4. Miscellaneous topics in risk theory

(4.1) Stochastic dominance

(4.2) Actuarial models and life tables

(4.3) Loss distribution and Value-at-Risk

(4.4) Default correlations

4.1 Stochastic dominance

- ⋆ Knowing the utility function, we have the full information on preference. Using the *maximum expected utility criterion*, we obtain a complete ordering of all the investments under consideration.

- ⋆ What happens if we have only partial information on preferences (say, prefer more to less and/or risk aversion)?

- ⋆ For example, in *the First Order Stochastic Dominance Rule*, we only consider the class of utility functions, call $\mathbb{U}_1$, such that $u' \geq 0$. This is a very general assumption and it does not assume any specific utility function.

Dominance in $\mathbb{U}_1$

Investment A dominates Investment B in $\mathbb{U}_1$ if for all utility functions such that $u \in \mathbb{U}_1, E_A u(x) \geq E_B u(x)$, or equivalently, $U(F_A) \geq U(F_B)$, and for at least one utility function, there is a strict inequality.

Efficient set in $\mathbb{U}_1$ (consists of investments that are not being dominated)

An investment is included in the efficient set if there is no other investment that dominates it.

Inefficient set in $\mathbb{U}_1$

The inefficient set includes all inefficient investments. An inefficient investment is that there is at least one investment in the efficient set that dominates it.

The partition into efficient and inefficient sets depends on the choice of the class of utility functions. In general, the smaller the efficient set relative to the feasible set, the easier for the investor to make decision.

First order stochastic dominance

Two Investment Alternatives: Outcomes and Associated Probabilities

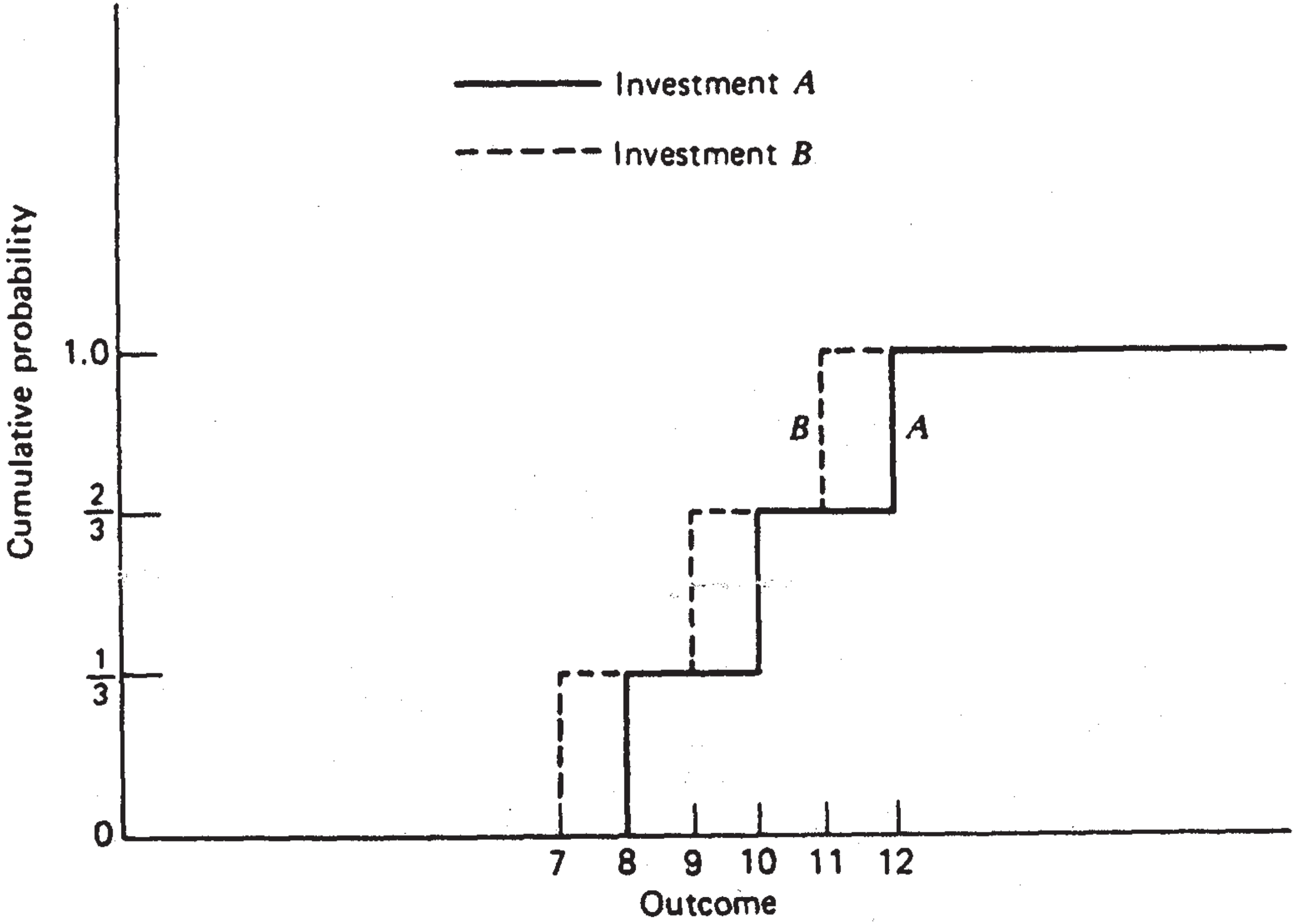
Investment A		Investment B	
Outcome	Probability	Outcome	Probability
12	1/3	11	1/3
10	1/3	9	1/3
8	1/3	7	1/3

Can we argue that Investment A is better than Investment B? It is still possible that the return from investing in B is 11% but the return is only 8% from investing in A.

⋆ By looking at the cumulative probability distribution, we observe that for all returns and the odds of obtaining that return or less, B consistently has a higher or same value.

A Cumulative Probability Distribution

Return	Odds of Obtaining a Return Equal to or Less Than That shown in Column 1	
	A	*B*
7	0	1/3
8	1/3	1/3
9	1/3	2/3
10	2/3	2/3
11	2/3	1
12	1	1



Cumulative frequency function for gambles *A* and *B*.

Recall that for each action $a \in A$, there is an induced probability distribution on C (the set of all consequences). To compare two choices of action, we examine their corresponding probability distribution.

Definition

A probability distribution F dominates another probability distribution G according to the first-order stochastic dominance if

$$F(x) \leq G(x) \quad \text{for all} \quad x \in C.$$

Lemma

F dominates G by FSD if and only if

$$\int_C u(x)\, dF(x) \geq \int_C u(x)\, dG(x)$$

for all strictly increasing expected utility indexes $u(x)$.

Proof

Let a and b be the smallest and largest values that F and G can take on. Consider

$$\int_a^b u(x)\,d[F(x)-G(x)] = \underbrace{u(x)[F(x)-G(x)]_a^b}_{\substack{\text{zero since } F(a)=G(a)=0 \\ \text{and } F(b)=G(b)=1}} - \int_a^b u'(x)[F(x)-G(x)]\,dx$$

$$\int_C u(x)\,dF(x) \geq \int_C u(x)\,dG(x) \Leftrightarrow -\int_a^b u'(x)[F(x)-G(x)]\,dx \geq 0.$$

Thus, for $u'(x) > 0$,

$$F(x) \leq G(x) \quad \Longleftrightarrow \quad \int_C u(x)\,dF(x) \geq \int_C u(x)\,dG(x).$$

Second order stochastic dominance

Two Investment Alternatives: Outcomes and Associated Probabilities

Investment A		Investment B	
Outcome	Probability	Outcome	Probability
6	1/4	5	1/4
8	1/4	9	1/4
10	1/4	10	1/4
12	1/4	12	1/4

If both investments turn out the worst, the investor obtains 6% from A and only 5% from B. If the second worst return occurs, the investor obtains 8% from A rather than 9% from B. If he is risk averse, then he should be willing to lose 1% in return at a higher level of return in order to obtain an extra 1% at a lower return level. If risk aversion is assumed, then A is preferred to B.

Definition

A probability distribution F dominates another probability distribution G according to the second order stochastic dominance if for all $x \in C$

$$\int_{-\infty}^{x} F(y)\,dy \leq \int_{-\infty}^{x} G(y)\,dy.$$

The Sum of the Cumulative Probability Distribution				
	Cumulative Probability		Sum of Cumulative Probability	
Return	A	B	A	B
4	0	0	0	0
5	0	1/4	0	1/4
6	1/4	1/4	1/4	1/2
7	1/4	1/4	1/2	3/4
8	1/2	1/4	1	1
9	1/2	1/2	1 1/2	1 1/2
10	3/4	3/4	2 1/4	2 1/4
11	3/4	3/4	3	3
12	1	1	4	4

According to SSD, A is preferred over B since the sum of cumulative probability for A is always less than or equal to that for B.

Theorem

If F dominates G by SSD, then

$$\int_C u(x)\, dF(x) \geq \int_C u(x)\, dG(x)$$

for all increasing and concave expected utility indexes $u(x)$.

Proof

$$\begin{aligned}
\int_a^b u(x)\, d[F(x) - G(x)] &= -\int_a^b u'(x)[F(x) - G(x)]\, dx \\
&= -u'(x) \int_a^x [F(y) - G(y)]\, dy \Big|_a^b \\
&\quad + \int_a^b u''(x) \int_a^x [F(y) - G(y)]\, dydx \\
&= -u'(b) \int_a^b [F(y) - G(y)]\, dy \\
&\quad + \int_a^b u''(x) \int_a^x [F(y) - G(y)]\, dydx.
\end{aligned}$$

Given that $u'(b) > 0$ and $u''(x) < 0$,

$$\int_C u(x)\, dF(x) \geq \int_C u(x)\, dG(x) \quad \text{if} \quad \int_a^x [F(y) - G(y)]\, dy \leq 0, \forall x.$$

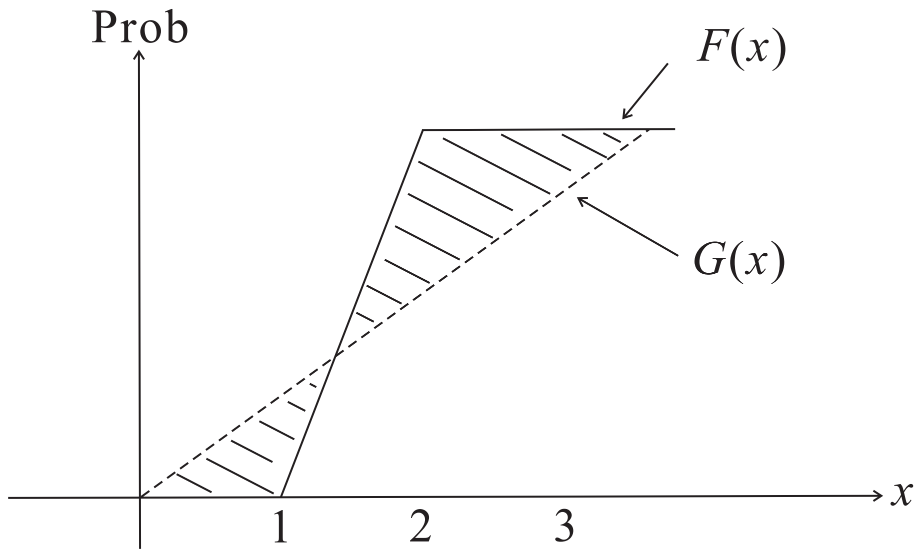
Example

$$F(x) = \begin{cases} 0 & \text{if } x < 1 \\ x-1 & \text{if } 1 \le x \le 2 \\ 1 & \text{if } x \ge 2 \end{cases}, \quad G(x) = \begin{cases} 0 & \text{if } x < 0 \\ x/3 & \text{if } 0 \le x \le 3 \\ 1 & \text{if } x \ge 3 \end{cases}.$$

F dominates G by SSD since

$$\int_{-\infty}^{x} F(y)\,dy \le \int_{-\infty}^{x} G(y)\,dy.$$

$F(x)$ is seen to be more concentrated (less dispersed).



Sufficient rules and necessary rules for second order stochastic dominance

Sufficient rule 1: FSD rule is sufficient for SSD

Proof: If F dominates G by FSD, then $F(x) \leq G(x), \forall x$.

This implies $\int_a^x [G(y) - F(y)]\, dy \geq 0$.

Remark

The efficient set according to SSD is larger than that of FSD. Since SSD rule requires risk aversion in addition to FSD rule, some elements in the inefficient set according to FSD may not stay again in the inefficient set of SSD.

Sufficient rule 2

$\text{Min}_F(x) > \text{Max}_G(x)$ is a sufficient rule for SSD.

Example

F		G	
x	$p(x)$	x	$p(x)$
5	1/2	2	3/4
10	1/2	4	1/4

$\text{Min}_F(x) = 5 \geq \text{Max}_G(x) = 4$ so that $F(x) \leq G(x)$. Hence, F dominates G.

$$\text{Min}_F(x) \geq \text{Max}_G(x) \Rightarrow \text{FSD} \Rightarrow \text{SSD} \Rightarrow E_F u(x) \geq E_G u(x), \forall u \in \mathbb{U}_2.$$

Necessary rule 1 (Geometric means)

Given a risky project with the distribution $(x_i, p_i), i = 1, \cdots, n$, the geometric mean, $\overline{X}_{geo}$, is defined as

$$\overline{X}_{geo} = x_1^{p_1} \cdots x_n^{p_n} = \prod_{i=1}^{n} x_i^{p_i}, \quad x_i \geq 0.$$

Taking logarithm on both sides

$$\ln \overline{X}_{geo} = \Sigma p_i \ln x_i = E[\ln X].$$

$\overline{X}_{geo}(F) \geq \overline{X}_{geo}(G)$ is a necessary condition for dominance of F over G by SSD.

Proof

Suppose F dominates G by SSD, we have

$$E_F u(x) \geq E_G u(x), \quad \forall u \in \mathbb{U}_2.$$

Since $\ln x = u(x) \in \mathbb{U}_2$,

$$E_F \ln x = \ln_F \overline{X}_{geo} \geq E_G \ln x = \ln_G \overline{X}_{geo};$$

we obtain $\ln \overline{X}_{geo}(F) \geq \ln \overline{X}_{geo}(G)$.

Since the logarithm function is an increasing function, we deduce $\overline{X}_{geo}(F) \geq \overline{X}_{geo}(G)$. Therefore, F dominates G by SSD $\Rightarrow \overline{X}_{geo}(F) \geq \overline{X}_{geo}(G)$.

Necessary rule 2 (left-tail rule)

Suppose F dominates G by SSD, then

$$\text{Min}_F(x) \geq \text{Min}_G(x),$$

that is, the left tail of G must be "thicker".

Proof by contradiction:

Suppose $\text{Min}_F(x) < \text{Min}_G(x)$, and write $x_k = \text{Min}_F(x)$. At x_k, G will still be zero but F will be positive. Observe that

$$\int_{-\infty}^{x_k} [G(y) - F(y)]\, dy = \int_{-\infty}^{x_k} [0 - F(y)]\, dy < 0,$$

implying F is not dominated by G by SSD. Hence, if F dominates G, then $\text{Min}_F(x) \geq \text{Min}_G(x)$.

4.2 Actuarial models and life tables

Future lifetime of a life aged x

Let $T(x)$ be the future lifetime of a life aged x so that $x + T$ will be the age at death of the person. The distribution function of the random variable T is

$$G(t) = P[T \leq t], \quad t \geq 0.$$

Assume that G is continuous and has a probability density

$$g(t) = G'(t)$$

so that

$$g(t)\, dt = P[t < T < t + dt].$$

Symbols

$$
{}_tq_x = G(t) \quad \text{and} \quad {}_tp_x = 1 - G(t)
$$

$$
{}_{s|t}q_x = P[s < T < s + t] = G(s + t) - G(s) = {}_{s+t}q_x - {}_sq_x.
$$

If $t = 1$, the index t is usually omitted in the symbols. For example, ${}_{s|}q_x$ is the probability of surviving s years and subsequently dying within one year.

Conditional probabilities

$$
\begin{aligned}
{}_t p_{x+s} &= P[T > s+t | T > s] = \frac{1 - G(s+t)}{1 - G(s)} \\
{}_t q_{x+s} &= p[T \leq s+t | T > s] = \frac{G(s+t) - G(s)}{1 - G(s)} \\
&= \text{conditional probability of dying with } t \text{ years,} \\
&\quad \text{given that the age of } x+s \text{ has been attained.}
\end{aligned}
$$

Expected remaining life-time of a life aged x

$$
\begin{aligned}
E[T] &= \mathring{e}_x = \int_0^\infty t g(t)\, dt \\
&= \int_0^\infty [1 - G(t)]\, dt = \int_0^\infty {}_t p_x \, dt.
\end{aligned}
$$

Force of mortality

The force of mortality of (x) at the age $x+t$ is

$$\mu_{x+t} = \frac{g(t)}{1-G(t)} = -\frac{d}{dt}\ln[1-G(t)]$$

or

$$\frac{P[t<T<t+dt]}{{}_tp_x} = \mu_{x+t}\,dt.$$

The expected future lifetime of (x)

$$\mathring{e}_x = \int_0^\infty {}_tp_x \mu_{x+t}\,dt.$$

Approximation: ${}_sq_{x+t} \approx \mu_{x+s}s$ for small values of s.

$$\mu_{x+t} = -\frac{d}{dt}\ln {}_tp_x$$

so that

$${}_tp_x = e^{-\int_0^t \mu_{x+s}\,ds}.$$

Analytic distribution of T

Gompertz

$$\mu_{x+t} = Bc^{x+t}, \quad t > 0$$

Makeham

$$\mu_{x+t} = A + Bc^{x+t}, \quad t > 0.$$

Setting $c = 1$ in Gompertz's distribution, force of mortality becomes constant. The probability distribution of T then becomes the exponential distribution.

By putting $m = B/\ln c$ in Makeham's distribution

$$_tp_x = \exp(-At - mc^x(c^t - 1)).$$

Weibull (1939) postulates

$$\mu_{x+t} = k(x+t)^n, \quad k > 0, n > 0,$$

then

$$_tp_x = \exp\left(-\frac{k}{n+1}\left[(x+t)^{n+1} - x^{n+1}\right]\right).$$

Curtate future lifetime of (x)

Define the random variables $K = K(x), S = S(x), S^{(m)} = S^{(m)}(x)$, all closely related to T.

(i) $K = [T]$ = number of completed future years lived by (x)

$$P[K = k] = P[k \leq T < k+1] = {}_kp_x q_{x+k}, \quad k = 0, 1, \cdots$$

$$e_x = E[K] = \sum_{k=1}^{\infty} kP[K = k] = \sum_{k=1}^{\infty} k\,{}_kp_x q_{x+k}$$

or

$$e_x = \sum_{k=1}^{\infty} P[K \geq k] = \sum_{k=1}^{\infty} {}_kp_x.$$

(ii) S = fraction of a year during which (x) is alive in the year of death

$$T = K + S.$$

The random variable S has a continuous distribution between 0 and 1.

$$\mathring{e}_x \approx e_x + \frac{1}{2}.$$

Assume that K and S are independent random variables

$$P[S \leq u | K = k] = \frac{{}_uq_{x+k}}{q_{x+k}}$$

will not depend on k. Hence, one can write

$${}_uq_{x+k} = H(u)q_{x+k}$$

for $k = 0, 1, \cdots$ and $0 \leq u \leq 1$, and some function $H(u)$.

If we take $H(u) = u$, then

$$\text{var}(T) = \text{var}(K) + \frac{1}{12}.$$

(iii) For positive integer m, define

$$S^{(m)} = \frac{1}{m}[mS + 1].$$

$S^{(m)}$ is derived from S by rounding to the next higher multiple of $\frac{1}{m}$. The distribution of $S^{(m)}$ has its mass in the points $\frac{1}{m}, \frac{2}{m}, \cdots, 1$.

- Independence between K and S implies independence between K and S implies independence between K and $S^{(m)}$.
- If S has a uniform distribution between 0 and 1, then $S^{(m)}$ has a discrete uniform distribution.

Life tables

- A life table is essentially a table of one-year death probabilities q_x, which then completely defines the distribution of K.
- Life tables are constructed from statistical data, categories for certain population groups, differentiated by factors such as sex, race, generation and insurance type.

Select life tables

The probabilities of death are graded according to the age of entry

- $q_{[x]+t}$ is the one-year probability of death for $x+t$ with x as entry age.
- $q_{[x]} < q_{[x-1]+1} < q_{[x-2]+2} < \cdots$

  since we expect that a person who has just bought insurance will be of better health than a person who bought insurance several years ago.

Probabilities of death for fractions of a year

The distribution of K and its related quantities may be calculated from a life table

$$_k p_x = p_x p_{x+1} \cdots p_{x-k+1}, \qquad k = 1, 2, \cdots .$$

To obtain the distribution of T by interpolation, assumptions are made regarding the pattern of the probabilities of death $_u q_x$ or the force of mortality μ_{x+u}, x is an integer and $0 < u < 1$.

Linearity of $_u q_x$

$$_u q_x = u q_x$$

then $_u p_x = 1 - u q_x$ and $\mu_{x+u} = \dfrac{q_x}{1 - u q_x}$.

μ_{x+u} *constant*

Denote the constant value of $\mu_{x+u}, 0 < u < 1$, by $\mu_{x+\frac{1}{2}}$, then

$$\mu_{x+\frac{1}{2}} = -\ln p_x.$$

we then obtain

$$_up_x = e^{-u\mu_{x+\frac{1}{2}}} - (p_x)^u,$$

and

$$P[S \leq u | K = k] = \frac{1 - p_{x+k}^u}{1 - p_{x+k}}.$$

The above conditional distribution is a truncated exponential distribution, depending on k.

Remark

The force of mortality is discontinuous at integer values.

Life insurance

Let Z be the present value of the payment (single) and v be the discount factor over one year.

Whole life insurance

Provides for payment of \$1 at the end of the year of death. Z is random since $K+1$ is random and

$$Z = v^{K+1}.$$

The distribution of Z is

$$P[Z = v^{k+1}] = P[K = k] = {}_kp_x q_{x+k}, \quad k = 0, 1, 2, \cdots .$$

The net single premium is A_x and

$$A_x = E[v^{K+1}] = \sum_{k=0}^{\infty} v^{k+1}{}_kp_x q_{x+k};$$

$$\text{var}(Z) = E[Z^2] - A_x^2.$$

Replacing v by $e^{-\delta}$, then $E[Z^2] = E[e^{-2\delta(K+1)}]$.

Term insurance of duration n

Provides for payment only if death occurs within n years

$$Z_1 \begin{cases} v^{K+1} & \text{for } K = 0, 1, \cdots, n-1 \\ 0 & \text{for } K = n, n+1, n+2, \cdots \end{cases} .$$

The net single premium is denoted by

$$A^{\ 1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} {}_k p_x q_{x+k}$$

Pure endowments

A pure endowment of duration n provides for payment of the sum incurred only if the insured is alive at the end of n years

$$Z_2 = \begin{cases} 0 & \text{for } K = 0, 1, \cdots, n-1 \\ v^n & \text{for } K = n, n+1, \cdots \end{cases} .$$

$$A_{x:\overline{n}|}^{\;\;1} = v^n \, {}_n p_x$$

and

$$\text{var}(Z_2) = v^{2n} {}_n p_x {}_n q_x.$$

Endowments

Payable at the end of the year of death, if this occurs within the first n years, otherwise at the end of the n^{th} year

$$Z = \begin{cases} v^{K+1} & \text{for } K = 0, 1, \cdots, n-1 \\ v^n & \text{for } K = n, n+1, n+2, \cdots \end{cases}.$$

Note that

$$Z = Z_1 + Z_2$$

so that

$$A_{x:\overline{n}|} = A^{\;1}_{x:n} + A_{x:n}^{\quad 1}$$

and

$$\text{var}(Z) = \text{var}(Z_1) + \text{var}(Z_2) + 2\text{cov}(Z_1, Z_2).$$

Since $Z_1Z_2 = 0$, hence

$$\mathrm{cov}(Z_1, Z_2) = E[Z_1Z_2] - E[Z_1]E[Z_2] = -A^{1}_{x:\overline{n}|} + A_{x:\overline{n}|}^{\;\;1}$$

$$\mathrm{var}(Z) = \mathrm{var}(Z_1) + \mathrm{var}(Z_2) - 2A^{1}_{x:n}A_{x:n}^{\;\;1}.$$

Thus, the risk in selling an endowment policy is less than that in selling a term insurance to one person and a pure endowment to another.

Insurances payable at the moment of death

Payable at the instant of death, T, then $Z = v^T$.

The net single premium

$$\overline{A}_x = \int_0^\infty v^x {}_tp_x \mu_{x+t}\, dt.$$

Write $T = K + S = (K+1) - (1-S)$, and assuming independence of K and S, we obtain

$$E\left[(1+i)^{1-s}\right] = \int_0^1 (1+i)^u\, du = \bar{s}_{\overline{1}|} = \frac{i}{\delta},$$

we find

$$\overline{A}_x = E[v^{K+1}]E[(1+i)^{1-S}] = \frac{i}{\delta} A_x.$$

For endowments, the factor i/δ is only used in the term insurance part.

4.3 Loan distribution and Value-at-Risk

Credit risk management of bank loans

- Charging an appropriate risk premium for every loan and collecting these risk premiums in an internal book account called *expected loss reserve* will create a capital cushion for covering losses arising from defaulting loans.
  - default probability (DP)
  - a loss fraction called the loss given default (LGD)
  - exposure at default (EAD)

Loss variable

$$\widetilde{L} = EAD \times LGD \times L$$

where $L = \mathbf{1}_D, P(D) = DP$. Here, D is the default event that the obligor defaults in a certain period of time.

For constant values of EAD and LGD, the *expected loss* (EL):

$$EL = E[\widetilde{L}] = EAD \times LGD \times DP.$$

- EAD and LGD may be the expectations of some underlying random variables

  e.g. $LGD = E[\text{severity}]$.

The above equation for EL remains valid when the exposure, severity and default event are independent.

Goal: Derive the portfolio risk based on information of individual risks and their correlations.

Default probability

- Calibration of default probabilities from market data
  - Expected default frequencies (EDF) from KMV
  - Based on the credit spreads of traded products e.g. corporate bonds, credit default swaps
- Calibration of default probabilities from ratings
  - Moody's Investors Services
  - Standard and Poor

External ratings do not react quick enough.

Drivers of firm's economic future

- future earnings and cashflows
- debts, short-term and long-term liabilities, and financial obligations
- capital structure (leverage)
- liquidity of the firm's assets
- political situations
- industrial situations
- management quality, company structure, etc.

Statistical analysis on financial variables plus soft factors.

TABLE 1.1 S & P Rating Categories

AAA	*best credit quality* *extremely reliable with regard to financial obligations*
AA	*very good credit quality* *very reliable*
A	*more susceptible to economic conditions* *still good credit quality*
BBB	*lowest rating in investment grade*
BB	*caution is necessary* *best sub-investment credit quality*
B	*vulnerable to changes in economic conditions* *currently showing the ability to meet its financial obligations*
CCC	*currently vulnerable to nonpayment* *dependent on favourable economic conditions*
CC	*highly vulnerable to a payment default*
C	*close to or already bankrupt* *payments on the obligation currently continued*
D	*payment default on some financial obligation has actually occurred*

TABLE 1.2: Moody's Historic Corporate Bond Default Frequencies.

Rating	1983	1984	1985	1986	1987	1988
Aaa	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Aa1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Aa2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Aa3	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
A1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
A2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
A3	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Baa1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Baa2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Baa3	0.00%	1.06%	0.00%	4.82%	0.00%	0.00%
Ba1	0.00%	1.16%	0.00%	0.88%	3.73%	0.00%
Ba2	0.00%	1.61%	1.63%	1.20%	0.95%	0.00%
Ba3	2.61%	0.00%	3.77%	3.44%	2.95%	2.59%
B1	0.00%	5.84%	4.38%	7.61%	4.93%	4.34%
B2	10.00%	18.75%	7.41%	16.67%	4.30%	6.90%
B3	17.91%	2.90%	13.86%	16.07%	10.37%	9.72%

Rating	1989	1990	1991	1992	1993	1994
Aaa	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Aa1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Aa2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Aa3	1.40%	0.00%	0.00%	0.00%	0.00%	0.00%
A1	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
A2	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
A3	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%
Baa1	0.00%	0.00%	0.76%	0.00%	0.00%	0.00%
Baa2	0.80%	0.00%	0.00%	0.00%	0.00%	0.00%
Baa3	1.07%	0.00%	0.00%	0.00%	0.00%	0.00%
Ba1	0.79%	2.67%	1.06%	0.00%	0.81%	0.00%
Ba2	1.82%	2.82%	0.00%	0.00%	0.00%	0.00%
Ba3	4.71%	3.92%	9.89%	0.74%	0.75%	0.59%
B1	6.24%	8.59%	6.04%	1.03%	3.32%	1.90%
B2	8.28%	22.09%	12.74%	1.54%	4.96%	3.66%
B3	19.55%	28.93%	28.42%	24.54%	11.48%	8.05%

The process of assigning a default probability to a rating is called a *calibration*. Rating $\longrightarrow DP$, e.g. $\{AAA, AA, \cdots, C\} \rightarrow [0, 1]$.

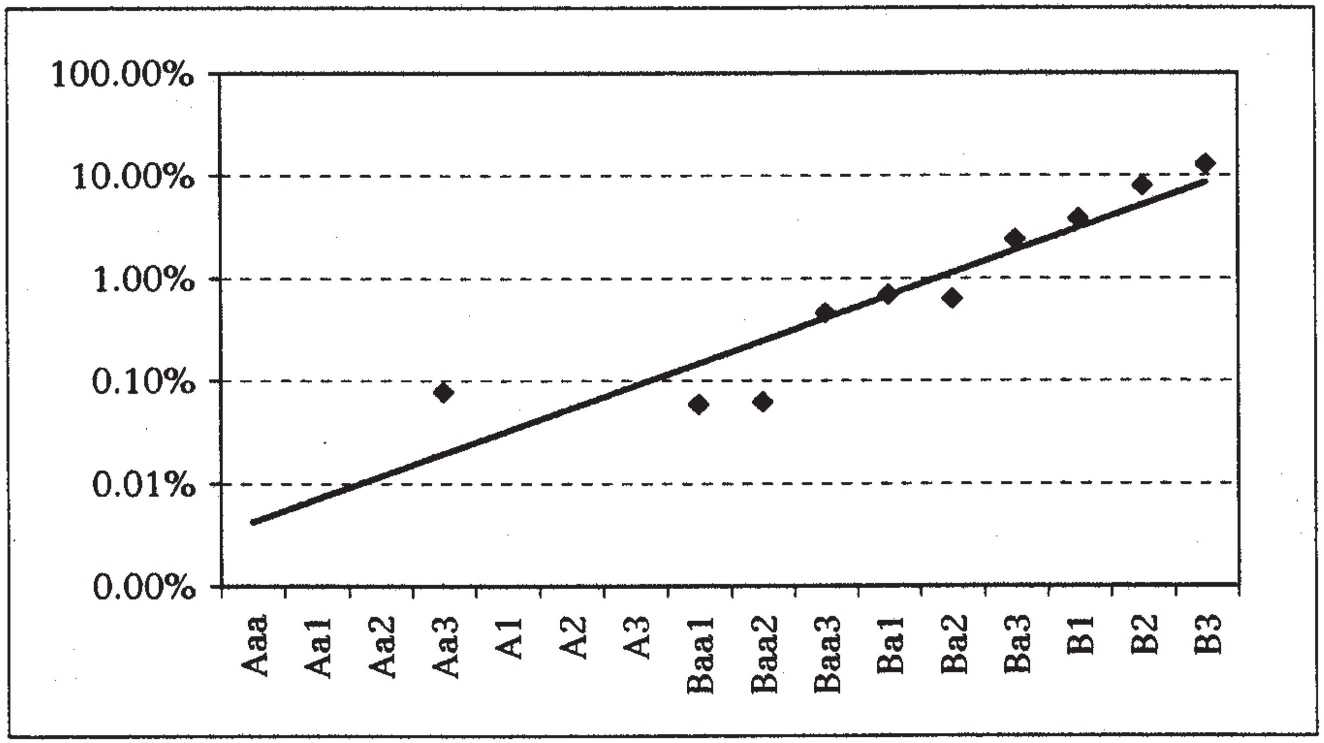


FIGURE 1.1: Calibration of Moody's Ratings to Default Probabilities

Exposure at default

- Outstanding and commitments

$$\swarrow \quad \searrow$$

drawn undrawn

$$EAD = OUTST + \gamma \times COMM$$

γ is the expected portion of the commitments likely to be drawn prior to default. Actually, γ is the expectation of the random variable modeling the utilization of the undrawn part of the commitments.

- Usually we take EAD to be deterministic so that we deal with the expectation γ rather than the underlying random variable.

- Commitment include various covenants – allowing bank to close committed lines triggered by some early default indication.

Loss given default

$LGD = 1-$ recovery rate

Driving factors

1. quality of collateral

2. seniority of bank's claim on borrower's assets

- How banks can share knowledge about their practical LGD experience?

- Better techniques for estimating LGD from historical data?

Unexpected loss

Holding capital as a cushion against expected losses is not enough. As a measure of the magnitude of the deviation of losses from the EL, a natural choice is the standard deviation of the loss variable $\widetilde{L}$.

$$\text{Unexpected loss } (UL) = \sqrt{\text{var}(\widetilde{L})} = \sqrt{\text{var}(EAD \times SEV \times L)}.$$

Under the assumption that the security and the default event D are independent,

$$UL = EAD \times \sqrt{\text{var}(SEV) \times DP + LGD^2 \times DP(1 - DP)}.$$

Proof

We make use of var$(X) = E[X^2] - E[X]^2$, var$(\mathbf{1}_D) = DP(1 - DP)$ and $E[\mathbf{1}_D^2] = E[\mathbf{1}_D] = DP$. Consider

$$
\begin{aligned}
\text{var}(SEV\,\mathbf{1}_D) &= E[SEV^2\mathbf{1}_D^2] = E[SEV\,\mathbf{1}_D]^2 \\
&= E[SEV^2]E[\mathbf{1}_D^2] - E[SEV]^2E[\mathbf{1}_D]^2 \\
&= E[SEV]^2E[\mathbf{1}_D^2] + \text{var}(SEV)E[\mathbf{1}_D] - E[SEV]^2E[\mathbf{1}_D]^2 \\
&= \text{var}(SEV) \times DP + LGD^2 \times DP(1 - DP).
\end{aligned}
$$

Remark

It is uncommon to have the situation where the severity of losses and the default events are random variables driven by a common set of underlying factors, hence the assumption of independence between SEV and $\mathbf{1}_D$ may be questionable.

Portfolio losses

Consider a portfolio of m loans

$$\widetilde{L}_i = EAD_i \times SEV_i \times \mathbf{1}_{D_i}, \quad i = 1, \cdots, m, \quad P[D_i] = E[\mathbf{1}_{D_i}] = DP_i.$$

portfolio loss:

$$\widetilde{L}_p = \sum_{i=1}^{m} \widehat{L}_i = \sum_{i=1}^{m} EAD_i \times SEV_i \times L_i, \quad L_i = \mathbf{1}_{D_i}.$$

Using the additivity of expectation

$$EL_p = \sum_{i=1}^{m} EL_i = \sum_{i=1}^{m} EAD_i \times LGD_i \times DP_i.$$

In the case UL, additivity holds if the loss variable $\widetilde{L}_i$ are pairwise uncorrelated.

$$\begin{aligned} UL_p &= \text{var}(\widetilde{L}_p) \\ &= \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m} EAD_i \times EAD_j \times \text{cov}(SEV_i \times L_i, SEV_j \times L_j]}. \end{aligned}$$

For a portfolio with constant severities

$$UL_p^2 = \sum_{i=1}^{m}\sum_{j=1}^{m} EAD_i \times EAD_j \times LGD_i \times LGD_j \times \sqrt{DP_i(1-DP_i)DP_j(1-DP_j)}\rho_{ij},$$

where $\rho_{ij} = \text{corr}(\mathbf{1}_{D_i}, \mathbf{1}_{D_j})$.

Example

Take $m = 2, LGD_i = EAD_i = 1, i = 1, 2$, then

$$UL_p^2 = p_1(1 - p_1) + p_2(1 - p_2) + 2\rho\sqrt{p_1(1 - p_1)p_2(1 - p_2)}.$$

(i) $\rho = 0$ – perfect diversification

(ii) $\rho > 0$ – default of one counterparty increases the likelihood that the other counterparty will also default

$$\begin{aligned} P[L_2 = 1 | L_1 = 1] &= \frac{P[L_1 = 1, L_2 = 1]}{P[L + 1 = 1]} = \frac{E[L_1 L_2]}{p_1} \\ &= \frac{p_1 p_2 + \text{cov}(L_1, L_2)}{p_1} = p_2 + \frac{\text{cov}(L_1 L_2)}{p_1}. \end{aligned}$$

Value-at-risk

VaR will be defined for a probability measure P and some confidence level α as the α-quantile of a loss random variable X

$$\text{var}_\alpha(X) = \inf\{x \geq 0 | P[X \leq x] \geq \alpha\}.$$

Banks should hold some capital cushion against unexpected losses. Using UL is not sufficient since there might be a significant likelihood that losses will exceed portfolio's EL by more than one standard deviation of the portfolio loss.

Take a *target level of statistical confidence* into account. For a given level of confidence α,

$$\begin{aligned} q_\alpha &= \alpha - \text{quantile of } \widetilde{L}_p \\ &= \inf\{q > 0 | P[\widetilde{L}_P \leq q] \geq \alpha\}. \end{aligned}$$

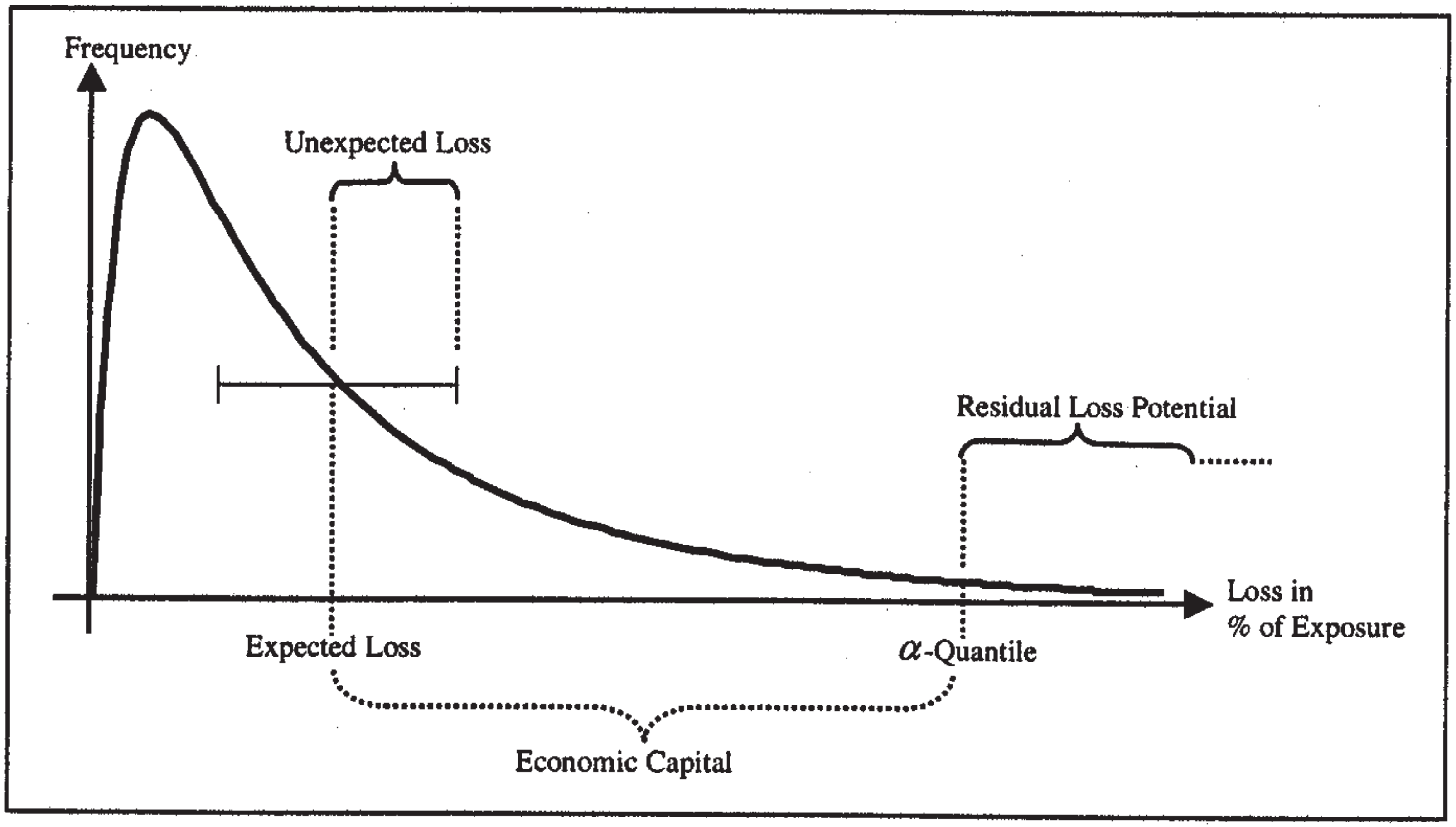
q_α is sometimes called the credit VaR. Define

$$EC_\alpha = \text{economic capital } = q_\alpha - EL_P.$$

Say, $\alpha = 99.98\%$, this would mean EC_α will be sufficient to cover unexpected losses in $9,998$ out of $10,000$ years, assuming a planning horizon of one year.

Why reducing the quantile q_α by the EL? This is the usual practice of decomposing the total risk capital into (i) expected loss (ii) cushion against unexpected losses.

Note that EL charges are *portfolio independent* (diversification has no impact) while EL charges are *portfolio dependent*. New loans may add a lot or little risk contributions (risk concentration).



The portfolio loss distribution

4.4 Default correlations

Modeling correlations by means of factor models

Why two companies experience a down- or up-turn at about the same time? Factor models – identifying underlying drivers of correlated defaults

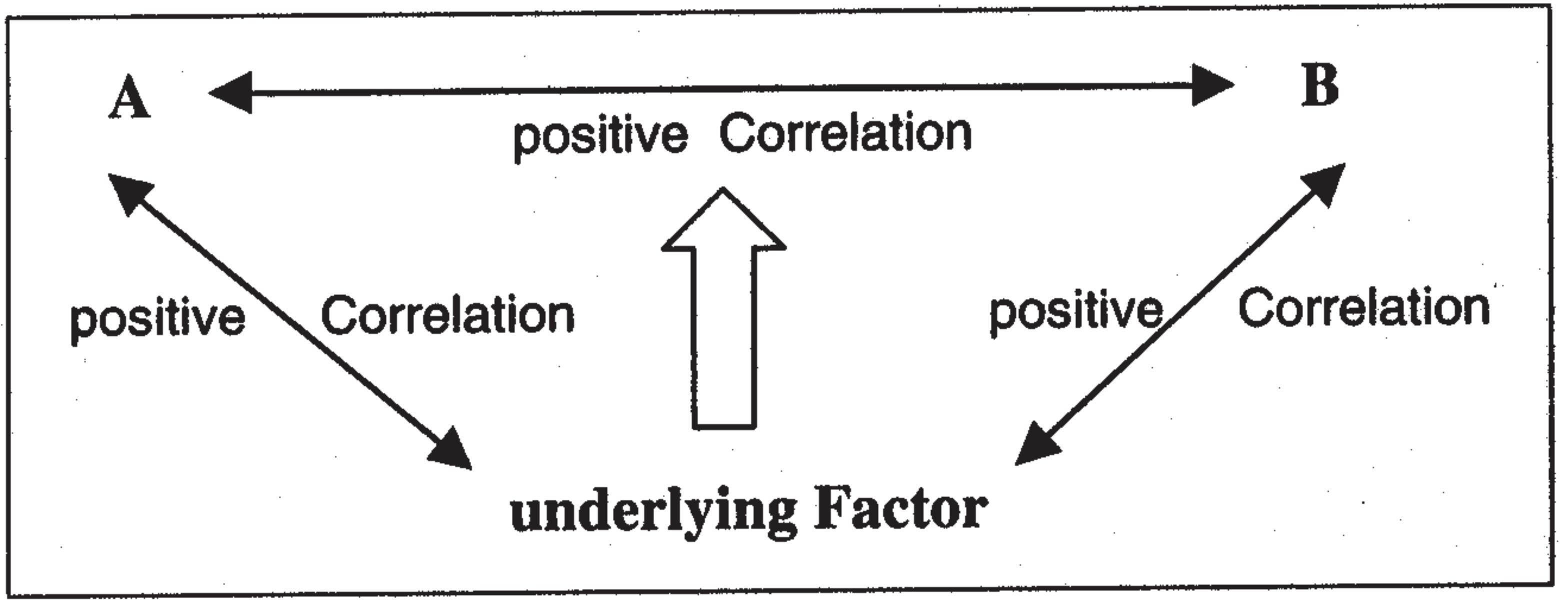
e.g. Chrysler and BMW

- automotive industry
- country factor

Volatility of a company's financial success is related to

(i) systematic factors like industries or countries

(ii) specific or idiosyncratic risk of the firm.



Correlation induced by an underlying factor

KMV's firm value model

- Every firm admits a process of asset value, such that default or survival of the firm depends on the state of the asset value at a certain planning horizon.

- If the process has fallen below a certain critical threshold, called the default point of the firm, then the company has defaulted.

- KMV models the dynamics of the firm value based on information of the price process of the firm's stock (for listed companies). The KMV model is forward looking since the stock price encompasses market view on the growth and creditworthiness of the firm instantly. This gives its competitive advantage in predicting defaults over rating methods (S&P and Moody's) which are backward looking.

KMV's factor model

The asset value log-returns r_i of counterparty $i, i = 1, \cdots, m$, at a certain planning horizon (typically one year) admits

$$r_i = \beta_i \Phi_i + \epsilon_i, \quad i = 1, \cdots, m.$$

- Φ_i is called the composite factor of firm i; Φ_i typically is a weighted sum of several factors.
- β_i, the sensitivity coefficient, captures the linear correlation of r_i and Φ_i. β_i may be called the beta of counterparty i.
- ϵ_i represents the residual part of r_i, which is the error one makes when substituting r_i by $\beta_i \Phi_i$.

Assumptions

1. $\boldsymbol{r} = (r_1 \cdots r_m)^T \sim N(\boldsymbol{\mu}, \Gamma)$ is a multivariate Gaussian with a correlation matrix Γ.
2. ϵ_i is independent of Φ_i, for every i; residuals $\epsilon_1, \cdots, \epsilon_m$ are assumed to be uncorrelated.

Φ_i – systematic part of r_i

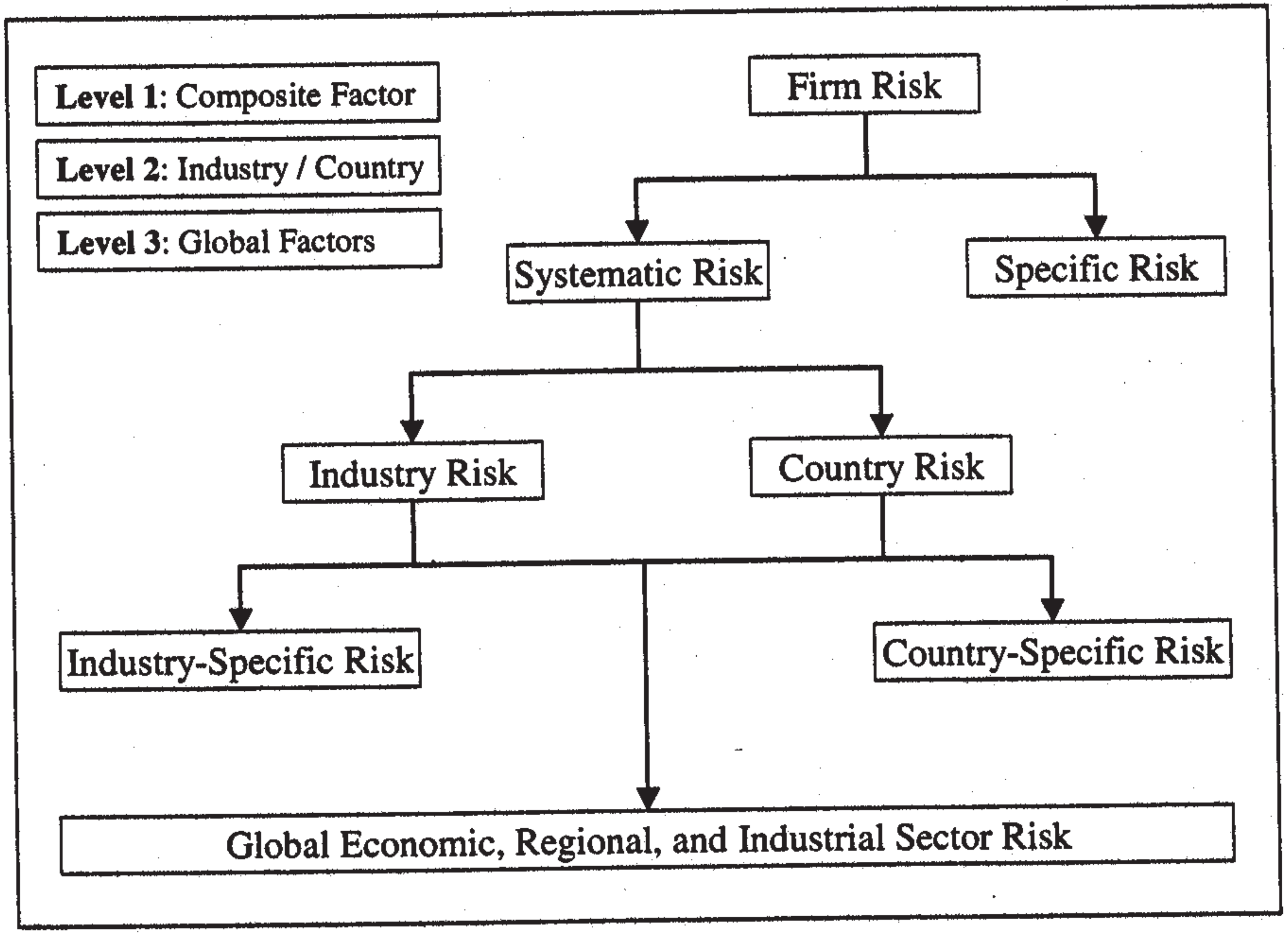
ϵ_i – random effect just relevant for counterparty i due to its independence from other involved variables

$$\text{var}(r_i) = \underbrace{\beta_i^2 \text{var}(\Phi_i)}_{\text{systematic}} + \underbrace{\text{var}(\epsilon_i)}_{\text{specific}}, \quad i = 1, \cdots, m.$$

Define

$$R_i^2 = \frac{\beta_i \text{var}(\phi_i)}{\text{var}(r_i)}$$

which quantifies how much of the variability of r_i can be explained by Φ_i. Then $1 - R_i^2$ gives the percentage value of the specific risk of counterparty i.



Define W to be the weight matrix, then

$$r = \beta W \boldsymbol{\psi} + \boldsymbol{\epsilon}, \quad \boldsymbol{\psi} = (\psi_1, \cdots, \psi_K)^T.$$

Write $\boldsymbol{\Phi} = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_m \end{pmatrix}, \boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_m \end{pmatrix}, \beta = \begin{pmatrix} \beta_1 & & 0 \\ & \ddots & \\ 0 & & \beta_m \end{pmatrix}$, then

$$\boldsymbol{r} = \beta\boldsymbol{\Phi} + \boldsymbol{\epsilon}.$$

For the second level, every Φ_i is decomposed with respect to industry and country breakdown

$$\Phi_i = \sum_{k=1}^{K} w_{ik}\psi_k$$

where $\psi_1, \cdots, \psi_{K_0}$ are industry indices and $\psi_{K_0+1}, \cdots, \psi_K$ are country indices, $w_{i,1}, \cdots, w_{i,K_0}$ are called the industry weights and w_{i,K_0+1}, $\cdots, w_{i,K}$ are called the country weights of counterparty i. Furthermore

$$\sum_{k=1}^{K_0} w_{ik} = \sum_{k=K_0+1}^{K} w_{ik} = 1, \quad i = 1, \cdots, m.$$

Define W to be the weight matrix, then

$$r = \beta W \psi + \epsilon, \quad \psi = (\psi_1, \cdots, \psi_K)^T.$$

At the third level, a representation by a weighted sum of *independent global factors* is constructed for representing industry and country indices

$$\psi_k = \sum_{n=1}^{N} b_{kn} \Gamma_n + \delta_k,$$

where δ_k denotes the ψ_k-specific residual. Write

$$\boldsymbol{\psi} = B\boldsymbol{\Gamma} + \boldsymbol{\delta}$$

so that

$$\boldsymbol{r} = BW(B\Gamma + \delta) + \boldsymbol{\epsilon}.$$

Now, the underlying factors are independent, so computation is more convenient.

Correlation calculations

Define the standardized asset value log-returns

$$\widetilde{r}_i = \frac{r_i - E[r_i]}{\sigma_i}, \quad i = 1, \cdots, m,$$

then

$$\widetilde{r}_i = \frac{\beta_i}{\sigma_i}\widetilde{\Phi}_i + \frac{\widetilde{\epsilon}_i}{\sigma_i} \quad \text{where} \quad E[\widetilde{\Phi}_i] = E[\widetilde{\epsilon}_i] = 0.$$

The asset correlation between two counterparties

$$\text{corr}(\widetilde{r}_i, \widetilde{r}_j) = E[\widetilde{r}_i \widetilde{r}_j] = \frac{\beta_i}{\sigma_i}\frac{\beta_j}{\sigma_j} E[\widetilde{\Phi}_i \widetilde{\Phi}_j]$$

since the residuals $\widetilde{\epsilon}_i$ are uncorrelated and independent of the composition factors.

It is preferable to express $\text{corr}(\widetilde{r}_j, \widetilde{r}_j)$ in terms of R_i. Observing

$$R_i^2 = \frac{\beta_i^2}{\sigma_i^2} \text{var}(\Phi_i) \quad \text{and} \quad \text{var}(\Phi_i) = \text{var}(\widetilde{\Phi}_i),$$

then

$$\text{corr}(\widetilde{r}_i, \widetilde{r}_j) = \frac{R_i}{\sqrt{\text{var}(\widetilde{\Phi}_i)}} \frac{R_j}{\sqrt{\text{var}(\widetilde{\Phi}_j)}} \; E[\widetilde{\Phi}_i \widetilde{\Phi}_j].$$

Bernuolli mixture model

Recall that loss variables are defined as indicators of default events. A vector of random variables $\boldsymbol{L} = (L_1, \cdots, L_m)$ is called a Bernuolli loss statistics if all marginal distributions of $\boldsymbol{L}$ are Bernuolli.

$$L_i \sim B(1; p_i), \text{ that is }, L_i = \begin{cases} 1 & \text{with probability } p_i \\ 0 & \text{with probability } 1 - p_i \end{cases}.$$

Absolute loss $L = \sum_{i=1}^{m} L_i$, percentage loss $= L/m$.

Easier cases: uniform default probability p and lack of dependency between counterparties

$$L_i \sim B(1; p) \quad \text{and} \quad (L_i)_{i=1,\cdots,m} \text{ independent.}$$

The absolute portfolio loss L is a sum of iid Bernuolli variables and so follows a binomial distribution with parameters m and p; $L \sim B(m; p)$.

Suppose the counterparties admit different default probabilities

$$L_i \sim B(1; p_i) \quad \text{and} \quad (L_i)_{i=1,\cdots,m} \text{ independent}$$

then

$$E[L] = \sum_{i=1}^{m} p_i \quad \text{and} \quad \text{var}(L) = \sum_{i=1}^{m} p_i(1 - p_i).$$

Next, we think of the loss probabilities as random variables

$$\boldsymbol{P} = (p_1 \cdots p_m)^T \sim \boldsymbol{F}$$

for some distribution function $\boldsymbol{F}$ with support in $[0,1]^m$. Conditional on a realization $\boldsymbol{p} = (p_1 \cdots p_m)$ of $\boldsymbol{P}$, the variables $L_1, \cdots, L_m$ are independent. The *conditional independence* of the losses can be expressed as

$$L_i | P_i = p_i \sim B(1; p_i), (L_i|_{\boldsymbol{P}=\boldsymbol{p}})_{i=1,\cdots,m} \text{ independent.}$$

The conditional joint distribution of the L_i's is

$$P[L_1 = \ell_1. \cdots, L_m = \ell_m] = \int_{[0,1]^m} \prod_{i=1}^{m} p_i^{\ell_i}(1-p_i)^{1-\ell_i}\, d\boldsymbol{F}(p_1, \cdots, p_m),$$

where $\ell_i \in \{0, 1\}$. Obviously, $E[L_i] = E[P_i]$. From

$$\begin{aligned} \text{var}(L_1) &= \text{var}(E[L_i|\boldsymbol{p}]) + E[\text{var}(L_i|\boldsymbol{p})] \\ &= \text{var}(P_i) + E[P_i(1-P_i)] = E[P_i](1 - E[P_i]). \end{aligned}$$

The covariance between single losses

$$\text{cov}(L_1, L_j) = E[L_i L_j] - E[L_i]E[L_j] = \text{cov}(P_i, P_j).$$

Default correlation in a Bernuolli mixture model is

$$\text{corr}(L_i, L_j) = \frac{\text{cov}(P_i, P_j)}{\sqrt{E[P_i](1 - E[P_i])}\sqrt{E[P_j](1 - E[P_j])}}$$

Uniform default probability and uniform correlation

For portfolios where all exposures are of approximately the same size and type in terms of risk, it makes sense to assume a uniform default probability and a uniform correlation among transactions in the portfolio.

$$P[L_1 = \ell_1, \cdots, L_m = \ell_m] = \int_0^1 p^k (1-p)^{m-k} \, dF(p)$$

where $k = \sum_{i=1}^{m} \ell_i$ and $\ell_i \in \{0, 1\}$.

The probability of exactly k defaults is

$$P[L = k] = {}_mC_k \int_0^1 p^k (1-p)^{m-k}\, dF(p).$$

The uniform default probability

$$\overline{p} = P[L_i = 1] = E[L_i] = \int_0^1 p\, dF(p).$$

Uniform default correlation

$$\begin{aligned} \mathrm{corr}(L_i, L_j) &= \frac{P[L_i = 1, L_j = 1] - \overline{p}^2}{\overline{p}(1-\overline{p})} = \frac{\int_0^1 p^2\, dF(p) - \overline{p}^2}{\overline{p}(1-\overline{p})} \\ &= \frac{\mathrm{var}(P)}{\overline{p}(1-\overline{p})}, \quad P \sim F. \end{aligned}$$

Coherent risk measures

Denote by $L^\infty = L^\infty(\Omega, \mathcal{F}, P)$ the space of bounded real random variables, defined on a probability space $(\Omega, \mathcal{F}, P)$. A mapping $\gamma : L^\infty(\Omega, \mathcal{F}, P) \to \mathbb{R}$ is called a coherent measure if the following properties hold:

(i) subadditivity

$$\forall X, Y \in L^\infty, \quad \gamma(X+Y) \leq \gamma(X) + \gamma(Y)$$

(ii) monotonicity

$$\forall X, Y \in L^\infty \text{ with } X \leq Y, \quad \gamma(X) \leq \gamma(Y)$$

(iii) positive homogeneity

$$\forall \lambda > 0, \forall X \in L^\infty, \quad \gamma(\lambda X) = \lambda \gamma(X)$$

(iv) translation risk invariance

$$\forall_x \in \mathbb{R}, \forall X \in L^\infty, \quad \gamma(X + x) = \gamma(X) + x.$$

This would imply $\gamma(X - \gamma(X)) = 0$ for every loss $X \in L^\infty$.

Expected Shortfall

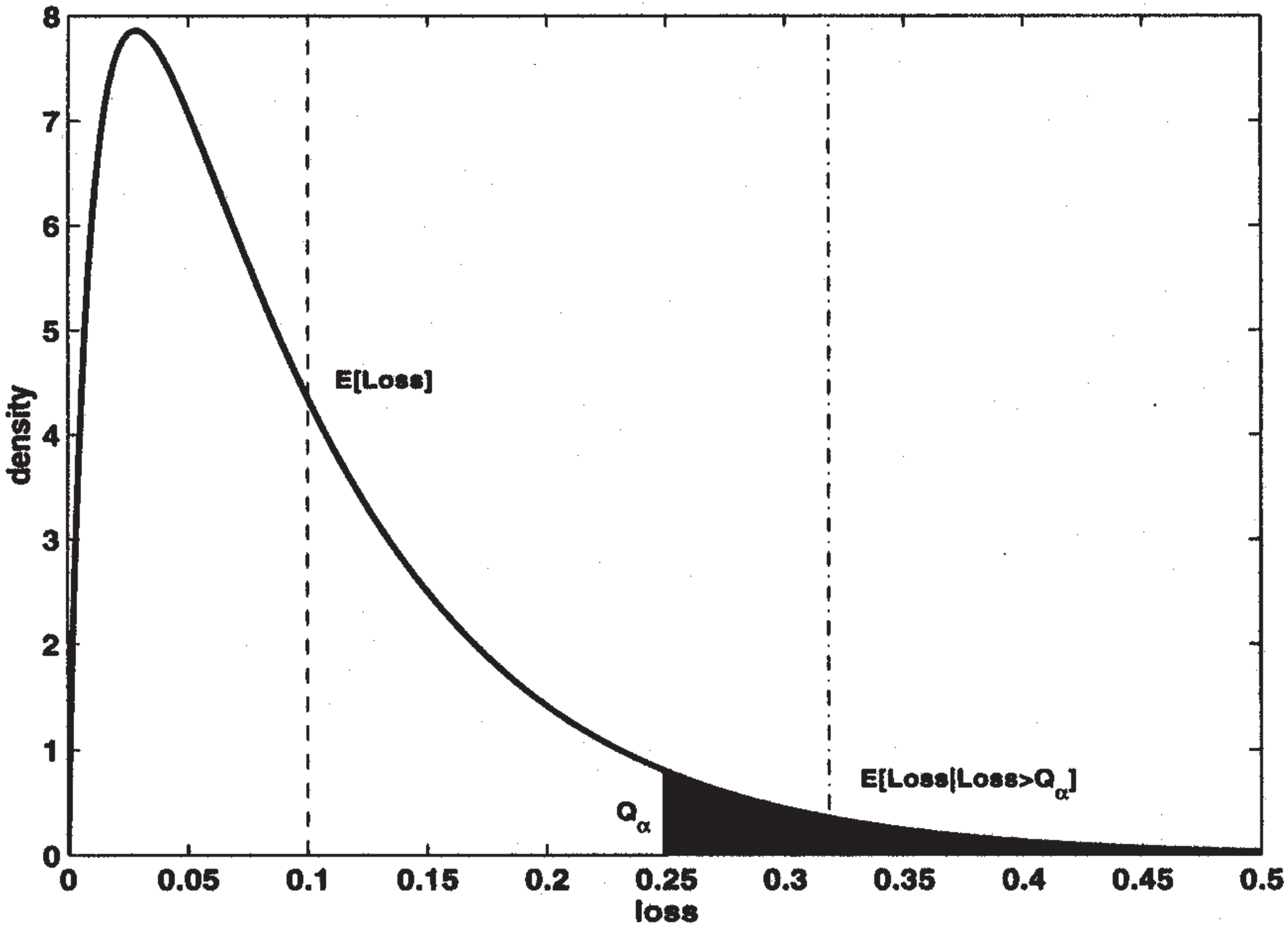
The *tail conditional expectation*, and *expected shortfall*, w.r.t. a confidence level α is defined as

$$\mathsf{TCE}_\alpha(X) = \mathbb{E}[X|X \geq \mathsf{VaR}_\alpha(X)].$$

From an insurance point of view, expected shortfall is a very reasonable measure: Defining by $c = \mathsf{VaR}_\alpha(X)$ a *critical loss threshold* corresponding to some confidence level α, expected shortfall capital provides a cushion against the mean value of losses exceeding the critical threshold c. TCE focusses on the expected loss in the tail, starting at c, of the portfolio's loss distribution.

Economic capital based on shortfall risk

$$EC_{TCE}(c) = E[X|X \geq c] - E[X].$$



Expected Shortfall $\mathbb{E}[X \mid X \geq \mathrm{VaR}_\alpha(X)]$.