374		53,219,368		53,219,368		48,219,368		48,219,368
num		**928,549,904**		**935,214,577**		**935,214,577**		**943,803,757**		**947,303,757**		**948,303,757**
Avg.	19.26		19.40		17.57		17.73		19.65		19.67	
Index	1000		1007.18		1007.18		1016.43		1016.43		1017.50	

4.6 Delivery

Sellers are required to deliver the securities within five business days of the trade. On the fifth day the cash and the shares will flow into the respective accounts.

4.7 Performance measures

Definition 4.7.1. The simple return on a financial instrument $P.$ is $R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$.

This definition has a number of caveats:

- The time from $t-1$ to t is one business day. Thus it is the daily return. We could also be interested in monthly returns, annual returns, etc.
- P_t is a price. Sometimes a conversion needs to be made to the raw data in order to achieve this. For example, if we start with bond yields y_t, it doesn't make much sense to focus on the return as formulated above. Why?
- We need to worry about other income sources, such as dividends, coupons, etc.
- The continuous return is $\ln \frac{P_t}{P_{t-1}}$. This has better mathematical and modelling properties than the simple return above. For example, it is what occurs in all financial modelling and (hence) is what we use for calibrating a historical volatility calculator.

Let

$$w_{t,i} = \frac{P_{t,i} W_{t,i}}{\sum_{i \in I} P_{t,i} W_{t,i}} \tag{4.4}$$

be the proportion of the wealth of the market at time t that is comprised by stock i.

For any one stock, this proportion will usually be small. A very big share is about 10 - 15%. (In Finland, Nokia makes up 50 - 55% of the index which makes it very volatile). Fund managers have to track indices, in order to do this, they need to own the shares in the correct proportions (which is why, for example, the demand for a South African share goes up before listing on a foreign exchange such as the FTSE - because FTSE index trackers need the share in order to track the index).

In order to calculate performance measures of the index as a function of the performance measures of the stocks, $w.$ becomes critical. Let us suppose (as is typical for most days) that between $t-1$

and t there were no corporate actions or other index rebalancing. Then

$$\begin{aligned}
R_{t,I} &= \frac{k_t \frac{\sum_{i=1}^n W_i P_{t,i}}{\sum_{i=1}^n W_i} - k_{t-1} \frac{\sum_{i=1}^n W_i P_{t-1,i}}{\sum_{i=1}^n W_i}}{k_{t-1} \frac{\sum_{i=1}^n W_i P_{t-1,i}}{\sum_{i=1}^n W_i}} \\
&= \frac{\sum_{i=1}^n W_i P_{t,i} - \sum_{i=1}^n W_i P_{t-1,i}}{\sum_{j=1}^n W_j P_{t-1,j}} \\
&= \frac{\sum_{i=1}^n W_i (P_{t,i} - P_{t-1,i})}{\sum_{j=1}^n W_j P_{t-1,j}} \\
&= \frac{\sum_{i=1}^n W_i P_{t-1,i} \frac{P_{t,i} - P_{t-1,i}}{P_{t-1,i}}}{\sum_{j=1}^n W_j P_{t-1,j}} \\
&= \frac{\sum_{i=1}^n W_i P_{t-1,i} R_{t,i}}{\sum_{j=1}^n W_j P_{t-1,j}} \\
&= \sum_{i=1}^n \left[\frac{W_i P_{t-1,i}}{\sum_{j=1}^n W_j P_{t-1,j}} \right] R_{t,i} \\
&= \sum_{i=1}^n w_{t-1,i} R_{t,i}
\end{aligned}$$

These are simple daily returns. A similar calculation involving continuous returns would not work because the log of a sum is not the sum of the logs. Nevertheless, volatility calculations are usually made using this assumption, which is valid to first order. See [J.P.Morgan and Reuters, December 18, 1996, TD4ePt_2.pdf, §4.1] for additional information.

4.8 Forwards, forwards and futures; index arbitrage

A forward is a legally binding agreement for the long party to buy an equity from the short party at a certain subsequent date at a price (the delivery price) that is agreed upon today.

Theorem 4.8.1. Suppose today is date t and the future date is date T. Suppose that a forward on an equity, which is not going to receive any dividends in the period $[t, T)$, is struck with a strike of K. Then, in the absence of arbitrage,

$$K = S_t C(t, T) \tag{4.5}$$

where time is measured in years.

Proof: suppose $K < S_t C(t, T)$. Then we go long the forward, borrow the stock and sell it, investing the proceeds. At time T we close the bank account, for an inflow of $S_t C(t, T)$, use K to buy the stock, and use this stock to close out the stock borrowing. There is an arbitrage profit of $S_t C(t, T) - K$.

Suppose $K > S_t C(t, T)$. Then we go short the forward, borrow S_t cash and buy the stock. At time T we deliver the stock for K and use $S_t C(t, T)$ to close the borrowing. There is an arbitrage profit of $K - S_t C(t, T)$. $\diamond$

This delivery value is known as the fair forward level. We will denote it $f_{t,T}$, or just f, if the dates are clear.

If we deal such a forward, it might have value 0 at inception, but the chances of it having value 0 at any time during its life again are small.

Theorem 4.8.2. Suppose today is date t and the future date is date T. Suppose that a forward on an equity, which is not going to receive any dividends, is struck (has already previously been dealt) with a strike of K. Then the fair valuation of the forward is

$$V = S_t - Z(t,T)K \tag{4.6}$$

where time is measured in years.

Proof: I am long the forward. I go short (possibly with another counterparty) a fair forward with a strike of $S_tC(t,T)$, from the above theorem, this has no cost. Thus, the flow of equity will cancel, but I will pay K cash in the first deal and recieve $S_tC(t,T)$ in the second deal. Hence the excess value of this is

$$Z(t,T)[S_tC(t,T) - K]$$

and we are done. $\diamond$

The arbitrage formula for an index is the same. How do you sell an index? You sell each share in proportion. That is why an index subject to arbitrage operations must comprise of a few liquid shares.

A forward is a deal struck between two legal entities in the open market, known as an Over The Counter deal (OTC). A future is something completely different. A future is a type of bet that is traded on an exchange, in South Africa on the South African Futures Exchange (SAFEX), which is now a branch of the JSE. Futures will be denoted $F_{t,T}$ or just F, if the dates are clear. $F_{t,T}$ is in some sense the market consensus or equilibrium view at time t of what the value of the stock price will be at time T i.e. the value $S(T)$ (of course unknown at time t). If a speculator believes that the value of $S(T)$ will in fact be higher than $F_{t,T}$ (in other words, he believes that the market is underestimating the forthcoming performance of the stock) then he will go long the future. If a speculator believes the value will be lower (in other words, the market is overestimating the forthcoming performance of the stock) then he will go short the future. This is because as the consensus value F goes up the long party makes money and the short party loses money, and as it goes down the short party makes money and the long party loses money. The consensus value is driven by the forces of supply and demand; going long or short is mechanistically exactly a buy/sell, so F truely is a consensus or equilibrium value.

In a sense which is not mathematically very solid, both f and F are predictions of where the market should be. Under certain assumptions, for example if interest rates are non-stochastic, $f = F$. This was first proved in Cox et al. [1981], see also [Hull, 2005, Appendix to Chapter 5]. Of course, the assumption that interest rates are non-stochastic is not practicable, but nevertheless, the not unreasonable assumption is usually made that $f \approx F$. A common strategy is the 'spot-futures arbitrage': to be long/short futures and short/long spot according to the above theorem, as if futures and forwards were the same. This strategy is not an arbitrage, it is a value play! It can be shown that this strategy can lead to a loss under unusual interest rate fluctuations.

In order to illustrate a neutral position, consider a position where at all times the futures level is the forward level. The portfolio we put on is

- buy stock at time 0, and sell it back at the termination date of the futures contract,
- short futures.

At every date we

- receive or pay margin on the futures,

- receive dividend income determined by the dividend yield of the stock.

All costs and income are carried to the termination date. If the futures price is equal to the forward price, then the p&l from this strategy will be close to 0 (not 0 because of risk free carry and dividend frictions). Thus, if the futures price is below/above the forward price, then we go long/short futures and short/long stock.

For an example, see the sheet SFArbitrage.xls

4.9 Dividends and dividend yields

Dividends are income paid by the company to shareholders. Companies pay dividends because they have to share profits with shareholders, otherwise the shares would not have long term value. However, some companies offer value to shareholders by promising growth in the share price (historically MicroSoft, and more recently, most .com enterprises); nevertheless the value of the company must eventually be realised by dividends or other income. In fact, one very ancient theory is that the value of the stock is equal to the sum of the present value of the dividends and other income, which is called the Rule of Present Worth, and was first formulated in Williams [1938].

If there are dividends whose value at time t is $Q[t,T)$,[2] then

$$f_{t,T} = (S_t - Q[t,T))C(t,T) \tag{4.7}$$

To prove this, we borrow against the value of the dividends. If there is a continuous dividend yield $q = q[t,T)$, then

$$f_{t,T} = S_t e^{-q(T-t)} C(t,T) \tag{4.8}$$

which is known as the Merton model for forwards. To prove this, arrange that dividends are instantly reinvested in the stock as they are paid. See [Hull, 2005, §5.5, §5.6]. Last, but not least

Theorem 4.9.1. Suppose today is date t and the future date is date T. Suppose that a forward on an equity, which is going to receive dividends, is struck (has already previously been dealt) with a strike of K. Suppose the present value of all the dividends to be received in the period is $Q[t,T)$. Then the fair valuation of the forward is

$$V = S_t - Q[t,T) - Z(t,T)K \tag{4.9}$$

where time is measured in years.

Proof: Experience (that is, an exam) has shown that the understanding of how to construct proofs of results like this is quite miserable. So, we give a proof in *every detail*.

Suppose the forward is trading at a price of $P < S_t - Q[t,T) - Z(t,T)K$. We construct an arbitrage. The forward is cheap, so I go long the forward for a cost of P. I borrow stock and sell it for an income of S_t. Suppose there are dividends D_i at various times $t_i \in [t,T)$. I deposit $D_i Z(t,t_i)$ for maturity date t_i; so in total I deposit $Q[t,T) = \sum_i D_i Z(t,t_i)$. Finally, I deposit $Z(t,T)K$ for maturity date T.

Income has been S_t and costs have been $P + Q[t,T) + Z(t,T)K$, so I am in net profit of $S_t - P - Q[t,T) - Z(t,T)K > 0$.

[2] This notation is appropriate, but this isn't obvious! Shares go ex-dividend at the close of the day of the last day to register (also called the last day to trade). Thus, the dividends involved in the forward include those at time t and exclude those at time T.

As each dividend becomes due, my deposits $D_iZ(t,t_i)$ mature with value D_i and I use these to make good the dividends with the party that I borrowed the stock from. At time T my deposit $Z(t,T)K$ matures with value K; I use this with the long forward to buy the stock. This stock I return to the lender.

Thus, besides the initial profit, there have been no net flows. This is an arbitrage profit.

Suppose the forward is trading at a price of $P > S_t - Q[t,T] - Z(t,T)K$. We construct an arbitrage. Now the forward is expensive, so I sell it for an income of P. I borrow $D_iZ(t,t_i)$ for maturity date t_i; so in total I borrow $Q[t,T] = \sum_i D_iZ(t,t_i)$. I borrow $Z(t,T)K$ for maturity date T. Finally, I buy the stock for S_t.

Income has been $P+Q[t,T]+Z(t,T)K$ and cost S_t, so I am in net profit of $P+Q[t,T]+Z(t,T)K-S_t > 0$.

As each dividend is received, my borrowing of $D_iZ(t,t_i)$ mature with value D_i and I use the dividend to eliminate what I have borrowed. At time T my borrowing of $Z(t,T)K$ matures with value K; I eliminate this by selling the stock for K in the short forward.

Thus, besides the initial profit, there have been no net flows. This is an arbitrage profit.

Assumptions made are that the dividend dates and sizes are known, even though they may not yet have been declared. Also in this proof we have ignored the value lag between the LDR date and the actual dividend receipt date, but the result can be modified to cater for this. ⋄

There may be a desire to formulate results in terms of a continuous dividend yield. This is most appropriate because option pricing may be formulated in this manner, and one can determine sensitivities w.r.t. q. Since $(S_t - Q[t,T])C(t,T) = S_te^{-q(T-t)}C(t,T)$, one has that

$$Q[t,T] = S_t[1 - e^{-q(T-t)}] \tag{4.10}$$

$$q[t,T] = \frac{-1}{T-t}\ln\frac{S_t - Q[t,T]}{S_t} \tag{4.11}$$

The value of q so obtained will be called the expected dividend yield and denoted $\mathbb{E}[q]$. Most importantly, if there is no dividend in the period under consideration then $\mathbb{E}[q] = 0$. In South Africa there are typically two dividends a year, the interim dividend consisting of about 30% of the dividend and the final dividend constituting the remainder. Special dividend payments may also be made.

However, unlike in this nice textbook theory, the share price does not drop at the very instant the dividend is paid. The share price drops the business day after the LDR (Last Day to Register). This is the last day to trade the stock cum the dividend. The LDR is usually a Friday. Some time later the dividend will be paid.

The LDR is the date by which securities must be lodged with the company's office to qualify for dividends rights or other corporate actions.

How much does the stock price drop? Intuitively, it should be about the present value of the dividend. We have seen it claimed that this relationship is exact, to avoid arbitrage: if t is an LDR date for a dividend of size D with pay date T then $S(t^-) = S(t^+) + Z(t,T)D$. Of course, this statement is completely absurd: it assumes pre-knowledge of the stock price $S(t^+)$ at the time t^- - an instant before. The stock price is not previsible, but forward prices are. The following would be correct:

Theorem 4.9.2. Suppose today is date t. Suppose two forwards for a date T are available for trade in the market (and we ignore market frictions such as bid-offer spreads). A dividend of size D has already been declared with a ldt of date T and a payment date of T_p. The one forward, with a strike of K_1 is for the stock cum-dividend, while the other with a strike of K_2 is for the stock ex-dividend. Prove that

$$K_1 - K_2 = DZ(t; T, T_p)$$

where $Z(t; T, T_p)$ is the forward discount factor as seen now for the period from T to T_p.

Proof: First suppose $K_1 - K_2 < DZ(t; T, T_p)$. Then K_1 is 'too low' relative to K_2, so I

(1) go long the forward with strike K_1.

(2) go short the forward with a strike of K_2.

(3) lend $Z(t, T)K_1$ for maturity T.

(4) borrow $Z(t, T)K_2$ for maturity T.

(5) borrow $Z(t, T_p)D$ for maturity T_p.

At time t there is a net flow of

$$\begin{aligned} -Z(t,T)K_1 + Z(t,T)K_2 + Z(t,T_p)D &= Z(t,T)\left[K_2 - K_1 + \frac{Z(t,T_p)}{Z(t,T)}D\right] \\ &= Z(t,T)\left[K_2 - K_1 + Z(t;T,T_p)D\right] \\ &> 0 \end{aligned}$$

At time T I receive in (3) K_1 which is use to pay for the stock in (1). I then deliver the stock and receive K_2 in (2). This money I use to repay (4). I now have the rights to the dividend D which I receive at T_p. This I use to repay (5).
Now suppose $K_1 - K_2 > DZ(t; T, T_p)$. I do the opposite, so

(1) go short the forward with strike K_1.

(2) go long the forward with a strike of K_2.

(3) borrow $Z(t, T)K_1$ for maturity T.

(4) lend $Z(t, T)K_2$ for maturity T.

(5) lend $Z(t, T_p)D$ for maturity T_p.

At time t there is a net flow of

$$\begin{aligned} Z(t,T)K_1 - Z(t,T)K_2 - Z(t,T_p)D &= Z(t,T)\left[K_1 - K_2 - \frac{Z(t,T_p)}{Z(t,T)}D\right] \\ &= Z(t,T)\left[K_1 - K_2 - Z(t;T,T_p)D\right] \\ &> 0 \end{aligned}$$

At time T I receive K_2 in (4). I use this to pay for the stock in (2). I then deliver the stock in (1) and receive K_1. This I use to repay (3). At time T_p I receive D in (5). This I use to manufacture a dividend payment for the long party in (1). $\diamond$
As a corollary, we have

Corollary 4.9.3. Suppose t is an LDR date for a dividend of size D with pay date T. Let $f(t^-, t^+)$ be the strike of a forward which is dealt at t^- and expires at time t^+ (effectively, a forward for immediate delivery, but of an ex stock). Then

$$S(t^-) = f(t^-, t^+) + Z(t, T)D$$

Proof: The forward for immediate delivery is just a purchase of stock, so has strike $S(t^-)$. $\diamond$

4.10 The term structure of forward prices

We are going to make extensive use of (4.7). In order to do so, we reformulate that equation so that we have a simple recursive procedure for determining the forward price for any date.
Let the dividend LDR dates be t_i. Let $f(t_i^-)$ be the forward for that date but prior to the stock going ex, and let $f(t_i^+)$ be the forward post going ex. Let the current date be t_0. Then

$$\begin{aligned} f(t_0) &= S(t_0) \\ f(t_i^-) &= f(t_{i-1}^+)C(t_0; t_{i-1}, t_i) \\ f(t_i^+) &= f(t_i^-) - D_i Z(t_0; t_i, t_i^{\text{pay}}) \end{aligned}$$

for $i = 1, 2, \ldots$.
Note that we can also perform a forward calculation into the past (a 'backward'?) It is a function of today's stock price, and the historical yield curves, but not the historical prices.

$$\begin{aligned} f(t_0) &= S(t_0) \\ f(t_i^-) &= f(t_i^+) + D_i Z(t_i, t_i^{\text{pay}}) \\ f(t_{i-1}^+) &= f(t_i^-)Z(t_{i-1}, t_i) \end{aligned}$$

for $i = -1, -2, \ldots$.

4.11 A simple model for long term dividends

We would like to construct a model that, given a valuation date t, and an expiry date T, will output a continuous dividend yield $q[t, T)$. Remember that continuous dividend yields are purely useful mathematical fictions - it is a number that can be used in comparing stocks very much in the same way that volatility can, and it is the q that will be used in the Black-Scholes formula.
Our intention is to model the evolution of the stock price and so the ex-dividend decrease in stock price is of higher concern than the actual valuation of dividend payments. Special dividends will be included in valuations and the calculation of dividend yields, but will not be used for prediction, so, of course, it will not be predicted that special dividends will reoccur.
Our model will take into account the period from one year BEFORE the valuation date until the expiry date.
Some dividends are known as cash amounts, for example, the known past, the short term known future (where the dividend amount has been declared by the company), or broker forecast future (where brokers have predicted the cash amount of the dividend). For example, there could be two

historic dividends, a special dividend, and two forecast dividends. Typically, of course, we will have dividends much further into the future, and these dividends that will require some creativity.

We first define the simple dividend yield rates q_i for the cash dividends D_i as follows:

$$q_i = \frac{Z(t;t_i,t_i^{\mathrm{pay}})D_i}{f(t_i)} \tag{4.12}$$

$$q_i = \frac{Z(t_i,t_i^{\mathrm{pay}})D_i}{f(t_i)} \tag{4.13}$$

for forward and backward dividends respectively. In the backwards case, this formulation models the observed practice that companies attempt to pay dividends which are consistent in Rand value, rather than consistent as a proportion of share price.

It is possible to have a model for dividend yields which is a 'mixed' function i.e. a function of both the cash dividends and the simple dividend yields. However, longer dated simple dividend yields are not known, so we need a model. For these dividends the fundamental modelling assumption will be that the LDR dates of regular (non-special) dividends will reoccur annually in the future and that the simple dividend yields as defined above of the last corresponding cash dividends will also reoccur ad infinitum.

In principle this model requires several yield curves: not only the valuation date curve, but also some historical curves. Empirically it turns out that the risk free rates are not a major factor in this model, and it usually suffices to take a flat risk free rate for all dates and all expiries.

That is our model. Everything that follows is model independent. Let PV_i be the present value of the i^{th} dividend. It now can be shown that

$$\mathrm{PV}_n = q_n\left(S_t - \sum_{i=1}^{n-1}\mathrm{PV}_i\right) \tag{4.14}$$

$$\mathrm{PV}_n = q_n\prod_{i=1}^{n-1}(1-q_i)S_t \tag{4.15}$$

$$\sum_{i=1}^{n}\mathrm{PV}_i = S_t\left(1-\prod_{i=1}^{n}(1-q_i)\right) \tag{4.16}$$

The first statement follows directly from a simple manipulation of (4.12) using Theorem 4.9.1. The second statement then follows by a careful induction argument. The third statement will again be induction; a trivial application of the second statement. (See tutorial.) This is consistent with the Rule of Present Worth.

Using the sum of the present values of the forecasted dividends for the stock that fall between today t, and the expiry date T we can calculate the NACC dividend yield $q = q[t,T)$ applicable over the period.

$$\begin{aligned}
S_t\exp(-q\tau) &= S_t - Q[t,T) \\
&= S_t\prod_{i=1}^{n}(1-q_i) \qquad (4.17) \\
\exp(-q\tau) &= \prod_{i=1}^{n}(1-q_i) \qquad (4.18) \\
q &= -\frac{1}{\tau}\sum_{i=1}^{n}\ln(1-q_i) \qquad (4.19)
\end{aligned}$$

(4.19) is the required dividend yield for the option, to be input into the appropriate pricing formula. (4.18) is the fundamental dividend yield equation, and is model independent i.e. it is still valid under other models of determining the q_i's. Either side represents the proportional loss in the value of the stock if the owner of the stock forego's the dividends in the period $[t, T)$.

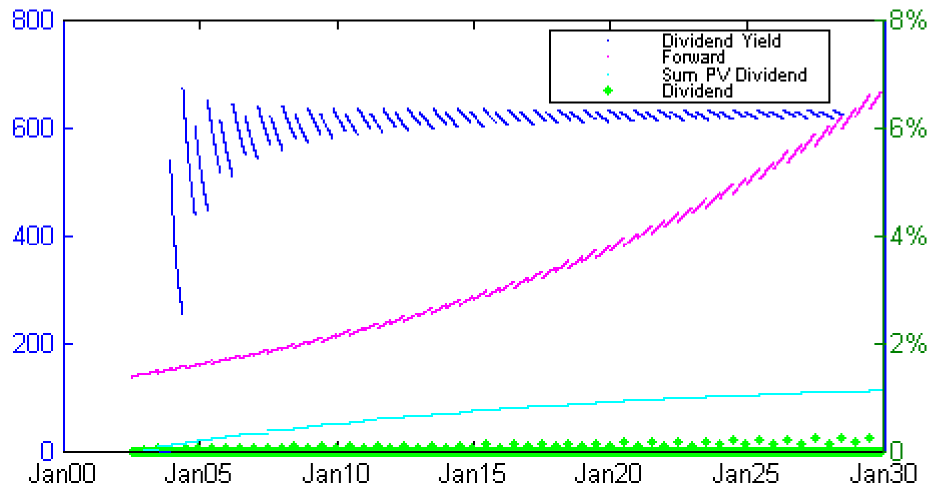


Figure 4.1: Dividends, values of dividends, forward values (left axis) and dividend yields (right axis)

The case of indices involves the same calculations in principle, taking into account the index constituents and their weights. In forward looking models the assumption will be that the constituents and their weights will be unchanged into the future.

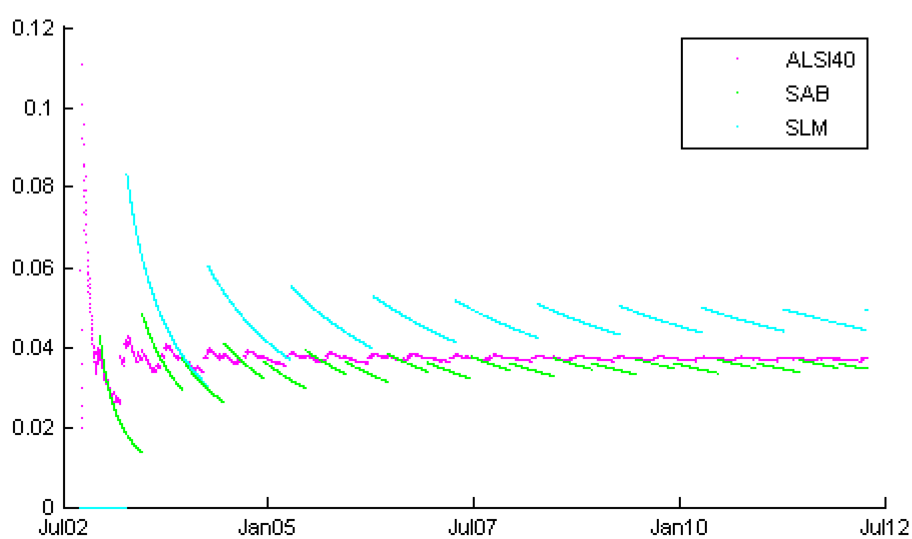


Figure 4.2: Dividend yields using the above model

These expected dividend yields need to be contrasted with the historic dividend yield, which is quoted by the JSE and all the data vendors, but is a completely useless piece of information. The historic dividend yield is the cash dividends in the last 12 months as a proportion of the most recent market value ie. the sum of all dividends in the past 12 months divided by the share price. The dividends are calculated relative to payment dates rather than ex-dividend dates.
The expected dividend yield and historic dividend yield are very different.

- $\mathbb{E}[q]$ is very often equal to 0 for short periods, and this, not the historical dividend yield needs to be used for option pricing. It can also be very large for short dated options over a LDR.
- Using the historical dividend yield can lead to severe mispricing. Shares pay 0, 1 or 2 dividends a year. Most shares have year end at December or June so the dividends occur at March and September. In other words we have a clustering of dividends at specific times. So the error of using the historical dividend yield does not wash out when we move from a single equity to an index.

4.12 Pairs trading

Pairs trading is a simultaneous position in two stocks: one long, one short. The trade is put on in the belief that there is a relative mispricing in the stock prices: the one held long is underpriced relative to the one held short. This belief may be transient (temporary supply and demand changes, large orders for the one stock but not the other, reaction to news about one of the companies) or it may be based on something more fundamental.
The position is held until the mispricing closes sufficiently to take profit.
Of course, there is a very simple problem with this strategy: stocks do not have a pull to par feature, so the mispricing can persist. It can even widen before it closes. In this case, the party that has put on the strategy either needs to fund the position, or cut their losses. This was the fundamental problem that caused the bankruptcy of Long-Term Capital Management: eventually, the trades that they had put on turned out to be mostly profitable for the parties that assumed them.

4.13 Program trading

With the advent of electronic trading, program trading becomes possible. These would be coded modules that facilitate the automation of rule-based execution into the JSE's trading engine.
However, program trading is fraught with dangers, and is typically blamed as the principal culprit of the Wall Street crash of 1987. Major portfolio holders typically hold large portfolios on the stock exchange and portfolio insurance positions on the derivatives exchange. When the portfolio insurance policy comprises a protective put position, no adjustment is required once the strategy is in place. However, when insurance is effected through equivalent dynamic hedging in index futures and risk free bills, it destabilises markets by supporting downward trends.
Dynamic hedging is the synthetic creation of a protective put at a given strike. To create a put sythetically (as in the derivation of the Black-Scholes formula) we have $Ke^{-r\tau}N(-d_2)$ in the money market account and we are short $N(-d_1)$ many of the stock.[3] Now, as the market moves, we rebalance these amounts on an ongoing basis (in the derivation of the formula, this rebalancing occurs continuously; in practice of course transaction costs mean that the rebalancing must occur

[3] Assume there is no dividend.

discretely, possibly according to some trigger rule). And if the market falls, the value of $N(-d_1)$ goes up.[4]

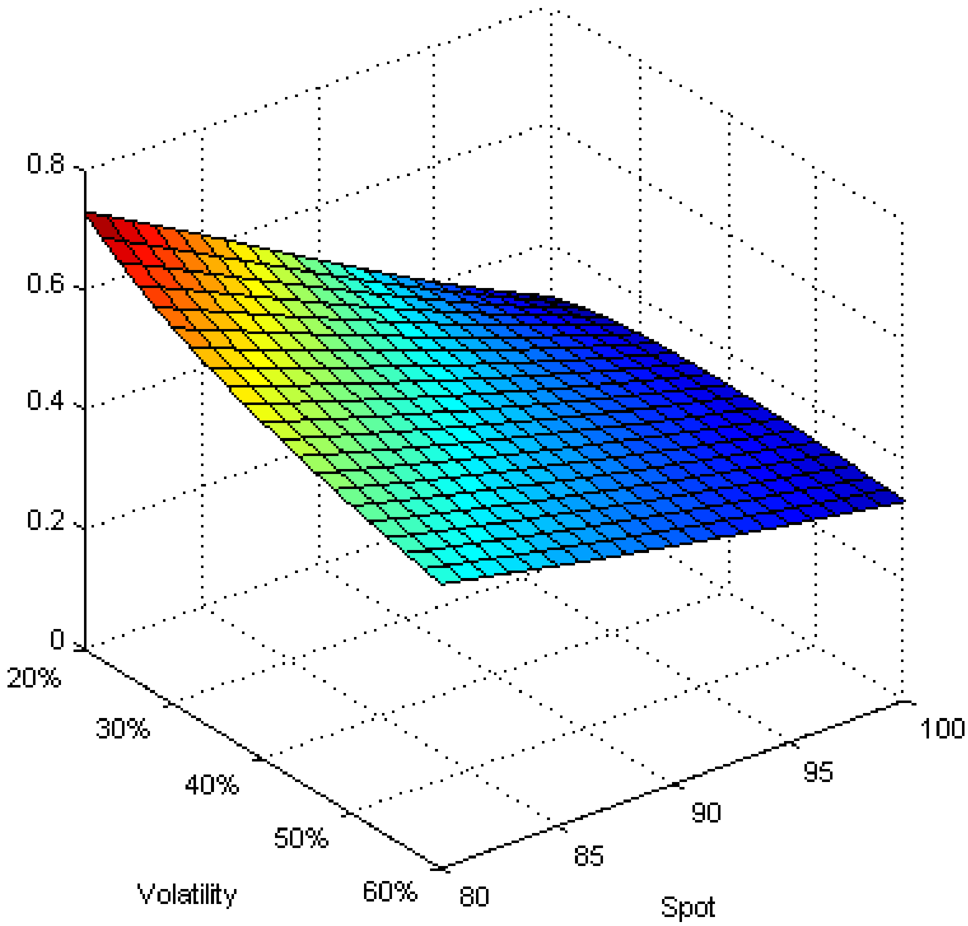


Figure 4.3: The value of $N(-d_1)$

Thus, we cause further downward momentum in the stock price. Alternatively, if we change our delta in the futures market, the prices of index futures will fall below their cost-of-carry value. Then index arbitrageurs step in to close the gap between the futures and the underlying stock market by buying futures and selling stocks through a sell program trade. Thus, either way, the sale of stocks gathers momentum.

For a very well written piece on program trading, see Furbush [2002].

[4]When the market falls, the volatility will typically go up, and this causes $N(-d_1)$ to decrease, but this effect is not as dramatic as the effect of the change in stock price.

4.14 Algorithmic and high frequency trading

See http://www.eci.com/knowledge-center/whitepapers/High%20Frequency%20Trading.pdf.
High-frequency traders use computerised algorithms to trade in and out of markets in a fraction of a second, in an attempt to profit from arbitrage. Furthermore, one class of strategies relies on the ability to see certain orders a few milliseconds before the rest of the market - these are known as flash orders. Such orders are used to determine dynamics of the order book.

Strategies are continuously evolving, with a particular strategy only profitable for a period of a few weeks, before other competing algorithms remove the margins being generated. As new strategies are thought up, they will be backtested, and if the backtest is sucessful, implemented.

IN 2011 high-frequency trading accounts for over 50% of US equity trading, while the level in Europe is about 35%. These numbers are down from the highs of 2008. In early 2011 industry experts reported a slowdown in high-frequency trading activity in the US and Europe, as they focus their efforts on stock exchanges in emerging markets. The fat has been removed from these markets through competition.

One element of trading that has yet to be fully realised is risk management. This is because an effective risk management system will identify when a trade exceeds risk boundaries set by the trader prior to it happening, and this adds latency to the execution of the strategy. Eventually, these risk controls may be forced on companies by regulation.

The Flash Crash of May 6 2010 is an example of how things can go wrong. The day was characterised by bearish sentiments and high volatility. A mutual fund initiated a large futures sell order as a hedge to an existing position; the algorithm executing the order was tuned to volume traded but mistakenly not to price. High frequency algorithms kicked in, first taking the long position, then offlaying it, then taking the long position again, as there was consistent volume available. As futures prices were driven down, so the price of the stock index was driven down by spot-futures arbitrage algorithms.

Some players exited because of stop-loss rules, but the trend continued as others were still active. The futures market had an automatic limit-down break (of five seconds!) and on resumption the market trended back to an equilibrium over the next 15 minutes or so.

The majority of high frequency trading firms are located in New York, Chicago and London at the moment, while other regions continue to show continued growth, particularly Singapore, Australia, South America and South Africa.

In April 2011 the JSE announced plans to allow for remote membership which opens the doors to international high-speed trading firms. Remote membership is prevalent among exchanges in developed countries and is favoured by high-frequency trading firms as they cut out the local broker middle man. They have also extended the technology deal with the LSE, and will offer co-location services i.e. trading firms can locate their servers in the same place as the matching engine of the exchange.

4.15 CFDs

Contracts for difference are OTC derivatives with a broker. With a long/short position, you have the difference in daily share prices paid into/taken out of your broker account.

The broker synthesises this position in the market. For a long position they borrow money, buy the stock, and thus can pay the difference in daily prices. When the position is unwound the broker sells the stock and pays back the loan. The interest required on the borrowed money is charged to the broker account. Similarly for a short position.

There is margin in the broker account, which earns interest. The broker makes money in all cases by making the interest rate slightly less favourable to the account holder than the rate they are actually obtaining. Of course the account is responsible for such things as MST and broker fees.

The party who holds a long/short position in a CFD on the last day to register of the underlying instrument will receive/pay an amount equal to the dividend.

The attraction of CFD's is the ability to achieve gearing. We see a far loss costly way to achieve more-or-less the same thing in Chapter 7 viz. trading in futures.

4.16 Exercises

1. As discussed in class, build a spreadsheet to calculate and graph the volatility of a time series of financial prices/yields/rates. Use:

   (a) Unweighted volatility calculation

   (b) 'unweighted window' volatility with N=90

   (c) EWMA volatility with $\lambda = 0.99$

2. Write vba code to price all varieties of the vanilla option pricing formulae. Use the symbols and the general approach from the notes.

3. Suppose that an index consists of the following shares:

Share	Price	ffSISS
AMS	22.00	6 722 223
BOC	4.52	40 000 000
MMD	3.20	22 331 700
ZZZ	0.45	17 010 000
ABC	7.20	10 000 555

   The index level at close of business today is 8005.22. Two events now occur:

   (a) AMS has a 5 for 1 share split.

   (b) ZZZ has deteriorated in terms of market capitalisation and is removed from the index. It is replaced by PAR which has a closing price of 12.02 and a ffSISS of 10 000 270.

   Calculate the old and the new basing constant (k Factor).

   Answer: 1602.7576 and 1566.8818.

4. Consider the recent history of dividends for the share, ABC:

LDT Date	Dividend Amount
18-Jun-07	4.50
18-Dec-07	3.75

   Using an interest rate of 12% NACC throughout, and the dividend model discussed in class, calculate for 26 February 2008,

   (a) the JSE quoted dividend yield

(b) the expected dividend yield for a 3 month period

(c) the expected dividend yield for a 6 month period

(d) the expected dividend yield for a 5 year period

The share price for ABC on 26 February 2008 was 122.00 and there were no corporate actions in the last year.

5. (UCT exam 2007) Suppose share ABC has the dividend information as follows:

LDR	Pay	Classification	Amount
08-Sep-06	15-Sep-06	Occurred	2.00
09-Mar-07	16-Mar-07	Occurred	1.50
06-Jul-07	13-Jul-07	Special, declared	1.00
07-Sep-07	14-Sep-07	Forecast	2.50

Today is 3 June 2007 and the stock price is R150. Using a risk free rate of 8% throughout, use the model discussed in class to calculate the dividend yield for an option which expires on 31 Dec 2011.

6. (UCT Exam 2008) Dividend forecasts on a stock are provided. Today is 31-Aug-08 and the stock price is 100.00.

LDR	Pay	Classification	Amount
30-Nov-07	07-Dec-07	Occurred	2.00
11-Apr-08	18-Apr-08	Occurred	1.50
03-Oct-08	10-Oct-08	Special, declared	0.50
28-Nov-08	05-Dec-08	Forecast	2.50

Using a risk free rate of 11% throughout, use the model discussed in class to calculate the dividend yield for an option which expires on 31 Dec 2012.

7. Suppose a share has the dividend information provided on the sheet 'Q5' of uct2009.xls. Note that in December 2009 two dividends will be paid simultaneously: a special dividend and an ordinary (usual) dividend.

   Today is 12 Nov 2009 and the stock price is R200. Using a risk free rate of 8% throughout, use the model discussed in class to calculate the dividend yield for an option which expires on 31 Dec 2011.

8. Suppose that we assume that, at least for moderate moves in stock price, the present value of dividends in the short term is unchanged. Derive a formula that shows, under these assumptions, how the dividend yield changes, as a function of the old dividend yield, the old stock price, and the new stock price.

   Why is the phrase 'in the short term' important?

9. Suppose a stock pays percentage dividends q_i at times t_i. What is the value today of the dividend payment at time t_2? How would you replicate this?

10. Prove equations (4.15) and (4.16) by induction.

11. Suppose a stock pays percentage dividends q_i at times t_i. What is the value today of the dividend payment at time t_2? How would you replicate this (no arbitrage replication)?

Chapter 5

Review of distributions and statistics

For this course, we will value equity options mostly in the Geometric Brownian Motion world where we have

$$dS = (r - q)S\,dt + \sigma S\,dZ \tag{5.1}$$

where the time is measured in years. One of the most important factors of this formulation is that the risk free rate, the dividend yield, and the volatility are all constant. Whilst the risk free and the dividend yield assumptions are not too problematic (in an equity derivative environment), the volatility assumption is. Volatility is certainly a function of time (this part is quite easy) but is also a function of how the stock price evolves: so one possibility is that $\sigma = \sigma(S, t)$, which is called the local volatility. Models of the volatility skew or smile are thus crucial. The development of the theory has branched into local volatility models and stochastic volatility models, with the latter now predominant theoretically but the former still in heavy use (although theoretically inferior, they are computationally almost instantaneous, whereas stochastic volatility pricing is more expensive, and Monte Carlo is sometimes the only possibility). The key evolution is Dupire [1994], Dupire [1997], Derman [1999], Derman and Kani [1998] for local volatility, and Hull and White [1987], Heston [1993], Hagan et al. [2002] for stochastic volatility.

In all cases, vanilla options and the vanilla skew are used to calibrate the model, which is then used for pricing of other more exotic options.

But, for the most part, we will assume that volatility is constant, or that it has a term structure. Only in some specific instances will we allow volatility to be dependent on the strike or on the evolution of spot, and we don't allow for jumps in the stock price (the stock price is a diffusion).

5.1 Distributional facts

A basic statistical result we shall use repeatedly is that if the random variable Z has probability density function f, and g is a suitably defined function then

$$\mathbb{E}\left[g(Z)\right] = \int f(s)g(s)\,ds \tag{5.2}$$

where the integration is done over the domain of f. This allows us to work out $\mathbb{E}[Z]$ and $\mathbb{E}\left[Z^2\right]$ for example, by putting $g(s) = s$ and $g(s) = s^2$ respectively. This result is known as 'the Law of the Unconscious Statistician'.
Now, note from statistics that if $X = \ln W \sim \phi(\Psi, \Sigma)$ [1] then the relevant probability density functions are

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi\Sigma}} \exp\left[-\tfrac{1}{2}\frac{(x-\Psi)^2}{\Sigma}\right] && (5.3) \\ f_W(x) &= \frac{1}{\sqrt{2\pi\Sigma}x} \exp\left[-\tfrac{1}{2}\frac{(\ln x-\Psi)^2}{\Sigma}\right] && (5.4) \end{aligned}$$

Of course the domain for f_X is $\mathbb{R}$ while the domain for f_W is $(0, \infty)$.
In the GBM formulation above, by Itô's lemma

$$X := \ln\left(\frac{S(T)}{S(t)}\right) \sim \phi\left(\left(r - q - \frac{\sigma^2}{2}\right)\tau, \sigma^2\tau\right) \tag{5.5}$$

Note now that $S(T) = S(t)e^X$: a very useful representation for European derivatives. Let

$$m_\pm = r - q \pm \frac{\sigma^2}{2} \tag{5.6}$$

So $X \sim \phi(m_-\tau, \sigma^2\tau)$ and so the probability density function for X is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}} \exp\left[-\tfrac{1}{2}\frac{(x-m_-\tau)^2}{\sigma^2\tau}\right] \tag{5.7}$$

and the probability distribution for $S(T)$ is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}x} \exp\left[-\tfrac{1}{2}\frac{(\ln x - \ln S(t) - m_-\tau)^2}{\sigma^2\tau}\right] \tag{5.8}$$

Now if $\ln Y \sim \phi(\Psi, \Sigma)$ then for $k > 0$

$$\begin{aligned} \mathbb{E}[Y^k] &= \frac{1}{\sqrt{2\pi\Sigma}} \int_{-\infty}^{\infty} e^{kx} \exp\left[-\tfrac{1}{2}\frac{(x-\Psi)^2}{\Sigma}\right] dx \\ &= \exp\left(k\Psi + \tfrac{1}{2}k^2\Sigma\right) && (5.9) \end{aligned}$$

Thus in the above risk neutral setting we have

$$\mathbb{E}_t^{\mathbb{Q}}\left[S(T)^k\right] = S(t)^k \exp\left(\left(k(r-q) + \tfrac{1}{2}(k^2-k)\sigma^2\right)\tau\right) \tag{5.10}$$

5.2 The cumulative normal function

Our concern here is the function $N(\cdot)$. The Cumulative Standard Normal Integral is the function:

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt \tag{5.11}$$

[1] By this we mean that the mean is Ψ and the variance is Σ. This could apply to more than one dimension too, in which case Ψ would be the mean vector and Σ the covariance matrix. Furthermore, in general we reserve the symbol σ for the annualised volatility, also known as the volatility measure, and do not use it as the standard deviation of some distribution.

A closed form solution does not exist for this integral, so a numerical approximation needs to be implemented. Most common is an approximation which involves an exponential and a fifth degree polynomial, given in Abramowitz and Stegun [1974], and repeated in [Hull, 2005, §13.9] and [Haug, 2007, §13.1.2], for example. The fifth degree function is used by most option exchanges for futures option pricing and margining, and hence may be preferred to better methods, in order to maintain consistency with the results from the exchange.

However, another option is one that first appears in Hart [1968]. This algorithm uses high degree rational functions to obtain the approximation. This function is accurate to double precision throughout the real line.

The first derivative of the cumulative normal

This is the closed form formula:

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{5.12}$$

The second derivative of the cumulative normal

This is again, given by a closed form formula:

$$N''(x) = -xN'(x) \tag{5.13}$$

5.3 The inverse of the cumulative normal function

Given an input y, the Inverse Standard Normal Integral gives the value of x for which $N(x) = y$, where $N(\cdot)$ denotes the Cumulative Standard Normal Integral.

The Moro transform Moro [1995] to find this function is the most well known algorithm. Having the ability to generate normally distributed variables from a (quasi) random uniform sample is clearly important in work involving any Monte Carlo experiments, and the Moro transformation is fast and accurate to about 10 decimal places.

For another approach, we can use our existing cumulant function and any version of Newton's method. As pointed out in Acklam [2004], having a double precision function has some rather pleasant spin-offs. Given a function that can compute the normal cumulative distribution function to double precision, the Moro approximation of the inverse normal cumulative distribution function can be refined to full machine precision, by a fairly straightforward application of Newton's method. In fact, higher degree methods such as Newton's second order method (sometimes called the Newton-Bailey method) or a third order method known as Halley's method will be the fastest, and are very amenable here, because the Gaussian function is so easily differentiated over and over - see Acklam [2004] and Acklam [2002].

The Newton-Bailey method would be as follows:

$$\begin{aligned}
x_{n+1} &= x_n - \frac{f(x_n) - y}{f'(x_n) - \frac{(f(x_n)-y)f''(x_n)}{2f'(x_n)}} \\
&= x_n - \frac{f(x_n) - y}{f'(x_n) + \frac{(f(x_n)-y)x_n f'(x_n)}{2f'(x_n)}} \\
&= x_n - \frac{f(x_n) - y}{f'(x_n) + \frac{1}{2}(f(x_n) - y)x_n}
\end{aligned}$$

Earlier versions of excel had an absurd error in the NORMSINV function: it would return impossible values for inputs within 0.0000003 of 1 or 0 respectively. Given that such values close to 0 or 1 on occasion are provided by uniform random number generators, this approach is to be avoided. Also note that the random number generator rand()/rnd() in excel/vba is absurd as it can (and does) return the value 0 (but not 1). This will cause either your own inverse function, or NORMSINV, to fail.

5.4 Bivariate cumulative normal

The probability density function of the bivariate normal distribution is

$$\phi_2(X, Y, \rho) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[\frac{-(X^2 - 2\rho XY + Y^2)}{2(1-\rho^2)}\right] \tag{5.14}$$

The cumulative bivariate normal distribution is the function

$$N_2\left(x,\ y,\ \rho\right) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{x} \int_{-\infty}^{y} \exp\left[\frac{-(X^2 - 2\rho XY + Y^2)}{2(1-\rho^2)}\right] dY\, dX \tag{5.15}$$

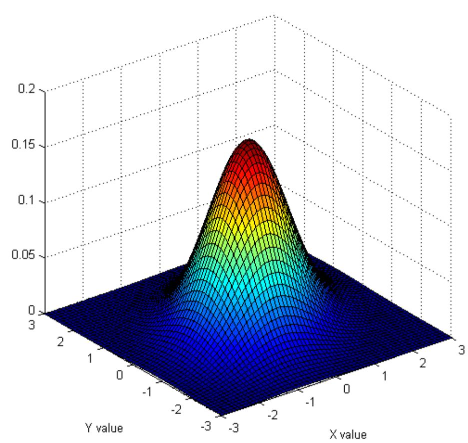


Figure 5.1: The bivariate normal pdf

Limiting cases are important for the bivariate cumulative normal. Note that in the sense of a limit

$$N_2(x, y, 1) = N(\min(x, y)) \tag{5.16}$$

$$N_2(x, y, -1) = \begin{cases} 0 & \text{if} \quad y \le -x \\ N(x) + N(y) - 1 & \text{if} \quad y > -x \end{cases} \tag{5.17}$$

5.5 Trivariate cumulative normal

The cumulative trivariate normal distribution is the function

$$N_3(x_1, x_2, x_3, \Sigma) = \frac{1}{(2\pi)^{3/2}\sqrt{|\Sigma|}} \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} \int_{-\infty}^{x_3} \exp\left(-\tfrac{1}{2}\underline{X}'\Sigma^{-1}\underline{X}\right) dX_3\, dX_2\, dX_1 \tag{5.18}$$

where Σ is the correlation matrix between standardised (scaled) variables X_1, X_2, X_3, and $|\cdot|$ denotes determinant. Denote by $N_3(x_1, x_2, x_3, \rho_{21}, \rho_{31}, \rho_{32})$ the function $N_3(x_1, x_2, x_3, \Sigma)$ where

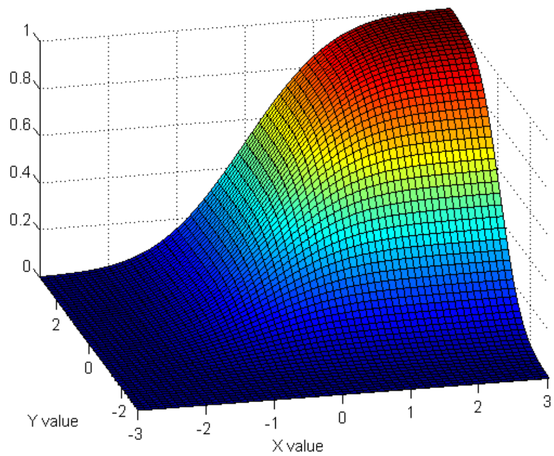


Figure 5.2: The bivariate cumulative normal function, $\rho = 50\%$

$\Sigma = \begin{bmatrix} 1 & \rho_{21} & \rho_{31} \\ \rho_{21} & 1 & \rho_{32} \\ \rho_{31} & \rho_{32} & 1 \end{bmatrix}$. Again, approximations are required. Code for the trivariate cumulative normal is not generally available. There are a few highly non-transparent publications, for example Schervish [1984], but this code is known to be faulty. We have used the algorithm in Genz [2004]. This has required extensive modifications because the algorithms are implemented in Fortran, using language properties which are not readily translated. The function in Genz [2004] returns the complementary probability, again, we have modified to return the usual probability that $X_i \leq x_i$ $(i = 1, 2, 3)$ given a correlation matrix. Again, it is claimed that this algorithm is double precision; high accuracy (of our vb and c++ translations) has been verified by testing against Niederreiter quasi-Monte Carlo integration (using the Matlab algorithm qsimvn.m, also at the website of Genz). As before, one can show that

$$N_3(x_1, x_2, x_3, \Sigma) = \int_{-\infty}^{x_3} N'(x) N_2 \left(\frac{x_1 - \rho_{13}x}{\sqrt{1-\rho_{13}^2}}, \frac{x_2 - \rho_{23}x}{\sqrt{1-\rho_{23}^2}}, \frac{\rho_{12} - \rho_{13}\rho_{23}}{\sqrt{1-\rho_{13}^2}\sqrt{1-\rho_{23}^2}} \right) dx \tag{5.19}$$

Many of the issues surrounding developing robust code for these cumulative functions are discussed in West [2005].

5.6 Exercises

1. Write vba code for the Newton-Bailey method of finding the cumnorm inverse function. Use the double precision cumnorm function provided. Use 'newx' below as your first estimate, where 'y' is the input:

```
r = Sqr(-2 * Log(Min(y, 1 - y)))
newx = r - (2.515517 + 0.802853 * r + 0.010328 * r ^ 2) /
           (1 + 1.432788 * r + 0.189269 * r ^ 2 + 0.001308 * r ^ 3)
If y < 0.5 Then newx = -newx
```

2. Show that if S is subject to GBM with drift μ and volatility σ,

$$\mathbb{E}[S(T)^k] = S(t)^k \exp\left(\left(k\mu + \tfrac{1}{2}(k^2 - k)\sigma^2\right)(T - t)\right)$$

3. Formally verify (5.16) and (5.17).

4. Verify (13.1), (13.2) and (13.3).

5. Find the integral $\int_\alpha^\infty \cdots$ in place of (13.2).

6. (exam 2004) Consider the bivariate normal cumulative function $N_2(x, y, \rho)$. Recall this is the probability that $X \leq x$, $Y \leq y$ where X and Y are normally distributed variables which are correlated with correlation coefficient ρ. So

$$N_2(x, y, \rho) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(X, Y, \rho)\, dY\, dX$$

where f is the relevant probability density function. Let $M_2(x, y, \rho)$ be the complementary probability i.e. it is the probability that $X \geq x$, $Y \geq y$. Also $N(\cdot)$ is the usual cumulative normal function. Prove that

$$N_2(x, y, \rho) = M_2(x, y, \rho) + N(x) + N(y) - 1$$

(Think before you dive in headfirst. Very simple, elegant proofs are possible.)

Chapter 6

Review of vanilla option pricing and associated statistical issues

6.1 Deriving the Black-Scholes formula

By the principle of risk-neutral valuation, the value of a European call option is

$$V = e^{-r\tau}\,\mathbb{E}_t^{\mathbb{Q}}\left[\max(Se^X - K, 0)\right] \tag{6.1}$$

where X has the meaning of (5.5). We now calculate:

$$\begin{aligned}
V &= e^{-r\tau}\,\mathbb{E}_t^{\mathbb{Q}}\left[\max(Se^X - K, 0)\right]\\
&= e^{-r\tau}\frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}}\int_{-\infty}^{\infty}\max(Se^x - K, 0)\exp\left[-\tfrac{1}{2}\left(\frac{x - m_-\tau}{\sigma\sqrt{\tau}}\right)^2\right]\,dx\\
&= e^{-r\tau}\frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}}\int_{\ln\frac{K}{S}}^{\infty}(Se^x - K)\exp\left[-\tfrac{1}{2}\left(\frac{x - m_-\tau}{\sigma\sqrt{\tau}}\right)^2\right]\,dx\\
&= e^{-r\tau}S\frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}}\int_{\ln\frac{K}{S}}^{\infty}e^x\exp\left[-\tfrac{1}{2}\left(\frac{x - m_-\tau}{\sigma\sqrt{\tau}}\right)^2\right]\,dx\\
&-\; e^{-r\tau}K\frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}}\int_{\ln\frac{K}{S}}^{\infty}\exp\left[-\tfrac{1}{2}\left(\frac{x - m_-\tau}{\sigma\sqrt{\tau}}\right)^2\right]\,dx
\end{aligned}$$

Now, for the first integral, we complete the square:

$$\begin{aligned}
x - \tfrac{1}{2}\left(\frac{x - m_-\tau}{\sigma\sqrt{\tau}}\right)^2 &= x - \tfrac{1}{2}\frac{x^2 - 2m_-\tau x + m_-^2\tau^2}{\sigma^2\tau} \\
&= -\tfrac{1}{2}\frac{x^2 - 2m_-\tau x - 2x\sigma^2\tau + m_-^2\tau^2}{\sigma^2\tau} \\
&= -\tfrac{1}{2}\frac{x^2 - 2m_+\tau x + m_-^2\tau^2}{\sigma^2\tau} \\
&= -\tfrac{1}{2}\frac{(x - m_+\tau)^2 - m_+^2\tau^2 + m_-^2\tau^2}{\sigma^2\tau} \\
&= -\tfrac{1}{2}\left(\frac{x - m_+\tau}{\sigma\sqrt{\tau}}\right)^2 + (r-q)\tau
\end{aligned}$$

so

$$\begin{aligned}
V &= e^{-q\tau}S\frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}}\int_{\ln\frac{K}{S}}^{\infty}\exp\left[-\tfrac{1}{2}\left(\frac{x-m_+\tau}{\sigma\sqrt{\tau}}\right)^2\right]dx \\
&\quad - e^{-r\tau}K\frac{1}{\sqrt{2\pi}\sigma\sqrt{\tau}}\int_{\ln\frac{K}{S}}^{\infty}\exp\left[-\tfrac{1}{2}\left(\frac{x-m_-\tau}{\sigma\sqrt{\tau}}\right)^2\right]dx \\
&= e^{-q\tau}SN\left(\frac{m_+\tau - \ln\frac{K}{S}}{\sigma\sqrt{\tau}}\right) - e^{-r\tau}KN\left(\frac{m_-\tau - \ln\frac{K}{S}}{\sigma\sqrt{\tau}}\right) \\
&= e^{-q\tau}SN(d_+) - e^{-r\tau}KN(d_-)
\end{aligned}$$

where the meaning of $d_+(\equiv d_1)$ and $d_-(\equiv d_2)$ will be established now.
The put formula follows by put-call parity, or by mimicking the argument.

6.2 A more general result

In full generality, we have the following result.

Lemma 6.2.1. Suppose we have a vanilla European call or put on a variable Y, strike K, where the terminal value of Y is lognormally distributed, $\log Y \sim \phi(\Psi, \Sigma)$. Then the option price is given by

$$V_\eta = e^{-r\tau}\eta\left[e^{\Psi+\frac{1}{2}\Sigma}N(\eta d_+) - KN(\eta d_-)\right] \tag{6.2}$$

$$d_+ = \frac{\Psi + \Sigma - \log K}{\sqrt{\Sigma}} \tag{6.3}$$

$$d_- = \frac{\Psi - \log K}{\sqrt{\Sigma}} \tag{6.4}$$

where $\eta = 1$ for a call and $\eta = -1$ for a put.
Check that the Black-Scholes formula follows as a special case of this, and be able to prove this result. (It is in the tutorial. Simply follow the scheme already seen for Black-Scholes.)

6.3 Pricing formulae

We will consider vanilla option pricing formulae. The inputs to these formulae will be (some of) spot S, future F, risk free rate r, dividend yield q, strike K, volatilility σ, valuation date t and expiry date T. So far, all of these symbols have been discussed, and we know how to derive them, with the exception of σ, which for the moment is just an input.

Black-Scholes

$$\begin{aligned} V_{\text{EC}} &= Se^{-q\tau}N(d_1) - Ke^{-r\tau}N(d_2) && (6.5) \\ V_{\text{EP}} &= Ke^{-r\tau}N(-d_2) - Se^{-q\tau}N(-d_1) && (6.6) \end{aligned}$$

where

$$d_{1,2} = \frac{\ln(S/K) + (r - q \pm \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \tag{6.7}$$

This formula is used for a European option on stock.

Forward form of Black-Scholes

$$\begin{aligned} V_{\text{EC}} &= e^{-r\tau}[fN(d_1) - KN(d_2)] && (6.8) \\ V_{\text{EP}} &= e^{-r\tau}[KN(-d_2) - fN(-d_1)] && (6.9) \end{aligned}$$

where

$$d_{1,2} = \frac{\ln(f/K) \pm (\sigma^2/2)\tau}{\sigma\sqrt{\tau}} \tag{6.10}$$

This is still the Black-Scholes formula, using the fact that $f = Se^{(r-q)\tau}$.

Standard Black

$$\begin{aligned} V_{\text{EC}} &= e^{-r\tau}[FN(d_1) - KN(d_2)] && (6.11) \\ V_{\text{EP}} &= e^{-r\tau}[KN(-d_2) - FN(-d_1)] && (6.12) \end{aligned}$$

where

$$d_{1,2} = \frac{\ln(F/K) \pm (\sigma^2/2)\tau}{\sigma\sqrt{\tau}} \tag{6.13}$$

This formula is used internationally for a European option on a future.

SAFEX Black

$$\begin{aligned} V_{\text{AC}} &= FN(d_1) - KN(d_2) && (6.14) \\ V_{\text{AP}} &= KN(-d_2) - FN(-d_1) && (6.15) \end{aligned}$$

where

$$d_{1,2} = \frac{\ln(F/K) \pm (\sigma^2/2)\tau}{\sigma\sqrt{\tau}} \tag{6.16}$$

This formula is used at SAFEX and the Sydney Futures Exchange for an American option on a future.

Forward version of SAFEX Black

If we assume $F = f$ ie. $F = Se^{(r-q)\tau}$ then we get

$$\begin{aligned} \mathrm{V}_{\mathrm{AC}} &= Se^{(r-q)\tau}N(d_1) - KN(d_2) && (6.17) \\ \mathrm{V}_{\mathrm{AP}} &= KN(-d_2) - Se^{(r-q)\tau}N(-d_1) && (6.18) \end{aligned}$$

where

$$d_{1,2} = \frac{\ln(S/K) + (r - q \pm \sigma^2/2)\tau}{\sigma\sqrt{\tau}} \qquad (6.19)$$

The unifying formula

Note that in all cases

$$\begin{aligned} V &= \xi\eta[\mathbb{f}N(\eta d_+) - KN(\eta d_-)] && (6.20) \\ d_\pm &= \frac{\ln(\mathbb{f}/K) \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} && (6.21) \end{aligned}$$

where

- ξ is $e^{-r\tau}$ for an European Equity Option and for Standard Black, and 1 for SAFEX Black Futures Options and SAFEX Black Forward Options.
- $\eta = 1$ for a call and $\eta = -1$ for a put,
- $\mathbb{f} = f = Se^{(r-q)\tau}$ is the forward value for an European Equity Option and for SAFEX Black Forward Options, and $\mathbb{f} = F$ is the futures value for Standard Black and SAFEX Black Futures Options.

In doing any calculations (greeks, for example) the following equations are key:

$$\begin{aligned} d_+ &= d_- + \sigma\sqrt{\tau} && (6.22) \\ d_+^2 &= d_-^2 + 2\ln\frac{\mathbb{f}}{K} && (6.23) \\ N'(x) &= N'(\eta x) && (6.24) \\ N'(x) &= \frac{1}{\sqrt{2\pi}}e^{-x^2/2} && (6.25) \\ \mathbb{f}\,N'(d_+) &= K\,N'(d_-) && (6.26) \end{aligned}$$

6.4 Risk Neutral Probabilities

We can speed up and simplify the calculation of the risk-neutral probabilities in vanilla option premium formulae. As usual in option pricing, we have

$$\begin{aligned} \tau &= T - t \\ d_\pm &= \frac{\ln\frac{\mathbb{f}}{K} \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \end{aligned}$$

where $\mathbb{f}$ denotes the forward level for spot-type options and the futures level for options involving futures.

Certain special cases apply, where the formula does not make sense in a pure sense, but can be made sense of mathematically by taking limits. This occurs if any of forward/future, strike, term or volatility are zero. The appropriate outcome in these cases (in the sense of a limit) is determined by testing:

- when the strike K is zero, $d_\pm = \infty$ which will give $N(d_\pm) = 1$ and $N'(d_\pm) = 0$,
- when $\mathbb{f}$ is zero, $d_\pm = -\infty$ which will give $N(d_\pm) = 0$ and $N'(d_\pm) = 0$,
- when either term or volatility are zero, and $\mathbb{f}$ is greater than the strike, $d_\pm = \infty$ which will give $N(d_\pm) = 1$ and $N'(d_\pm) = 0$,
- when either term or volatility are zero, and $\mathbb{f}$ is less than the strike, $d_\pm = -\infty$ which will give $N(d_\pm) = 0$ and $N'(d_\pm) = 0$.

6.5 What is Volatility?

Volatility is a measure of the jumpiness of the time series (price, rate or yield) of data we have. However, there are two very different notions of volatility:

- implied volatility: the volatility which can be backed out of an appropriate pricing formula. Such a volatility may be explicitly traded if there is agreement on the appropriate pricing formula, or one may imply it by taking the price and determining the appropriate volatility using an algorithm such as Newton's or Brent's method. There is no claim whatsoever that the underlying obeys a process (such as Geometric Brownian motion) at this level of volatility.

  The trader charges for the option what he thinks the market will bear. This notion is forward looking, based on investor sentiment.

  We will say much more about implied volatility in §8.1.

- realised or historic volatility. This is backward looking. It is a measure of the 'jumpiness' that has occurred. Its calculation will be premised on the assumption that the process follows geometric Brownian motion or some variant thereof.

Implied volatility includes market sentiment and nervousness, not to mention fees, so will usually be higher than historical volatility.

6.6 Calculating historic volatility

In the simplest cases, we suppose that the underlying follows Geometric Brownian motion. Then

$$dx = \mu x\, dt + \sigma x\, dZ$$

where the time is measured in years, and so by Itô's lemma

$$\ln\left(\frac{x_T}{x_t}\right) \sim \phi\left(\left(\mu - \frac{\sigma^2}{2}\right)(T-t), \sigma^2(T-t)\right).$$

and

$$p_{t,T} := \frac{1}{\sqrt{T-t}} \ln\left(\frac{x_T}{x_t}\right) \sim \phi\left(\left(\mu - \frac{\sigma^2}{2}\right)\sqrt{T-t}, \sigma^2\right). \tag{6.27}$$

In particular,

$$p_t := \ln\left(\frac{x_t}{x_{t-1d}}\right) \sim \phi\left(\left(\mu - \frac{\sigma^2}{2}\right)\frac{1}{250}, \frac{\sigma^2}{250}\right). \tag{6.28}$$

where the data $x.$ we are taking are daily closes of prices/rates/yields. This follows from the fact that there are 250 trading days in the SA market in the year, and we are assuming they are equally spaced. The number 252 is typical in international texts such as Hull [2005].[1]

Hence:

- σ is estimated to be $\sqrt{250}$ times the standard deviation of $\{p_t\}$. This is the 'unweighted volatility' method. The problem here is that we consider the entire sample, much of which can be old, and stale.

- The second and only slightly less naive method is the N-window moving average method, where statistics are based on the last N observations. N can take any value, typically 30, 60, 90, 180 etc. Thus, on day $t-1$, we calculate s_{t-1} using p_i for $i = t-N, \ldots, t-1$, and on date t we drop p_{t-N} and include the new observation p_t, thus calculating s_t using p_i for $i = t-N+1, \ldots, t$. This allows for the dropping off of old data, because the input data is never older than N observations. However, this drop off is very discrete, and at the time of dropping off, can cause an otherwise unjustified spike down in the volatility estimate.

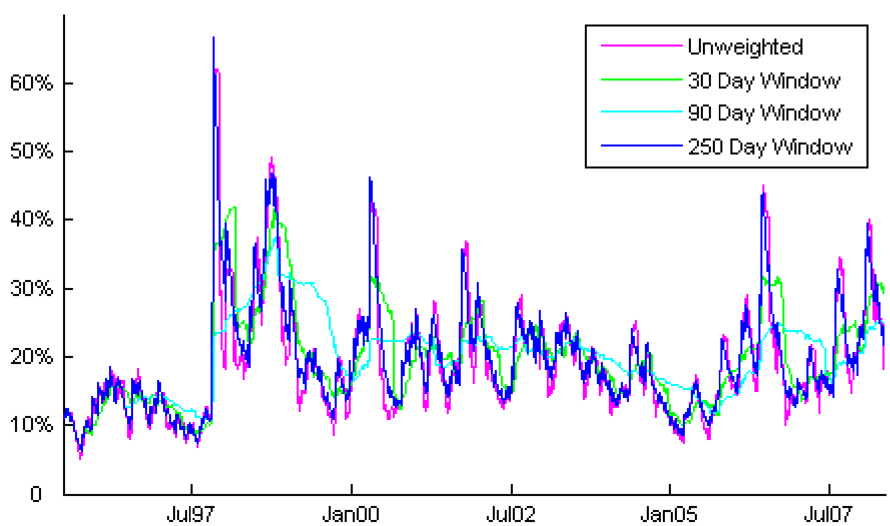


Figure 6.1: Unweighted rolling window volatilities for the ALSI40/TOPI

We have $N = 30$, $N = 90$, $N = 250$. The volatility graphs described above are labeled unweighted window volatility graphs. The reason for this is that all N observations play an equally important role in the calculation - they are not weighted by perceived importance. This is certainly a problem. Notice that at the October 1997 crash, the volatility shoots up, and then N business days later, it falls off. This is because the October observation is in the

[1] If we were to take monthly rates, we would replace 250/252 with 12, for example, because a month is $\frac{1}{12}$ of a year.

window until N business days after it occurred, and then the next day it suddenly disappears. The implicit assumption in this calculation is that an observation from N business days ago is equally relevant as yesterday's, but an observation from $N+1$ business days ago is completely irrelevant. It is the size of the crash observation that is having such an enormous effect on the data, drowning out the informational content of the other $N-1$ observations.

- A common feature of a variance estimate would be that $\Sigma_t^2 = \sum_{i \le t} \alpha_i p_i^2$ where $\sum_i \alpha_i = 1$.[2] The sum in general will be a finite one - namely, the entire (but finite) history, or the last N observations. Now let us make the sum infinite, at least in principal, so

$$\begin{aligned} \sigma(t)^2 &= 250 \sum_{i=0}^{\infty} \alpha_{t-i} p_{t-i}^2 \\ \sum_{i=0}^{\infty} \alpha_{t-i} &= 1 \end{aligned}$$

We have used the fact that $\sigma^2 = 250\Sigma^2$, ie the square of (annualised) volatility is 250 times the variance of the daily LPRs.

What property would we like? A nice property would be that the importance of an observation is only λ times the importance of the observation which is one day more recent than it. Here λ is called the weight, typically $0.9 < \lambda$, and certainly $\lambda < 1$. But if we put $\alpha_{t-i} = \lambda^i$, we would have $\sum_{i=0}^{\infty} \alpha_{t-i} = \sum_{i=0}^{\infty} \lambda^i = \frac{1}{1-\lambda}$. So instead, let us put $\alpha_{t-i} = (1-\lambda)\lambda^i$. Thus

$$\begin{aligned} \sigma^2(t) &= 250(1-\lambda)\left(p_t^2 + \lambda p_{t-1}^2 + \lambda^2 p_{t-2}^2 + \lambda^3 p_{t-3}^2 + \cdots\right) \\ &= 250(1-\lambda)p_t^2 + \lambda\, 250(1-\lambda)\left(p_{t-1}^2 + \lambda p_{t-2}^2 + \lambda^2 p_{t-3}^2 + \cdots\right) \\ &= 250(1-\lambda)p_t^2 + \lambda\sigma^2(t-1) \end{aligned} \tag{6.29}$$

which is the fundamental updating equation for the EWMA method: Exponentially Weighted Moving Average. If we were to want daily volatilities instead (i.e. not to annualise) we could divide the final answer by 250. Easier, we simply drop the 250 from all of the above calculations.

The smaller λ is, the more quickly it forgets past data and the more jumpy it becomes. Large λ forgets past more slowly; in the limit the graph becomes straight. How one chooses λ is a major problem.

However, some comments are in order. It follows immediately that in order to attempt to mimic short term volatility, one should use a lower value of lambda: probably never lower than 0.9, and typically in the regions of 0.95 or so. For longer dated volatility, one will use a higher value of lambda, say 0.99. One can compare these historical measures of volatility with implied volatility; as already discussed, the one is NOT a model for the other, and if historical volatility is to be used as a surrogate for implied volatility, one must take into account additional factors (that implied volatility has a risk price built in). Roughly, however, historical and implied volatility should be cointegrated.

How do we start the inductive process for determining these EWMA volatilities? The theory developed requires an infinite history, in reality, we do not have this history. And we should

[2] To understand this, the standard formula would be $\Sigma^2 = \frac{1}{n-1}\sum_{i=1}^{n}(p_i - \overline{p})^2$. First, accept that we will assume that the population mean is zero, so we gain a degree of freedom, and $\Sigma^2 = \frac{1}{n}\sum_{i=1}^{n} p_i^2$. Now we have the above formulation, where $\alpha_i = \frac{1}{n}$ for all i.

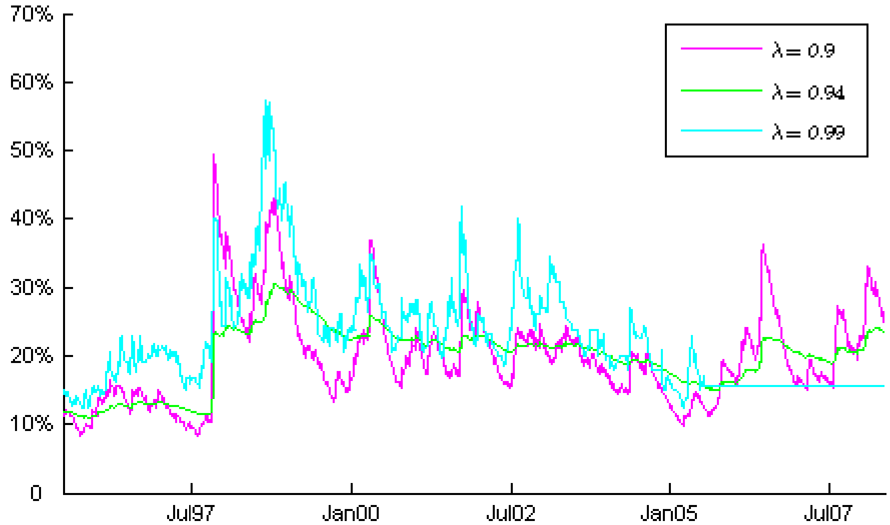


Figure 6.2: Various EWMA volatility measures of ALSI40/TOPI

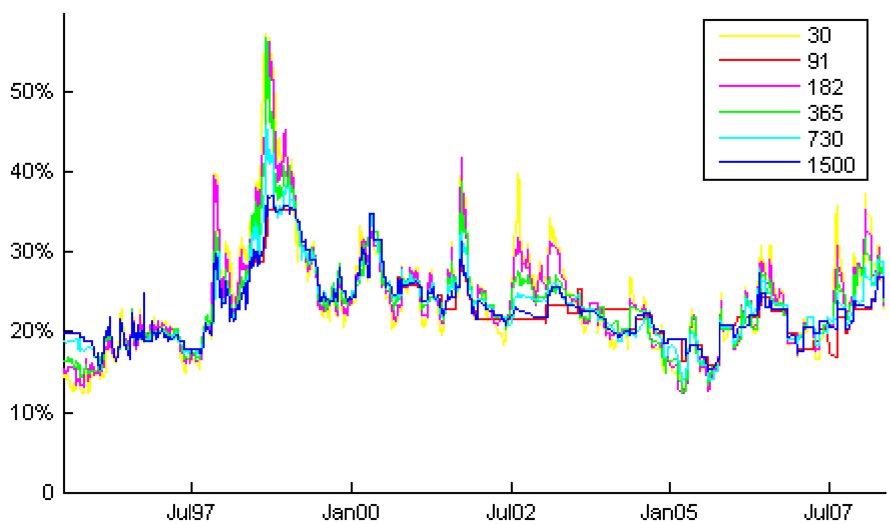


Figure 6.3: Various implied ATM volatilities for ALSI40/TOPI. The key is the number of days to expiry, interpolation between quoted expiries is applied.

not start our estimation at 0.[3] A usable workaround is to take the average of the first few squared returns for the estimate of σ_0^2. We have taken the average of the first 25 observations. Hence the rolling calculator for volatility is

[3]This problem is completely ignored in the texts.

The data available is $x_0,\ x_1,\ \ldots,\ x_t$:

$$p_i = \ln \frac{x_i}{x_{i-1}} \quad (1 \le i \le t) \tag{6.30}$$

$$\sigma(0) = \sqrt{10 \sum_{i=1}^{25} p_i^2} \tag{6.31}$$

$$\sigma(i) = \sqrt{\lambda\sigma^2(i-1) + (1-\lambda)p_i^2 250} \quad (1 \le i \le t) \tag{6.32}$$

- Finally, there are the Generalised Autoregressive Conditional Heteroskedasticity (GARCH) methods. These are logical estensions to EWMA methods (in fact EWMA is a so called baby-GARCH method) but where the process selects its own exponential weighting parameter, on a daily basis, using Maximum Likelihood Estimation techniques. An advantage of these methods is that they have built in mean reversion properties; the EWMA method does not display mean reversion.

  It is known that great care has to be taken to ensure that a GARCH process is sufficiently stable to be meaningful, and even to always converge.

6.7 Other statistical measures

We just consider the EWMA case.

The rolling calculator for covolatility (i.e. annualised covariance) - see [Hull, 2005, §19.7] - is

$$\begin{aligned} \mathrm{covol}_0(x,y) &= \left(\sum_{i=1}^{25} p_i(x)p_i(y)\right) 10 & (6.33)\\ \mathrm{covol}_i(x,y) &= \lambda \mathrm{covol}_{i-1}(x,y) + (1-\lambda)p_i(x)p_i(y)250 \quad (1 \le i \le t) & (6.34) \end{aligned}$$

Following on from this, the derived calculators are

$$\begin{aligned} \rho_i(x,y) &= \frac{\mathrm{covol}_i(x,y)}{\sigma_i(x)\sigma_i(y)} & (6.35)\\ \beta_i(x,y) &= \frac{\mathrm{covol}_i(x,y)}{\sigma_i(x)^2} & (6.36) \end{aligned}$$

the latter since the CAP-M β is the linear coefficient in the regression equation in which y is the dependent variable and x is the independent variable. The CAP-M intercept coefficient α has to be found via rolling calculators. Thus

$$\begin{aligned} \overline{p_1(x)} &= 10 \sum_{i=1}^{25} p_i(x) & (6.37)\\ \overline{p_i(x)} &= \lambda \overline{p_{i-1}(x)} + (1-\lambda)p_i(x)250 \quad (1 \le i \le t) & (6.38) \end{aligned}$$

and likewise for $p(y)$. Then

$$\alpha_i(x,y) = \overline{p_i(y)} - \beta_i(x,y)\overline{p_i(x)} \tag{6.39}$$

6.8 Rational bounds for the premium

Clearly, the search for an implied volatility can only return a volatility in the bounds $[0, \infty)$. These bounds correspond exactly to (model independent) arbitrage bounds on option pricing formulae.

Thus, the arbitrage bound minimum price for an option is

$$V_{\min} = \begin{cases} \xi\eta[fN(\eta\infty) - KN(\eta\infty)] & \text{if} \quad f > K \\ \xi\eta[fN(-\eta\infty) - KN(-\eta\infty)] & \text{if} \quad f < K \end{cases} \tag{6.40}$$

and the arbitrage bound maximum price for an option is

$$V_{\max} = \xi\eta[fN(\eta\infty) - KN(-\eta\infty)] \tag{6.41}$$

Code which searches for an implied volatility should first check if the premium lies in $(V_{\min}, V_{\max})$, and if not, the code should return an error. This needs to be trapped as an immediate application of Brent's method, for example, will fail to converge.

6.9 Implied volatility

For any of the 4 option types we will on occasion know all of the inputs except the volatility, and know the premium, and require the volatility that, when input, will return the correct premium. Such a volatility is known as the implied volatility. It can be found using the Newton-Rhapson method, although one has to be careful, because an injudicious seed value will cause this method to not converge. In Manaster and Koehler [1982], a seed value of the implied volatility is given which guarantees convergence.

The argument in Manaster and Koehler [1982] is unnecessarily complicated, and can easily be understood as follows: premium as a function of volatility is an increasing function, bounded below by the intrinsic value and above by the price of the underlying. It is initially convex up and subsequently convex down. Thus, choosing the point of inflection as the seed value, guarantees convergence, no matter which way the iteration, which will be monotone and quadratic in speed, will go. By simple calculus, one finds this point of inflection, for any of the four methods, to be

$$\sigma = \sqrt{\frac{2}{\tau} \left| \ln \frac{f}{K} \right|} \tag{6.42}$$

If the option is at the money forward then the point of inflection is at 0, so we start the iteration there.

An alternative to use the first estimate of Corrado and Miller [1996], modified to ensure valid computation. This estimate is the root of a quadratic, but a naïve application will run into the problem of having complex roots. Thus, a first estimate which is always valid is:

$$\begin{aligned} \sigma = & \frac{\sqrt{2\pi}}{\xi(f+K)\sqrt{\tau}} \\ & \cdot \left[V - \frac{\xi\eta(f-K)}{2} + \sqrt{\left(\left(V - \frac{\xi\eta(f-K)}{2} \right)^2 - \frac{(\xi(f-K))^2}{\pi} \right)^+} \right] \end{aligned} \tag{6.43}$$

The code will then expand this point to an interval in which the root must lie, and then use Brent's algorithm.

6.10 Calculation of forward parameters

Forward quantities are calculated as follows:

$$r(0; T_1, T_2) = \frac{r_2 T_2 - r_1 T_1}{T_2 - T_1} \tag{6.44}$$

$$q(0; T_1, T_2) = \frac{q_2 T_2 - q_1 T_1}{T_2 - T_1} \tag{6.45}$$

$$\sigma(0; T_1, T_2) = \sqrt{\frac{\sigma_2^2 T_2 - \sigma_1^2 T_1}{T_2 - T_1}} \tag{6.46}$$

where time is measured in years. Alternatively, for dividends, we may simply calculate the forward values or the present value of the forward values. For the volatility, this is the at the money volatility. Inclusion of the skew is always tricky and requires additional assumptions.
Sometimes, if the notation is getting heavy, we will just notate these quantities $r(i,j)$, $q(i,j)$ and $\sigma(i,j)$.

6.11 Exercises

1. As discussed in class, build a spreadsheet to calculate and graph the volatility of a time series of financial prices/yields/rates. Use:
   (a) Unweighted volatility calculation
   (b) 'unweighted window' volatility with N=90
   (c) EWMA volatility with $\lambda = 0.99$
2. Write vba code to price all varieties of the vanilla option pricing formulae. Use the symbols and the general approach from the notes.
3. Repeat the derivation of the Black-Scholes formula, this time for puts.
4. Prove Lemma 6.2.1.
5. Verify that, with the usual notation, $\mathbb{f}\ N'(d_1) = K\ N'(d_2)$. Torturous, long, solutions are problematic. That does not mean leave out details!
6. (a) Make sure your cumnorm function is working. Approximately, on what domain does it return values which are different from 0 or 1? Why is this not the whole real line?
   (b) Write a d_1 and a d_2 function. Be sure to accommodate special cases (term or strike or future/forward being 0).
   (c) Write a SAFEX Black option pricing function (inputs F, K, σ, valuation date, expiry date and style).
   (d) Make sure that the function works for the special cases already discussed. This work should be done by the d_i functions, not by the option pricing functions.
   (e) Draw graphs of the option values for varying spot/future and varying time to expiry.
   (f) Extend to a Black-Scholes option pricing function (inputs S, r, q, K, σ, valuation date, expiry date and style).

7. (exam 2004) A supershare option entitles the holder to a payoff of $\frac{S(T)}{X_L}$ if $X_L \le S(T) \le X_H$, and 0 otherwise. The price of a supershare option is given by

$$\begin{aligned} V &= \frac{S}{X_L} e^{-q\tau}[N(d_1) - N(d_2)] \\ d_1 &= \frac{\ln \frac{f}{X_L} + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \\ d_2 &= \frac{\ln \frac{f}{X_H} + \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \end{aligned}$$

Create a option pricing calculator in excel, referring to a pricing function written in vba. The time input will be in years i.e. don't use dates. Draw a spot profile of the value of the derivative.

8. (exam 2004) The Standard Black call option pricing formula is

$$\begin{aligned} V &= e^{-r\tau}(FN(d_1) - KN(d_2)) \\ \Delta &= e^{-r\tau}N(d_1) \\ \Gamma &= e^{-r\tau}N'(d_1)\frac{1}{F\sigma\sqrt{\tau}} \\ d_{1,2} &= \frac{\ln \frac{F}{K} \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}} \end{aligned}$$

   (a) Write code to price, and provide Greeks for, a call option using the Standard Black formula. The last input of your list of inputs to the pricing formula will be an optional string parameter. The default will be "p" (for premium). Have "d" (for delta) and "g" (for gamma) other possibilities. Fix the strike, the volatility, the risk free rate, and the term (which will be in years i.e. don't use dates).

   (b) For a range of futures prices, draw graphs (separate sheets for each) of the value, the delta, and the gamma. On each of these above sheets, illustrate the effect of time on each profile by drawing the graphs for 6 months, 1 month and 1 week to expiry.

9. (exam 2003) A chooser option is one that expires after term τ_2. After term $\tau_1 < \tau_2$, however, the holder must decide if the option is a put or a call (European, with identical strikes X). Use put-call parity to find the value (in terms of vanilla options) of this option at the inception of the product. As usual, assume constant term structures of r, q and σ.

10. (a) Write code to price, and provide Greeks for, a European call option using Black-Scholes. The last input of your list of inputs to the pricing formula will be an optional string parameter. The default will be "p" (for premium). Have "d" (for delta) and "g" (for gamma) other possibilities.

   (b) For a range of spot prices, draw graphs (separate sheets for each) of the value, the delta, and the gamma.

   (c) On each of the above sheets, draw several graphs, illustrating the effect of time on each profile i.e. draw the graphs for 1 year to expiry, 6 months, 1 month, 1 week, etc.

Chapter 7

The South African Futures Exchange

7.1 The SAFEX setup

Organised exchanges are a method of bringing market participants together. Participants in futures exchanges are hedgers, arbitrageurs, speculators, and curve traders. The exchange takes margin as a measure to ensure the contracts are honoured ie. mitigate against credit risk. As an entity the exchange makes money through the use of this margin on deposit. The exchange does not take positions itself.

See SAFEX [2011].

SAFEX was first organised by RMB in April 1987 and formalised in September 1988.

In order to open positions at SAFEX, one has to pay an initial margin. The market takes this margin as a measure to ensure contracts are honoured (mitigate credit risk). If contracts are not honoured, the contract is closed out by the exchange and this initial margin is used to make good any shortfall.

In order for this strategy to be effective, initial margin is obviously quite a significant amount. Thus it has to earn interest. This is a good interest rate, but not quite as good as the rate earned by SAFEX when they deposit all of the various margin deposits. Thus, as an entity SAFEX makes money through the use of margin on deposit.

If the total margin paid in falls below the maintenance margin then the account has to be topped up to the initial margin.

This is all completely standard, see [Hull, 2005, Chapter 2].

Futures are always fully margined on any exchange. However, what is unusual is that SAFEX futures options are fully margined. No premium is paid up front!! All that is paid up front is the initial margin. However, on a daily basis one pays/receives the change in valuation (MtM). They are called fully margined futures options.

SAFEX trades futures and futures options on indices and individual stocks; on JIBAR rate, bonds etc. We will concentrate on futures and futures options on TOP40, INDI25.

Futures are screen traded via a bid and offer, and are very liquid.

Futures options are slightly less liquid, as they are typically negotiated via brokers and then traded on the screen, so the screen can be stale. Furthermore, these trades can be for packages ie. a price (implied volatility) is set for a set of options, without determining what the implied volatility for each option is.

Input Data		term (days)	735
Value Date	15-Mar-07	term (years)	2.014
Future	TOPI	d_1	1.0254
Futures Spot	25902	d_2	0.7310
Strike	20000	$N(d_1)$	0.8474
Option Expiry	19-Mar-09	$N(d_2)$	0.7676
Volatility	20.75%	$N(-d_1)$	0.1526
		$N(-d_2)$	0.2324
Prices		Rand Price	
Future	25902.00	R 259,020.00	
Call Price	6597.80	R 65,978.00	
Put Price	695.80	R 6,958.00	
rpp	10.00		

Table 7.1: MtM of SAFEX option

Rounding:

- For indices: futures levels and strikes are always whole point numbers. Unrounded option prices are calculated. The result (for futures or futures options) is multiplied by 10, and then rounded to the nearest rand.

- For equities: futures levels and strikes are in rands and cents. Unrounded option prices are calculated. The result (for futures or futures options) is multiplied by 100, and then rounded to the nearest rand.

Contracts vary by strike and expiry.

- Strike: for indices, discrete strikes allowed, in units of 50 points minimum. The strike levels need to have a reasonable dispersion for liquidity, so that one can find the opposite side for a trade. Too many strikes means low liquidity at each strike.

- Expiry: as far out as there is interest (currently December 2014) expiring 3rd Thursday of March, June, Sept, Dec. If this is a holiday then scroll backwards. The March following (currently March 2012) is always the most liquid, for tax reasons.

The volatility we have used is implied, not realised. At the end of each day, SAFEX uses the last implied traded volatility for an at or near the money option for margining purposes. All options for the ALSI40 contracts for that expiry are then margined at a skew, which is dependent on that at the money volatility. All other contracts are insufficiently liquid to have a skew built for them. We will study the skew later, but in the absence of any specific information, we will assume there is no skew.

JSE FINANCIAL DERIVATIVES CLOSING PRICES | **DATE:** 17-Aug-2010 | **Market** 1

CONTRACT	SPOT	BID	OFFER	M-T-M	VOL. CHANGES	PREV. VOLS.
FTSE/JSE TOP40 INDEX FUTURE (ALSI)						
16-Sep-2010	24586	24565	24569	24567	24.50	25.00
15-Dec-2010	24586	24805	24859	24832	24.75	25.00
17-Mar-2011	24586	24902	24982	24942	24.75	25.00
15-Jun-2011	24586	25226	25326	25276	25.00	25.25
15-Dec-2011	24586	25722	25842	25782	25.00	25.25
15-Mar-2012	24586	25966	26106	26036	25.25	25.50
20-Dec-2012	24586	27146	27306	27226	25.50	25.75
20-Mar-2013	24586	27377	27557	27467	25.75	26.00
18-Dec-2014	24586	30676	30876	30776		26.25
MINI FTSE/JSE TOP40 INDEX - ALMI (ALMI)						
16-Sep-2010	24586	24567	24567	24567		30.00
15-Dec-2010	24586	24832	24832	24832		30.00
FTSE/JSE Top 40 Index Dividend Future (ALSF)						
16-Sep-2010	158.51	158.51	158.51	158.51		30.00
15-Dec-2010	266.99	266.99	266.99	266.99		30.00
FTSE/JSE African Banks Index (BANK)						
16-Sep-2010	37783	37599	37599	37599		30.00
FTSE/JSE Capped Top 40 Index (CTOP)						
16-Sep-2010	12859	12852	12852	12852		30.00
15-Dec-2010	12859	12999	12999	12999		30.00
FTSE/JSE DIVIDEND PLUS INDEX (DIVI)						
16-Sep-2010	153.73	154.04	154.04	154.04		30.00

Figure 7.1: The SAFEX MtM page. The volatilities quoted are at-the-money volatilities; the skew follows from that in a way we will see shortly.

7.2 Are fully margined options free?

No. A legendary selling point for paid-up options is that the downside of the investment is at most the premium paid. This is seen as being preferable for risk adverse players to futures, for example, where the gearing is very high and the potential downside is enormous. One can think initially that fully margined options are free, because there is no cost to entering such options[1]. Alternatively, one can think that these options could become unreasonably expensive - in terms of the margin requirements - if, for example, volatility goes up.

Neither viewpoint is correct. Suppose an option is dealt at $V(t)$ and has an initial MtM of $M(t)$ - these will probably not coincide - we buy intraday, we are a price maker or taker, and the SAFEX margin skew is not the same as the traded skew (indeed, for some contracts, SAFEX do not have a skew). Suppose the expiry date MtM is $M(T) = V(T)$. If the options were fully paid up then there would be a flow of $-V(t)$ at date t and a flow of $+V(T)$ on date T.

Now suppose we have the SAFEX situation, where options are fully margined. Suppose the business days in $[t, T]$ are $t = t_0,\ t_1,\ \ldots,\ t_N = T$. Let the MtM on date t_i be denoted $M(t_i)$. The margin flow in the morning of date t_1 is $M(t_0) - V(t_0)$, and the margin flow in the morning of date t_{i+1} is $M(t_i) - M(t_{i-1})$ for $i \geq 1$. Thus, the total margin flow is

$$M(t_0) - V(t_0) + \sum_{i=1}^{N} M(t_i) - M(t_{i-1}) = V(T) - V(t)$$

as a telescoping series. Hence, the net effect is the same: it is merely the timing of cash flows that will be different.

Thus, the downside of the investment is still at most the initial 'cost'. However, there may be funding requirements in the interim which are not necessarily commensurate with that initial cost.

[1] We ignore the initial margin requirements because, whatever they are, they earn a competitive rate of interest at SAFEX.

7.3 Deriving the option pricing formula

1. Let F be the futures price, μ the drift, and σ the volatility. F is subject to Geometric Brownian Motion:

$$dF = \mu F dt + \sigma F dz$$

Suppose $g = g(F, \tau)$ is a derivative. By Itô's lemma,

$$dg = \left(\frac{\partial g}{\partial F} \mu F + \frac{\partial g}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 g}{\partial F^2} \right) dt + \frac{\partial g}{\partial F} \sigma F \, dz. \tag{7.1}$$

Let π be the portfolio which is long g and short $\frac{\partial g}{\partial F}$ of F.

$$\begin{aligned} d\pi &= dg - \frac{\partial g}{\partial F} dF \\ &= \left(\frac{\partial g}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 g}{\partial F^2} \right) dt \end{aligned}$$

To write $\pi = g - \frac{\partial g}{\partial F} F$ or $\pi = g$ would be highly misleading. In fact, $\pi = 0$ is the initial MONETARY investment into the portfolio. So $d\pi = 0$ and so

$$\frac{\partial g}{\partial t} + \frac{1}{2} \sigma^2 F^2 \frac{\partial^2 g}{\partial F^2} = 0 \tag{7.2}$$

Even now one can show that g is a martingale. r has disappeared because there is no (hedged and riskless) up front investment requiring a return of r.[2] Compare to the original B/S DE (for options on equity)

$$\frac{\partial g}{\partial t} + (r - q) S \frac{\partial g}{\partial S} + \tfrac{1}{2} \sigma^2 S^2 \frac{\partial^2 g}{\partial S^2} = rg$$

and Standard Black DE (for paid up European options on futures)

$$\frac{\partial g}{\partial t} + \tfrac{1}{2} \sigma^2 F^2 \frac{\partial^2 g}{\partial F^2} = rg.$$

2. We solve the DE with the usual European boundary conditions:

   - Call: $V(F, t) \to \infty$ as $F(t) \to \infty$, $V(0, t) = 0$, $V(F(T), T) = \max\{F(T) - K, 0\}$.
   - Put: $V(F, t) \to 0$ as $F(t) \to \infty$, $V(0, t) = K$, $V(F(T), T) = \max\{K - F(T), 0\}$.

   For European vanilla options, get

$$V_c = FN(d_1) - KN(d_2) \tag{7.3}$$

$$V_p = KN(-d_2) - FN(-d_1) \tag{7.4}$$

$$d_{1,2} = \frac{\ln \frac{F}{K} \pm \frac{1}{2} \sigma^2 \tau}{\sigma \sqrt{\tau}} \tag{7.5}$$

$$\tau = T - t \tag{7.6}$$

[2] That is quite different to saying that $r = 0$: it isn't. After all, F is of the order of $e^{r\tau}$. However, as a mathematical DE problem, we have eliminated r. This is a subtle point, but rather obvious once pointed out. Nevertheless, there are some otherwise quite sophisticated players in the market who don't understand this point.

3. However, options are American. If not exercised early, and in the money, then there is automatic exercise on the expiry date.

   To get the American condition, we need to worry about the free boundary condition $V_c \geq \max\{F(t) - K, 0\}$ for a call, $V_p \geq \max\{K - F(t), 0\}$ for a put.

   To see that $V > 0$, show $V_c(0) = 0$ and show $\frac{\partial V_c}{\partial F} > 0$; $V_p(\infty) = 0$ and $\frac{\partial V_p}{\partial F} < 0$.[3]

   For a call,

$$\begin{aligned} V_c - (F - K) &= FN(d_1) - KN(d_2) - F + K \\ &= K[1 - N(d_2)] - F[1 - N(d_1)] \\ &= KN(-d_2) - FN(-d_1) \\ &= V_p \\ &> 0. \end{aligned}$$

   In particular, we have put-call parity:

$$V_c + K = V_p + F \tag{7.7}$$

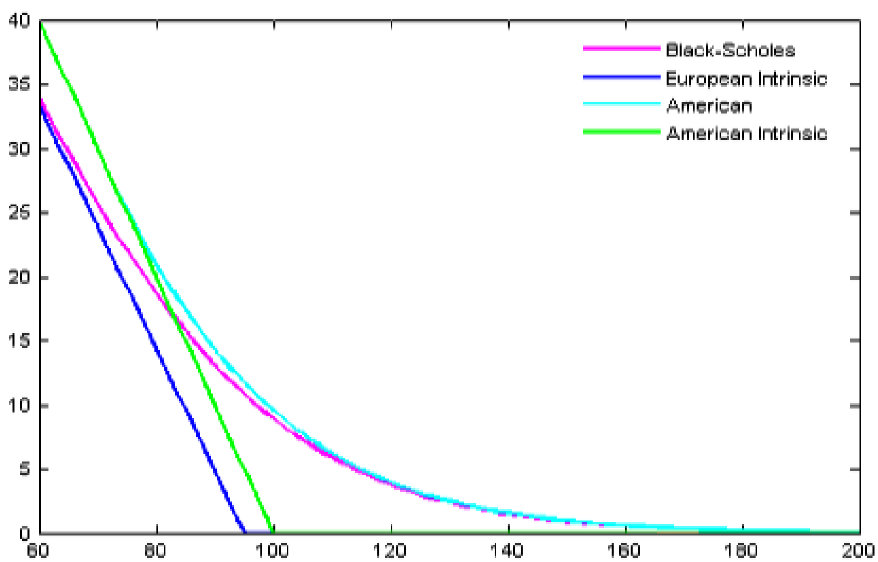


Figure 7.2: The European premium for an equity option violates the American free boundary condition.

Similarly for the put. So the free boundary is never touched. The American pricing formula is the same as the European one, and it is never optimal to exercise early. If someone chooses to

[3]There are some tricks here. Firstly, the differentiation is not completely trivial, as there are some disguised dependences: d_i are functions of F!. Secondly, the proof that $V_p(\infty) = 0$ needs l'Hopital's rule. One may not use 'known' properties of option prices when what we are trying to do is establish that we have a valid option pricing formula.

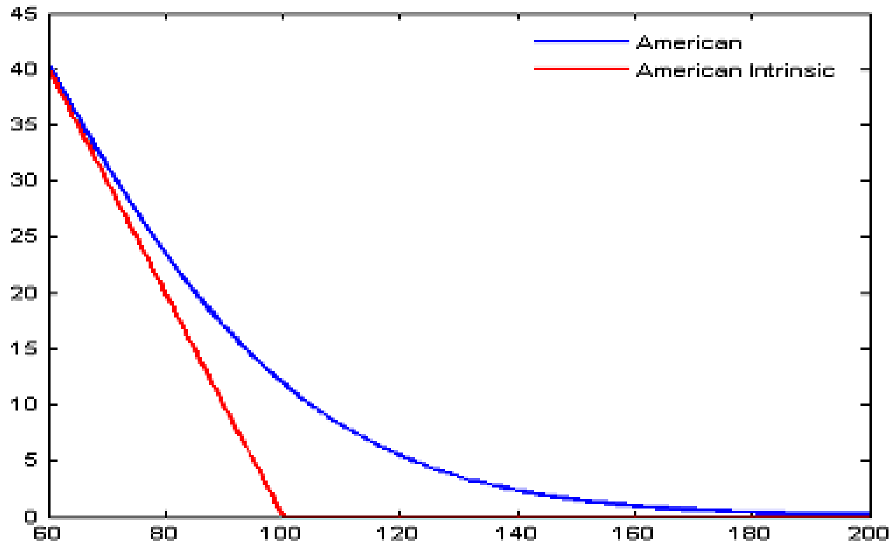


Figure 7.3: The SAFEX Black premium (European formula!) does not violate the free boundary, so is valid for American options.

exercise, SAFEX have a random drawing of individuals who are to be exercised against - lucky people because they are happy because they make money out of it (ie. the person exercising should not have).

See also Etheredge and West [1999].

7.4 Delivery, what happens at closeout?

On closeout day the futures price is averaged over the closing period (average of a per minute observation for the last 100 minutes of trading). The following day positions are margined as per usual, with the final variation margin being as usual.

If there is physical settlement, then the trade is executed (brokers are allocated positions at random) and these trades will have settlement period one day less than usual, so the net effect is as if the trade had been done on closeout day. In this case, initial margin is returned only upon completion of that trade execution.

In the case of options, the final variation margin will reflect changes in the value of the option, and settlement will be into a future, and what happens then is as above.

Chapter 8

The implied volatility skew

8.1 The skew

Why is there a skew? Volatility is not constant. For example, when the stock market falls, index option implied volatility tends to move up, and vice-versa. The reason is as follows: as the stock price decreases, the fear that the price may decrease even further translates into a higher volatility. This is known as the leverage effect as was first noted in Black [1976]. Yet, for option prices to follow the classical form of the Black family of models, volatility must be a constant - the leverage effect shouldn't exist. Thus we have a contradiction, and this explains why there has to be a mechanism for modifying the output of the Black models.

A portfolio hedger will be concerned about how a market fall might happen. Hypothetically, if the fall was guaranteed to be fairly steady, then they can hedge continuously by selling some stock on the way down. In reality, the fall might instead be a sharp jump that cannot be hedged by trading shares. Because of this possibility, they are willing to pay relatively more for those low strike puts than for the high strike ones. The sellers must charge more for the puts because they too will endeavor - and fail under the above scenario - to hedge the puts. On the other hand, there will be relatively less hedge slippage of the high strike options when the market falls. See Figure 8.1.

When the market goes up, it usually does so in an orderly fashion, and hedging is relatively easy (for all options). This is why options struck above the money might be relatively cheap, because there is not much demand for them. Thus the market expects negative skewness of the distribution of returns.

Furthermore, there is the so called 'fat tail' effect - the market expects the tail events to occur with a far greater probability than is implied by normally distributed returns: the distribution of returns has excess kurtosis. Thus options which pay off in these circumstances need to be priced higher. The expected smooth hedging if the market goes up seems to be a more significant factor than the fat tail there.

Because a vanilla option value is a monotonic increasing function of the volatility, implied volatility for traded options with different strikes and maturities can always be calculated. In fact, option prices are often quoted by stating this implied volatility σ_{imp}, if there is exact agreement on the correct option pricing formula to use.

Different types of skews are seen, and some typical shapes appear:

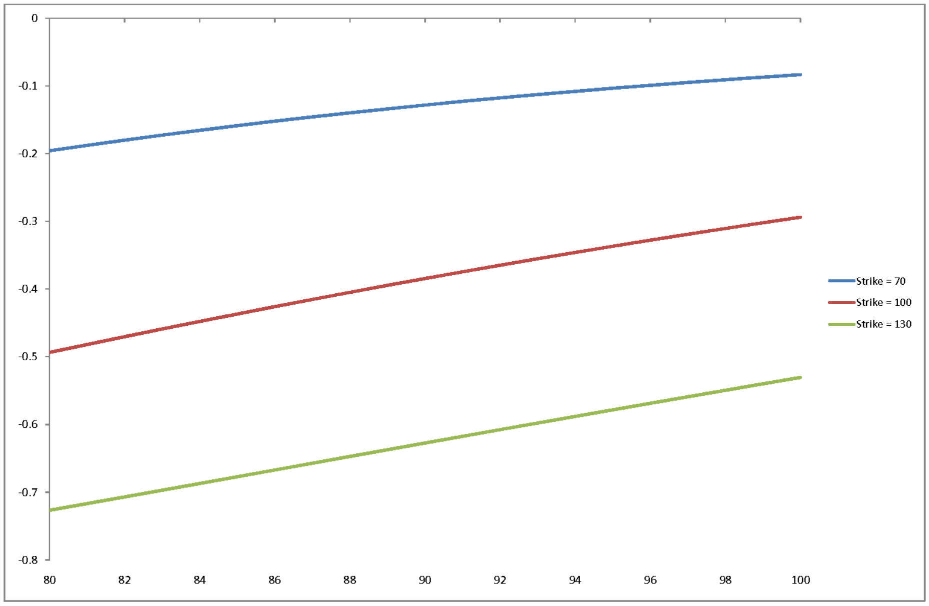


Figure 8.1: The deltas of a low, at-the-money and high strike put option. As the market moves, the delta of the low strike option is impacted most.

- Equities have downward sloping skews: implied volatility decreases as the strike increases.
- Currencies have more symmetrical curves, with implied volatility lowest at-the-money, and higher volatilities in both wings.
- Commodities often higher implied volatility for higher strikes.

Typically, although not always, the word skew is reserved for the slope of the volatility/strike function, and smile for its curvature.

Furthermore, as many options approach expiration, their smiles become very steep. These skews are explained by the fact that such options can only go in the money if the stock price jumps to a new level (rather than diffuse there) and in this case hedging will certainly fail. To compensate for this risk of failure, sellers charge more.

Because there is a volatility surface the Black-Scholes model cannot be an accurate approximation of reality. Nevertheless, the Black-Scholes PDE and Black-Scholes formula are still used extensively in practice. A typical approach is to have the volatility surface as a fact about the market, and use an implied volatility from it in a Black-Scholes valuation model. According to Rebonato, this is using 'the wrong number in the wrong formula to get the right price'. Nevertheless, it is done, because

- volatilities are all in the same dimension, premiums are not. Thus different options can be compared.
- the Greeks can be calculated (and can be made skew aware).
- the skew displays stylised facts of the market and enables the calibration of more sophisticated models.

One way to consider these issues is to consider models with a (negative) correlation between stock prices changes and volatility changes. A negative correlation means that volatility tends to move up when stock price moves down and vice-versa. The models of Hull and White [1987] and Heston [1993] are the original benchmark models in this class. Recently the model in Hagan et al. [2002] has become the standard model for quoting many options, if not for actual trading purposes.

8.2 The SAFEX margining skew

For the TOPI, SAFEX publish a skew every two weeks or so. They publish a distinct skew for each traded expiry. For example, on 23 Feb 2010 the skew in Figure 8.1 for March 2011 was published. Data available at `http://www.jse.co.za/volskewindices`.

EXPIRY	17-Mar-2011		
LOWEST STRIKE	18000	VOL	33.93
STRIKE	20550	VOL	30.46
STRIKE	23100	VOL	27.25
STRIKE	24400	VOL	25.71
STRIKE	25700	VOL	24.25
STRIKE	26950	VOL	22.91
STRIKE	28250	VOL	21.58
STRIKE	30800	VOL	19.17
HIGHEST STRIKE	33400	VOL	16.99
FUTURE PRICE	25700		
BASE VOLATILITY	24.25		
MAX VOLATILITY	65.00		
MIN VOLATILITY	5.00		

Table 8.1: Skew for TOPI; volatility skew changes with effect Tuesday 23 Feb 2010 for settlement on Wednesday, 24 Feb 2010

The MtM futures price was 25,700 and the atm volatility was 24.25% (this appears both as the 'base volatility' and in the table itself). We then calculate two vectors of moneyedness and skew. The i^{th} moneyedness number is $\frac{K_i - F}{F}$, where K_i denotes the strike and F the futures level. This number is rounded to 4 decimal places. Thus, the first moneyedness value is -0.2996 in this example. The i^{th} skew number is $\sigma_i - \sigma_{\mathrm{atm}}$; in this case the first is 9.68% (again, to four decimal places, but this would seem superfluous, given that the original quotes have four decimal places). Thus, we obtain a pair of vectors as in Figure 8.2.

This table is valid until the next publication of the auction skew.

To find the skew volatility that needs to be used for any option is now straightforward. Suppose, some time later (but while this skew is valid) $F = 26,010$ and $\sigma_{\mathrm{atm}} = 25.00\%$. Suppose I want to find the Black volatility for an option with this expiry and with a strike of $K = 25,000$.

First, we calculate the moneyedness of this option. This is $\frac{25{,}000-26{,}010}{26{,}010} = -0.0388312$ (we can be grateful that rounding seems to have been forgotten for the moment). By linear interpolation on the moneyedness and skew vectors we get a skew of 0.0112079. Then, this is added to the ATM volatility to get the skew volatility; in this case 26.12%.

Volatilities are always constrained to be between the min volatility and the max volatility quoted,

Moneyedness	Skew
-0.2996	0.0968
-0.2004	0.0621
-0.1012	0.03
-0.0506	0.0146
0	0
0.0486	-0.0134
0.0992	-0.0267
0.1984	-0.0508
0.2996	-0.0726

Figure 8.2: The results of calibrating the skew

here 5% and 65%.

It follows from this that all we need to do is construct scenarios for the atm volatility, the skew volatility to be used in the Black formula follows mechanistically. It will be found by locating which skew is to be used, and then adding the appropriate number of skew points.

However, it is well know that the SAFEX skew is not market related, as it is determined by an opt-in request for data, rather than by an auction, and could be stale or subject to gaming.

8.3 The relationship between the skew and the risk neutral density of prices

If there was no skew then the market believes that the underlying is lognormally distributed. In reality the market believes the distribution is leptokurtotic (excessively peaked and fat-tailed). This modification is brought in via the skew. There is a precise mathematical relationship between any distribution function and any skew - Breeden and Litzenberger [1978], Li [2000/1].

To determine this relationship, we proceed as follows: for a SAFEX Black contract, the put option price is given by

$$P = \int_0^K (K - F) f(F)\, dF \tag{8.1}$$

where K is the strike and $f(F)$ is the TRUE risk neutral probability density function for the expiry of the option. If $f(F)$ was log-normal, it is a straightforward calculus exercise to derive the Black formula for option pricing. However, f is the actual distribution - with fat tails and so on. Differentiating with respect to K, we get

$$\frac{\partial P}{\partial K} = \int_0^K f(F)\, dF \quad [1] \tag{8.2}$$

and again, we get

$$\frac{\partial^2 P}{\partial K^2} = f(K) \tag{8.3}$$

[1] Here we are using the Leibnitz rule, for differentiation of a definite integral with respect to a parameter [National Institute of Standards and Technology, 2010, §1.5(iv)]:

$$\frac{d}{d\xi}\int_{\psi(\xi)}^{\phi(\xi)} f(\psi,\xi)\, d\psi = f(\phi(\xi),\xi)\frac{d\phi(\xi)}{d\xi} - f(\psi(\xi),\xi)\frac{d\psi(\xi)}{d\xi} + \int_{\psi(\xi)}^{\phi(\xi)} \frac{d}{d\xi} f(\psi,\xi)\, d\psi$$

(You can carefully repeat the whole argument with calls instead of puts and get $\frac{\partial^2 C}{\partial K^2} = f(K)$.) Thus, the risk neutral probability density function as a function of strike is given by $\frac{\partial^2 P}{\partial K^2}$, and the cumulative distribution is given by $\frac{\partial P}{\partial K}$. This of course, is a theoretical result, in that it assumes that a continuum of quoted strikes are available. In reality, only a discrete set of strikes is available. To obtain the continuum, one needs to interpolate. It is known that the most amenable interpolation is natural cubic interpolation on the put prices.

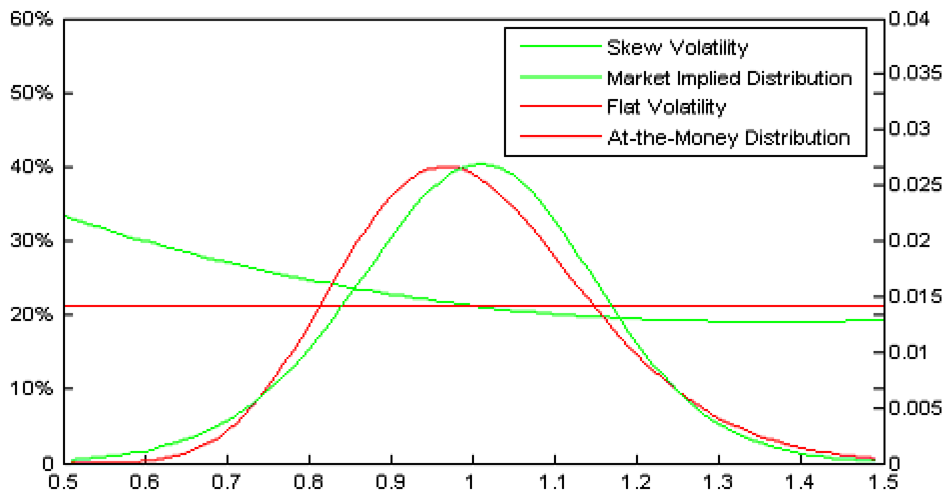


Figure 8.3: The skew and at-the-money volatilities, and the skew and lognormal distributions corresponding to these volatilities. We have a skew, so a thick left tail and a thin right tail.

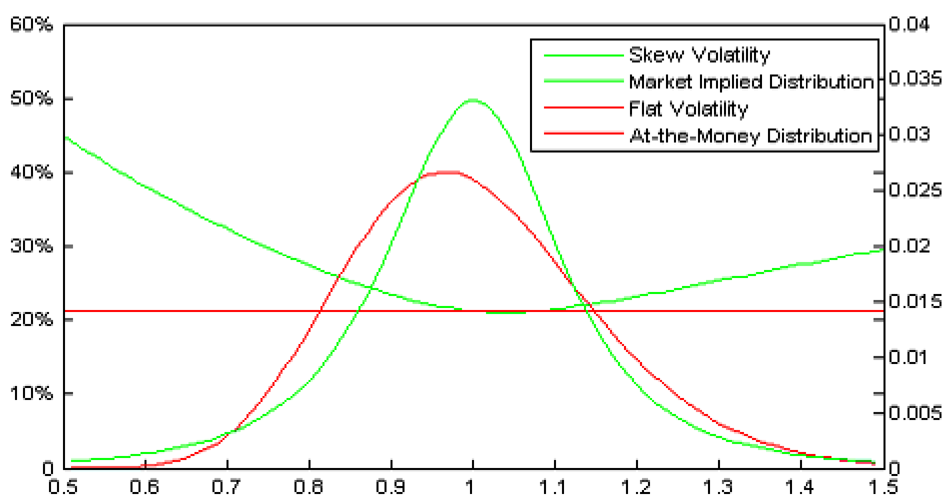


Figure 8.4: Another set of skew and at-the-money volatilities, and the skew and lognormal distributions corresponding to these volatilities. We have a smile, so both tails are thick.

8.4 Real Africa Durolink

They had a margining call of about R200m when the new SAFEX margining system began in April 2001, using a skew for the margining of SAFEX futures options. Until then the SAFEX margining system and the RAD MtM method had been on a flat volatility. In fact, RAD had paid large bonuses to dealers based on this flawed MtM method.

Furthermore, their exposure was huge. "Even for a big bank, RAD's exposure to equity derivatives would have been excessive. For a small bank, it is simply unthinkable. At the very least there should have been limits on the size of the open position the traders could take." (Anonymous source, Business Day, April 12 2001.) I have been told that the executive at RAD instructed that positions be cut, but this was ignored, and the situation persisted.

One of the reasons for the failure of RAD was a lack of skills and understanding at the bank. Probably the only person at the bank who knew of the existence of a skew was the dealer James McIntyre (who had no reason to share his knowledge, because this would expose the flaw in his strategy). RAD failed to replace skilled Risk Managers who had left the bank earlier in the year.

The book was bought by PSG bank. This deal seemed to typify the PSG approach of buying assets cheaply: PSG acted as the undertaker of the South African banking sector.

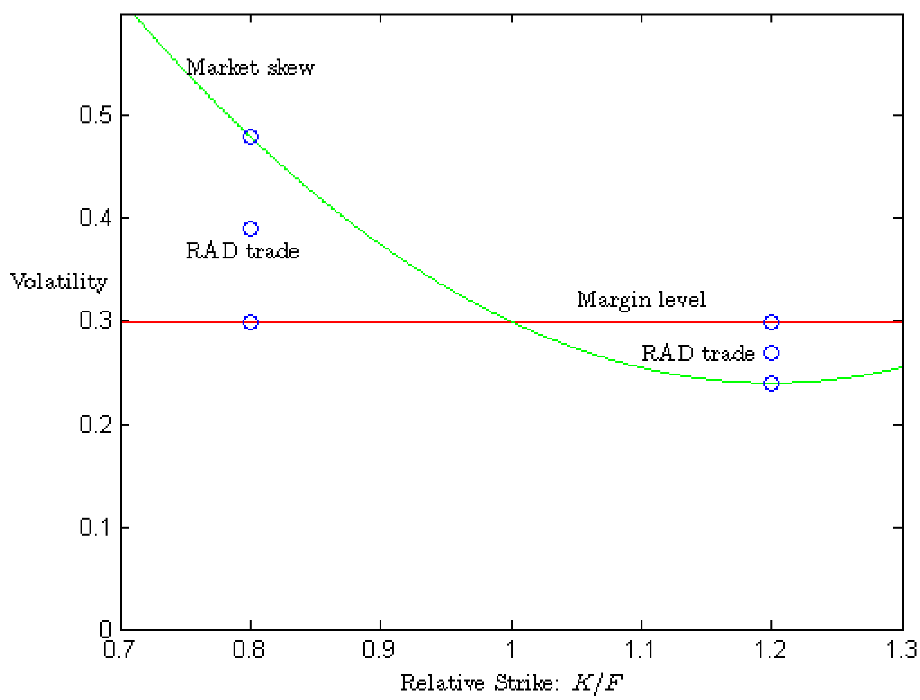


Figure 8.5: The problem with the RAD trading method

8.5 Exercises

1. Find the rand premium for a Dec08 futures put option with the following criteria:

- Valuation Date: 1-Jun-08
- Underlying: abt
- Strike: 100.00
- Volatility: 42.00%
- Futures price: 105.00

2. You are managing a portfolio of futures and futures options. If you are short 255 Mar09 ALSI40 futures call options with a strike of 33000.00 and long 150 Mar09 ALSI40 futures what is the mark-to-market of your portfolio (in Rands) on 30-Jun-08? The futures price is 29000.00 and the volatility is 33.00%.

3. (Wits exam 2005) Write a vba function to price SAFEX futures options. For the cumulative normal function, use the cumnorm6 function provided, calling this function from within vba.

   Suppose I am long 10000 TOP40 futures call options for expiry March 2006 on 26 and 28 April 05, strike 12050 and short 5000 futures for the same expiry. By using the information on the SAFEX MtM sheets provided, calculate the MtM of my portfolio on 26 and 28 April 05, and hence determine the change in portfolio value between those two days.

   Answer the following questions:

   (a) What is the delta of the call option on the first day?

   (b) Hence, say why the futures position is a delta hedge for the call position.

   (c) Nevertheless, if your working is correct, you will find that the p&l between the two days is nearly R7m. Why has that happened? Is it because the hedge is only approximate, because the gains are over two days rather than one, or what?

4. (Wits exam 2005) Explain heuristically how volatility skews and smiles translate to fat or thin tails of the pdf of prices, when we move from an analysis of implied volatility skew to an analysis of the pdf.

   Breeden and Litzenberger showed how this relationship could be made completely precise, by calculating the exact mathematical relationship between the volatility skew, the SAFEX option prices at those volatilities, and the probability density function for the futures level on the expiry date. Carefully state and prove their main result.

   Suppose I have an opinion on what the expiry date pdf should look like. Explain how this could be translated into a skew for trading, and how I would exploit differences between my skew and the skew trading in the market.

5. (UCT exam 2008) Suppose on 1 Jul 2008 the futures spot for expiry 19 March 2009 is 30190. A derivatives dealer provides a collar to a client (thus, the dealer is long the call and short the put). The dealers MtM details are as follows:

	position1	position2
style	call	put
size	100	-100
strike	34000	28000
skew volatility	25%	29%

(a) Build a vba function that will price a SAFEX futures option. The inputs to the function will be: futures spot, strike, valuation date, expiry date, skew volatility, style (± 1), and an optional required as string. The required could be 'p' (for premium) or 'd' (for delta).

(b) Decide on the appropriate futures hedge for the dealer (so that their aggregate delta will be approximately 0). Of course, you can only trade whole numbers of futures.

(c) The dealer does the futures hedge. What is the MtM of his portfolio? (Check: about -20.6m.)

6. (UCT exam 2009) Suppose I am long 10000 TOP40 futures put options for expiry Dec 2009 on 11 and 12 Aug 09, strike 22000 and long 4000 futures for the same expiry. MtM information is as follows:

Value Date	11-Aug-09	12-Aug-09
Futures Spot	22596.00	22496.00
Option Expiry	17-Dec-09	17-Dec-09
Volatility	25.00%	23.75%

(a) Write a vba function that can price and calculate the delta of SAFEX options.

(b) Calculate the MtM of my portfolio on 11 and 12 Aug 09.

(c) Determine the change in portfolio value between those two days.

(d) What is the delta of the put option on the first day? Hence, say why the futures position is a delta hedge for the put position.

(e) Nevertheless, if your working is correct, you will find that the p&l between the two days is about -R7m. Why has that happened? Has the hedge performed poorly because it is only approximate, or are there other reasons?

7. (Exam 2010)

(a) Write a vba function that can price SAFEX futures options. Inputs to the function should be: future, strike, term, volatility, style (could be a string or an integer - the η value as discussed).

(b) A trader is using a skew for SAFEX futures options of a 2 year tenor, part of which is in Table 8.2.

Calculate and graph the cumulative distribution and the probability distribution implied by this skew. Hence explain why their strategy is flawed i.e. they will be arbitraged by counterparties.

(c) Explicitly write down (and price) at least one arbitrage.

0.7	54.56%
0.71	53.31%
0.72	52.07%
0.73	50.84%
0.74	49.62%
0.75	48.42%
0.76	47.23%
0.77	46.04%
0.78	44.87%
0.79	43.70%
0.8	42.55%
0.81	41.40%
0.82	40.26%
0.83	39.13%
0.84	38.00%
0.85	36.89%
0.86	35.78%
0.87	34.67%
0.88	33.57%
0.89	32.48%
0.9	31.40%
0.91	30.32%
0.92	29.25%
0.93	28.19%
0.94	27.14%
0.95	26.10%
0.96	25.08%
0.97	24.07%
0.98	23.10%
0.99	22.15%
1	21.25%
1.01	20.41%
1.02	19.64%
1.03	18.97%
1.04	18.40%
1.05	17.97%
1.06	17.67%
1.07	17.51%
1.08	17.46%
1.09	17.51%
1.1	17.65%

Table 8.2: A bogus skew.

Chapter 9

Structures

Any piecewise linear payoff can be decomposed into some linear combination of calls and puts, and if there discontinuities simply add asset or nothing and cash or nothing options to the mix. Although there are theorems that deal with this, their notational complexity conceals the fact that the procedure one needs to invoke is fairly routine.

What follows are some named payoffs:

Spreads

A spread (vertical) has options at two strikes at the same expiry date on the same stock. The options are either both calls or both puts, with one long and the other short.

- A bull call spread is long the call at the lower strike and short the call at the higher strike.
- A bear call spread is short the call at the lower strike and long the call at the higher strike.
- A bull put spread is long the put at the lower strike and short the put at the higher strike.
- A bear put spread is short the put at the lower strike and long the put at the higher strike.

Collars

Suppose we have a long position in stock. We might want to avoid massive losses in the event that the stock price falls dramatically by buying an out the money put. Rather than paying for the put, we sell an out the money call to the same counterparty. This structure is called a collar. If the net premium is zero, it is alled a zero-cost collar.

Similarly if we have a short position in the stock we might go long an out the money call and short an out the money put.

Collars are also called risk-reversals.

These instruments also enable one to take a bullish or bearish position without trading in the actual underlying. Thus, they allow gearing.

Straddles and strangles

- A straddle is long 1 call and long 1 put at the same strike price and expiration and on the same stock.
- A strangle is long 1 call at a higher strike and long 1 put at a lower strike in the same expiration and on the same stock.

Such long positions makes money if the stock price moves up or down well past the strike prices of the strangle. Long straddles and strangles have limited risk but unlimited profit potential.
Such short positions makes money if the stock price stays at or about the strike(s). Short straddles and strangles have unlimited risk and limited profit potential.
Butterflies and condors

- A butterfly is long a call at strike X_1, short two calls at X_2, and long a call at X_3, with $X_3 - X_2 = X_2 - X_1$.
- A condor (wingspread, flat topped butterfly) has options at four strikes, with the same distance between the each wing strike and the lower or higher of the body strikes. Thus, a call is long a call at strike X_1, short one call at X_2, short a call at X_3, and long a call at X_4, with $X_4 - X_3 = X_2 - X_1$ and $X_3 > X_2$.

The identical structure can be manufactured with puts instead of calls!
Fences Having a long position in a stock, a fence then consists of the following elements:

- long put with a strike near the money;
- short put with a lower strike e.g. at 80% of the money;
- short call with a higher strike e.g. at 120% of the money.

By varying the number of puts and calls one varies the amount of participation in market moves. Then it is called a ratio fence.

9.1 Exercises

1. What is the connection between bull and bear spreads and collars?
2. How would expectations of volatility impact the choice between a straddle and a strangle?

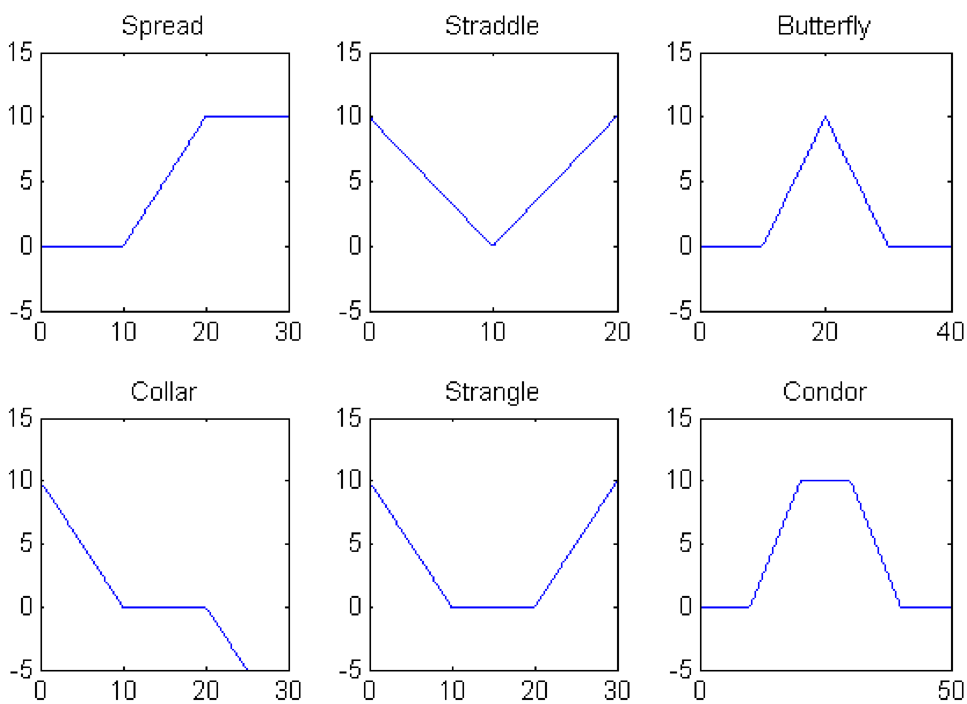


Figure 9.1: The terminal payoff of some of the structures discussed here.

Chapter 10

European option pricing in the presence of dividends

Typically European equity options are priced using the Black-Scholes model Black and Scholes [1973] or that model adjusted for dividends by calculating a continuous dividend yield. This has the effect of spreading the dividend payment throughout the life of the option. This is most attractive where the option is on an index (where the index is paying out several dividends, spread out through the period of optionality).

In the case of only a few dividend payments this approach is not satisfactory. The dividends occur at one or a few discrete times, but we are spreading dividend payment out throughout the life of the option by making this assumption, and this has a material effect on the stochastic process for the stock price.

Another standard approach (for the European case) is to reduce the stock value by the present value of dividends (the escrowed dividend method), or to increase the strike by the future value of dividends. Again this can turn out to be unsatisfactory approaches as they affect the stochastic process on the equity fairly significantly. See Frishling [2002], Bos and Vandermark [2002], Haug et al. [2003]. If the dividend is closer to the start than the end of the option, the spot adjustment method is superior to the strike adjustment method, and conversely.

See 'mispricing options from dividend assumptions.xls'

This comment also applies to the classic binomial tree approach for pricing American options developed in Cox et al. [1979]. Although a binomial tree can be modified to allow for a term structure of interest rates, it can't handle discrete dividends: because having discrete dividends will make the tree no longer recombine, which is computationally a disaster.

It is fairly routine to price options under the assumption that the dividends are a known proportion of the stock price on the dividend payment date: just reduce the stock price as in 4.17. See Björk [1998], for example. However, to use this approach alone is academic fiction: companies and brokers think of, predict, and eventually declare the reasonably short dated dividends in a currency unit. Furthermore, companies are very much loathe to reduce the dividend amount year on year, as a significant proportion of stock holders hold the stock purely for the purpose of receiving annuity revenue from the dividends (for example, retirees, who intend living on the dividends, and leaving the stock to their inheritors) and may transfer their holding to another stock if the dividend was decreased significantly (or even, was not keeping up with inflation). Thus, even if the stock price has decreased somewhat, the company will attempt to maintain dividend levels at more or less the same currency level, at least for a while. Thus, the model that dividends are a known proportion of share price is not practicable.

Again, a simple expression for the price of an option on a stock that pays one known cash dividend D at time t_D is available. If the stock price at t_D is below the dividend amount is D, we assume the company pays the dividend and declares bankruptcy. Then using the law of the Unconscious Statistician, the value of the option is

$$e^{-rt_D}\int_D^{\infty} \mathrm{BS}(S-D, T-t_D) f_{\mathrm{LN}}(S; \ln S(0)+m_- t_D, \sigma^2 t_D)\, dS + \frac{1-\eta}{2} e^{-rT} K N\left(\frac{\ln \frac{D}{S(0)} - m_- t_D}{\sigma\sqrt{t_D}}\right)$$

The second part of the expression represents the certain payoff if the company declares bankruptcy. Of course this expression does not have a closed form expression, but it is amenable to evaluation using a numerical technique. If there is exactly one known cash dividend and the rest are proportional, then it is straightforward to concatenate the above evaluation techniques.

However, typically more than one dividend is known (forecast). Our preferred approach is as follows: use broker/analyst forecasts in the short and medium term, and then forecast percentage dividends in the long term using the model of §4.10. Alternatively, if there are no broker forecasts (for a smaller stock, or simply because we are operating under informational constraints) then all forecasts are percentage dividends based on history.

Thus it would appear that even for European options with a few dividends one should probably prefer to use a finite difference scheme for pricing. This finite difference scheme easily accommodates the discrete jumps of dividends, and both the cash and proportion formulation. One can use the finite difference approach for any number of dividends if prepared to input them. Another clever approximate approach will be covered in §10.1. As the number of dividends increases, the benefits of these approaches are outweighed by the superior speed of using a continuous dividend yield in the Black-Scholes formula, and the fact that as this occurs the objections described above are continuously ameliorated.

10.1 Pricing European options by Moment Matching

Let $t = t_0$ with dividends occurring on $t_1, \ \ldots, \ t_n$ and $T = t_{n+1}$. Now if we have a cash dividend D_i on t_i then

$$\begin{aligned} S(t_i) &= S(t_{i-1})\, e^{X_i} - D_i \\ \Rightarrow S(t_i)^2 &= S(t_{i-1})^2 e^{2X_i} - 2D_i S(t_{i-1})\, e^{X_i} + D_i^2 \end{aligned}$$

Here the price at t_i is the cum price, that is, we are notation t_i^- as t_i throughout. Then

$$\begin{aligned} \mathbb{E}[S(t_i)] &= \mathbb{E}[S(t_{i-1})]\,\mathbb{E}\left[e^{X_i}\right] - D_i \\ \mathbb{E}\left[S(t_i)^2\right] &= \mathbb{E}\left[S(t_{i-1})^2\right]\mathbb{E}\left[e^{2X_i}\right] - 2D_i \mathbb{E}[S(t_{i-1})]\,\mathbb{E}\left[e^{X_i}\right] + D_i^2. \end{aligned}$$

Here $X_i = \ln\left(\frac{S(t_i^-)}{S(t_{i-1})}\right)$ and we know its distribution as in (5.5).

Otherwise, if we have a simple dividend yield d_i, then

$$\begin{aligned} S(t_i) &= S(t_{i-1})\, e^{X_i}(1-d_i) \\ \Rightarrow S(t_i)^2 &= S(t_{i-1})^2 e^{2X_i}(1-d_i)^2 \end{aligned}$$

so that

$$\mathbb{E}\left[S\left(t_i\right)\right] = \mathbb{E}\left[S\left(t_{i-1}\right)\right]\mathbb{E}\left[e^{X_i}\right]\left(1-d_i\right)$$
$$\mathbb{E}\left[S\left(t_i\right)^2\right] = \mathbb{E}\left[S\left(t_{i-1}\right)^2\right]\mathbb{E}\left[e^{2X_i}\right]\left(1-d_i\right)^2.$$

Here, in both cases, $e^{X_i} = e^{\left(r-\frac{1}{2}\sigma^2\right)\tau_i+\sigma\sqrt{\tau_i}Z}$ and $\tau_i = t_i - t_{i-1}$ and hence

$$\mathbb{E}\left[e^{X_i}\right] = e^{r\tau_i}$$
$$\mathbb{E}\left[e^{2X_i}\right] = e^{\left(2r+\sigma^2\right)\tau_i} \text{ from (5.10).}$$

We then proceed by induction starting with $\mathbb{E}\left[S\left(t_0\right)\right] = S$ and $\mathbb{E}\left[S\left(t_0\right)^2\right] = S^2$ until we reach $\mathbb{E}\left[S\left(t_{n+1}\right)\right]$ and $\mathbb{E}\left[S\left(t_{n+1}\right)^2\right]$.

Now assume that S is lognormally distributed at time T. Clearly this assumption is not mathematically correct, but is is known that the error is not severe unless the dividends are very large. If $\ln S \sim \phi(\Psi, \Sigma)$, where Ψ and Σ are not known a priori, then from (5.9) we have

$$\mathbb{E}[S] = e^{\Psi+\frac{1}{2}\Sigma} \tag{10.1}$$
$$\mathbb{E}[S^2] = e^{2\Psi+2\Sigma} \tag{10.2}$$

Hence, given $\mathbb{E}[S]$ and $\mathbb{E}[S^2]$, we can easily solve simultaneously for Ψ and Σ. It follows in our application of these facts that

$$\Sigma = \ln\frac{\mathbb{E}_t^{\mathbb{Q}}\left[S^2\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[S\right]^2} \tag{10.3}$$
$$\Psi = \ln\mathbb{E}_t^{\mathbb{Q}}\left[S\right] - \tfrac{1}{2}\Sigma \tag{10.4}$$

Now, $\sqrt{\Sigma}$ is to be thought of as the volatility for the period. In other words, $\ln S \sim \phi(\Psi, \sigma^2(t_n - t))$ where σ is the annualised volatility measure, or $\Sigma = \sigma^2(t_n - t)$. Hence

$$\sigma^2 = \frac{1}{t_n - t}\left[\ln\frac{\mathbb{E}_t^{\mathbb{Q}}\left[S^2\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[S\right]^2}\right] \tag{10.5}$$

where σ denotes the appropriate volatility measure to use in a Black model option valuation. We use Lemma 6.2.1: easier to implement from existing models, one is using Black's model with

- a futures spot of $\mathbb{E}_t^{\mathbb{Q}}\left[S\right]$,
- a strike of K,
- a volatility of σ as in (10.5),
- a risk free rate of r_n,
- a term of $t_n - t$.

10.2 Exercises

1. Use vba to price a European option using the method explained in this chapter. Inputs are the spot, etc. and a flat risk free rate. Input is also a table with dividend LDR dates, sizes, and whether the dividend is cash or simple yield. Looping in the table occurs from effective date to expiry date.

2. What is the EXACT Black-Scholes formula if all the dividends are simple yields? Check that your calculator above agrees with these answers.

Chapter 11

The Pricing of BEE Share Purchase Schemes

A version of this chapter appears as West and West [2009].

11.1 Introduction and history

Black Economic Empowerment (BEE) transactions are very topical, with companies that have to date failed to put in place meaningful BEE schemes under serious pressure from various interested parties. What is involved is the sale of a stake in the business to suitably qualified partners.

The Broad-Based Black Economic Empowerment Act (2003) was introduced in order to assist in ensuring a more equitable distribution of wealth amongst the people of South Africa. This act defines 'black people' and introduces legal mechanisms for their economic upliftment. The core feature of these transactions is that the seller (vendor company) exchanges company value for BEE credentials. The buyer (BEE partner) provides

- the legal requirement
- avenues to new business, in particular statal and parastatal businesses
- promotion of the vendor company, in particular as a BEE compliant company
- assistance for the vendor company in staffing, affirmative action and its social responsibility programmes.

Under most circumstances the designated partner does not have the resources to pay cash for their stake. Thus, structures need to be put in place to facilitate the purchase of the designated stake. In the earliest stages of BEE these transfers were achieved somewhat cynically, by means of fronting. More or less any transfer of equity would be as a gift, in return for the privilege that the vendor would then enjoy of having a black name on letterheads. The next stage is the vendor financing stage. Typically the structure consists of the following scheme: shares are transferred to the BEE partners by the vendor company at or at about market value. To pay for this, typically a small cash payment (usually 1% to 5%) is made, but the great majority of the payment is set up as debt issued by the BEE partners. The vendor company is the party that buys this debt.

During the debt period the transfer of shares is a legal transfer of ownership; in particular the BEE partner has voting rights.

Over time, the debt rolls up with interest, and rolls down with dividend flows that are received from the shares. At termination, if the share value is higher than the outstanding debt, the BEE partners keep their shares and pay off the outstanding debt, or surrender sufficient shares to pay off the debt. If lower, the BEE partners also surrender their shares, but walk away from the debt.

Thus, what the BEE partner owns is a European call option on the shares with the strike being the level of debt. This option is somewhat exotic because the strike of the otherwise vanilla call option is not known in advance. Reading statements made by participants in these schemes can make for mildly amusing reading given this understanding. When the vendor company wishes to trumpet the successful creation of a BEE structure, they announce that such-and-such a percentage of the company is now held by black partners. On the other hand, if they are being pressured about a grant which some parties - such as the financial press, for example - are viewing as too generous, then they will brazenly retort that the partner owns precisely nothing until such time as the debt has been paid down. As we now see, neither statement is correct: the truth lies somewhere in between.

In the next stage, institutional financing comes into play. In these cases, real money is needed to facilitate the transaction - for example, minorities may need to be bought out. In this case equity is purchased using financing from banks and other financial institutions. The financial institution structures their asset into a cascade, with senior, mezzanine and equity components. The senior and/or mezzanine tranches receive a spread above typical interest rates in the market, and will enjoy covenants on the equity. The financial institution might buy the senior tranche in its entirety, and will participate in equity upside.

Currently most common is a mix of the vendor and institutional financing models. We are now starting to see a few straight purchases: BEE companies will have enjoyed income from previous transactions that they can use to enter into new trades. This cuts out the institutions who are acting as quite pricey middle men here. We believe that this type of transaction will become more and more common.

11.2 Stages in the History of BEE Transactions

BEE has moved through the following evolutionary stages:

- names on letterheads, gifts of equity
- fronting.

Currently, BEE is in the stages of

- vendor financing: equity in companies is sold on credit to the BEE partners, to be repaid out of later dividends and/or profits from the performance of the equity
- institutional financing: equity is purchased using financing from banks and other financial institutions who will earn a spread on the financing and also will share in the equity upside through securitisation.

In future there will be straight purchases: BEE companies will have money from previous transactions in order to enter into new trades.

11.3 The BEE Transaction is a Call Option

The vendor company transfers shares to the BEE partner. The BEE partner pays a small deposit, but is unable to pay the full amount. The BEE partner then issues a bond and uses the proceeds to pay the remainder. The vendor company (vendor financing) or a financial institution(s) (institutional financing) or some combination subscribes to this bond.

To fix ideas let us focus for starters on the vendor financing model.

Over time (a period of about 10 years, say) this debt

- rolls up at a fixed rate or is linked to some floating rate such as Prime or JIBAR or the inflation rate
- is paid down by dividends.

The trade sits in an SPV, ring-fenced from any other transactions the BEE company makes. At maturity of the transaction, if the SPV's value of equity holding is

- greater than the debt level, the BEE partner uses the equity to pay down the debt and keep the remainder
- less than the debt level, the BEE partner (technically, the SPV) walks away from the deal.

The only asset of the SPV is these shares, and the only liability this debt. Some vendor companies operate under the erroneous impression that at termination they will have legal recourse to ensure that the BEE party will honour the full debt amount that has accrued. Even in the case where a BEE company owns several assets, one will find that each asset is held by a separate SPV.

Thus, what the BEE partners own (technically, what the SPV own) is a European call option on the shares with the strike being the level of debt. (Sometimes this debt is paid down to zero before the termination date of the deal, in which case there is early termination.) However, as we have already noted, we cannot apply a classic option pricing formula because the strike is not known in advance.

As the vendor company has given the BEE partners the option, it needs to be expensed at inception in accordance with IFRS2 IASCF [2006]. Even in the case where, for example, majority shareholders of the vendor company facilitate the BEE deal on behalf of the company, the company itself will have to recognise the expense so as to be compliant with 'push down' accounting regulations. The expense will be the fair mark-to-market value of the option. The only income they enjoy is the small physical cash payment that was made by the BEE partners. The bond issued by the BEE partners can obviously not be recognised as income.

As we will argue here, to provide appropriate valuation of this option we need to use a Monte Carlo valuation technique. Monte Carlo will pose no implementation problems as the option itself is European.

Roughly speaking, in each Monte Carlo sample, at maturity we test the value of the proportion of the company granted to the BEE partners versus that of the accrued loan; the terminal value of the deal to the BEE partners is the difference if positive. See Figure 11.1 and Figure 11.2.

11.4 Naïve use of Black-Scholes

Many participants in this market price this option using the Black-Scholes approach, with the strike of the option being set as the forward level of the debt. However, as we shall see, this approach is

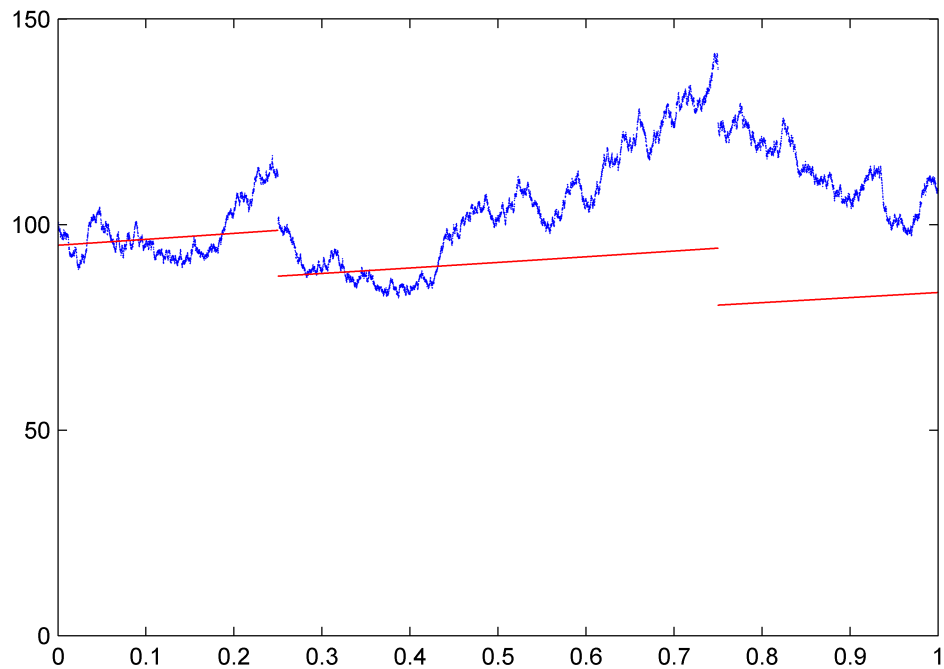


Figure 11.1: A path ending in the money

invalid, due to the path dependency of that level of debt, and can lead to severe mispricing of the option.

When using the Black-Scholes formula for valuation of these options, the strike is calculated as the forward level of the debt. Here the terminal strike price has to be determined "outside the model" (this terminology appears in van der Merwe [2008]) and entered as a fixed input. This treatment ignores the fact that the true final debt amount will depend on the dividends the share pays over the life of the transaction, thus, on the share price evolution: the option is path dependent. The final level of the debt may not be equal to the forward level of the debt.

Suppose as an example a tenor of 10 years, a spot price of 1, the initial debt of 0.95 (thus, the deposit was 5%), a volatility of 30%, a flat risk free rate of 10% (applying both to equity growth and to the debt), and dividends as in Table 11.1.

Term (years)	Amount	Type
0.5	0.02	cash
0.9	0.01	cash
1.5	0.02	cash
1.9, 2.9, ...	1%	yield
2.5, 3.5, ...	2%	yield

Table 11.1: Dividend Schedule

We find the forward level of debt to be 2.08.

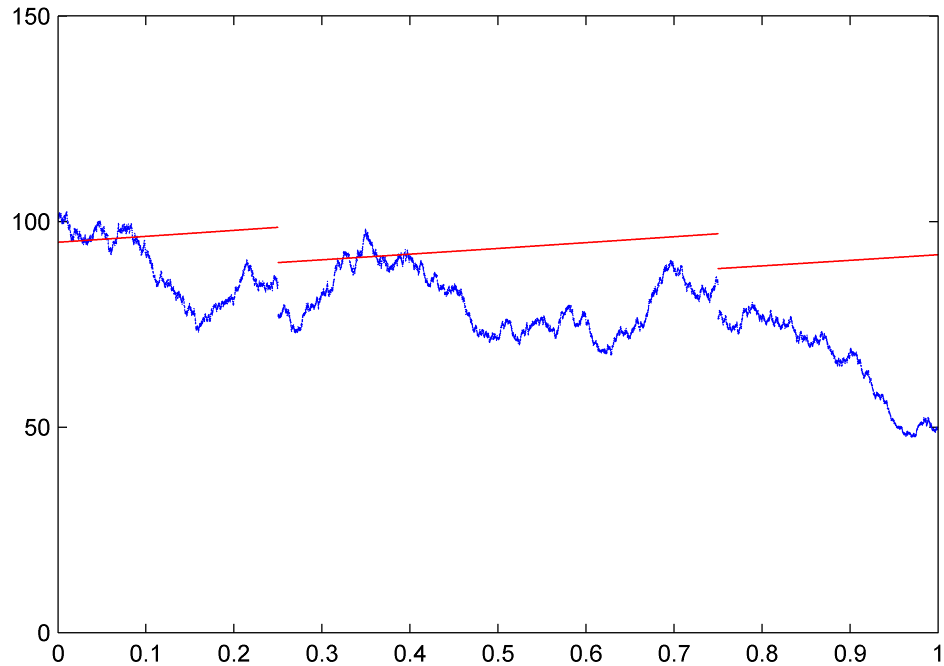


Figure 11.2: A path ending out of the money

The incorrect Black-Scholes approach using the "out of model" strike of 2.08 gives the value of the option of 0.277. With a finite difference approach with the "out of model" strike we get a valuation of 0.266.

The correct approach using Monte Carlo gives a value of 0.331.

A common variation is to declare a large first dividend, to give the debt paydown 'a helping hand'. Of course, this reduces the valuation of the option being granted (which from the vendor point of view will be a good thing). For example, if in this example the first dividend is instead 0.07 cash, with all other dividends unchanged, then the "out of model" strike is 1.97, the Black-Scholes valuation is 0.267, the finite difference valuation is 0.254, and the Monte Carlo valuation is 0.317.

- t = 31-Dec-07
- S = 97.9m
- Dividend every year on 31-Dec starting on 31-Dec-11 of $q = 1\%$ of the stock price at that time.
- "Out of Model" K = 169.4m
- $\sigma = 47.80\%$
- T = 07-Jul-17

Using the incorrect approach, that is, the Black-Scholes formula with "out of model" strike gives a value of 54.6m. The correct approach (using Monte Carlo) gives a value of 57.4m.

We thus have a 5% difference between the two results.

11.5 Valid use of Black-Scholes

Nevertheless, there are two instances where using the Black-Scholes formula is valid.

Suppose all the dividends are known, or can confidently be forecast as cash amounts. (This is quite unrealistic except when a deal nears maturity.) Then the 'out-of-model' calculation of the strike is valid, and so Black-Scholes can be used, or a moment-matching method if necessary.

The only practical occasion in which the Black-Scholes approach is valid is when dividends are paid using the stock, and that stock is paid into the basket of assets, with the debt never being paid down. In this case we use the Black-Scholes with a zero dividend yield.

In preparation for the Nedbank and Mutual and Federal BEE transactions, those companies introduced the option for all shareholders to receive dividends in cash or as stock. (A shareholder's broker will contact them each dividend LDR to determine their election.) However, the BEE partner contractually elected to receive dividends as stock.

The Nedbank deal was restructured in May 2008, as it was felt that too much dilution of the stock was occurring. The modification is that the SPV will receive cash dividends, and will be obligated to buy shares in the open market with that cash, for inclusion in the basket of assets. Thus the valuation approach is unchanged, but the dilution problem is avoided.

11.6 Naïve use of Binomial Trees

Let us now consider pricing such an option in a two step binomial tree. We will explicitly see a numerical example of what can go wrong. Because of the path dependency, the debt tree will not recombine, even if we can ensure the stock price tree recombines.

At time $t = 0$, let $S = 20$, $r = 9\%$, $\sigma = 30\%$, simple dividend after one year of $q = 6\%$ of the stock price at that time, initial debt level $D = 17$, and initial deposit $= 3$.

We consider a two year option and build a two step tree (steps of one year each).

From Cox et al. [1979] we find the up factor $u = 1.350$, the down factor $d = 0.741$, and the risk neutral probability of an up move $\pi = 0.580$.

We calculate the expected dividend

$$q\mathbb{E}\left[S(1)\right] = qSe^{r}.$$

The forward level of debt is calculated as

$$\left(De^{r} - qSe^{r}\right)e^{r} = \left(D - qS\right)e^{2r} = 18.916.$$

However, notice how in Figure 11.3 the debt is path dependent. If the stock goes up and then down, the debt is 18.58; if it goes down and then up, the debt is 19.38.

- We can price incorrectly, using a call strike of 18.916 throughout, and discounting through the tree in the usual way. Then value then is $e^{-2r}[\pi^2 15.34] = 4.313$. All other paths end out of the money.

- We can price incorrectly, using a call strike of 18.916 throughout, and applying the Black-Scholes formula. We first need to find the dividend yield continuous per annum: this is the

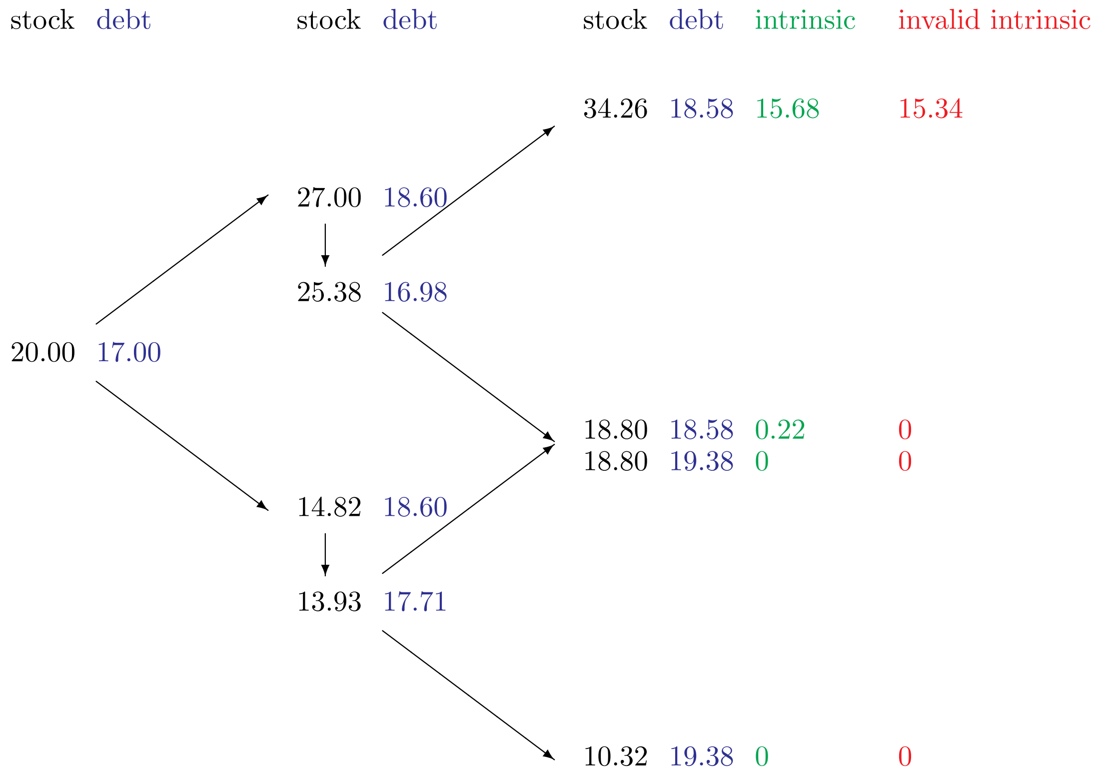


Figure 11.3: The two step binomial tree

value y in $1 - 0.06 = e^{-2y}$, giving $y = 3.094\%$. Then using the Black-Scholes formula we get an option value of 4.643.

Remember the Black-Scholes price is the limit, as we increase the number of time steps, of the price found using a binomial tree.

- Pricing correctly (within this binomial world) we discount along every path one at a time, gives a value of $e^{-2r}[\pi^2 15.68 + \pi(1-\pi)0.22] = 4.452$.

- Pricing correctly using a Monte Carlo approach gives a value of approximately 4.744.

It may be possible to use a many-stepped tree in the spirit of this example with a path dependent extra factor (being the level of debt) as in [Hull, 2005, §24.4]. However, this approach only works while the underlying tree of stock prices is a recombining tree. Usually this doesn't occur: it only occurred even in our simple two step example, because there was a simple dividend yield.

The problem is if we want to model that the stock pays some discrete dividends, and not only percentage dividends. This will almost always be the case, as broker forecasts will be of cash amounts for the next two or three years, say. Furthermore there will often be other complicating features, which make Monte Carlo pricing almost a forced approach.

Our preferred approach is as follows: use broker forecasts in the short and medium term, and then forecast percentage dividends in the long term using the model of §4.10.

11.7 The institutional financing style of BEE transaction

It has become more common for institutions to be involved in the financing of BEE transactions. In this case, the institution buys the bond with real money which is passed to the vendor. In the simplest such cases, the vendor receives full and immediate value for their sale, so they are not required to pass an expense as in §11.3.

Broadly speaking, the institution has bought a bond which will roll up with a contractually specified interest rate, and will roll down with dividends that they receive. At the termination date any residual value of the bond will be extinguished with equity, and the residual equity will accrue to the BEE partner.

Of course, reality will be far more complicated. The equity of the BEE partner is serving as the guarantee for the performance of the bond. Thus the bond will typically have covenants attached to it. If certain asset coverage ratios (ACRs) are not achieved then there might be contractual requirements to pay down the debt by liquidating part of the equity holding. In extreme cases, the deal might be unwound in its entirety. We have seen this occurrence in the market turbulence of early 2008.

Furthermore, it is typical for the bond to be structured as a typical securitised vehicle, with senior and mezzanine debt. In this case the covenants of the senior tranche take precedence over those of the mezzanine tranche. The portion belonging to the BEE partner will be the equity of the vehicle. Typically, the institution will share in this equity as well (the so called 'kicker').

Since the economic downturn vendor financing has become popular again, because in many of the institutional financed cases the covenants have been called which is bad news for both the vendor - they have lost their BEE status, and have to start from scratch to regain it - and the BEE partner.

11.8 Our approach

We have focused on the simplest issues that arise. There are much more complicated BEE transactions; some of the features that arise are

- trickle dividends (dividends that are received and are spent, leaving the system)
- the proportion of the dividends paid as trickle vs. those used to pay down debt is a function of some price or earnings target
- American/Bermudan features
- the need to have models of the forward prime curve or forward inflation curve. The debt rollup is often associated with one of these curves. We then have separate curves for the equity growth and for the debt growth.
- the covenants in the institutional financing case can be quite complicated.
- possibly the need to model the evolution of the asset rather than the equity.

These features can be handled by a careful use of Monte Carlo.

We take the usual approach in valuing equity derivatives: we assume that the various interest rates (risk free curve, prime curve, inflation curve) will evolve according to their forward levels. Only the stock price (and possibly its volatility) will evolve randomly.

We bootstrap our own yield curves using market rates and the methods in Hagan and West [2006], Hagan and West [2008].

For major stocks we use implied volatilities provided by a prominent market participant and dividend forecasts from a major broker. For smaller stocks we use historical volatility estimates and note the dividends that have occurred recently. In either case we then forecast percentage dividends further out in time using the simple 'repeating-yield' model of §4.10.

In these deals there are several important dates namely

- the commencement date, the effective date, the maturity date
- dividend declare, LDR and payment dates
- debt rollup and paydown dates
- trickle dividend dates.

There might be as many as 150 dates here, although 10-40 is most common. As already discussed, because of the path dependency, we are using the Monte Carlo method to price the transaction. Thus we need low discrepancy sequences in high dimensions. Sobol' sequences are most suitable here, see [Glasserman, 2004, §5.2.3].

11.9 Exercises

1. (exam 2009) In the lecture on BEE, it was argued heuristically that a decrease in spot by a certain amount and a simultaneous decrease in strike by the same amount will decrease the value of the call option. This question aims to prove that claim, at least in a simple case.

   (a) First, write down the value V of a European call option under the Black-Scholes model. Use the usual symbols for all the terms.

   (b) Now, write down the value V_α of the call if the spot is changed from S to $S+\alpha$ and the strike is changed from K to $K+\alpha$. (Be careful with the $d_{1,2}$ terms.)

   (c) Now calculate $\frac{\partial V}{\partial \alpha}$. (Once the smoke has cleared, the answer will have only two terms.)

   (d) Show that if $q < r$ then $\frac{\partial V}{\partial \alpha} > 0$.

   (e) Why does this finish the problem?

Chapter 12

Variance swaps

12.1 Contractual details

A long party in a variance swap will receive realised variance, and pay fixed. Realised variance is defined as

$$\Sigma = \frac{d}{N} \sum_{i=1}^{N} \left(\ln \frac{S_i}{S_{i-1}} \right)^2$$

where $S_0,\ S_1,\ \ldots,\ S_N$ are the stock closing prices on contractually specified days $t_0,\ t_1,\ \ldots,\ t_N$, and d is the number of contractually specified trade days in the year (so, 252 or 250 or suchlike).

The definition of log returns might or might not be adjusted for dividends.

Clearly a variance swap is a natural hedge for volatility exposure in an options portfolio.

Very often the payoff to a variance swap will be capped. The default seems to be at a cap level which corresponds to $2.5K$, where K is the strike in volatility terms. As such, these caps are irrelevant (worthless) in the case of index variance swaps. They may have relevance in the pricing of single equity variance swaps.

Note that a position in a capped swap is the same as a position in a swap and a short position in a call.

Thus, we restrict attention to the the case where there is no cap.

Variance swaps were introduced in Neuberger [1994], Dupire [1993]. The payoff of a variance swap is dynamically replicated by trading in futures and a static position in a log contract. This log contract can theoretically be replicated using a continuum of option positions. The only difficulty is that to replicate we require a continuum of options. In reality of course this is impossible. Thus one would choose a super-replication strategy using the options available for trade in the market.

These notes are based on Demeterfi et al. [March 1999] which is the standard reference on this topic.

12.2 Replicating almost arbitrary European payoffs

Let S_* be some fixed point in $(0, \infty)$. Suppose we have some payoff $g(S)$. We assume that g is a twice differentiable function on $(0, \infty)$. Let f be the pdf of the terminal distribution of S, we use the Breeden and Litzenberger [1978] result[1], put-call parity, and integration by parts. Thus

$$
\begin{aligned}
& e^{-r\tau}\mathbb{E}^{\mathbb{Q}}\left[g(S(T))\right] \\
&= e^{-r\tau}\int_0^\infty g(K)f(K)\,dK \\
&= \int_0^{S_*} g(K)p''(K)\,dK + \int_{S_*}^\infty g(K)c''(K)\,dK \\
&= \left[g(K)p'(K)\right]_0^{S_*} - \int_0^{S_*} g'(K)p'(K)\,dK + \left[g(K)c'(K)\right]_{S_*}^\infty - \int_{S_*}^\infty g'(K)c'(K)\,dK \\
&= \left[g(K)p'(K) - g'(K)p(K)\right]_0^{S_*} + \int_0^{S_*} g''(K)p(K)\,dK + \left[g(K)c'(K) - g'(K)c(K)\right]_{S_*}^\infty + \int_{S_*}^\infty g''(K)c(K)\,dK \\
&= \int_0^{S_*} g''(K)p(K)\,dK + \int_{S_*}^\infty g''(K)c(K)\,dK + e^{-r\tau}\left[g(S_*) + g'(S_*)(\mathbb{E}^{\mathbb{Q}}\left[S\right] - S_*)\right] \qquad (12.1)
\end{aligned}
$$

This also shows that we can replicate any sufficiently smooth payoff $g(S)$ by trading

- a continuum of puts with strikes from 0 to S_* with weights $g''(K)$,
- a continuum of calls with strikes from S_* to ∞ with weights $g''(K)$,
- a zero coupon bond with payoff $g(S_*) - g'(S_*)S_*$,
- a stock position of size $e^{-q\tau}g'(S_*)$.

This clearly motivates us to choose S^* to be a point below which we prefer to use puts, and above which we prefer to use calls for replication: in this we prefer to use out-the-money options at all times, because they are more liquid than in-the-money options. So S_* might be the liquid strike on the skew closest to the forward level of S, for example. If S_* was equal to the forward, then the last term in (12.1) disappears, and we do not need to hold a position in stock, just the zero coupon bond with payoff $g(S_*)$.

12.3 The theoretical pricing model

In a diffusion model, the realized variance for a given evolution of the stock price is the integral

$$
\Sigma = \frac{1}{T}\int_0^T \sigma^2(t)\,dt \qquad (12.2)
$$

This is a good approximation to the contractually defined variance above. Not only will we find the risk neutral expectation of Σ, but we will actually replicate this payoff.

The value of the pay fixed leg of the variance swap with volatility strike K is the expected present value of the payoff in the risk-neutral world

$$
V = e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[\Sigma - K^2\right] \qquad (12.3)
$$

Here r is the risk-free rate for expiration T. Also, let q be the dividend yield.

Then

$$
\begin{aligned}
\frac{dS}{S} &= (r-q)dt + \sigma(t)\,dZ \\
d\ln S &= \left(r - q - \tfrac{1}{2}\sigma^2(t)\right)\,dt + \sigma(t)\,dZ
\end{aligned}
$$

[1] In this setting this result states that $\frac{\partial^2 P}{\partial K^2} = e^{-r(T-t)}f(K) = \frac{\partial^2 c}{\partial K^2}$.

so taking differences and rearranging we get

$$\sigma^2(t)\,dt = 2\left(\frac{dS}{S} - d\ln S\right)$$

Writing this in the integral form we have

$$\frac{1}{T}\int_0^T \sigma^2(t)\,dt = \frac{2}{T}\int_0^T \frac{dS}{S} - \frac{2}{T}\ln\frac{S(T)}{S(0)}$$

and this is the position we want to replicate.
Let us first define another similar exotic payoff f, and calculate its derivatives:

$$f(x) := \frac{x - S_*}{S_*} - \ln\frac{x}{S_*} \tag{12.4}$$

$$f'(x) = \frac{1}{S_*} - \frac{1}{x} \tag{12.5}$$

$$f''(x) = \frac{1}{x^2} \tag{12.6}$$

We see immediately from (12.1) that

$$e^{-r\tau}\mathbb{E}_0^{\mathbb{Q}}\left[f(S(T))\right] = \int_0^{S_*} \frac{1}{K^2}p(K)\,dK + \int_{S_*}^{\infty} \frac{1}{K^2}c(K)\,dK \tag{12.7}$$

so the payoff $f(S(T))$ will be delivered by a static position in

- a continuum of puts with strikes from 0 to S_* with weights $\frac{1}{K^2}$,
- a continuum of calls with strikes from S_* to ∞ with weights $\frac{1}{K^2}$,

Thus

$$\begin{aligned}
\tfrac{1}{2}\int_0^T \sigma^2(t)\,dt &= \int_0^T \frac{dS}{S} - \ln\frac{S(T)}{S(0)} \\
&= \int_0^T \frac{dS}{S} + f(S(T)) - \frac{S(T) - S_*}{S_*} - \ln\frac{S_*}{S(0)} \\
&= \int_0^T \frac{dS}{S} + f(S(T)) - \frac{S(T)}{S_*} + \left[1 - \ln\frac{S_*}{S(0)}\right]
\end{aligned}$$

We consider the four positions we need to take in turn:

1. $\int_0^T \frac{dS}{S}$ is dynamically replicated by trading in stock - continuously rebalancing the portfolio to be long $\frac{e^{-r\tau}}{S(t)}$ units of stock. Such a position has a delta of $\frac{e^{-r\tau}}{S(t)}$.
2. We enter into the continuum of puts and continuum of calls. Currently it is a static unhedged position in options. The delta of this position we find by analysing (12.4):

$$\Delta = \frac{e^{-q\tau}}{S_*} - \frac{e^{-r\tau}}{S(t)} \tag{12.8}$$

3. We short $\frac{e^{-q\tau}}{S_*}$ of stock. Such a position has a delta of $-\frac{e^{-q\tau}}{S_*}$.
4. We buy and hold a zero coupon bond with face value $1 - \ln\frac{S_*}{S(0)}$.

This decomposition shows that the total delta of the right hand side is 0, as expected. However, there is a simpler approach to achieving this: we know the aggregate delta of the option position from (12.8). If we then hedge the portfolio of options, it will automatically achieve the total delta target of 0 as required.

In terms of valuation, we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\Sigma\right] &= \frac{2}{T}\mathbb{E}^{\mathbb{Q}}\left[\int_0^T \frac{dS}{S} + f(S(T)) - \frac{S(T)}{S_*} + \left[1 - \ln\frac{S_*}{S(0)}\right]\right] \\
&= 2(r-q) + \frac{2e^{rT}}{T}\left[\int_0^{S_*} \frac{1}{K^2}p(K)\,dK + \int_{S_*}^{\infty} \frac{1}{K^2}c(K)\,dK\right] - \frac{2}{T}\frac{S(0)e^{(r-q)T}}{S_*} + \frac{2}{T}\left[1 - \ln\frac{S_*}{S(0)}\right]
\end{aligned}$$

This argument is not only valid in the Black-Scholes world. It is also valid in the local or stochastic (without jumps) volatility world.

When in §12.5 we approximate the continuum of options with a discrete set of options, we replace the integrals above with a discrete sum. Moreover, the hedge is of that discrete set of options, again for an aggregate delta of 0. Thus, the trader will enter into $\frac{2}{T}$ many of the strip of puts and calls, and will dynamically hedge those options, thus capturing an approximation for the realised variance.

12.4 Implementing the theoretical pricing model

We calculate S_* to be that strike which is the smallest strike larger or equal to the forward level.

We have an array $K_1,\ K_2,\ \ldots,\ K_n$ of strikes less than or equal to S_* and an array $\sigma_1,\ \sigma_2,\ \ldots,\ \sigma_n$ of volatilities. We calculate the put prices $p_1,\ p_2,\ \ldots,\ p_n$ associated with those volatilities.

We perform interpolation on the interval $[K_1, K_n]$, $K_n = S_*$ in the put prices. The interpolation algorithm we use is a natural cubic spline. The method of Stineman [1980] might be preferable, but the implementation requirements would be quite onerous.

Now we need to calculate, for $1 \le i < n$, $\int_{K_i}^{K_{i+1}} \frac{1}{K^2}p(K)\,dK$. Suppose the interpolation formula is given by

$$\begin{aligned}
p(K) &= a_i + b_i(K-K_i) + c_i(K-K_i)^2 + d_i(K-K_i)^3 \\
&= (a_i - b_iK_i + c_iK_i^2 - d_iK_i^3) + (b_i - 2c_iK_i + 3d_iK_i^2)K + (c_i - 3d_iK_i)K^2 + d_iK^3 \\
&:= A_i + B_iK + C_iK^2 + D_iK^3
\end{aligned}$$

on the interval $[K_i, K_{i+1}]$. Then

$$\begin{aligned}
\int_{K_i}^{K_{i+1}} \frac{1}{K^2}p(K)\,dK &= \int_{K_i}^{K_{i+1}} \frac{1}{K^2}(A_i + B_iK + C_iK^2 + D_iK^3)\,dK \\
&= \int_{K_i}^{K_{i+1}} A_iK^{-2} + B_iK^{-1} + C_i + D_iK\,dK \\
&= \left[-\frac{A_i}{K} + B_i\ln(K) + C_iK + \frac{D_i}{2}K^2\right]_{K_i}^{K_{i+1}} \\
&= A_i\left[\frac{1}{K_i} - \frac{1}{K_{i+1}}\right] + B_i\ln\frac{K_{i+1}}{K_i} + C_i(K_{i+1}-K_i) + \frac{D_i}{2}(K_{i+1}^2 - K_i^2)
\end{aligned}$$

The same argument applies in the case of the calls. This time the array starts at S_*.

Thus enables us to calculate the relevant integrals $\int_{K_1}^{S_*}$ and $\int_{S_*}^{K_m}$, where K_m is the largest (call) strike, not the requisite $\int_0^{S_*}$ and $\int_{S_*}^{\infty}$. However, the error is not material as long as K_1 and K_m are reasonably small/large.

12.5 Super-replication on a large corridor

The only difficulty is that to replicate we require a continuum of options. In reality of course this is impossible. Thus we seek a super-replication strategy.
Of course it is the payoff of f that needs to be super-replicated, everything else can be done straightforwardly. On the region $[K_1, K_m]$ we can super-replicate the payoff f with a portfolio of options: exactly those options used to calculate the theoretical variance.

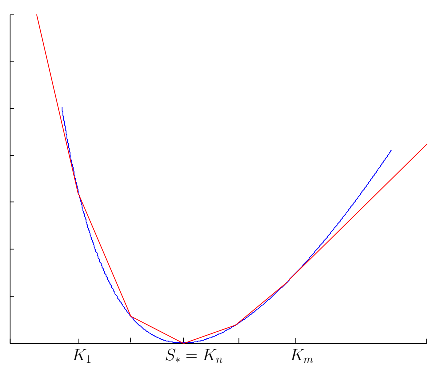


Figure 12.1: The function f and the options that super-replicate on a corridor, and sub-replicate in the wings.

For the region $[K_n, K_m]$ we choose the following calls:

- $\frac{f(K_{n+1})-f(K_n)}{K_{n+1}-K_n}$ many calls struck at K_n, plus
- $\frac{f(K_{j+1})-f(K_j)}{K_{j+1}-K_j} - \frac{f(K_j)-f(K_{j-1})}{K_j-K_{j-1}}$ many calls struck at K_j for $j = n+1, \ \ldots, \ m-1$.

This super-replicates in the region $[K_n, K_m]$ and sub-replicates in the region $[K_m, \infty)$. We choose in addition

- $f'(K_m) - \frac{f(K_m)-f(K_{m-1})}{K_m-K_{m-1}}$ many calls struck at K_m.

which improves the sub-replication in the region $[K_m, \infty)$. Note that super-replication in that region is impossible.
For the region $[K_1, K_n]$ we choose the following puts:

- $\frac{f(K_{n-1})-f(K_n)}{K_n-K_{n-1}}$ many puts struck at K_n, plus
- $\frac{f(K_{j-1})-f(K_j)}{K_j-K_{j-1}} - \frac{f(K_j)-f(K_{j+1})}{K_{j+1}-K_j}$ many puts struck at K_j for $j = n-1, \ \ldots, \ 2$.

This super-replicates in the region $[K_1, K_n]$ and sub-replicates in the region $[0, K_1]$. We choose in addition

- $-f'(K_1) - \frac{f(K_1)-f(K_2)}{K_2-K_1}$ many puts struck at K_1.

which improves the sub-replication in the region $[0, K_1]$. Note that super-replication in that region is impossible.

12.6 Sub-replication

In Davis et al. [2010] for the first time the optimal sub-replication strategy for variance swaps has been demonstrated. There is no clearly graphically obvious best choice of options for sub-replication. However, [Davis et al., 2010, Proposition 3.1] show that the weights of the various options should be chosen as the solution of a dynamic programming problem.

12.7 Off the run valuation

Suppose at some earlier date we dealt a variance swap for expiry date T. Suppose there are h many daily returns that have already occurred and there are f many still to occur. (Thus, the first observation occured at t_{-h}, the first return was $\ln \frac{S_{-h+1}}{S_{-h}}$, as before now is t_0). Let $\lambda = \frac{f}{h+f}$. Then

$$\Sigma = (1-\lambda)\Sigma_h + \lambda\Sigma_f$$

where

$$\Sigma_h = \frac{d}{h}\sum_{i=-h+1}^{0}\left(\ln\frac{S_i}{S_{i-1}}\right)^2 \Sigma_f \qquad = \frac{d}{f}\sum_{i=1}^{f}\left(\ln\frac{S_i}{S_{i-1}}\right)^2$$

Thus the value of the swap is

$$\begin{aligned} V &= e^{-rT}\left[(1-\lambda)\Sigma_h + \lambda\mathbb{E}\left[\Sigma_f\right] - K^2\right] \\ V &= e^{-rT}(1-\lambda)\left[\Sigma_h - K^2\right] + \lambda V_0 \end{aligned} \tag{12.9}$$

where V_0 is the value of the just started variance swap.

What about the case of intra-day valuation? Let's say that it is now during (but not at the close) of the trading day, and the market has moved since last night's close. The historical variance calculation ends at last night's close, so it doesn't know about the move. The unrealised variance is a (forward looking) function of the skew, so this doesn't know either. So - if the move was a big one - the MtM of the (receiver) swap will be too low.

A simple fudge to this problem for intra-day valuations is to set (in an anticipating fashion) today's closing price equal to the current spot price. In this way, jumps in the valuation occur at the opening, not at the close.

12.8 Exercises

1. Using risk-neutral expectations, calculate in the GBM world the value of a derivative which pays $\ln S(T)$.

2. Prove (12.7) and (12.8).

3. What is the price of an off-the-run variance swap i.e. one that has already started? Write it as a function of history and the price of a just starting variance swap.

4. The notional of the variance swap is usually quoted in 'vega notional' $\mathcal{V}_N$, where the vega notional and the cash (variance) notional $\$_N$ are related by $\$_N = \frac{\mathcal{V}_N}{2K}$. Show that with this definition, if the realised volatility is $K+\Delta$, where Δ is small, then the payoff is approximately $\mathcal{V}_N\Delta$.

Chapter 13

Compound options

13.1 Introduction

A compound option is an option in which the underlying is an option. Thus, there is a first exercise date, and if exercised, a vanilla option is born, for a (necessarily subsequent) exercise date.

Compound options have a high delta, so this makes them a popular speculative tool in the retail market. On the other hand, compound options might be used to hedge foreign exchange or index risk for a business event that may not occur.

The standard analysis is based on the work of Geske [1979] who investigated equity options and hypothesised that an equity itself behaved like an option. Geske's model used the assumptions of the Black-Scholes model: constant volatility, constant interest rates, no dividend yield, no transaction costs etc. Logical adjustments to Geske's model allow the three term structures of volatility, interest rates and dividend yields to be reflected in the premium.

13.2 Some facts about bivariate normal distributions

We have that

$$
\begin{aligned}
\frac{\partial}{\partial x} N_2\left(x,\ b,\ \rho\right) &= \frac{\partial}{\partial x} \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{x} \int_{-\infty}^{b} \exp\left[\frac{-(X^2 - 2\rho XY + Y^2)}{2(1-\rho^2)}\right] dY\, dX \\
&= \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{b} \exp\left[\frac{-(x^2 - 2\rho xY + Y^2)}{2(1-\rho^2)}\right] dY \\
&= N'(x) N\left(\frac{b-\rho x}{\sqrt{1-\rho^2}}\right) \qquad (13.1)
\end{aligned}
$$

and hence by the Fundamental Theorem of Calculus

$$\int_{-\infty}^{a} N'(x) N\left(\frac{b-\rho x}{\sqrt{1-\rho^2}}\right) dx = N_2(a, b, \rho)$$

by manipulating with the constants we get

$$\int_{-\infty}^{a} N\left(K + Lx\right) N'(x)\, dx = N_2\left(a, \frac{K}{\sqrt{L^2+1}}, \frac{-L}{\sqrt{L^2+1}}\right) \qquad (13.2)$$

It follows by completing the square from this that

$$\int_{-\infty}^{a} e^{Ax} N\left(K+Lx\right) N'(x)\,dx = e^{\frac{A^2}{2}} N_2\left(a-A,\ \frac{K+AL}{\sqrt{L^2+1}}, \frac{-L}{\sqrt{L^2+1}}\right) \tag{13.3}$$

13.3 The theoretical model

$$\begin{aligned}
T_{1,2} &\sim \text{maturity of compound/underlying option.}\\
t &\sim \text{valuation date of the compound option.}\\
\tau_2 &= T_2 - T_1,\ \text{in years.}\\
\tau_1 &= T_1 - t,\ \text{in years.}\\
\tau &= T_2 - t,\ \text{in years.}\\
K_{1,2} &\sim \text{strike of compound/underlying option.}\\
r &\sim \text{risk free rate}\\
\sigma &\sim \text{volatility}\\
q &\sim \text{expected dividend yield}\\
S_t &\sim \text{spot of underlying stock at time } t.
\end{aligned}$$

The value of $S(T_1)$ that causes the value of the underlying option to equal K_1 at time τ_1 is a key value - it is a boundary value that will determine whether or not the compound option is exercised or allowed to lapse.
Rather we will perform the analysis in terms of returns. Let y be the draw from the normal distribution, so the stock price at time T_1 is $S(y)$ as below. Let us perform the analysis for a call on a call. The other cases are all similar. Let $\mathrm{BS}(y)$ be the value of the underlying option at time T_1 that corresponds to $S(y)$. Thus

$$\begin{aligned}
S(y) &= Se^{m_-\tau_1+\sigma\sqrt{\tau_1}y} &(13.4)\\
\mathrm{BS}(y) &= S(y)e^{-q\tau_2}N(d_+) - K_2e^{-r\tau_2}N(d_-) &(13.5)\\
d_\pm &= \frac{\ln\frac{S(y)}{K_2} + m_\pm\tau_2}{\sigma\sqrt{\tau_2}}\\
&= \frac{\ln\frac{S}{K_2} + m_-\tau_1 + m_\pm\tau_2 + \sigma\sqrt{\tau_1}y}{\sigma\sqrt{\tau_2}}\\
&= \frac{\ln\frac{S}{K_2} + m_-\tau_1 + m_\pm\tau_2}{\sigma\sqrt{\tau_2}} + \sqrt{\frac{\tau_1}{\tau_2}}y &(13.6)
\end{aligned}$$

Let y_* be such that $\mathrm{BS}(y_*) = K_1$. y_* is found using Newton's method, with

$$\begin{aligned}
y_0 &= 0\\
y_{n+1} &= y_n - \frac{\mathrm{BS}(y_n) - K_1}{\frac{\partial}{\partial y}\mathrm{BS}(y)|y_n}
\end{aligned}$$

y^* is that return which is at the margin between exercise and non-exercise of the underlying option on the first expiry date. Then

$$
\begin{aligned}
V &= e^{-r\tau_1}\int_{y=-\infty}^{\infty}[\mathrm{BS}(y)-K_1]^+ N'(y)\,dy \\
&= e^{-r\tau_1}\int_{y=y_*}^{\infty}(\mathrm{BS}(y)-K_1)N'(y)\,dy \\
&= e^{-r\tau_1}\int_{y=y_*}^{\infty}(S(y)e^{-q\tau_2}N(d_+)-K_2e^{-r\tau_2}N(d_-)-K_1)N'(y)\,dy \\
&= e^{-r\tau_1}Se^{m_-\tau_1}e^{-q\tau_2}\int_{y=y_*}^{\infty}e^{\sigma\sqrt{\tau_1}y}N(d_+)N'(y)\,dy \qquad (13.7)\\
&- e^{-r\tau_1}K_2e^{-r\tau_2}\int_{y=y_*}^{\infty}N(d_-)N'(y)\,dy \qquad (13.8)\\
&- e^{-r\tau_1}K_1\int_{y=y_*}^{\infty}N'(y)\,dy \qquad (13.9)
\end{aligned}
$$

(13.7) follows from a careful application of (13.3), and (13.8) follows from a careful application of (13.2). Of course (13.9) is trivial.

Let $\eta=\pm1$ if the underlying option is a call/put and $\zeta=\pm1$ if the compound option is a call/put. Then the value of the compound option is

$$
\begin{aligned}
V &= \eta\zeta Se^{-q\tau}N_2\left(-\eta\zeta(y_*-\sigma\sqrt{\tau_1}),\ \eta d_+,\ \zeta\sqrt{\frac{\tau_1}{\tau}}\right) \\
&\quad -\eta\zeta K_2e^{-r\tau}N_2\left(-\eta\zeta y_*,\ \eta d_-,\ \zeta\sqrt{\frac{\tau_1}{\tau}}\right) \\
&\quad -\zeta K_1e^{-r\tau_1}N(-\eta\zeta y_*) \qquad (13.10)\\
d_\pm &= \frac{\ln\frac{S}{K_2}+m_\pm\tau}{\sigma\sqrt{\tau}} \qquad (13.11)
\end{aligned}
$$

13.4 Exercises

1. Find $\frac{\partial}{\partial y}\mathrm{BS}(y)$.

2. Fill in the missing details in the notes for getting the final pricing formula for a compound call on call option.

3. (exam 2003) Suppose we wish to price a compound call on call, where all of risk free, dividend yield and volatility have a term structure. (There is however no skew structure to volatility.) Let the compound option expire after time τ_1 (in years) with strike X_1 and let the relevant rates be r_1, q_1 and σ_1. Let the vanilla option expire after time τ_2 (in years) with strike X_2 and let the relevant rates be r_2, q_2 and σ_2.

   (a) What are the forward rates for the period from τ_1 to τ_2 in terms of the ordinary rates for τ_1 and τ_2? Denote them henceforth as r_f, q_f and σ_f,

   (b) Now price such an option, following the following hints which come from the notes:

   - Let $S(y)$ be the value of the underlying and $\mathrm{BS}(y)$ be the value of the underlying option at time τ_1 if the draw from the $\phi(0,1)$ distribution has been y. Thus $S(y)=$ .... and $\mathrm{BS}(y)=$ ....

- Let y_* be such that $\mathrm{BS}(y_*) = X_1$. y_* is found using Newton's method, with .... (include the differentiation).
- Then $V = ...$

For the final part, you may use without proof the following two facts

$$\begin{aligned}\int_\alpha^\infty N(K+Lx)\,N'(x)\,dx &= N_2\left(-\alpha,\ \frac{K}{\sqrt{L^2+1}},\ \frac{L}{\sqrt{L^2+1}}\right)\\ \int_\alpha^\infty e^{Ax}N(K+Lx)\,N'(x)\,dx &= e^{\frac{A^2}{2}}N_2\left(-\alpha+A,\ \frac{K+AL}{\sqrt{L^2+1}},\ \frac{L}{\sqrt{L^2+1}}\right)\end{aligned}$$

4. (exam 2008) This question concerns pricing a complex chooser option. Such an option is valued today t_0; at time t_1 the owner has to choose between owning

   - A call with a maturity date t_C and strike K_C;
   - A put with a maturity date t_P and strike K_P.

   Volatility, dividend yield, and the risk free rate are all constant.

   (a) I can write the value of $S(t_1)$ as $S(t_0)e^{\cdots z}$ where z is a random sample from a normal distribution with mean 0 and standard deviation 1. Complete the $\cdots$ here.

   (b) Hence write the value AT TIME t_1 of the call as a function $\mathrm{BS}^C(z)$ and the value of the put as a function $\mathrm{BS}^P(z)$, being careful to distinguish between $d_\pm^C$ for the call and $d_\pm^P$ for the put.

   (c) Explain why there will be a value of z, call it z^*, where we will be indifferent to selecting the put or the call. A diagram might be useful. Say how z^* will be found, although you are not required to give any details.

   (d) Write down the value of the option in terms of integrals.

   (e) How will these integrals be evaluated? What will the final answer look like? (You are NOT being asked to perform the calculation - or even to write down any formulae.)

5. (Exam 2010) This question concerns pricing an extendable call option. Such an option is valued today t_0; at time t_1 the owner has to choose between

   - Exercising the call that has strike K_1;
   - Paying a further premium A and extending the option to a new maturity date t_2 with a new strike K_2;
   - Letting the option expire unexercised.

   Volatility, dividend yield, and the risk free rate are all constant. Assume GBM dynamics.

   (a) Write the value at time t_1 of the extendible call as a maximum of three terms.

   (b) Draw a graph of the value of the (maximum of the) three terms with $S = S(t_1)$ on the horizontal axis. Identify on your graph two critical values, S_1 and S_2, with $S_1 < S_2$, such that the exercise strategy divides into three regions: one strategy where $S < S_1$, one where $S_1 < S < S_2$, and one where $S > S_2$. Explicitly write down what the strategy is.

(c) Say how S_1 and S_2 will be found, giving full details of the scheme. I can write the value of $S(t_1)$ as $S(t_0)e^{\cdots z}$ where z is a random sample from a normal distribution with mean 0 and standard deviation 1. Complete the $\cdots$ here. Associate z_i with S_i for $i = 1,\ 2$. Thus, write down z_i as a function of S_i.

(d) Write down the value of the option in terms of two integrals.

(e) Do some expansion/simplification of these integrals. Continue until the integrand only involves products of exponential functions, normal cdfs and normal pdfs (all expressed as functions of the variable of integration). There will be no credit for the purely mechanical steps beyond this point, so really truly please stop at this point.

Chapter 14

Asian options

(European) Asian or Average Options are options for which the payoff depends on an average of the price of the underlying over some (contractually) specified time interval, or at some discrete times. In reality, only the latter makes sense, and the discrete times could be a quasi-interval in the sense that they are closing prices over a period, such as a month.
There are two fundamental categories of (European) Asian Options. Letting A be a weighted average of historic prices of the underlying:

(a) an average price/rate option has payoff $\max(\eta(A-K), 0)$; K is a contractually specified strike).

(b) an average strike options has payoff $\max(\eta(S(T) - A), 0)$

For standard Asian Options, all the weights in the weighted average are equal. We will only consider this case.
Asian options were first seen in Asian markets as an attempt to prevent option traders from manipulating the price of the underlying security at exercise time. Moreover, there is a far greater degree of certainly (less volatility) to the payoff as the maturity approaches.

14.1 Geometric Average Price Options

The Geometric Average is defined as

$$A = \sqrt[n]{\prod_{i=1}^{n} S_{t_i}}$$

where the observation dates are $t_1,\ t_2,\ \ldots,\ t_n$.
There is also the notion of a continuous average over an interval of observation $[t^*, T]$, in which case the average is given by

$$\exp\left(\frac{1}{T-t^*}\int_{t^*}^{T} \log S_t dt\right)$$

where $[t^*, T]$ is the interval of observation. However, this is just academic nonsense. Such an average does not exist.

Lemma 14.1.1. If U_i, $i = 1,\ 2,\ \ldots,\ q$ is a finite set of independent normal random variables, $U_i \sim \phi(m_i, s_i^2)$, then $\sum_{i=1}^{q} a_i U_i$ is a normal random variable, $\sum_{i=1}^{q} a_i U_i \sim \phi\left(\sum_{i=1}^{q} a_i m_i, \sum_{i=1}^{q} a_i^2 s_i^2\right)$.

Suppose none of the observations have yet been made. Put $S = S_0$, $t = t_0$. By the properties of standard Brownian Motion, $\left\{\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right\}_{i=1,\ 2,\ \ldots,\ n}$ is a set of independent random variables, with

$$\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right) \sim \phi\left(m_-(0;t_{i-1},t_i)(t_i - t_{i-1}), \sigma^2(0;t_{i-1},t_i)(t_i - t_{i-1})\right)$$

where

$$m_-(0;t_{i-1},t_i) = r(0;t_{i-1},t_i) - q(0;t_{i-1},t_i) - \tfrac{1}{2}\sigma^2(0;t_{i-1},t_i)$$

Therefore, using the lemma,

$$\begin{aligned}
\log\left(\frac{A}{S}\right) &= \frac{1}{n}\sum_{i=1}^{n}\ln S_i - \ln S \\
&=^{[1]} \frac{1}{n}\sum_{i=1}^{n}(n-i+1)\ln\frac{S_i}{S_{i-1}} \\
&\sim \phi\left(\frac{1}{n}\sum_{i=1}^{n}(n-i+1)m_-(0;t_{i-1},t_i)(t_i - t_{i-1}), \frac{1}{n^2}\sum_{i=1}^{n}(n-i+1)^2\sigma^2(0;t_{i-1},t_i)(t_i - t_{i-1})\right)
\end{aligned}$$

and so $\log A :\sim \phi\left(\Psi, \Sigma\right)$ where

$$\begin{aligned}
\Psi &= \ln S + \frac{1}{n}\sum_{i=1}^{n}(n-i+1)m_-(0;t_{i-1},t_i)(t_i - t_{i-1}) \\
\Sigma &= \frac{1}{n^2}\sum_{i=1}^{n}(n-i+1)^2\sigma^2(0;t_{i-1},t_i)(t_i - t_{i-1})
\end{aligned}$$

Thus from Lemma 6.2.1

$$V_\eta = e^{-r\tau}\eta\left[e^{\Psi+\frac{1}{2}\Sigma}N(\eta d_+) - KN(\eta d_-)\right] \tag{14.1}$$

$$d_+ = \frac{\Psi + \Sigma - \log K}{\sqrt{\Sigma}} \tag{14.2}$$

$$d_- = \frac{\Psi - \log K}{\sqrt{\Sigma}} \tag{14.3}$$

Suppose some observations at time indices $1,\ 2,\ \ldots,\ p$ have already been made. Let the first rate σ_{p+1} actually be denoted $\sigma(0;t_{p+1},t_p)$ and likewise for the other variables i.e. they have a forward notation even though they are spot variables. Then

$$\begin{aligned}
\log\left(\frac{A}{S}\right) &= \frac{1}{n}\sum_{i=1}^{n}\ln S_i - \ln S \\
&= \frac{n-p}{n}\left[\frac{1}{n-p}\sum_{i=p+1}^{n}\ln S_i - \ln S\right] + \frac{1}{n}\sum_{i=1}^{p}\ln S_i - \frac{p}{n}\ln S
\end{aligned}$$

[1]This is the crucial observation, seen as true by a diagram, and proved by induction.

and so $\log A :\sim \phi(\Psi, \Sigma)$ where

$$\Psi = \frac{1}{n}\sum_{i=p+1}^{n}(n-i+1)m_-(0;t_{i-1},t_i)(t_i-t_{i-1}) + \frac{1}{n}\sum_{i=1}^{p}\ln S_i + \frac{n-p}{n}\ln S$$
$$\Sigma = \frac{1}{n^2}\sum_{i=p+1}^{n}(n-i+1)^2\sigma^2(0;t_{i-1},t_i)(t_i-t_{i-1})$$

and we finish as before.

14.2 Geometric Average Strike Options

Standard (European) Geometric Average Strike Options are in principle priced similarly, although it is quite tricky, as we have to determine the joint distribution of $S(T)$ and A.

14.3 Arithmetic average Asian formula

The payoff of an average price call is $\max\{A-K, 0\}$ and that of an average price put is $\max\{K-A, 0\}$, where K is the strike price and A is the observed average. A is calculated at a predetermined discrete set of dates, or daily over a certain interval (which may, practically be seen as equivalent to continuous averaging). Of course, the notion of continuous averaging is purely of academic interest.

As there is no closed form pricing formula for such arithmetic averages, a number of approximations have emerged in the literature. The two moment approximation has become the most popular model in use. It makes use of the fact that the distribution under arithmetic averaging is approximately lognormal, and they use the first and second moments of the average to find that lognormal distribution which best matches the arithmetic average in order to price the option.

That the arithmetic average is approximately lognormal can be tested using Monte Carlo simulation.

The average calculated can either be under the assumption of discrete averaging or continuous averaging. It is very important to distinguish between these two cases. Much academic work focuses on the continuous averaging case. Typically 'cheap' software, such as that included in Hull [2005] or Haug [2007], or free add-ins found online, assume that averaging is performed continuously from some specified averaging start date, until expiry date. (In this case there is still no closed form solution, but the data manipulation requirements in code are more straightforward.) However, such a model is quite far removed from reality, as in reality averaging will be on some discrete basis. Even if averaging is daily, the error in assuming continuous averaging will be quite significant towards the end of the life of the option, where weekend and such effects become significant. More typically the contract will specify averaging at the last day for the last few weeks, or some specified day in the month for the last few months.

An attempt to improve the 2-moment model appears in Turnbull and Wakeman [1991] where the model is extended to 4 moments. This model has some serious flaws (for larger volatilities and terms) and should never be used.

Kemna and Vorst [1990] use the closed form pricing solution to geometric averaging options as a control variate within a Monte Carlo simulation framework. Another approach is that of Curran [1992] which develops an approximation based on a geometric conditioning approach. Levy [1992] proposes another analytical approximation to compete with Turnbull and Wakeman [1991], with the results being almost identical from a numeric point of view. In these models there is the assumption of equally spaced, or continuous, averaging from the time of first observation to the termination date of the option.

Most recently there is the work done by the academic actuaries at KU Leuven; `http://econ.kuleuven.be/insurance/research.htm`. The mathematics of their approach is quite challenging, and the notation is exceptionally challenging. Nevertheless, it appears that this is the optimal closed-form approximation approach. The main references are Dhaene et al. [2002b], Dhaene et al. [2002a] and Vyncke et al. [2004]; other papers by the same authors and collaborators are cited as well.

14.4 Pricing by Moment Matching

We follow almost the same process as in §10.1, so let's first calculate the first two moments of the average. Define the following variables:

$$
\begin{aligned}
t &\sim \text{valuation date,} \\
n &\sim \text{number of asset price observations used in the averaging,} \\
S_i &\sim \text{asset price at time } t_i, \{t_1 \leq t_2 \leq \cdots \leq t_n = T\} \\
f_i &\sim \text{forward price if } t_i > t\text{; or the observed price } S_i \text{ if } t_i \leq t \\
\sigma_i &\sim \text{implied volatility for time } t_i > t;\ 0 \text{ if } t_i \leq t \\
r_i &\sim \text{risk-free rate of interest for time } t_i > t;\ 0 \text{ if } t_i \leq t \\
q_i &\sim \text{expected dividend yield for time } t_i > t;\ 0 \text{ if } t_i \leq t
\end{aligned}
$$

Note that

$$
\begin{aligned}
A &= \frac{1}{n}\sum_{i=1}^{n} S_i && (14.4) \\
A^2 &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} S_i S_j && (14.5)
\end{aligned}
$$

and so

$$
\mathbb{E}_t^{\mathbb{Q}}[A] = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{1}{n}\sum_{i=1}^{n} S_i\right] = \frac{1}{n}\sum_{i=1}^{n}\mathbb{E}_t^{\mathbb{Q}}[S_i] = \frac{1}{n}\sum_{i=1}^{n} f_i \qquad (14.6)
$$

$$
\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}\left[A^2\right] &= \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}_t^{\mathbb{Q}}[S_i S_j] \\
&= \frac{1}{n^2}\sum_{i=1}^{n}\mathbb{E}_t^{\mathbb{Q}}\left[S_i^2\right] + \frac{2}{n^2}\sum_{j=1}^{n}\sum_{i=1}^{j-1}\mathbb{E}_t^{\mathbb{Q}}[S_i S_j] \qquad (14.7)
\end{aligned}
$$

Consider $\mathbb{E}_t^{\mathbb{Q}}[S_i S_j]$, by the tower property for expectations, with $i < j$ we have (in this case $t_i < t_j$):

$$\begin{aligned}
\mathbb{E}_t^{\mathbb{Q}}[S_i S_j] &= \mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{E}_{t_i}^{Q}[S_i S_j]\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[S_i\, \mathbb{E}_{t_i}^{\mathbb{Q}}[S_j]\right] \\
&= \mathbb{E}_t^{\mathbb{Q}}\left[S_i^2\, e^{(r(t;t_i,t_j)-q(t;t_i,t_j))(t_j-t_i)}\right] \\
&= S_t e^{2m_-(t_i)(t_i-t)+2\sigma^2(t_i)(t_i-t)+(r(t;t_i,t_j)-q(t;t_i,t_j))(t_j-t_i)} \\
&= f_i f_j\, e^{\sigma_i^2(t_i-t)}
\end{aligned}$$

Hence

$$\mathbb{E}_t^{\mathbb{Q}}\left[A^2\right] \quad = \quad \frac{1}{n^2}\sum_{i=1}^{n} f_i^2\, e^{\sigma_i^2(t_i-t)} + \frac{2}{n^2}\sum_{j=2}^{n} f_j \sum_{i=1}^{j-1} f_i\, e^{\sigma_i^2(t_i-t)} \tag{14.8}$$

Now, as in §10.1, we assume that A is lognormally distributed at time t. To be explicit, we assume $\ln A \sim \phi(\Psi, \Sigma)$, and so

$$\Sigma = \ln \frac{\mathbb{E}_t^{\mathbb{Q}}[A^2]}{\mathbb{E}_t^{\mathbb{Q}}[A]^2} \tag{14.9}$$

$$\Psi = \ln \mathbb{E}_t^{\mathbb{Q}}[A] - \tfrac{1}{2}\Sigma \tag{14.10}$$

and now use Lemma 6.2.1.

Equivalently, and easier to implement from existing models, one is using Black's model with

- a futures spot of $\mathbb{E}_t^{\mathbb{Q}}[A]$,
- a strike of K,
- a volatility of

$$\sigma = \sqrt{\frac{1}{t_n - t}\left[\ln \frac{\mathbb{E}_t^{\mathbb{Q}}[A^2]}{\mathbb{E}_t^{\mathbb{Q}}[A]^2}\right]}$$

- a risk free rate of r_n,
- a term of $t_n - t$.

It seems that it would be exceptionally challenging to work this model to use simple and cash dividends as in §10.1, rather than the dividend yields we have here.

14.5 The solution of the DTEW group at KU Leuven

We consider the comonotonicity solution of Dhaene et al. [2002b], Dhaene et al. [2002a] and Vyncke et al. [2004].

In Dhaene et al. [2002b] the distribution function of a sum of random variables is analysed. The application these authors have in mind is actuarial. Such a sum often appears in an insurance or actuarial context, when for example considering aggregate claims of a portfolio. Sometimes an assumption of mutual independence between the components of the sum is made; this can be very convenient from a computational point of view but can be completely unrealistic. Their goal is to determine approximations for sums of random variables, when the distributions of the individual terms are known but the dependence structure is unknown or is too difficult to calculate with.

They first introduce the concept of convex ordering of random variables: $X \leq_{\text{sl}} Y$ (X precedes Y in the stop-loss order sense) if for all $K \in \mathbb{R}$ we have

$$\mathbb{E}\left[(X-K)^+\right] \leq \mathbb{E}\left[(Y-K)^+\right] \tag{14.11}$$

The idea is that any risk-averse decision maker would prefer to have an obligation to pay X instead of Y, which implies that acting as if the obligations were Y instead of X leads to conservative decisions. Y will be called less favourable (for the payer). The idea is to replace a random payment X by a less favorable random payment Y, for which the distribution function is easier to obtain.

Note that if $X \leq_{\text{sl}} Y$ then $\mathbb{E}[X] \leq \mathbb{E}[Y]$; the tightest approximations will occur in the borderline case where there is equality. This leads to the so-called convex order: $X \leq_{\text{cx}} Y$ (X precedes Y in the convex order sense) if $X \leq_{\text{sl}} Y$ and $\mathbb{E}[X] = \mathbb{E}[Y]$. By a fairly simple argument (not unrelated to put-call parity) we find that a necessary and sufficient characterisation is that

$$\mathbb{E}\left[(\eta(X-K))^+\right] \leq \mathbb{E}\left[(\eta(Y-K))^+\right]$$

for $\eta = \pm 1$. See [Dhaene et al., 2002b, §2].

Suppose we are interested in an average or weighted average $X = \sum_i w_i X_i$. In the geometric Brownian motion setting the vector X_i can typically be described by a vector U_i of draws from a uniform distribution. For example, uniformly distributed numbers are drawn at random, the cumulative normal inverse applied, and then a stock price path is generated. [Dhaene et al., 2002b, Theorem 5] establishes an inequality in the convex order which can exactly be applied in the Asian setting. It shows that the distribution which is obtained if the same uniform draw is made for each of the U_i dominates the distribution X in the convex order. However, this new distribution is one dimensional. In the applications we have in mind, options can be priced using a version of the Jamshidian trick and a Black-Scholes formula. See Dhaene et al. [2002a].

Similarly if we can find a more favourable random variable, for which the distribution can be determined, we will get an idea of the degree of overestimation of the real risk. [Dhaene et al., 2002b, Theorem 10] establishes a more favourable random variable for X. It shows that

$$\sum_i \mathbb{E}\left[w_i X_i | \Lambda\right] \leq_{\text{cx}} \sum_i w_i X_i$$

for any random variable Λ. The trick is to make a choice of Λ which is very much like X. The choice that will be made makes $\sum_i \mathbb{E}\left[w_i X_i | \Lambda\right]$ again one dimensional, and the same Jamshidian trick will apply. See Dhaene et al. [2002a].

In our setting, we have found both a more and a less favourable random variable, and we can calculate the moments of both of these variables and of the original one. Thus, by moment matching the first two moments (the first is matched already, so this reduces to matching the second moment) we find a two-dimensional variable which is intermediate between the more and less favourable variables. This improvement was introduced in Vyncke et al. [2004].

14.6 Pricing Asian Forwards

The payoff of a long position in an Asian forward is $A - K$ where A is the average and K is the strike. The value is

$$\begin{aligned} V &= e^{-r_n(t_n - t)} \mathbb{E}_t^{\mathbb{Q}} [A - K] \\ &= e^{-r_n(t_n - t)} \left[\mathbb{E}_t^{\mathbb{Q}} [A] - K \right] \end{aligned} \tag{14.12}$$

14.7 The case where some observations have already been made

Suppose $t_p \le t < t_{p+1}$, so we have observed asset prices at $\{t_1, t_2, \cdots, t_p\}$. Let $A_h = \frac{1}{p} \sum_{i=1}^{p} S_i$. Let A_f be the (unknown) average of the observations still to be made i.e. A_f applies to $\{t_{p+1}, t_{p+2}, \ldots, t_n\}$. Since $A = \frac{pA_h + (n-p)A_f}{n}$, we have that

$$A - K = \frac{pA_h + (n-p)A_f}{n} - K \tag{14.13}$$

$$= \frac{n-p}{n}(A_f - K^*) \tag{14.14}$$

$$K^* = \frac{n}{n-p} K - \frac{p}{n-p} A_h \tag{14.15}$$

which shows (as in [Hull, 2005, §22.10]) that the option can now be seen as equivalent to $\frac{n-p}{n}$ newly issued Asian options with a strike of K^*. This observation is model independent.

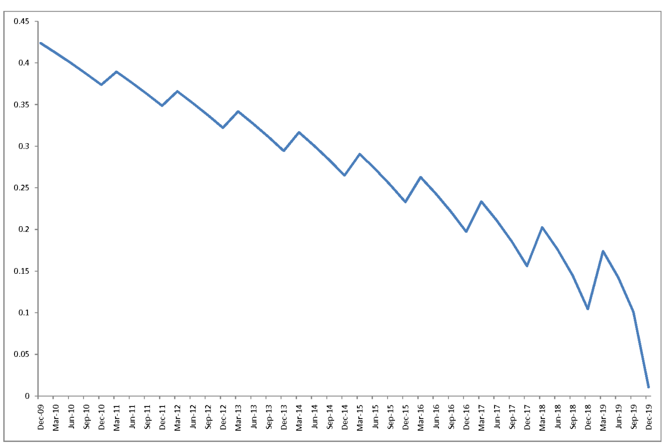


Figure 14.1: The standard deviation of the logarithm of the (unknown) average. The expiry date is 1 Jan 2020, averaging observations occur annually. On the horizontal axis are valuation dates.

If $pA_h \ge nK$ ie. $K^* \le 0$ then the option is guaranteed to be exercised if a call, and is guaranteed to be worthless if a put. The call can be valued as a type of forward, as in §14.6

$$V = e^{-r_n(t_n - t)} \left(\mathbb{E}_t^{\mathbb{Q}} [A] - K \right) \tag{14.16}$$

$$= e^{-r_n(t_n - t)} \frac{n-p}{n} \left(\mathbb{E}_t^{\mathbb{Q}} [A_f] - K^* \right) \tag{14.17}$$

14.8 Exercises

1. (Exam 2008) Write a VBA function to price a just started Asian call option using the 2-moment method of Turnbull and Wakemann. After preliminary calculations the function can call the Standard Black function already built.

   The function should be passed the spot, strike, style etc. and then also a vertical range of observation dates, a vertical range of risk free rates, a vertical range of volatilities, and a vertical range of dividend yields. The number of observations is variable, and is determined by the function as the length of the vertical ranges (which you may assume the user is bright enough to make equal). You may assume the expiry date is the last observation date.

2. Complete the calculation for finding $\mathbb{E}^{\mathbb{Q}}[S_i S_j]$.

3. (Exam 2010) In this problem, we will develop a possible pricing model at initiation date for a product called a Constant Annuity Guarantee. This product is traded by life assurance companies.

   Let the initiation date be t_0. At every date $t_1, t_2, \ldots, t_n$ the client will pay an amount of I (the installment). With this money the life assurance company will buy stock S and include this stock in a portfolio. At termination date, t_n, the stock portfolio is liquidated. The proceeds are paid over to the client. There is a guarantee that the payment that will be made is at least G.

   Assume there are no transaction costs and that dividend income is reinvested in the stock as it occurs (equivalently, there are no dividends). Assume GBM dynamics. There are term structures of risk-free rates and volatility.

   (a) Why is the assumption that dividend income is reinvested in the stock as it occurs equivalent to there being no dividends?

   (b) At time t_i how many stocks are bought with the installment I? Thus, at t_n, what is the value of the portfolio and what is the value of the guarantee?

   (c) Your payoff is a function of the forward relative prices $\frac{S_n}{S_i}$. How is the logarithm of these quantities distributed?

   (d) Your payoff is in fact a type of Asian option. Let A denote the thing that is an average price (the stochastic variable that the option is written on). What is $\mathbb{E}_t^{\mathbb{Q}}[A]$?

   (e) What is $\mathbb{E}_t^{\mathbb{Q}}\left[A^2\right]$? (Hint: rewrite $\frac{S_n}{S_i}\frac{S_n}{S_j}$ as the product of two serial events.)

   (f) What would be an appropriate Black-style closed form (approximate) option value that uses $\mathbb{E}_t^{\mathbb{Q}}[A]$ and $\mathbb{E}_t^{\mathbb{Q}}\left[A^2\right]$?

Chapter 15

Barrier options

A barrier option is an otherwise vanilla call or put option with a strike of K but with an extra parameter B, the barrier: the option only comes into existence (is knocked in) or is terminated (is knocked out) if the spot price crosses the barrier during the life of the option. Because there is a positive probability (in either case) of worthlessness, these options are cheaper than the corresponding vanilla option, and hence possibly more attractive to the speculator.
Even though very much path-dependent, closed form Black-Scholes type formulae for all the possible types of vanilla barrier option were developed in Rubinstein and Reiner [1991] for a stock following geometric Brownian motion and with the barrier continuously monitored. There are eight types: the barrier could be above or below the initial value of S (up or down); the barrier could cause the birth or death of the vanilla option (in or out) and the option could be a call or a put.

15.1 Closed form formulas: continual monitoring of the barrier

We develop the machinery necessary to price vanilla barrier options under the usual Black-Scholes assumptions. The final step - the calculation of the option pricing formula using risk-neutral expectations - is quite fearsome, so we only do it for one case.

The reflection principle

Suppose X is an arithmetic Brownian motion, define the running maximum and minimum by

$$M_t(X) = \max_{s \leq t} X(s) \tag{15.1}$$

$$m_t(X) = \min_{s \leq t} X(s) \tag{15.2}$$

Note that $M_t(X) \geq X(t)$ and $m_t(X) \leq X(t)$.
Suppose we have $k < b$. For every path that ends below k but previously reached b, there is another path that goes above $2b - k$: we simply reflect the path in a mirror at the level b. (Remember, it is arithmetic Brownian motion, and there is no drift.) This is the reflection principle (easy to believe

but hard to prove):

$$\mathbb{P}\left[X(t) < k, M_t(X) > b\right] = \mathbb{P}\left[X(t) > 2b - k\right] = 1 - N\left(\frac{2b-k}{\sqrt{t}}\right) \tag{15.3}$$

An aside: hitting times

Define

$$T_b(X) = \inf_{t \geq 0}\{X(t) = b\}$$

is the hitting or stopping time: the first time that the process X reached the level b. Suppose $b > 0$. If the process never reaches b, then the hitting time is ∞ (no problem here as $\inf \emptyset = \infty$). Note that $M_t(X) > b \Leftrightarrow T_b(X) < t$. We examine the probability of hitting by time t:

$$\begin{aligned}
\mathbb{P}\left[T_b(X) \leq t\right] &= \mathbb{P}\left[T_b(X) \leq t,\ X(t) < b\right] + \mathbb{P}\left[T_b(X) \leq t,\ X(t) > b\right] \\
&= 2\mathbb{P}\left[T_b(X) \leq t,\ X(t) > b\right] \\
&= 2\mathbb{P}\left[X(t) > b\right] \\
&= 2N\left(\frac{-b}{\sqrt{t}}\right).
\end{aligned}$$

This shows that we eventually hit a.s., although (perhaps paradoxically) one can show that the expected hitting time is infinite. By differentiation, the pdf of $T_b(X)$ is given by $p(b,t) = \frac{b}{t^{3/2}} N'\left(\frac{b}{\sqrt{t}}\right)$.

The joint distribution of the Brownian motion and its running maximum

Let the joint distribution of $X(t)$ and M_t be $f(x,m)$. $f(x,m) = 0$ for $x > m$ and $m < 0$. Now

$$\begin{aligned}
\int_M^\infty \int_{-\infty}^X f(x,m)\, dx\, dm &= \mathbb{P}\left[X(t) < X, M_t(X) > M\right] \\
&= \frac{1}{\sqrt{2\pi t}} \int_{2M-X}^\infty e^{-\frac{x^2}{2t}}\, dx
\end{aligned}$$

from (15.3).
First differentiate w.r.t. M

$$-\int_{-\infty}^X f(x,M)\, dx = \frac{-2}{\sqrt{2\pi t}} \exp\left(-\frac{(2M-X)^2}{2t}\right)$$

Now differentiate w.r.t. X:

$$\begin{aligned}
-f(X,M) &= \frac{-2}{\sqrt{2\pi t}} \cdot \exp\left(-\frac{(2M-X)^2}{2t}\right) \cdot \frac{-2(2M-X)}{2t} \cdot -1 \\
f(X,M) &= \frac{2(2M-X)}{\sqrt{2\pi} t^{3/2}} \exp\left(-\frac{(2M-X)^2}{2t}\right) \qquad (15.4)
\end{aligned}$$

for $X < M$ and $M > 0$, and 0 otherwise. This is [Shreve, 2004, Theorem 3.7.3].

The joint distribution where the Brownian motion has drift

Let $X(t)$ be as before, a Brownian motion w.r.t. the measure $\widetilde{\mathbb{P}}$ and let $\widehat{X}(t) = X(t) + \alpha t$, a Brownian motion with drift. We want to know what is the joint density of $\widehat{X}$ and its maximum. Girsanov's theorem tells us that $\widehat{X}$ is driftless under $\widehat{\mathbb{P}}$ where $\frac{\partial \widehat{\mathbb{P}}}{\partial \widetilde{\mathbb{P}}} = \exp\left(-\alpha X(t) - \frac{1}{2}\alpha^2 t\right) = \exp\left(-\alpha \widehat{X}(t) + \frac{1}{2}\alpha^2 t\right)$.

Then

$$\begin{aligned}
\widetilde{\mathbb{P}}\left[\widehat{X}(t) < X, M_t(\widehat{X}) < M\right] &= \widetilde{\mathbb{P}}\left[\mathbb{1}_{\{\widehat{X}(t)<X, M_t(\widehat{X})<M\}}\right] \\
&= \widehat{\mathbb{P}}\left[\frac{\partial\widetilde{\mathbb{P}}}{\partial\widehat{\mathbb{P}}}\mathbb{1}_{\{\widehat{X}(t)<X, M_t(\widehat{X})<M\}}\right] \\
&= \widehat{\mathbb{P}}\left[\exp\left(\alpha\widehat{X}(t) - \tfrac{1}{2}\alpha^2 t\right)\mathbb{1}_{\{\widehat{X}(t)<X, M_t(\widehat{X})<M\}}\right] \\
&= \int_{-\infty}^{X}\int_0^M \exp\left(\alpha x - \tfrac{1}{2}\alpha^2 t\right) f(x,m)dm\,dx
\end{aligned}$$

Thus, the required pdf is

$$g(X,M) = \exp\left(\alpha X - \tfrac{1}{2}\alpha^2 t\right)\frac{2(2M-X)}{\sqrt{2\pi}t^{3/2}}\exp\left(-\frac{(2M-X)^2}{2t}\right)$$

for $X < M$ and $M > 0$, and 0 otherwise. This is [Shreve, 2004, Theorem 7.2.1].

Aside II: hitting times of Brownian motion with drift

Now we have

$$\begin{aligned}
\widetilde{\mathbb{P}}\left[M_t(\widehat{X}) > M\right] &= \widetilde{\mathbb{P}}\left[\mathbb{1}_{\{M_t(\widehat{X})>M\}}\right] \\
&= \widehat{\mathbb{P}}\left[\frac{\partial\widetilde{\mathbb{P}}}{\partial\widehat{\mathbb{P}}}\mathbb{1}_{\{M_t(\widehat{X})>M\}}\right] \\
&= \int_M^{\infty}\int_{-\infty}^{m}\exp\left(\alpha x - \tfrac{1}{2}\alpha^2 t\right) f(x,m)dx\,dm \\
&= 1 - N\left(\frac{M-\alpha t}{\sqrt{t}}\right) + e^{2\alpha M}N\left(\frac{-M-\alpha t}{\sqrt{t}}\right)
\end{aligned}$$

the last line being an exercise strictly for masochists, see [Shreve, 2004, Corollary 7.2.2] for example. As this quantity is equal to $\widetilde{\mathbb{P}}\left[T_M(\widehat{X}) < t\right]$ we have that differentiation w.r.t. t gives us the density for the first hitting time. Thus the pdf of $T_M(\widehat{X})$ is given by

$$\begin{aligned}
&-N'\left(\frac{M-\alpha t}{\sqrt{t}}\right)\left[\frac{-M}{2t^{3/2}} - \frac{\alpha}{2t^{1/2}}\right] + e^{2\alpha M}N'\left(\frac{M+\alpha t}{\sqrt{t}}\right)\left[\frac{M}{2t^{3/2}} - \frac{\alpha}{2t^{1/2}}\right] \\
&= -N'\left(\frac{M-\alpha t}{\sqrt{t}}\right)\left[\frac{-M}{2t^{3/2}} - \frac{\alpha}{2t^{1/2}}\right] + N'\left(\frac{M-\alpha t}{\sqrt{t}}\right)\left[\frac{M}{2t^{3/2}} - \frac{\alpha}{2t^{1/2}}\right] \\
&= \frac{M}{t^{3/2}}N'\left(\frac{M-\alpha t}{\sqrt{t}}\right)
\end{aligned}$$

Aside III: the cumulative distribution of the maximum process

Suppose we consider t to be fixed, and the maximum attained the variable. Then

$$\widetilde{\mathbb{P}}\left[M_t(\widehat{X}) \leq M\right] = N\left(\frac{M-\alpha t}{\sqrt{t}}\right) - e^{2\alpha M}N\left(\frac{-M-\alpha t}{\sqrt{t}}\right) \tag{15.5}$$

so this is the cumulant of the maximum attained. If we were to differentiate this expression, we would get the density function of the maximum attained by $\widehat{X}$ by time t.

Pricing a barrier option

We consider one special case: pricing an up and out call, with barrier B, strike K, with $S < K < B$. We have $\ln \frac{S(t)}{S} = \left(r - \frac{1}{2}\sigma^2\right) t + \sigma X(t)$ where $X(t)$ is driftless Brownian motion. Put $\alpha = \frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right)$ and $\widehat{X}(t) = \alpha t + X(t)$, so $S(t) = Se^{\sigma \widehat{X}(t)}$.
Note that $S(t) = B \Leftrightarrow \widehat{X}(t) = \frac{1}{\sigma}\ln\frac{B}{S} := b$, and $S(t) = K \Leftrightarrow \widehat{X}(t) = \frac{1}{\sigma}\ln\frac{K}{S} := k$. Thus, the value of the option is

$$
\begin{aligned}
V &= e^{-rT}\mathbb{E}\left[(S(T) - K)\mathbb{1}_{\{M_T(\widehat{X})<b,\, k<\widehat{X}(T)\}}\right] \\
&= e^{-rT}\int_k^b \int_y^b (Se^{\sigma y} - K)\exp\left(-\tfrac{1}{2}\alpha^2 T + \alpha y\right) \frac{2(2m-y)}{\sqrt{2\pi}T^{3/2}} \exp\left(-\frac{(2m-y)^2}{2T}\right) dm\, dy \\
&= e^{-rT}\int_k^b (Se^{\sigma y} - K)\exp\left(-\tfrac{1}{2}\alpha^2 T + \alpha y\right) \frac{1}{\sqrt{2\pi}T^{3/2}} \int_y^b 2(2m-y)\exp\left(-\frac{(2m-y)^2}{2T}\right) dm\, dy \\
&= e^{-rT}\int_k^b (Se^{\sigma y} - K)\exp\left(-\tfrac{1}{2}\alpha^2 T + \alpha y\right) \frac{1}{\sqrt{2\pi}T^{3/2}} \left[-T\exp\left(-\frac{(2m-y)^2}{2T}\right)\right]_{m=y}^{m=b} dy \\
&= e^{-rT-\frac{1}{2}\alpha^2 T}\frac{1}{\sqrt{2\pi T}}\int_k^b (Se^{\sigma y} - K)e^{\alpha y}\left[\exp\left(-\frac{y^2}{2T}\right) - \exp\left(-\frac{(2b-y)^2}{2T}\right)\right] dy
\end{aligned}
$$

The rest is straightforward (if irritating). There are 4 definite integrals, each integrating the exponent of a quadratic with leading term $-\frac{1}{2T}$. We complete the square and evaluate each to get

$$
\begin{aligned}
V = S&\left[N\left(\delta_+\left(\frac{S}{K}\right)\right) - N\left(\delta_+\left(\frac{S}{B}\right)\right)\right] \\
&- e^{-rT}K\left[N\left(\delta_-\left(\frac{S}{K}\right)\right) - N\left(\delta_-\left(\frac{S}{B}\right)\right)\right] \\
&- B\left(\frac{S}{B}\right)^{-2r/\sigma^2}\left[N\left(\delta_+\left(\frac{B^2}{KS}\right)\right) - N\left(\delta_+\left(\frac{B}{S}\right)\right)\right] \\
&+ e^{-rT}K\left(\frac{S}{B}\right)^{-2r/\sigma^2+1}\left[N\left(\delta_-\left(\frac{B^2}{KS}\right)\right) - N\left(\delta_-\left(\frac{B}{S}\right)\right)\right]
\end{aligned}
$$

where

$$\delta_\pm(s) = \frac{\ln s + (r \pm \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}$$

15.2 Obtaining the price using a discrete approach

Discrete tree type models for barrier options have been in existence for some time, but have in general been quite slow, because of the need to induct backwards through the tree, much like a tree model for an American option. (Some arguments can be invoked to reduce this problem.) Moreover, the payoff can be arbitrarily exotic.
However, the issue of when and how often we can hit the barrier is relevant. The price of a barrier option found using a lattice is sensitively dependent on the location of the barrier within the lattice. Given a lattice (a value of N), the price of a particular style of barrier option with a particular strike, will be the same for the barrier being anywhere between two layers of nodes in the spatial dimension. Define the *inner barrier* as the layer formed by nodes on the inside of the true barrier

and the *outer barrier* as the layer formed by nodes just outside the true barrier. A binomial tree assumes the outer barrier to be the true barrier because the barrier conditions are first used by these nodes. This causes option specification error where the option is priced at a different barrier to the one specified by the contract.

Thus, approaches have been developed to ensure that the barrier coincides withe the outer barrier (or is just to the right side of it). Boyle and Lau [1994] suggest using the standard binomial model of Cox et al. [1979] but with the number of time steps equal to

$$N(i) = \frac{i^2\sigma^2(T-t)}{\left(\ln \frac{S}{B}\right)^2} \tag{15.6}$$

for some choice of i. As Figure 15.1 shows, if the value of N is not chosen to be one of these $N(i)$, it is better to be under rather than over.

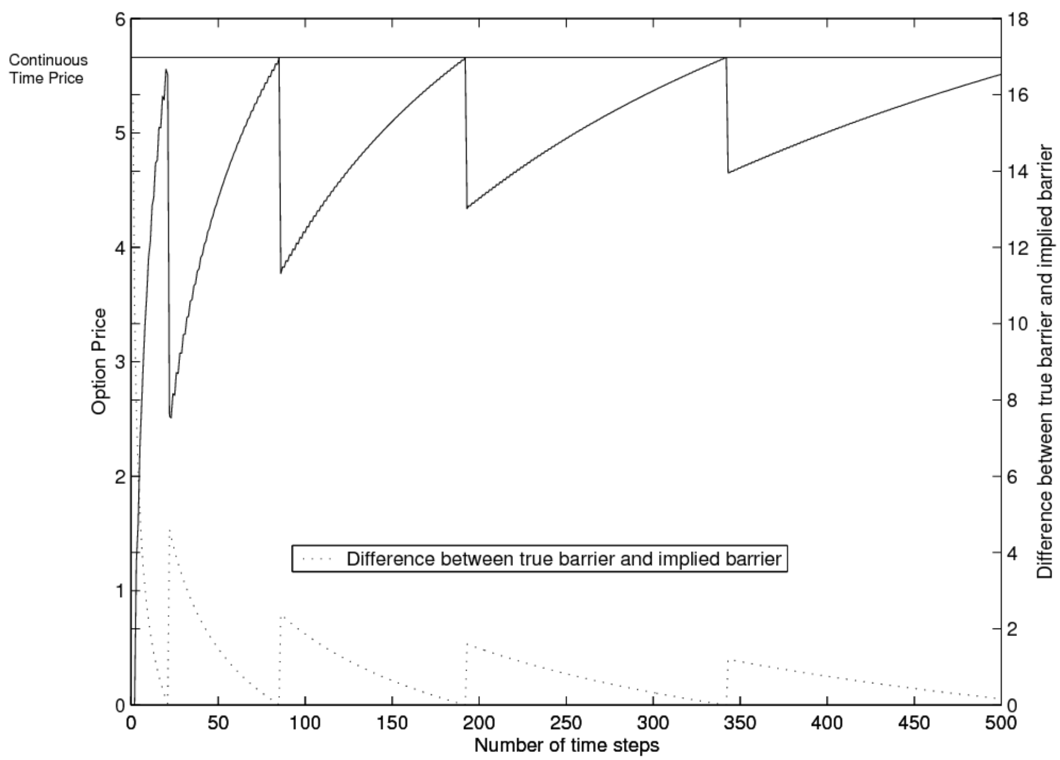


Figure 15.1: Tree model misspecification versus the 'true' price

15.3 The discrete problem

In reality, the maximum or minimum will be calculated based on a discrete set of times, not on continuous monitoring. Thus we will have a sequence of dates $t_1, t_2, \ldots, t_N = T$ and then the maximum or minimum specified in the contracted is that of $S_1, S_2, \ldots, S_N$, where $S_i = S(t_i)$.

There is an obvious reason why continuous-time solutions for discretely monitored path dependent options such as barrier or lookback options may incorrectly value the option. Continuous-time solutions assume that the relevant measurement (such as the barrier being breached in the case of a barrier option, or the maximum or minimum being found in the case of a lookback option) is being made continuously. In the real world, these measurements are typically determined by discrete, daily or monthly, closes. It may be technically difficult to determine if in continuous time a barrier has been breached, and legally there will be a question of whether or not this has actually happened, because of the problem of having to record tick data. Thus in many countries regulators require a discrete monitoring, especially if the end product is in the retail market, for example, exchange traded barrier warrants.

In a geometric Brownian motion world, with continuous monitoring of the stock, the barrier has greater chance of being pierced than in the real world, or the maximum/minimum stock price will be higher/lower, than with discrete observations. For this reason, the analytical formulae will consistently misprice options compared to an appropriate discrete model. The bias is always to one side.

In contrast, continuous formulae for Asian options are not severely away from the discrete formulae as the bias can occur in either direction, and this bias offsets.

15.4 Discrete approaches: discrete monitoring of the barrier

Note that if an option is subject to discrete observations with N observation points before expiry remaining, then in principle the option price can be calculated exactly via N-variate cumulative normals. For $N \leq 3$ fast, double precision algorithms are available West [2005], for $N > 3$ they are not, and any method of evaluating the cumulative probabilities (numerical or quasi-random integration techniques) will be no quicker - in fact, materially slower - than direct Monte Carlo evaluation of the original product.

The two most feasible approaches are the method of 'moving the barrier' as a function of the frequency of observations as in Broadie et al. [1997], and a naïve interpolation approach of Levy and Mantion [1997].

15.4.1 Adjusting continuous time formulae for the frequency of observation

In Broadie et al. [1997] the formulae of Rubinstein and Reiner [1991] are considered and a correction for the discreteness is made. They price these options by applying a continuity correction to the barrier. This correction shifts the barrier away from spot by a multiplicative factor of $\exp(\pm\beta\sigma\sqrt{\tau})$, where τ is the time between monitoring (so typically one day) and

$$\beta = -\frac{\zeta(\frac{1}{2})}{\sqrt{2\pi}} \approx 0.5826$$

is determined from the Riemann zeta function.

More formally, let the option which has N remaining equally distributed observations to be made have (unknown) value V_N. Thus, the valuation date is t, the expiry date is T, and there are observations at t_i for $i = 1, 2, \ldots, N$ where $t = t_0 < t_1 < \cdots < t_N = T$, and $t_i - t_{i-1} = \tau$ is constant. Then

$$V_N(B) = V_\infty(Be^{\pm\beta\sigma\sqrt{\tau}}) + o\left(\sqrt{\tau}\right) \tag{15.7}$$

where $\pm$ corresponds to an up/down option.

This approach is highly attractive, not least because it is of course very fast, however, the solution is not as universally accurate as is often thought: it is quite inaccurate under many combinations of pricing inputs - in particular if we are close to expiry, or if the spot is close to the barrier.
This approach was generalised to other discretely monitored path dependent options in Broadie et al. [1999].

15.4.2 Interpolation approaches

The attractive idea of Levy and Mantion [1997] is to find a price for the N-observations remaining option via interpolation, between the known closed form formula (infinitely many observations remaining), and the computed prices for a low number of observations remaining.
The known closed form price is denoted V_∞. The Levy and Mantion [1997] *ansatz* is

$$V_N = V_\infty + aN^{-1/2} + bN^{-1} \tag{15.8}$$

a type of second-order Taylor series expansion. The requirement in each case then is to determine a and b. We have

$$V_1 = V_\infty + a + b \tag{15.9}$$

$$V_2 = V_\infty + a\sqrt{\tfrac{1}{2}} + b\tfrac{1}{2} \tag{15.10}$$

and so a and b are found using Cramer's rule. V_1 and V_2 are found as functions of univariates and bivariates respectively. Nowadays one would extend the ansatz to use V_3 as well.
As an example, let us take this approach (second order) for the up and out call as before.
For V_1 we have payoff if $S(T) \in (K, B)$. This can be evaluated mechanically, or we note that this value is the value of a call struck at K, less the value of a call struck at B, less a cash-or-nothing call struck at B.
For V_2: now is time t_0, observations occur at t_1 and $t_2 = T$, the expiry. If $S(t_1) < B$ and $S(t_2) < B$ then the payoff is $\max(S(t_2) - K, 0)$, 0 otherwise. Currently, the option is in i.e. $S(t_0) < B$.
Let X_1, X_2 be the random $\phi(0,1)$ normal variables that are the 'stock return draws' for the two periods. In other words,

$$\begin{aligned} m_\pm &= r - q \pm \tfrac{1}{2}\sigma^2 \\ S(t_i) &= S(t_{i-1})\exp(m_-\tau + \sigma\sqrt{\tau}X_i) \quad (i = 1,\ 2) \end{aligned}$$

where $\tau = t_1 - t_0 = t_2 - t_1$, measured in years. For there to be a payoff we require

$$\begin{aligned} X_1 &\in \left(-\infty, \frac{\ln\frac{B}{S(t_0)} - m_-\tau}{\sigma\sqrt{\tau}}\right) \\ X_1 + X_2 &\in \left(\frac{\ln\frac{K}{S(t_0)} - 2m_-\tau}{\sigma\sqrt{\tau}}, \frac{\ln\frac{B}{S(t_0)} - 2m_-\tau}{\sigma\sqrt{\tau}}\right) \end{aligned}$$

The crucial point is that because the returns are serial, they are independent, and so (X_1, X_2) are distributed bivariate normal with zero correlation. Thus, the value is

$$
\begin{aligned}
V_2 &= e^{-2r\,\tau} \int_{-\infty}^{x_1^-} \int_{{}^K x_2^- - X_1}^{{}^B x_2^- - X_1} (S(t_0)\exp(2m_-\tau + \sigma\sqrt{\tau}(X_1+X_2)) - K) N'(X_1) N'(X_2)\, dX_2\, dX_1 \\
x_1^{\pm} &= \frac{\ln \frac{B}{S(t_0)} - m_{\pm}\tau}{\sigma\sqrt{\tau}} \\
{}^Y x_2^{\pm} &= \frac{\ln \frac{Y}{S(t_0)} - 2m_{\pm}\tau}{\sigma\sqrt{\tau}}
\end{aligned}
$$

which can be routinely evaluated explicitly using the bivariate cumulative normal function, to get

$$
\begin{aligned}
V_2 = S(t_0)e^{-2q\tau} &\left[N_2\left(x_1^+, \frac{{}^B x_2^+}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) - N_2\left(x_1^+, \frac{{}^K x_2^+}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right] \\
- e^{-2r\tau} K &\left[N_2\left(x_1^-, \frac{{}^B x_2^-}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) - N_2\left(x_1^-, \frac{{}^K x_2^-}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right]
\end{aligned}
$$

15.5 Exercises

1. $\boxed{\text{Exercises required.}}$

Chapter 16

Forward starting options

A forward starting option is an option purchased today which will start at some date in the future with the strike being a function of some growth factor. This growth factor we denote with α; typically $\alpha \approx 1$ or $\alpha \approx e^{r(0;T_1,T_2)(T_2-T_1)}$ where T_1 is the forward start date, T_2 is the termination date. Here, for the moment, we will assume that there is no skew or forward skew in volatility.

16.1 Constant spot

The simplest variation is the 'constant spot' one: here $V(T_2) = \max\left(\eta\left(\frac{S(T_2)}{S(T_1)} - \alpha\right), 0\right)$. Clearly

$$\begin{aligned} V &= e^{-r_2(T_2-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\max\left(\eta\left(\frac{S(T_2)}{S(T_1)} - \alpha\right), 0\right)\right] \\ &= e^{-r_2(T_2-t)}\,\mathrm{SBf}(S=1, K=\alpha, r(0;T_1,T_2), q(0;T_1,T_2), \sigma(0;T_1,T_2), T_1, T_2, \eta) \\ &= e^{-r_1(T_1-t)}\,\mathrm{BS}(S=1, K=\alpha, r(0;T_1,T_2), q(0;T_1,T_2), \sigma(0;T_1,T_2), T_1, T_2, \eta) \end{aligned}$$

where $\mathrm{SBf}(S, K, r, q, \sigma, t, T, \eta)$ is the SAFEX-Black forward price of an option with spot S, strike K, risk free rate r, dividend yield q, volatility σ, valuation date t, expiry date T, and style η (+1 for a call and -1 for a put).
$\mathrm{BS}(\cdots)$ is the similarly defined Black-Scholes formula. Here we have used the fact that $\mathrm{SBf}(S, K, r, q, \sigma, t, T, \eta) = e^{r(T-t)}\mathrm{BS}(S, K, r, q, \sigma, t, T, \eta)$.
This option is called constant spot because at T_1 the option is on a spot asset worth 1. So we might have a position in $S(0)$-many of these options, for example.

16.2 Standard forward starting options

The next, slightly more complicated version has payoff $V(T_2) = \max(\eta(S(T_2) - \alpha S(T_1)), 0)$. Superficially it is like the previous case, but now having $S(T_1)$ many of the options, but this is not a constant. The value of the option is, from risk neutral valuation, clearly

$$\begin{aligned} V &= e^{-r_2(T_2-t)}\mathbb{E}_t^{\mathbb{Q}}\left[V(T_2)\right] \\ &= e^{-r_2(T_2-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\max(\eta(S(T_2) - \alpha S(T_1)), 0)\right] \end{aligned} \quad (16.1)$$

We make a key observation which allows us to proceed to deal with the 'stochastic number of options'. The returns in the period $[t, T_1]$ and $[T_1, T_2]$ are serial, hence independent. Thus

$$
\begin{aligned}
V &= e^{-r_2(T_2-t)}\mathbb{E}_t^{\mathbb{Q}}\left[S(T_1)\max\left(\eta\left(\frac{S(T_2)}{S(T_1)}-\alpha\right),0\right)\right] \\
&= e^{-r_2(T_2-t)}\mathbb{E}_t^{\mathbb{Q}}\left[S(T_1)\right]\mathbb{E}_t^{\mathbb{Q}}\left[\max\left(\eta\left(\frac{S(T_2)}{S(T_1)}-\alpha\right),0\right)\right] \\
&= e^{-r_2(T_2-t)}e^{(r_1-q_1)(T_1-t)}S(t)\mathbb{E}_t^{\mathbb{Q}}\left[\max\left(\eta\left(\frac{S(T_2)}{S(T_1)}-\alpha\right),0\right)\right] \\
&= e^{-r_2(T_2-t)}e^{(r_1-q_1)(T_1-t)}\,\mathrm{SBf}(S(t), K=\alpha S(t), r(0;T_1,T_2), q(0;T_1,T_2), \sigma(0;T_1,T_2), T_1, T_2, \eta) \\
&= S(t)e^{-q_1(T_1-t)}\,\mathrm{BS}(\mathrm{Spot}=1, K=\alpha, r(0;T_1,T_2), q(0;T_1,T_2), \sigma(0;T_1,T_2), T_1, T_2, \eta)
\end{aligned}
$$

16.3 Additive (ordinary) cliquets

Very often a series of these options are traded; the i^{th} element of the series is active in the interval $[T_{i-1}, T_i]$ where $t = T_0 < T_1 < \cdots < T_n$. Each interval is called a tranche. The entire set of options is called a cliquet[1] Very often a merchant bank will sell the following product to an asset manager: a cliquet of puts struck at $\alpha = 1$ (the floor), and buy from them a cliquet of calls struck at $\omega = 1.20$ (the cap), say. This might be calculated to have net nil premium. Thus, the asset manager has bought a series of forward starting collars. This ensures IN EACH TRANCHE that the asset manager's portfolio is protected against a reduction in the nominal value of their portfolio. In return, they give away the potential performance above a certain level.

The valuation of the cliquet(s) is simply a matter of carefully performing sums with the appropriate α's/ω's, forward rates, $\pm$'s, η's, etc.

16.4 Multiplicative cliquets

Cliquet structures are very common. However, as is usually the case, there are common variations, particularly in South Africa, one of which is clearly superior to the above product: the MULTIPLICATIVE variation, rather than the ADDITIVE variation. The idea is to protect the portfolio against a fall throughout its life, not just at each stage of its life. Suppose for illustration that an unprotected portfolio falls by 10% in each tranche of a five tranche product. A cliquet of protective puts as above would give the option holder a payoff of 10% in the first year, 9% in the second year, 8.1% in the third year, and so on. These cash payoffs are reinvested elsewhere. In the alternative, the payoff is reinvested into the basket, and the protection level is restored for the next tranche. Conversely, if the portfolio outperforms the cap, the asset manager liquidates a portion of the basket down to the cap level and the protection level is continued at the cap level.

If α is the floor level and ω is the cap level then the portfolio is guaranteed to terminate with value in $[\alpha^n, \omega^n]$. Furthermore, the burden of raising or reinvesting cash is removed from the asset manager. Thus, the vanilla product is a cliquet of options against the spot. The above variation is a cliquet of options against the basket, which may be discretely adjusted up or down at each reset date. Thus,

[1] A French word for pawl. Lest that translation does not help, a pawl is (according to the Oxford English dictionary) 'a pivoted bar or lever whose free end engages with the teeth of a cogwheel or ratchet, allowing it to move or turn in one direction only'.

although the spot is subject to geometric Brownian motion, the basket is only subject to GBM in the interval (T_{i-1}, T_i), at each reset, it may jump discontinuously.
We will discuss pricing not only at initiation but during the life of the product.

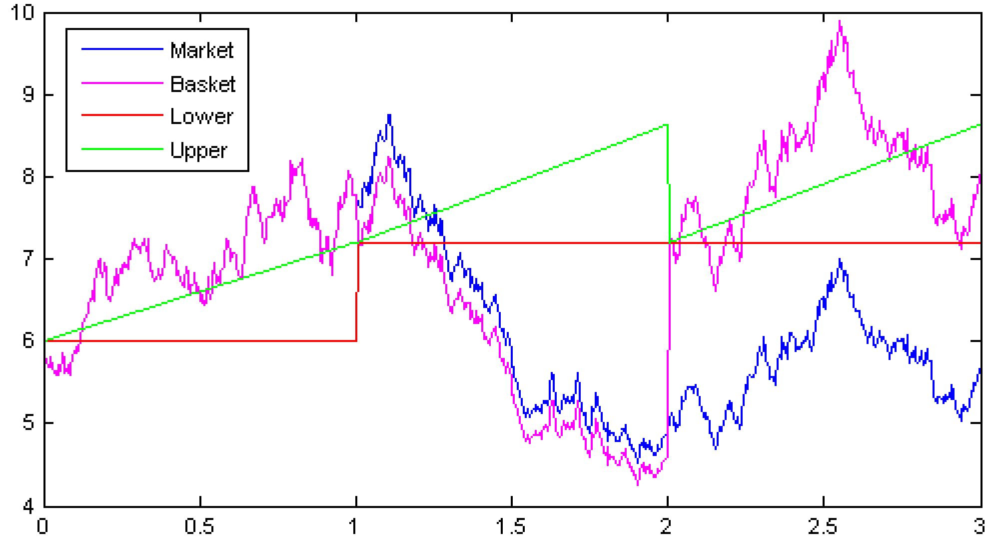


Figure 16.1: The evolution of the market, the basket, and the protection 'triangles'

Define (with $\sim$) and notice (with $=$) the following terms:

$$\begin{array}{rcl}
\alpha,\ \omega & \sim & \text{lower, upper strike} \\
B_0^+ & \sim & 1 \\
B_i^{-,+} & \sim & \text{expected basket value immediately before/after the } i^{th} \text{ resetting} \\
S_0 & \sim & \text{initial spot} \\
S_i & \sim & \text{spot at the termination of the } i^{th} \text{ tranche} \\
S & \sim & \text{current spot} \\
a & \sim & \text{current tranche} \\
S_i/S_{i-1} & = & \text{performance of the stock in the } i^{th} \text{ tranche} \\
P_i & \sim & \max\left(\alpha, \min\left(S_i/S_{i-1}, \omega\right)\right) \\
B_i^- & = & B_{i-1}^+ S_i/S_{i-1} \\
B_i^+ & = & B_{i-1}^+ P_i
\end{array}$$

These quantities are built inductively, either for known past data, or in the sense of expectations. We only need to consider the case of expectations. In this, it is key to note that by the independence of serial market events, the expectation of serial market events is the product of their expectations.

Thus

$$\mathbb{E}_t^{\mathbb{Q}}\left[B_i^-\right] = \mathbb{E}_t^{\mathbb{Q}}\left[B_{i-1}^+\right]\frac{\mathbb{E}_t^{\mathbb{Q}}\left[S_i\right]}{\mathbb{E}_t^{\mathbb{Q}}\left[S_{i-1}\right]} \tag{16.2}$$

$$\mathbb{E}_t^{\mathbb{Q}}\left[B_i^+\right] = \mathbb{E}_t^{\mathbb{Q}}\left[B_{i-1}^+\right]\mathbb{E}_t^{\mathbb{Q}}\left[P_i\right] \tag{16.3}$$

Firstly the expected values of the S_i are determined from the current spot S and risk free rates r_i and dividend yields q_i. Thus, it all boils down to determining $\mathbb{E}_t^{\mathbb{Q}}\left[P_i\right]$. For this,

$$\begin{aligned} P_i &= \max\left(\alpha, \min\left(\frac{S_i}{S_{i-1}}, \omega\right)\right) \\ &= \alpha + \max\left(\frac{S_i}{S_{i-1}} - \alpha, 0\right) - \max\left(\frac{S_i}{S_{i-1}} - \omega, 0\right) \end{aligned} \tag{16.4}$$

Hence

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}\left[P_i\right] &= \alpha + \mathbb{E}_t^{\mathbb{Q}}\left[\max\left(\frac{S_i}{S_{i-1}} - \alpha, 0\right)\right] - \mathbb{E}_t^{\mathbb{Q}}\left[\max\left(\frac{S_i}{S_{i-1}} - \omega, 0\right)\right] \\ &= \alpha + \mathrm{SBf}\left(1, \alpha, r(0;t_{i-1},t_i), q(0;t_{i-1},t_i), \sigma(0;t_{i-1},t_i), t_{i-1}, t_i, 1\right) \\ &- \mathrm{SBf}\left(1, \omega, r(0;t_{i-1},t_i), q(0;t_{i-1},t_i), \sigma(0;t_{i-1},t_i), t_{i-1}, t_i, 1\right) \end{aligned} \tag{16.5}$$

where $\mathrm{SBf}(S, K, r, q, \sigma, t, T, \eta)$ is the SAFEX Black forward option price with spot S, strike K, risk free rate r, dividend yield q, volatility σ, valuation date t, expiry date T, and style η (+1 for a call and -1 for a put).

The above calculation is for a tranche that is truly in the future. For the active tranche, we would have

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}\left[P_a\right] &= \alpha + \mathrm{SBf}\left(\frac{S}{S_{a-1}}, \alpha, r(t_a), q(t_a), \sigma(t_a), t, t_a, 1\right) \\ &- \mathrm{SBf}\left(\frac{S}{S_{a-1}}, \omega, r(t_a), q(t_a), \sigma(t_a), t, t_a, 1\right) \end{aligned} \tag{16.6}$$

To find the price, we require the sum of the present values of the expected payoffs at each tranche expiry. This is given by

$$\begin{aligned} V(t) &= \sum_{i=a}^{n} e^{-r_i(t_i-t)}\mathbb{E}_t^{\mathbb{Q}}\left[B_i^+ - B_i^-\right] \\ &= \sum_{i=a}^{n} e^{-r_i(t_i-t)}\left[\mathbb{E}_t^{\mathbb{Q}}\left[B_i^+\right] - \mathbb{E}_t^{\mathbb{Q}}\left[B_i^-\right]\right] \end{aligned} \tag{16.7}$$

The summation is quite intuitive: at time t_i the basket holder gives away the basket B_i^- and receives the basket B_i^+ in its place.

Another variation is that the 'basket corrections' do not take place at the end of each tranche, rather, a single correction take place at the end of the life of the product (the DEFERRED case, as opposed to the canonical NON-DEFERRED case). In this case

$$\begin{aligned} V(t) &= e^{-r_n(t_n-t)}\mathbb{E}_t^{\mathbb{Q}}\left[B_n^+ - \frac{S(t_n)}{S(t_0)}\right] \\ &= e^{-r_n(t_n-t)}\mathbb{E}_t^{\mathbb{Q}}\left[B_n^+\right] - e^{-q_n(t_n-t)} \end{aligned} \tag{16.8}$$

In these options, the nature of implied volatility is crucial. Even to have working models of the skew itself can be a difficult exercise, see, for example, Hagan et al. [2002]. Not only is there a skew in volatility, but because there are forward starting options, there are forward starting skews. To obtain forward starting skews is non-trivial.

A simple and commonly chosen option is as follows: assume that the forward skew for the period from T_1 to T_2 is equal to the skew for term $T_2 - T_1$ (that is, for time $t + T_2 - T_1$). Variations are possible: for example, in the case of the SABR skew model, it might be assumed that the unobserved parameters of the skew are constant, rather that the actual skew is constant. The forward atm volatility will be used.

In general, it can be shown that it is impossible to find a static hedge of the forward smile. With no way to lock in this forward smile, the product is model-dependent, i.e, two models that fit the smile would not necessarily give the same value for the product. So, it is important to check any given model's forward smile behaviour. See Quessette [2002].

16.5 Exercises

1. (exam 2004) In the course of pricing forward starting cliquet options, we developed pricing models that depended entirely on the risk neutral expected 'collared performance', which in the i^{th} tranche is denoted P_i. Suppose an asset manager is prepared to participate in the performance of their basket of stocks in the following way: suppose $\beta > \alpha > \gamma$ are constants.

   If $x = \frac{S(t_i)}{S(t_{i-1})}$, then the collared performance will be $P_i = \begin{cases} x + \alpha - \gamma & \text{if} \quad x < \gamma \\ \alpha & \text{if} \quad \gamma \leq x < \alpha \\ x & \text{if} \quad \alpha \leq x < \beta \\ \beta & \text{if} \quad \beta \leq x \end{cases}$

   Draw the graph of P_i, and then write P_i as a combination of factors which look like option payoffs. Hence determine $\mathbb{E}_t^{\mathbb{Q}}[P_i]$.

2. (exam 2005) Suppose an employer has a pension fund for its employees. The assets of the pension fund are invested in a diverse basket of assets with a volatility of σ and dividend yield of q. Volatility, dividend yield, and the risk free rate are all constant.

   The basket currently has value L. All future income from the assets will be immediately reinvested in the basket as it is received.

   The pension fund will now be closed, that is, there will be no more members added, and no more member contributions received. The fund will now be allowed to 'die out': assume that withdrawals from the pension fund are only made at the end of each year, and only because of death, and that it is estimated (via actuarial considerations) that at the start of the i^{th} year there will be a proportion of p_i members still in the fund.

   The employer has guaranteed the pension fund returns of α per annum: that is, if the assets do not increase by $\alpha\%$ in any year then the employer will inject cash into the scheme to that level. They have undertaken to do this until the fund closes completely (until the last member dies).

   Find the value of the obligation that the employer has.

3. (exam 2008) This question concerns a variation on the deferred forward starter product we have seen. Suppose now is time t_0, with annual anniversaries $t_1, t_2, \ldots, t_n$. All term structures are flat. For avoidance of doubt, all the time periods $t_i - t_{i-1}$ are of equal length.

   At the end of n years, a derivative will pay $\prod_{i=1}^{n} p_i - \frac{S(t_n)}{S(t_0)}$, where $p_i = \max\left(1, \beta \frac{S(t_i)}{S(t_{i-1})}\right)$. Find the value of β that makes the product have 0 price.

4. (exam 2009) Suppose we have traded an ordinary (non-deferred) multiplicative cliquet. Recall that this means the following (there is no new information here, any variation from the class notes, or any other tricks):

   - The option is on a basket whose initial value is R1. Throughout the basket consists of amounts of a stock whose price at time t is denoted $S(t)$.
   - Two relative strike values α and ω are contracted.
   - Option payoff times $t_1,\ t_2,\ \ldots,\ t_n$ are contracted. The date of initiation of the contract is $t = t_0$, today.
   - We set $B_0^+ = 1$. At time t_i we calculate
     - $P_i = \max\left(\alpha, \min\left(\frac{S_i}{S_{i-1}}, \omega\right)\right)$. Here, $S_i = S(t_i)$.
     - $B_i^- = B_{i-1}^+ \frac{S_i}{S_{i-1}}$
     - $B_i^+ = B_{i-1}^+ P_i$

     and the long party receives a payment of $B_i^+ - B_i^-$.

   In this question we will calculate the delta of the multiplicative cliquet at time t.

   (a) Using risk neutral expectations, write down an expression for the value V to the long party of the product. This expression should take the form $V = \sum_{i=1}^{n} V_i$ where V_i is the value of the payment on date t_i.

   (b) Find a formula for the risk neutral expected value of $B_i^+ - B_i^-$ as a function of the risk neutral expected value of $P_1,\ P_2,\ \ldots,\ P_i$ and the risk neutral expected value of $\frac{S_i}{S_{i-1}}$. (It must be a function of those terms and no other terms.)

   (c) What is $\frac{\partial}{\partial S}\mathbb{E}_t^{\mathbb{Q}}[P_i]$ for $i > 1$ and for $i = 1$? (They are different.)

   (d) What is $\frac{\partial}{\partial S}\mathbb{E}_t^{\mathbb{Q}}\left[\frac{S_i}{S_{i-1}}\right]$ for $i > 1$ and for $i = 1$? (They are different.)

   (e) Hence explain how to find $\frac{\partial V_i}{\partial S}$ and $\frac{\partial V}{\partial S}$, referring to the previous steps. Do the cases $i > 1$ and $i = 1$ separately.

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