FAIR VALUATION OF INSURANCE LIABILITIES

Pierre DEVOLDER
Université Catholique de Louvain
03/ 09/2004
Fair value of insurance liabilities

1. INTRODUCTION TO FAIR VALUE
2. RISK NEUTRAL PRICING AND DEFLATORS
3. EXAMPLES: THE BINOMIAL and THE BLACK/ SCHOLES CASES
4. FAIR VALUE OF LIFE INSURANCE PARTICIPATING CONTRACTS
5. FAIR VALUE OF VARIABLE ANNUITIES
6. CONCLUSION
1. Introduction to FAIR VALUE

✓ International norms IAS / IFRS for all financial institutions in Europe soon (…)

✓ …A lot of discussions linked to accounting principles and artificial volatility

✓ General principle of Fair valuation of elements for assets as well as for liabilities: market values instead of historical values
1. Introduction to FAIR VALUE

- Basic principle: from an historical or statutory accounting point of view to fair value bases

- **Fair value**: price at which an instrument would be traded if a liquid market existed for this instrument

- **ASSETS**: market values

- **LIABILITIES**: ???
  - If no market value: principle of estimation of future cash flows properly discounted and taking into account the different kinds of risk
1. Introduction to FAIR VALUE

- Need to develop good models of valuation especially for actuarial liabilities where there is no market price
- Consistency between modern financial pricing theory and classical actuarial models

- Important example: guarantees in life insurance and pension: one of the most challenging risk nowadays

- Even if for competition reasons methods of pricing could remain very classical, fair valuation will require new insights taking into account modern finance
2. Risk neutral pricing and Deflators

Purpose:

- introduction to the modern financial paradigm of *risk neutral pricing*

- link with *deflator* methodology

- link with *classical actuarial principle* of discounting

- development in a simple discrete market model
2. Risk neutral pricing and Deflators

Paradigm of *risk neutral pricing*:

*Purpose*: to compute the present price of a future stochastic cash flow correlated with financial market in an uncertain environment

What we could expect: *price = discounted expected value of the future cash flow*:

\[
L(0) = \frac{1}{(1 + i)^T} \ E(L(T))
\]

NO!!!
2. Risk neutral pricing and Deflators

You have to change either the probability measure, either the discounting factor

**Change of probability measure**: *risk neutral method*:
You can stay with a classical discounting factor but the expectation of the cash flows must be done using another probability measure than the real one

**Change of discounting factor**: *deflators method*:
You can use the real probability measure but the discounting factor has to be changed and becomes stochastic

Pierre Devolder  09/2004
2. Risk neutral pricing and Deflators

Risk neutral pricing:

\[
L(0) = \frac{1}{(1+r)^T} \mathbb{E}_Q(L(T))
\]

\(Q\) = risk neutral measure

Deflator method:

\[
L(0) = \mathbb{E}(D(T) L(T))
\]

\(D(T)\) = deflator = stochastic discounting
2. Risk neutral pricing and Deflators

**Single period market model**: 
✓ one riskless asset:

\[ S_0(1) = S_0(0)(1 + r) \]  \text{ with } r = \text{riskfree rate}

✓ d risky assets defined on a probability space:

\[ \Omega = \{\omega_1, \omega_2, ..., \omega_N\} \] \text{ with } p_j = P(\{\omega_j\}) \text{ } j = 1, ..., N

\[ S_i(1) = (S_i(1, \omega_1), S_i(1, \omega_2), ..., S_i(1, \omega_N)) \text{ } i = 1, ..., d \]

✓ classical assumption of absence of arbitrage opportunities
2. Risk neutral pricing and Deflators

**STATE PRICE** : Basic property :
if the market is without arbitrage there exists
a random variable \( \psi \) such that for any asset :

\[
S_i(0) = \sum_{j=1}^{N} \Psi_j S_i(1, \omega_j) \quad \text{with} \quad \Psi_j = \Psi(\omega_j) \rangle 0
\]

If the market is complete, the state price is unique.

Consequence : for asset \( i = 0 \) (risk free asset)

\[
\overline{\Psi} = \sum_{j=1}^{N} \Psi_j = \frac{1}{1 + r}
\]
2. Risk neutral pricing and Deflators

RISK NEUTRAL MEASURE

Definition: artificial probability measure given by:

\[ q_j = Q(\omega = \omega_j) = \frac{\Psi_j}{\Psi} = (1 + r)\Psi_j \]

Properties:

1°) \( 0 \leq q_j \leq 1 \) and \( \sum_{j=1}^{N} q_j = 1 \)

2°) for each asset: mean return = risk free rate

\[ S_i(0) = \frac{1}{(1 + r)} \sum_{j=1}^{N} q_j S_i(1, \omega_j) \]
2. Risk neutral pricing and Deflators

✓ **DEFLATOR** : random variable \( D \) defined by:

\[
D(\omega_j) = D_j = \frac{\Psi_j}{p_j}
\]

**Properties**:

i. \( \sum_{j=1}^{N} p_j D_j = E(D) = \frac{1}{1+r} \)

( expected value of the deflator = classical discount )

ii. \( S_i(0) = \sum_{j=1}^{N} p_j D_j S_i(1,\omega_j) \)
2. Risk neutral pricing and Deflators

Generalization: if $X$ is a financial instrument on this market (replicable by the underlying assets) and giving for scenario $j$ a cash flow $X(1,j)$ then the initial value of this instrument can be written as:

i. State price vision: $X(0) = \sum_{j=1}^{N} \Psi_j X(1, j)$

ii. Risk neutral vision: $X(0) = \frac{1}{1+r} \sum_{j=1}^{N} q_j X(1, j) = \frac{1}{1+r} \mathbb{E}_Q(X(1))$

iii. Deflator vision: $X(0) = \sum_{j=1}^{N} p_j D_j X(1, j) = \mathbb{E}(DX(1))$
2. Risk neutral pricing and Deflators

**Multiple periods model**: discrete time model (t=0,1,…, T)

✓ Riskfree asset:

\[ S_0(t) = S_0(0)(1+r)^t \quad \text{with} \quad r = \text{riskfree rate} \]

✓ Risky assets:

\[ S_i(t) = (S_i(t,\omega_1), S_i(t,\omega_2),..., S_i(t,\omega_N)) \quad i = 1,...,d \]

✓ STATE PRICE :

\[ S_i(0) = \sum_{j=1}^{N} \Psi_j(t) S_i(t,\omega_j) \quad \text{with} \quad \Psi_j(t) = \Psi(\omega_j, t) \big\rangle 0 \]
2. Risk neutral pricing and Deflators

✓ **DEFLATOR**: 

\[ D_j(t) = \frac{\Psi_j(t)}{p_j} = \text{discount factor from } t \text{ to } 0 \text{ if scenario } j \]

✓ **Pricing**: if X is a financial replicable instrument on this market generating successive stochastic cash flows:

\[ \{C(t, \omega); t = 1, \ldots, T; \omega \in \Omega\} \]

Then the initial price of X can be written alternatively:

\[ X(0) = \sum_{t=1}^{T} \sum_{j=1}^{N} C(t, \omega_j) \Psi_j(t) \]
2. Risk neutral pricing and Deflators

Or with risk *neutral measure*:

\[
X(0) = \sum_{t=1}^{T} \frac{1}{(1 + r)^t} \sum_{j=1}^{N} q_j C(t, \omega_j)
\]

Or with *deflators*:

\[
X(0) = \sum_{t=1}^{T} \sum_{j=1}^{N} p_j C(t, \omega_j) D_j(t) = \sum_{t=1}^{T} E(D(t)C(t))
\]
3.1. THE BINOMIAL CASE

**Single period model:**

✓ Risky asset:

\[ S_1(1) = S_1(0) \cdot u \]  with probability \( p \)

\[ = S_1(0) \cdot d \]  with probability \( 1-p \)

Absence of arbitrage opportunities if:

\[ 0 < d < 1 + r < u \]

Other form of the risky asset:

\[ u = 1 + r + \lambda + \mu \]

\[ d = 1 + r + \lambda - \mu \]
3.1. THE BINOMIAL CASE

With condition: \( 0 < \lambda < \mu \)

\[ \lambda = \text{risk premium} \quad \mu = \text{volatility} \]

Equations of the STATE PRICE:

For \( i = 0 \):
\[ (1 + r)\Psi_1 + (1 + r)\Psi_2 = 1 \]

For \( i = 1 \):
\[ u\Psi_1 + d\Psi_2 = 1 \]
3.1. THE BINOMIAL CASE

Solution for the STATE PRICE:

\[ \Psi_1 = \frac{1 + r - d}{(1 + r)(u - d)} = \frac{\mu - \lambda}{2\mu(1 + r)} \]

\[ \Psi_2 = \frac{u - (1 + r)}{(1 + r)(u - d)} = \frac{\mu + \lambda}{2\mu(1 + r)} \]

Safety principle:

\[ \Psi_2 = \Psi_1 \text{ if } \lambda = 0 \]

\[ \Psi_2 > \Psi_1 \text{ if } \lambda > 0 \text{ (normal case)} \]
3.1. THE BINOMIAL CASE

Fair value in a binomial environment – single period:

If $X$ is a financial instrument on this market with future stochastic cash flows given respectively by:

$$X(1, \omega_1) = X_1 \quad \text{and} \quad X(1, \omega_2) = X_2$$

Then the initial fair value of $X$ is given by:

$$X(0) = X_1 \Psi_1 + X_2 \Psi_2$$

Or:

$$X(0) = \frac{1}{2} (X_1 + X_2) \frac{1}{1+r} + \frac{1}{2} (X_2 - X_1) \frac{\lambda}{\mu(1+r)}$$
3.1. THE BINOMIAL CASE

Multiple periods model:

Risky asset

\[
\begin{array}{c}
1 \\
\downarrow \\
d \\
\downarrow \\
u \\
\uparrow \\
u^2 \\
\downarrow \\
u \cdot d \\
\downarrow \\
d^2
\end{array}
\]
3.1. THE BINOMIAL CASE

Structure of STATE PRICES in multiple periods:

Assumption: financial product having successive cash flows depending only on the current situation of the market (no path dependant).

\[ \Psi_{t,j} = \text{state price at time } t \text{ if scenario } j \]

State price diagram:

- \( \Psi_{11} \)
- \( \Psi_{21} \)
- \( \Psi_{12} \)
- \( \Psi_{22} \)
- \( \Psi_{13} \)
- \( \Psi_{23} \)
3.1. THE BINOMIAL CASE

Value of the STATE PRICES:

\[ \Psi_{t,j} = C_t^{j-1} \Psi_1^{j-1} \Psi_2^{t-j-1} \]

Where \( j-1 = \text{number of up (} j=1,..,t+1) \)
\( t-j-1 = \text{number of down} \)

And: \( C_t^{j-1} \) is the number of paths in the tree with \( j-1 \) up in \( t \) periods
3.1. THE BINOMIAL CASE

Fair valuation in multiple periods – binomial:

If $X$ is a financial instrument having successive cash flows in the tree given by:

$$X_{t_j} = \text{cash flow at time } t \text{ if scenario } j$$

Then the initial fair value of $X$ is given by:

$$X(0) = \sum_{t=1}^{T} \sum_{j=1}^{t+1} X_{t_j} \Psi_{t_j}$$
3.2. THE BLACK / SCHOLES CASE

Continuous extension: Black and Scholes model

Riskless asset:

\[ dS_0(t) = rS_0(t) \, dt \]

Risky asset:

\[ dS_1(t) = \delta S_1(t) dt + \sigma S_1(t) \, dw(t) \]

Where:
- \( w \) is a standard brownian motion
- \( \delta \) is the mean return of the asset (\( \delta > r \))
- \( \sigma \) is the volatility of this return
3.2. THE BLACK / SCHOLES CASE

Risk neutral measure:

\[ dS_1(t) = rS_1(t)dt + S_1(t)((\delta - r)dt + \sigma dw(t)) \]
\[ = rS_1(t)dt + \sigma S_1(t)dw^*(t) \]

with: \( w^*(t) = w(t) + \frac{\delta - r}{\sigma} t \)

\( Q = \) risk neutral measure (Girsanov Theorem)
Under \( Q \) the process \( w^* \) is a standard brownian motion
3.2. THE BLACK / SCHOLES CASE

**Deflators:** stochastic process \( D \) such that for \( i=1 \) (risk free asset) and for \( i=2 \) (risky asset):

\[
S_i(0) = E(D(t)S_i(t))
\]

**Solution:**

\[
D(t) = e^{-rt} e^{-(\frac{\delta-r}{\sigma})w(t) - \frac{1}{2}(\frac{\delta-r}{\sigma})^2 t}
\]

**Safety principle:**

Lower values of « \( w \) » give higher values for deflator
4. Fair value of participating contracts

4.1. Liability side:
   Life insurance contract with profit:
   guaranteed interest rate + participation

4.2. Asset side:
   Strategic asset allocation:
   Cash + Bonds + Stocks

4.3. Valuation of the contract:
   Fair value and equilibrium condition
4. Fair value of participating contracts

Need for a consistent ALM approach:
Double link between asset and liability in this kind of product:

Liability $\rightarrow$ Asset:
Investment strategy must take into account the specificities of the underlying liability

Asset $\rightarrow$ Liability:
Participation liability linked with investment results
4.1. Liability side

**Pure Endowment contract:**
- initial age at t=0 : \( x \)
- maturity : \( N \)
- Benefit if alive at time \( t=N \) : \( 1 \)
- Benefit in case of death before \( N \) : \( 0 \)
- Contract with single or periodical premiums (pure premium)
- Technical parameters :
  - mortality table: \( \{l_x\} \)
  - guaranteed technical rate : \( i \)
  - participation rate on surplus: \( \eta \)
4.1. Liability side

Bonus systems:

general definition: percentage of the surplus
(Assets – Liabilities)

Three possible schemes:

Terminal bonus: bonus computed at maturity

Reversionary bonus: bonus computed each year and fully integrated in the contract as additional premium

Cash bonus: bonus computed each year and paid directly to the client
4.2. Asset side

**Cash – Bonds- Stocks model (CBS model):**
The underlying portfolio of the insurer is supposed to be invested in three big classes of asset:

- **Cash**: short term position (money account)
- **Bonds**: zero coupon bonds with a maturity not necessarily matched with the duration of the contract
- **Stocks**: stock index
4.2. Asset side

**Cash model:**

Money market account:

\[ d\beta(t) = r(t) \beta(t) \, dt \]

with \( r(t) = \text{risk free rate} \)

Risk free rate: Ornstein-Uhlenbeck process

\[ dr(t) = a(b - r(t)) \, dt + \sigma_r \, dw_1(t) \]

with \( w_1 = \text{standard brownian motion} \)
4.2. Asset side

**Bond model**

\[ P(t,T) = \text{price at time } t \text{ of a zero coupon with maturity } T \]

General evolution equation:

\[ dP(t,T) = P(t,T)\mu(t,T)dt - P(t,T)\sigma(t,T)dw_1(t) \]

**Particular case**: VASICEK Model

\[ \mu(t,T) = r(t) + \lambda \sigma_r B(T - t) \]

\[ \sigma(t,T) = \sigma_r B(T - t) \]

with: \( B(s) = \frac{1}{a} (1 - e^{-as}) \)
4.2. Asset side

**Stocks model:**

S(t) = value at time t of a stock index

\[ dS(t) = S(t) \left( \mu \, dt + \sigma_s (\rho \, dw_1 (t) + \sqrt{1 - \rho^2} \, dw_2 (t)) \right) \]

with \( \rho = \text{correlation stocks/interest rates} \)

\( w_2 = \text{standard brownian motion independant of } w_1 \)
4.2. Asset side

**Portfolio:**

\[ \alpha_C = \text{proportion in cash} \]
\[ \alpha_B = \text{proportion in bonds} \]
\[ \alpha_S = \text{proportion in stocks} \]
\[ \alpha_C + \alpha_B + \alpha_S = 1 \]

2 main assumptions:

- proportions remain constant
  - (continuous rebalancing)
- self financed strategy
4.2. Asset side

**Bond strategy:**

Assumption: at each time $t$ zero coupon bonds of only one maturity can be held but the maturity has not necessarily to match the duration of the liability (*price of mismatching-long duration of life insurance contract*)

Strategy 1: matched strategy: $T=N$
Strategy 2: shorter maturity of the bond and successive reinvestments till end of the contract
4.2. Asset side

Bond strategy

\[ s = 0, 1, ..., n : \text{number of reinvestments} \]
\[ t_0 = 0, t_1, t_2, ..., t_n : \text{instances of reinvestment} \]

Particular case: matched strategy: \( n = 0 \)
4.2. Asset side

**Evolution of the portfolio:**

\[ V(t) = \text{market value of the underlying portfolio} \]

**Evolution equation:**

\[
\frac{dV(t)}{V(t)} = \alpha_S \frac{dS(t)}{S(t)} + \alpha_B \frac{dP(t, t_i)}{P(t, t_i)} + \alpha_C \frac{d\beta(t)}{\beta(t)}
\]

(for \( t_{i-1} < t < t_i \))

Pierre Devolder 09/2004
4.2. Asset side

Value at maturity of the assets:

\[ \ln V(N) - \ln V(0) = \sum_{s=0}^{n} (\ln V(t_{s+1}) - \ln V(t_{s})) \]

with each term \((\ln V(t_{s+1}) - \ln V(t_{s})) = \) normally distributed

\[ \mu(N) = E(\ln V(N) - \ln V(0)) \]
\[ \sigma^2(N) = \text{var}(\ln V(N) - \ln V(0)) \]
4.2. Asset side

Explicit form:

\[
\ln V(t_{s+1}) - \ln V(t_s) = \int_{t_s}^{t_{s+1}} (\alpha_S \mu + (\alpha_B + \alpha_C) r(u) + \alpha_B \lambda \sigma_r B(t_{s+1} - u)) \, du
\]

\[
+ \int_{t_s}^{t_{s+1}} \alpha_S \sigma_S \sqrt{1 - \rho^2} \, dw_2(u) + \int_{t_s}^{t_{s+1}} (\alpha_S \sigma_S \rho - \alpha_B \sigma_r B(t_{s+1} - u)) \, dw_1(u)
\]

\[
- \frac{1}{2} \int_{t_s}^{t_{s+1}} (\alpha_S^2 \sigma_S^2 + \alpha_B^2 \sigma_B^2 B^2(t_{s+1} - u) - 2\alpha_S \alpha_B \sigma_S \sigma_r \rho B(t_{s+1} - u)) \, du
\]
4.3. Valuation of the contract

Example of a contract with single premium and terminal bonus:

**Fair value at maturity** = pay off of the contract

\[ FV(N) = (1 + i)^N + \eta \max(V(N) - (1 + i)^N; 0) \]

**Initial fair value** : (?) In line with the single premium

\[ FV(0) = \sum_{x} ((1 + i)^N P(0, N) + \eta \text{Call}(V; N;(1 + i)^N)) \]
4.3. Valuation of the contract

Equilibrium condition:

\[(1 + i)^N P(0, N) + \eta \text{ call}(V; N; (1 + i)^N) = 1\]

Consequences:

1°) \(i < R(0, N)\)

\[P(0, N) = \frac{1}{(1 + R(0, N))^N}\]

2°) depends on the investment strategy through the value of \(V(N)\)
4.3. Valuation of the contract

3°) implicit relation for the technical rate $i$;  
4°) explicit relation for the participation rate $\eta$:

$$\eta = \frac{1 - (1 + i)^N P(0, N)}{\text{call}(V; N; (1 + i)^N)}$$

An explicit formula of the call can be obtained in the CBS model presented before.
4.3. Valuation of the contract

Risk forward neutral measure method:

\[
\text{call} = P(0, N) E_{Q_N} \left( \max \left( (V(N) - (1 + i)^T); 0 \right) \right)
\]

where \( Q_N = \text{forward risk neutral measure} \)

In the CBS model we have:

\[
\text{call} = \Phi(D_+(i)) - (1 + i)^N P(0, N) \Phi(D_-(i))
\]

with:

\[
D_{\pm}(i) = \frac{-N \ln(1 + i) - \ln P(0, T) \pm 1/2 \sigma^2}{\sigma}
\]
4.3. Valuation of the contract

With for instance for the matched strategy:

\[ \nu^2 = \alpha_s^2 \sigma_s^2 N + (1 - \alpha_B)^2 \sigma_r^2 B_2(N) + 2\alpha_s (1 - \alpha_B) \sigma_s \sigma_r \rho B_1(N) \]

with:

\[ B_1(N) = \frac{N}{a} - \frac{1}{a^2} (1 - e^{-aN}) \]

\[ B_2(N) = \frac{N}{a^2} + \frac{1}{2a^3} (1 - e^{-2aN}) - \frac{2}{a^3} (1 - e^{-aN}) \]
5. FAIR VALUE OF VARIABLE ANNUITIES

**Purposes**:

- How to valuate pension annuities not in terms of technical basis but in terms of market fair values;

- Influence of reversionary bonus (variable annuities) on the level of provision;

- Sensitivity of the provision with respect to financial parameters;

- How to fix the technical interest rate.
5.1. Liability side

- Immediate lifetime annuity for an affiliate to a pension fund
- x : initial age at time t=0
- Liability to pay: 2 cases :

  1) **fixed annuity** : \( L \)
  2) **variable annuity** :

     \( L_{t,j} \) = amount to pay at time \( t \) for scenario \( j \)

     ( possibility to increase yearly the pension depending on the financial performances – asset side)

- Payment at the end of the year till death or during a fixed period of \( n \) years
5.1. Liability side

Actuarial first order bases :

\[ i = \text{technical discount rate} \]

\[ t p_x = \text{survival probability at time } t \]

Technical provision for a constant pension (case 1):

\[ L_{t,j} = L \]

\[ n V_x = L a_x |_{n} = L \sum_{t=1}^{n} t p_x \frac{1}{(1+i)^t} \]
5.2. Asset side

Binomial model:
mixed financial strategy of the pension fund
between riskless asset ($r =$ riskfree rate)
and risky asset (binomial model $u / d$)

$\gamma$: part invested in the risky asset

$1 - \gamma$: part invested in the riskless asset

$(0 \leq \gamma \leq 1)$
5.3. Bonus scheme

Definition of the reversionary bonus for variable annuities

Used rule of bonus: comparison each year between the effective return of the assets and the riskfree rate; a part of this surplus is given back to the affiliate:

\[ 0 \leq \beta \leq 1 \quad : \text{participation rate} \]
5.3. Bonus scheme

Yearly rate of increase of the pension:

- **If the risky asset is up:**
  
  \[
  1 + k = 1 + \beta \left( \frac{\gamma u + (1 - \gamma)(1 + r)}{(1 + r)} - 1 \right)^+ 
  \]

  or

  \[
  1 + k = 1 + \beta \gamma \left( \frac{\lambda + \mu}{1 + r} \right) 
  \]

- **If the risky asset is down:**

  \[
  1 + l = 1 + \beta \left( \frac{\gamma d + (1 - \gamma)(1 + r)}{(1 + r)} - 1 \right)^+ = 1 
  \]
5.3. Bonus scheme

Final form of the liabilities of the variable annuity:

\[ L_{t,j} = L \cdot (1 + k)^{t-j+1} \]

Where \( t-j+1 \) is the number of times of up permitting to give a bonus.

As expected
THE LIABILITY DEPENDS ON TIME AND IS STOCHASTIC
5.4. Valuation of the contract

Computation of the fair value of the liabilities:
(fixed annuity)

\[ FV(L)_{x,n} = \sum_{t=1}^{n} p_x \left[ \sum_{j=1}^{t+1} L \Psi_{tj} \right] \]

\[ = L \sum_{t=1}^{n} p_x \left[ \sum_{j=1}^{t+1} \Psi_{tj} \right] \]

\[ = L \sum_{t=1}^{n} p_x \left( \frac{1}{1+r} \right)^t = L a^{r \cdot x \cdot n} \]
5.4. Valuation of the contract

Computation of the fair value of the liabilities
(variable annuity)

- Actuarial valuation: not so simple: liabilities not deterministic
- Fair valuation: general formula of valuation:

$$ FV(L_k)_{x,n} = \sum_{t=1}^{n} t \cdot p_x \left[ \sum_{j=1}^{t+1} L_{t,j} \Psi_{tj} \right] $$

$$ = L \sum_{t=1}^{n} t \cdot p_x \left[ \sum_{j=1}^{t+1} C_t^{j-1} \Psi_{2}^{j-1} \left( \Psi_{1} (1 + k) \right)^{t-j+1} \right] $$

$$ = L \sum_{t=1}^{n} t \cdot p_x \left[ (\Psi_{2} + \Psi_{1} (1 + k))^t \right] $$
5.4. Valuation of the contract

Computation of the fair value of the liabilities
(variable annuity)

\[
FV(L_k)_{x,n} = L \sum_{t=1}^{n} t p_x \left[ \frac{1}{1+r} \right]^t \left[ 1 + \beta \gamma \frac{\mu^2 - \lambda^2}{2\mu(1+r)} \right]^t
\]

\[
= L \sum_{t=1}^{n} t p_x \left( \frac{1}{1+i^*} \right)^t = L \bar{a}^{i^*}_{x,n}
\]
5.4. Valuation of the contract

Equilibrium relation:

\[ i^* = \text{equilibrium constant discount rate given by:} \]

\[ i^* = \left( r - \beta \gamma \frac{\mu^2 - \lambda^2}{2 \mu (1 + r)} \right) / \left( 1 + \beta \gamma \frac{\mu^2 - \lambda^2}{2 \mu (1 + r)} \right) \]

\[ \rightarrow \text{if } \beta = 0 \text{ or } \gamma = 0 : i^* = r \]

\[ \rightarrow \text{if } \beta > 0 \text{ and } \gamma > 0 : i^* < r \]
5.5. Numerical illustration

Central scenario:
\[ u = 1.1 \quad d = 0.99 \]
\[ r = 0.03 \]
\[ i = 0.025 \]
Mortality: GRM 95
\[ n = 5 \]

Risk premium: \( \lambda = 0.02 \)
Volatility: \( \mu = 0.06 \)
5.5. Numerical illustration

The graph illustrates the performance of different investment strategies over various fixed annuity periods. The strategies range from 0% to 100% stocks, with each strategy represented by a different color. The x-axis represents the fixed annuity period, while the y-axis shows the total return. The legend at the bottom of the graph indicates the percentage of stocks in each strategy.
5.5. Numerical illustration

Volatility sensitivity analysis: (60% in risky asset)
5.5. Numerical illustration

Value of the equilibrium discount rate : central scenario

1° for $\beta = 0.5$ and $\gamma = 0.6$ :
   $i^* = 2.21\%$

2° for $\beta = 1$ and $\gamma = 0.6$ :
   $i^* = 1.42\%$

3° for $\beta = 0.9$ and $\gamma = 0.4$ :
   $i^* = 2.05\%$

4° for $\beta = 1$ and $\gamma = 1$ :
   $i^* = 0.4\%$

$\lambda = 2\%$
$\mu = 6\%$
$r = 3\%$
$i = 2.5\%$
5.5. Numerical illustration

Value of the equilibrium discount rate: other scenario
(less volatile)

1° for $\beta = 0.5$ and $\gamma = 0.6$ :
  $i^* = 2.60\%$ versus $2.21\%$

2° for $\beta = 1$ and $\gamma = 0.6$ :
  $i^* = 2.21\%$ versus $1.42\%$

3° for $\beta = 0.9$ and $\gamma = 0.4$ :
  $i^* = 2.52\%$ versus $2.05\%$

4° for $\beta = 1$ and $\gamma = 1$ :
  $i^* = 1.68\%$ versus $0.4\%$
5.5. Numerical illustration

Equilibrium technical rate

![Equilibrium technical rate graph]

Legend:
- 100%stocks
- 80%stocks
- 60%stocks
- 40%stocks
- 20%stocks
- 0%stocks

Pierre Devolder 09/2004
5.5. Numerical illustration

EQUILIBRIUM TECHNICAL RATE(II)
6. CONCLUSION

- State prices, risk neutral measures and deflators are easy tools in order to valuate stochastic future cash flows correlated to future financial markets.

- This is exactly the situation of life insurance products.

- One of the most important result is the way to valuate bonus and to define properly a technical interest rate in a complete ALM framework.
devolder@fin.ucl.ac.be