CHAPTER 5
American Options

The distinctive feature of an American option is its early exercise privilege, that is, the holder can exercise the option prior to the date of expiration. Since the additional right should not be worthless, we expect an American option to be worth more than its European counterpart. The extra premium is called the *early exercise premium*.

First, we would like to recall some of the pricing properties of American options discussed in Sec. 1.2. The early exercise of either an American call or American put leads to the loss of insurance value associated with holding of the option. For an American call, the holder gains on the dividend yield from the asset but loses on the time value of the strike price. There is no advantage to exercise an American call prematurely when the asset received upon early exercise does not pay dividends. In this case, the American call has the same value as that of its European counterpart. By dominance argument, we have shown that an American option must be worth at least its corresponding intrinsic value, namely, $\max(S - X, 0)$ for a call and $\max(X - S, 0)$ for a put, where S and X are the asset price and strike price, respectively. While put-call parity relation exists for European options, we can only obtain lower and upper bounds on the difference of American call and put option values.

When the underlying asset is dividend paying, it may become optimal for the holder to exercise prematurely an American call option when the asset price S rises to some critical asset value, called the *optimal exercise price*. Since the loss of insurance value and time value of the strike price is time dependent, the optimal exercise price depends on time to expiry. For a longer-lived American call option, the optimal exercise price should assume a higher value so that larger dividends are received to compensate for the greater loss on time value of strike. When the underlying asset pays continuous dividend yield, the collection of these optimal exercise prices for all times constitutes a continuous curve, which is commonly called the *optimal exercise boundary*. For an American put option, the early exercise leads to some gain on time value of strike. Therefore, when the riskless interest rate is positive, there always exists an optimal exercise price below which it becomes optimal to exercise the American put prematurely.

The optimal exercise boundary of an American option is not known in advance but has to be determined as part of the solution process of the pricing model. Since the boundary of the domain of an American option model is a free boundary, the valuation problem constitutes a *free boundary value problem.* In Sec. 5.1, we present the characterization of the optimal exercise boundary at infinite time to expiry and at the moment immediately prior to expiry. The optimality condition in the form of smooth pasting of the option value curve with the intrinsic value line is derived. When the underlying asset pays discrete dividends, the early exercise of the American call may become optimal only at time right before a dividend date. Since the early exercise policy becomes relatively simple, we manage to derive closed form price formulas for American calls on an asset that pays discrete dividends. We also discuss the optimal exercise policy of American put options on a discrete dividend paying asset.

In Sec. 5.2, we present two pricing formulations of American options, namely, the linear complementarity formulaton and the optimal stopping formulation. We show how the early exercise premium can be expressed in terms of the exercise boundary in the form of an integral and examine how the determination of the optimal exercise boundary is resorted to the solution of an integral equation. The early exercise premium can be interpreted as the compensation paid to the holder when the early exercise right is forfeited. The early exercise feature can be combined with other path dependent features in an option contract. We examine the impact of the barrier feature on the early exercise policies of American barrier options. Also, we obtain the analytic price formula for the Russian option, which is essentially a perpetual American lookback option.

In general, analytic price formulas are not available for American options, except for a few special types. In Sec. 5.3, we present several analytic approximation methods for estimating the price of an American option. One approximation approach is to limit the exercise right such that the American option is exercisable only at a finite number of time instants. The other method is the solution of the integral equation of the exercise boundary by a recursive integraton method. The third method, called the quadratic approximation approach, is based on the reduction of the Black-Scholes equation to an ordinary differential equation whose domain boundary is determined by maximizing the value of the option.

The modeling of a financial derivative with voluntary right on resetting certain terms in the contract, like resetting the strike price to the prevailing asset price, also constitutes a free boundary value problem. In Sec. 5.4, we construct the pricing model for the reset-strike put option and examine the optimal reset strategy adopted by the option holder. Unlike the American early exercise right, the right to reset may not be limited to only one time. We also examine the pricing behaviors of multi-reset put options. Interestingly, when the right to reset is allowed to be infinitely often, the multi-reset put option becomes a European lookback option.

5.1 Characterization of the optimal exercise boundaries

The characteristics of the optimal early exercise policies of American options depend critically on whether the underlying asset is non-dividend paying or dividend paying (discrete or continuous). Throughout our discussion, we assume that the dividends are known in advance, both in amount and time of payment. In this section, we would like to give some detailed quantitative analysis of the properties of the early exercise boundary. We show that the optimal exercise boundary of an American put, with continuous dividend yield or zero dividend, is a *continuous decreasing* function of time of expiry τ. However, the optimal exercise boundary for an American put on an asset which pays discrete dividends may or may not have *jumps of discontinuity*, depending on the size of the discrete dividend payments. For an American call on an asset which pays a continuous dividend yield, we explain why it becomes optimal to exercise the call at sufficiently high value of S. The corresponding optimal exercise boundary is a *continuous increasing* function of τ. When the underlying asset of an American call pays discrete dividends, optimal early exercise of the American call may occur only at those times immediately before the asset goes ex-dividend. Additional conditions required for optimal early exercise include (i) the discrete dividend is sufficiently large relative to the strike price, (ii) the ex-dividend date is fairly close to expiry and (iii) the asset price level prior to the dividend date is higher than some threshold value. Since exercise possibilities are limited to a few discrete dividend dates, the price formula for an American call on an asset paying known discrete dividends can be obtained by relating the American call option to a European compound option.

Besides the value matching condition of the American option value across the optimal exercise boundary, the delta of the option value are also continuous across the boundary. This *smooth pasting condition* is a result derived from maximizing the American option value among all possible early exercise policies (see Sec. 5.1.2).

5.1.1 American options on an asset paying dividend yield

First, we consider the effects of continuous dividend yield (at the constant yield $q > 0$) on the early exercise policy of an American call. When the asset value S is exceedingly high, it is almost certain that the European call option on a continuous dividend paying asset will be in-the-money at expiry. Its value then behaves almost like the asset but without its dividend income minus the present value of the strike price X. When the call is sufficiently deep in-the-money, by observing that

$$N(\hat{d}_1) \sim 1 \quad \text{and} \quad N(\hat{d}_2) \sim 1$$

in the European call price formula (3.4.7a), we obtain

$$c(S,\tau) \sim e^{-q\tau}S - e^{-r\tau}X \qquad \text{when} \quad S \gg X. \tag{5.1.1}$$

The price of this European call may be below the intrinsic value $S - X$ at a sufficiently high asset value, due to the presence of the factor $e^{-q\tau}$ in front of S. While it is possible that the value of a European option stays below its intrinsic value, the holder of an American option with embedded early exercise right would not allow the value of his option to become lower than the intrinsic value. Hence, at a sufficiently high asset value, it becomes optimal for the American option on a continuous dividend paying asset to be exercised prior to expiry, avoiding its value to drop below the intrinsic value if unexercised.

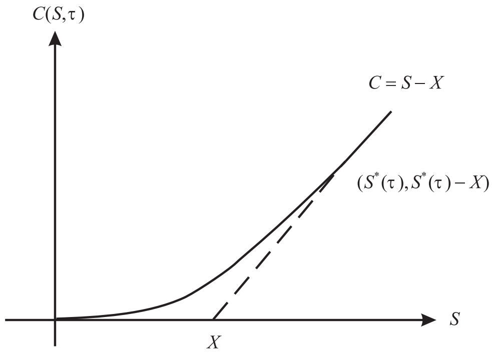


Fig. 5.1 The solid curve shows the price function $C(S,\tau)$ of an American call on an asset paying continuous dividend yield. The price curve touches the dotted intrinsic value line tangentially at the point $(S^*(\tau), S^*(\tau) - X)$, where $S^*(\tau)$ is the optimal exercise price. When $S \geq S^*(\tau)$, the American call value becomes $S - X$.

In Fig. 5.1, the American call option price curve $C(S,\tau)$ touches *tangentially* the dotted line representing the intrinsic value of the call at some optimal exericse price $S^*(\tau)$. Note that the optimal exericse price has dependence on τ, the time to expiry. The tangency behavior of the American price curve at $S^*(\tau)$ (continuity of delta value) will be explained in the next subsection. When $S \geq S^*(\tau)$, the American call value is equal to its intrinsic value $S - X$. The collection of all these points $(S^*(\tau), \tau)$, for all $\tau \in (0, T]$, in the (S,τ)-plane constitutes the optimal exercise boundary. The American call option remains alive only within the *continuation region* $\{(S,\tau) : 0 \leq S < S^*(\tau), 0 < \tau \leq T\}$. The complement is called the *stopping region*, inside which the American call should be optimally exercised (see Fig. 5.2).

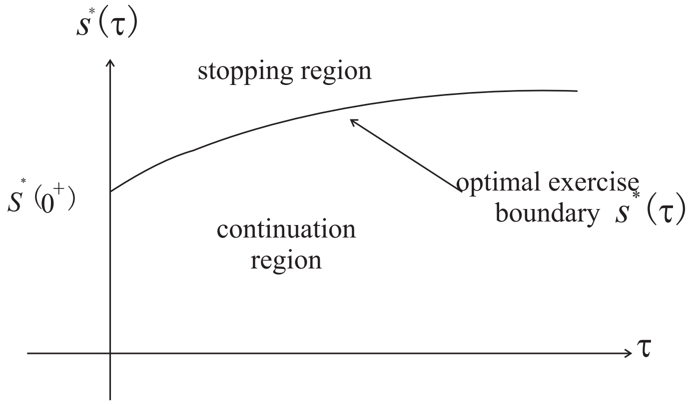


Fig. 5.2 An American call on an asset paying continuous dividend yield remains alive inside the continuation region $\{(S,\tau): S \in [0, S^*(\tau)), \tau \in (0,T]\}$. The optimal exercise boundary $S^*(\tau)$ is a continuous increasing function of τ.

Under the assumption of continuity of the asset price path and dividend yield, we expect that the optimal exercise boundary should also be a continuous function of τ, for $\tau > 0$. While a rigorous proof of the continuity of $S^*(\tau)$ is rather technical, a heuristic argument is provided below. Assume the contrary, suppose $S^*(\tau)$ has a downward jump as τ decreases across the time instant $\widehat{\tau}$. Assume that the asset price S at $\widehat{\tau}$ satisfies $S^*(\widehat{\tau}^-) < S < S^*(\widehat{\tau}^+)$, the American call option value is strictly above the intrinsic value $S - X$ at $\widehat{\tau}^+$ since $S < S^*(\widehat{\tau}^+)$ and becomes equal to the intrinsic value $S - X$ at $\widehat{\tau}^-$ since $S > S^*(\widehat{\tau}^-)$. The discrete downward jump in option value across $\widehat{\tau}$ would lead to an arbitrage opportunity.

5.1.2 Smooth pasting condition

We would like to examine the smooth pasting condition (tangency condition) along the optimal exercise boundary for an American call on a continuous dividend paying asset. At $S = S^*(\tau)$, the value of the exercised American call is $S^*(\tau) - X$. This is termed value matching condition:

$$C(S^*(\tau), \tau) = S^*(\tau) - X. \tag{5.1.2}$$

Suppose $S^*(\tau)$ were a known continuous function, the pricing model becomes a boundary value problem with a time dependent boundary. However, in the American call option model, $S^*(\tau)$ is not known in advance. Rather, it must be determined as part of the solution. An additional auxiliary condition has to be prescribed along $S^*(\tau)$ so as to reflect the nature of optimality of the exercise right embedded in the American option.

We follow Merton's (1973; Chap. 1) argument to show the continuity of the delta of option value of an American call at the optimal exercise price $S^*(\tau)$. Let $f(S,\tau;b(\tau))$ denote the solution to the Black-Scholes equation in the domain $\{(S,\tau)\colon\ S \in (0,b(\tau)), \tau \in (0,T]\}$, where $b(\tau)$ is a known boundary. The holder of the American call chooses an early exercise policy which maximizes the value of the call. Using such argument, the American call value is given by

$$C(S,\tau) = \max_{\{b(\tau)\}} f(S,\tau;b(\tau)) \tag{5.1.3}$$

for all possible continuous functions $b(\tau)$. For fixed τ, for convenience, we write $f(S,\tau;\, b(\tau))$ as $F(S,b)$, where $0 \le S \le b$. It is observed that $F(S,b)$ is a differentiable function, concave in its second argument. Further, we write $h(b) = F(b,b)$ which is assumed to be a differentiable function of b. For usual American call option, $h(b) = b - X$. The total derivative of F with respect to b along the boundary $S = b$ is given by

$$\frac{dF}{db} = \frac{dh}{db} = \frac{\partial F}{\partial S}(S,b)\bigg|_{S=b} + \frac{\partial F}{\partial b}(S,b)\bigg|_{S=b}, \tag{5.1.4}$$

where the property $\dfrac{\partial S}{\partial b} = 1$ along $S = b$ has been incorporated. Let b^* be the critical value of b which maximizes F. When $b = b^*$, we have $\dfrac{\partial F}{\partial b}(S,b^*) = 0$ as the first derivative condition at a maximum point. On the other hand, from the exercise payoff function of the American call option, we have

$$\frac{dh}{db}\bigg|_{b=b^*} = \frac{d}{db}(b - X)\bigg|_{b=b^*} = 1. \tag{5.1.5}$$

Putting the results together, we obtain

$$\frac{\partial F}{\partial S}(S,b^*)\bigg|_{S=b^*} = 1. \tag{5.1.6}$$

Note that the optimal choice $b^*(\tau)$ is just the optimal exercise price $S^*(\tau)$. The above condition can be expressed in an alternative form as

$$\frac{\partial C}{\partial S}(S^*(\tau),\tau) = 1. \tag{5.1.7}$$

Condition (5.1.7) is commonly called the *smooth pasting* or *tangency condition*. The two conditions (5.1.2) and (5.1.7), respectively, reveal that $C(S,\tau)$ and $\dfrac{\partial C}{\partial S}(S,\tau)$ are continuous across the optimal exercise boundary (see Fig. 5.1).

The smooth pasting condition is applicable to all types of American put options. For an American put option, the slope of the intrinsic value line is -1. The continuity of the delta of the American put value at $S = S^*(\tau)$ gives

$$\frac{\partial P}{\partial S}(S^*(\tau), \tau) = -1. \tag{5.1.8}$$

An alternative proof of the above smooth pasting condition is outlined in Problem 5.5.

5.1.3 Optimal exercise boundary for an American call

Consider an American call on a continuous dividend paying asset, the optimal exercise boundary $S^*(\tau)$ is a continuous increasing function of τ. The increasing property stems from the fact that the loss of time value of strike is more significant for a longer-lived American call so that the call must be deeper-in-the-money in order to induce early exercise decision. In addition, the compensation from the dividend received from the asset is higher whereas the loss of insurance value associated with holding of the call option becomes lower (chance of expiring out-of-the-money becomes lower). Hence, the American call should be exercised at a higher optimal exericse price $S^*(\tau)$ compared to its shorter-lived counterpart.

The increasing property of $S^*(\tau)$ can also be explained by relating to the increasing property of the price curve $C(S, \tau)$ as a function of τ [see Eq. (1.2.5a)]. The option price curve of a longer-lived American call plotted against S always stays above that of its shorter-lived counterpart. The upper price curve corresponding to the longer-lived option cuts the intrinsic value line tangentially at a higher critical asset value $S^*(\tau)$.

Moreover, it is obvious from Fig. 5.1 that the price curve of an American call always cuts the intrinsic value line at a critical asset value greater than X. Hence, we have $S^*(\tau) \geq X$ for $\tau \geq 0$. Alternatively, assume the contrary, suppose $S^*(\tau) < X$, then the early exercise proceed $S^*(\tau) - X$ becomes negative. Since the early exercise privilege cannot be a liability, the possibility $S^*(\tau) < X$ is ruled out and so $S^*(\tau) \geq X$.

Next, we present the analysis of the asymptotic behaviors of $S^*(\tau)$ at $\tau \to 0^+$ and $\tau \to \infty$.

Asymptotic behavior of $S^*(\tau)$ close to expiry

When $\tau \to 0^+$ and $S > X$, by the continuity of the call price function, the call value tends to the terminal payoff value so that $C(S, 0^+) = S - X$. If the American call is alive, then the call value satisfies the Black-Scholes equation. By substituting the above call value into the Black-Scholes equation, given that (S, τ) lies in the continuation region, we have

$$\begin{aligned} \left.\frac{\partial C}{\partial \tau}\right|_{\tau=0^+} &= \frac{\sigma^2}{2} S^2 \left.\frac{\partial^2 C}{\partial S^2}\right|_{\tau=0^+} + (r-q)S \left.\frac{\partial C}{\partial S}\right|_{\tau=0^+} - rC\Big|_{\tau=0^+} \\ &= (r-q)S - r(S-X) = rX - qS. \end{aligned} \tag{5.1.9}$$

Suppose $\frac{\partial C}{\partial \tau}(S, 0^+) < 0$, $C(S, \tau)$ becomes less than $C(S, 0) = S - X$ (intrinsic value of the American call) immediately prior to expiry. This leads to a contradiction since the American call value is always above the intrinsic value. Therefore, we must have $\frac{\partial C}{\partial \tau}(S, 0^+) \geq 0$ in order that the American call is kept alive until the time close to expiry. The value of S at which $\frac{\partial C}{\partial \tau}(S, 0^+)$ changes sign is $S = \frac{r}{q}X$. Also, $\frac{r}{q}X$ lies in the interval $S > X$ only when $q < r$. We consider the two separate cases, $q < r$ and $q \geq r$.

1. $q < r$
   At time immediately prior to expiry, we argue that the American call will be kept alive when $S < \frac{r}{q}X$. This is because within a short time interval δt prior to expiry, the dividend $qS\delta t$ earned from holding the asset is less than the interest $rX\delta t$ earned from depositing the amount X in a bank at the riskless interest rate r. The above observation is consistent with positivity of $\frac{\partial C}{\partial \tau}(S, 0^+)$ when $S < \frac{r}{q}X$. When $S > \frac{r}{q}X$, the American call should be exercised since the negativity of $\frac{\partial C}{\partial \tau}(S, 0^+)$ would lead to the violation of the condition that the American call value must be above the intrinsic value $S - X$. Hence, for $q < r$, the optimal exercise price $S^*(0^+)$ is given by the asset value at which $\frac{\partial C}{\partial \tau}(S, 0^+)$ changes sign. We then obtain
   $$S^*(0^+) = \frac{r}{q}X. \qquad (5.1.10a)$$
   In particular, when $q = 0$, $S^*(0^+)$ becomes infinite. Furthermore, since $S^*(\tau)$ is known to be a monotonically increasing function of τ, we then deduce that $S^*(\tau) \to \infty$ for all values of τ. This result is consistent with the well known fact that it is always non-optimal to exercise an American call on a non-dividend paying asset prior to expiry.

2. $q \geq r$
   When $q \geq r$, $\frac{r}{q}X$ becomes less than X and so the above argument has to be modified. First, we show that $S^*(0^+)$ cannot be greater than X. Assume the contrary, suppose $S^*(0^+) > X$ so that the American call is still alive when $X < S < S^*(0^+)$ at time close to expiry. Given the combined conditions: $q \geq r$ and $S > X$, it is observed that the loss in dividend amount $qS\delta t$ not earned is more than the interest amount $rX\delta t$ earned if the American call is not exercised within a short time interval δt prior to expiry. This represents a non-optimal early exercise policy. Hence, we must have $S^*(0^+) \leq X$. Together with the properties that $S^*(\tau) \geq X$ for $\tau > 0$ and $S^*(\tau)$ is a continuous increasing function of τ, we deduce that for $q \geq r$,
   $$S^*(0^+) = X. \qquad (5.1.10b)$$

In summary, the optimal exercise price $S^*(\tau)$ of an American call on a continuous dividend paying asset at time close to expiry is given by

$$\lim_{\tau \to 0^+} S^*(\tau) = \begin{cases} \frac{r}{q}X & q < r \\ X & q \geq r \end{cases} = X \max\left(1, \frac{r}{q}\right). \tag{5.1.11}$$

At expiry $\tau = 0$, the American call option will be exercised whenever $S \geq X$ and so $S^*(0) = X$. Hence, for $q < r$, there is a jump of discontinuity of $S^*(\tau)$ at $\tau = 0$.

Asymptotic behavior of $S^*(\tau)$ at infinite time to expiry

Since $S^*(\tau)$ is a monotonic increasing function of τ, the lower bound for the optimal exercise boundary $S^*(\tau)$ for $\tau > 0$ is given by $\lim\limits_{\tau \to 0^+} S^*(\tau)$. It would be interesting to explore whether $\lim\limits_{\tau \to \infty} S^*(\tau)$ has a finite bound or otherwise. An option with infinite time to expiration is called a *perpetual option.* The determination of $\lim\limits_{\tau \to \infty} S^*(\tau)$ is related to the analysis of the price function of corresponding perpetual American option.

Let $C_\infty(S; X, q)$ denote the price of an American perpetual call option with strike price X and on an asset which pays a continuous dividend yield q. The value of a perpetual option is seen to be insensitive to temporal rate of change so that the Black-Scholes equation is reduced to the following ordinary differential equation

$$\frac{\sigma^2}{2} S^2 \frac{d^2 C_\infty}{dS^2} + (r - q)S \frac{dC_\infty}{dS} - rC_\infty = 0, \quad 0 < S < S^*_\infty, \tag{5.1.12a}$$

where S^*_∞ is the optimal exercise price at which the perpetual American call option should be exercised. Note that S^*_∞ is independent of τ and it is simply the asymptotic value $\lim\limits_{\tau \to \infty} S^*(\tau)$. The boundary conditions for the pricing model of the perpetual American call are

$$C_\infty(0) = 0 \quad \text{and} \quad C_\infty(S^*_\infty) = S^*_\infty - X. \tag{5.1.12b}$$

We let $f(S; S^*_\infty)$ denote the solution to Eqs. (5.1.12a,b) for a given value of S^*_∞. Since Eq. (5.1.12a) is a linear equi-dimensional ordinary differential equation, its general solution is of the form

$$f(S; S^*_\infty) = c_1 S^{\mu_+} + c_2 S^{\mu_-}, \tag{5.1.13}$$

where c_1 and c_2 are arbitrary constants, μ_+ and μ_- are the respective positive and negative roots of the auxiliary equation:

$$\frac{\sigma^2}{2} \mu^2 + (r - q - \frac{\sigma^2}{2})\mu - r = 0. \tag{5.1.14}$$

Since $f(0; S^*_\infty) = 0$, we must have $c_2 = 0$. Applying the boundary condition at S^*_∞, we have

$$f(S^*_\infty; S^*_\infty) = c_1 S^{*\mu_+}_\infty = S^*_\infty - X, \tag{5.1.15}$$

thus giving

$$c_1 = \frac{S^*_\infty - X}{S^{*\mu_+}_\infty}. \tag{5.1.16}$$

The solution $f(S; S^*_\infty)$ is now reduced to the form

$$f(S; S^*_\infty) = (S^*_\infty - X) \left(\frac{S}{S^*_\infty} \right)^{\mu_+}, \tag{5.1.17a}$$

where

$$\mu_+ = \frac{-(r - q - \frac{\sigma^2}{2}) + \sqrt{(r - q - \frac{\sigma^2}{2})^2 + 2\sigma^2 r}}{\sigma^2} > 0. \tag{5.1.17b}$$

To complete the solution, S^*_∞ has yet to be determined. We find S^*_∞ by maximizing the value of the perpetual American call option among all possible optimal exercise prices, that is,

$$C_\infty(S; X, q) = \max_{\{S^*_\infty\}} \left\{ (S^*_\infty - X) \left(\frac{S}{S^*_\infty} \right)^{\mu_+} \right\}. \tag{5.1.18}$$

The use of calculus shows that $f(S; S^*_\infty)$ is maximized when

$$S^*_\infty = \frac{\mu_+}{\mu_+ - 1} X. \tag{5.1.19}$$

Suppose we write $S^*_{\infty,C} = \dfrac{\mu_+}{\mu_+ - 1} X$, then the value of the perpetual American call takes the form

$$C_\infty(S; X, q) = \left(\frac{S^*_{\infty,C}}{\mu_+} \right) \left(\frac{S}{S^*_{\infty,C}} \right)^{\mu_+}. \tag{5.1.20}$$

It can be easily verified that the above solution also satisfies the smooth pasting condition:

$$\left. \frac{dC_\infty}{dS} \right|_{S = S^*_{\infty,C}} = 1. \tag{5.1.21}$$

One may solve for $S^*_{\infty,C}$ by applying the smooth pasting condition directly without going through the above maximization procedure. Indeed, the application of the smooth pasting condition implicity incorporates the procedure of taking the maximum of the option values among all possible choices of $S^*_{\infty,C}$.

5.1.4 Put-call symmetry relations

The behaviors of the optimal exercise boundary for an American put option on a continuous dividend paying asset can be inferred from those of the call counterpart once the put-call symmetry relations between their price functions and optimal exercise prices are established. The plot of the price function $P(S, \tau)$ of an American put against S is shown in Fig. 5.3.

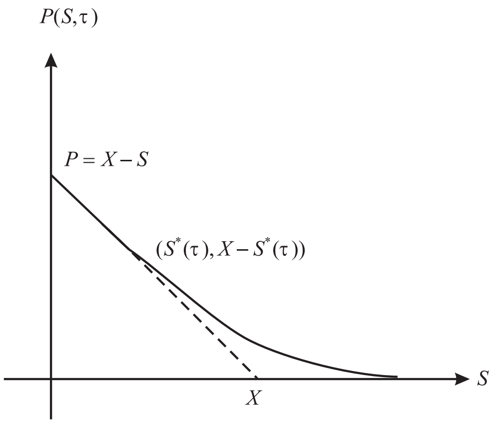


Fig. 5.3 The solid curve shows the price function of an American put at a given time to expiry τ. The price curve touches the dotted intrinsic value line tangentially at the point $(S^*(\tau), X - S^*(\tau))$, where $S^*(\tau)$ is the optimal exercise price. When $S \leq S^*(\tau)$, the American put value becomes $X - S$.

We may consider an American call option as providing the right to exchange X dollars of cash for one unit of stock which is worth S dollars at any time during the option's life. If we take asset one to be the stock, asset two to be the cash, then asset one and asset two have dividend yield q and r, respectively. The above call option can be considered as an exchange option which exchanges asset two for asset one. Similarly, we may consider an American put option as providing the right to exchange one unit of stock which is worth S dollars for X dollars of cash at any time. What would happen if we interchange the role of stock and cash in the American put option? Now, this new American put can be considered to be equivalent to the usual American call since both options confer the same right of exchanging cash for stock to their holders. If we use $P(S, \tau; X, r, q)$ to denote the price function of the usual American put, then the price function of the modified American put (after interchanging the role of stock and cash) is given by $P(X, \tau; S, q, r)$, where S

and X are interchanged and so do r and q. Since the modified American put is equivalent to the usual American call, we then have

$$C(S,\tau;X,r,q) = P(X,\tau;S,q,r). \tag{5.1.22}$$

This symmetry between the price functions of American call and put is called the *put-call symmetry relation*.

Next, we would like to establish the put-call symmetry relation for the optimal exercise prices for American put and call options. Let $S_P^*(\tau;r,q)$ and $S_C^*(\tau;r,q)$ denote the optimal exercise boundary for the American put and call options on a continuous dividend paying stock, respectively. When $S = S_C^*(\tau;r,q)$, the call owner is willing to exchange X dollars of cash for one unit of stock which is worth S_C^* dollars or *one dollar* of cash for $\frac{1}{X}$ units of stock which is worth $\dfrac{S_C^*}{X}$ dollars. Similarly, when $S = S_P^*(\tau;r,q)$, the put owner is willing to exchange $\dfrac{1}{S_P^*}$ units of stock which is worth *one dollar* for $\dfrac{X}{S_P^*}$ dollars of cash. If both of these American call and put options can be considered as exchange options and the roles of cash and stock are interchangeable, then the corresponding put-call symmetry relation for the optimal exercise prices is deduced to be

$$S_C^*(\tau;r,q) = \frac{X^2}{S_P^*(\tau;q,r)}. \tag{5.1.23}$$

A mathematical proof of symmetry relation (5.1.22) can be established quite easily (see Problem 5.7). Indeed, more complicated symmetry relations between the price functions of American call and put options can be derived (see Problems 5.8–5.9).

Behavior of $S_P^*(\tau)$ near expiry

From Eq. (5.1.23) and the monotonically increasing property of $S_C^*(\tau)$, we can deduce that $S_P^*(\tau)$ is a monotonically decreasing function of τ. Since Eq. (5.1.23) remains valid as $\tau \to 0^+$, the lower bound for $S_P^*(\tau)$ is given by

$$\lim_{\tau\to 0^+} S_P^*(\tau;r,q) = \frac{X^2}{\displaystyle\lim_{\tau\to 0^+} S_C^*(\tau;q,r)} = \frac{X^2}{X\max\left(1,\frac{q}{r}\right)} = X\min\left(1,\frac{r}{q}\right). \tag{5.1.30}$$

From Eq. (5.1.30), we observe that when $q \le r$, we have $\displaystyle\lim_{\tau\to 0^+} S_P^*(\tau) = X$. Now, even when $q = 0$, $S_P^*(\tau)$ is non-zero since $S_P^*(\tau)$ is a continuous decreasing function of τ for $\tau > 0$ and its lower bound equals X. Hence, it is always optimal to exercise an American put even when the underlying asset pays no dividend. On the other hand, at zero interest rate, $\displaystyle\lim_{\tau\to 0^+} S_P^*(\tau)$ becomes zero. It then follows that $S_P^*(\tau) = 0$ for $\tau > 0$ since $S_P^*(\tau)$ is a decreasing function of τ. Therefore, it is never optimal to exercise an American put prematurely when the interest rate is zero. From financial intuition, such a conclusion is obvious since there is no time value gained on X from the early exercise of the American put when there is null interest.

The quest for more refined asymptotic behaviors of $S_P^*(\tau)$ when $\tau \to 0^+$ poses great mathematical challenges. Evans *et al*. (2002) show that at time close to expiry the optimal exercise boundary is parabolic when $q > r$ but it becomes parabolic-logarithmic when $q \leq r$. The asymptotic expansion of $S_P^*(\tau)$ as $\tau \to 0^+$ takes the following forms

(i) $0 \leq q < r$

$$S_P^*(\tau) \sim X - X\sigma\sqrt{\tau \ln\left(\frac{\sigma^2}{8\pi\tau(r-q)^2}\right)} \tag{5.1.31a}$$

(ii) $q = r$

$$S_P^*(\tau) \sim X - X\sigma\sqrt{2\tau \ln\left(\frac{1}{4\sqrt{\pi}q\tau}\right)} \tag{5.1.31b}$$

(iii) $q > r$

$$S_P^*(\tau) \sim \frac{r}{q}X(1 - \sigma\alpha\sqrt{2\tau}). \tag{5.1.31c}$$

Here, α is a numerical constant which satisfies the following transcendental equation

$$-\alpha^3 e^{\alpha^2}\int_\alpha^\infty e^{-u^2}\,du = \frac{1-2\alpha^2}{4}. \tag{5.1.31d}$$

Behavior of $S_P^*(\tau)$ at infinite time to expiry

Following a similar derivation procedure as that for the perpetual American call option, the price of the perpetual American put option can be deduced to be

$$P_\infty(S; X, q) = -\frac{S_{\infty,P}^*}{\mu_-}\left(\frac{S}{S_{\infty,P}^*}\right)^{\mu_-}. \tag{5.1.32}$$

Here, $S_{\infty,P}^*$ denotes the optimal exercise price at infinite time to expiry and its value is equal to

$$S_{\infty,P}^* = \frac{\mu_-}{\mu_- - 1}X, \tag{5.1.33a}$$

where

$$\mu_- = \frac{-(r-q-\frac{\sigma^2}{2}) - \sqrt{(r-q-\frac{\sigma^2}{2})^2 + 2\sigma^2 r}}{\sigma^2} < 0. \tag{5.1.33b}$$

One can verify easily that

$$S^*_{\infty,P}(r,q) = \frac{X^2}{S^*_{\infty,C}(q,r)}, \tag{5.1.34}$$

a result that is consistent with the relation given in Eq. (5.1.23).

5.1.5 American call options on an asset paying single dividend

It has been explained in Sec. 1.2 that when an asset pays discrete dividend payments, the asset price declines by the same amount as the dividend right after the dividend date if there are no other factors affecting the income proceeds. Empirical studies show that the relative decline of the stock price as a proportion of the amount of the dividend is shown to be not meaningfully different from one. For simplicity, we assume that the asset price falls by the same amount as the discrete dividend. An option is said to be *dividend protected* if the value of the option is invariant of the choice of the dividend policy. This is done by adjusting the strike price in relation to the dividend amount. Here, we consider the effects of discrete dividends on the early exercise policy of American options which are not protected against the dividend, that is, the strike price is not marked down (for calls) or marked up (for puts) by the same amount as the dividend.

Early exercise policies

Since the holder of an American call on an asset paying discrete dividends will not receive any dividends between successive dividend dates, it is never optimal to exercise the American call on any non-dividend paying date. For those times between dividend dates, the early exercise right is non-effective. If the American call were exercised at all, the possible choices of exercise times are those instants immediately before the asset goes ex-dividend. As a result, he owns the asset right before the asset goes ex-dividend and receives the dividend in the next instant. We explore the conditions under which the holder of such American call would optimally choose to exercise his option.

In the following discussion, it is more convenient to characterize the time dependence of the optimal exercise boundary using the calendar time t. We consider an American call on an asset which pays only one discrete dividend of deterministic amount D at the known dividend date t_d. The generalization to multi-dividend models can be found in Problems 5.15-17. Let S_d^- (S_d^+) denote the asset price at time t_d^- (t_d^+) which is immediately before (after) the discrete dividend date t_d. If the American call is exercised at t_d^-, the call value becomes $S_d^- - X$. Otherwise, the asset price drops to $S_d^+ = S_d^- - D$ right after the asset goes ex-dividend. Since there is no further discrete dividend after time t_d, the American price function behaves like that of its European counterpart for $t > t_d^+$. To preclude arbitrage opportunities, the call price function must be continuous across the ex-dividend instant since the holder of the call option does not receive any dividend payment on the dividend date (unlike holding the asset).

From Eq. (1.2.11), the lower bound of the American call value at t_d^+ is $S_d^+ - Xe^{-r(T-t_d^+)}$, where $T - t_d^+$ is the time to expiry. As far as time to expiry is concerned, the quantities $T-t_d, T-t_d^+$ and $T-t_d^-$ are considered equal. By virtue of the continuity of the call value across the dividend date, the lower bound for the call value at time t_d^- should also be equal to $S_d^+ - Xe^{-r(T-t_d)} = (S_d^- - D) - Xe^{-r(T-t_d)}$. Note that the lower bound for the call value at t_d^- is driven down by D in anticipation of the known discrete dividend amount D in the next instant. Now, it may occur that the lower bound value at t_d^- becomes less than the exercise payoff of $S_d^- - X$ when D is sufficiently large. We compare the following two quantities: exercise payoff $E = S_d^- - X$ and lower bound of the call value $B = (S_d^- - D) - Xe^{-r(T-t_d)}$. Suppose $E \leq B$, that is

$$S_d^- - X \leq (S_d^- - D) - Xe^{-r(T-t_d)} \quad \text{or} \quad D \leq X\,[1 - e^{-r(T-t_d)}], \quad (5.1.35)$$

then it is never optimal to exercise the American call. This is because at any value of asset price S_d^- the call is worth more when it is held than exercised. However, when the discrete dividend D is deep enough, in particular $D > X[1 - e^{-r(T-t_d)}]$, then it may become optimal to exercise at t_d^- when the asset price S_d^- is above some threshold value. This requirement on D gives one of the necessary conditions for the commencement of early exercise. The dividend amount D must be sufficiently deep to offset the loss in the time value of the strike price, where the loss is given by $X[1 - e^{-r(T-t_d)}]$.

Let $C_d(S,t)$ denote the price function of the one-dividend American call option with the calendar time t as the time variable. By virtue of the continuity property of the call value across the dividend date, we have

$$C_d(S_d^-, t_d^-) = c(S_d^- - D, t_d^+), \quad (5.1.36)$$

where $c(S_d^- - D, t_d^+)$ is the European call price given by the Black-Scholes formula with asset price $S_d^- - D$ and calendar time t_d^+. To better understand the decision of early exercise at t_d^-, we plot the call price function, the exercise payoff E (corresponds to line ℓ_1: $E = S_d^- - X$) and the lower bound value B (corresponds to line ℓ_2: $B = S_d^- - D - Xe^{-r(T-t_d)}$) versus the asset price S_d^- (see Fig. 5.4). The exercise payoff line l_1 lies to the left of the lower bound value line l_2 when $D > X[1 - e^{-r(T-t_d)}]$. Now, the call price curve may intersect (not tangentially) the exercise payoff line l_1 at some critical asset price S_d^*, which is given by the solution to the following algebraic equation

$$c(S_d^- - D, t_d) = S_d^- - X. \quad (5.1.37)$$

It can be shown mathematically that when $D \leq X[1 - e^{-r(T-t_d)}]$, there is no solution to Eq. (5.1.37), a result that is consistent with the necessary condition on D discussed earlier (see also Problem 5.13). When the discrete dividend is sufficiently deep such that $D > X[1 - e^{-r(T-t_d)}]$, the American call remains alive beyond the dividend date only if $S_d^- < S_d^*$. When S_d^- is at or above S_d^*, the call should be optimally exercised at t_d^-. Hence, the American call price at time t_d^- is given by

$$C_d(S_d^-, t_d^-) = \begin{cases} c(S_d^- - D, t_d^+) & \text{when} \quad S_d^- < S_d^* \\ S_d^- - X & \text{when} \quad S_d^- \geq S_d^*. \end{cases} \tag{5.1.38}$$

If the American call is not optimally exercised at t_d^-, then its value remains unchanged as time lapses across the dividend date. Note that S_d^* depends on D, which decreases in value when D increases (see Problem 5.13). This agrees with the financial intuition that the propensity of optimal early exercise becomes higher (corresponding to a lower value of S_d^*) with deeper discrete dividend payment.

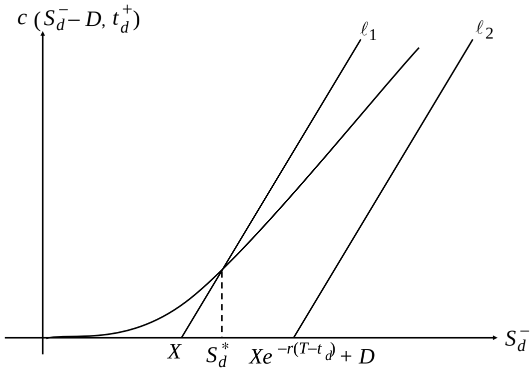


Fig. 5.4 The curve representing the European call price function $V = c(S_d^- - D, t_d^+)$ falls below the exercise payoff line $\ell_1 : E = S_d^- - X$ when ℓ_1 lies to the left of the lower bound value line $\ell_2 : B = S_d^- - D - Xe^{-r(T-t_d)}$. Here, S_d^* is the value of S_d^- at which the European call price curve cuts the exercise payoff line ℓ_1.

In summary, the holder of an American call option on an asset paying single discrete dividend will exercise the call optimally only at the instant immediately prior to the dividend date, provided that $S_d^- \geq S_d^*$, where S_d^* satisfies Eq. (5.1.37). Also, S_d^* exists only when $D > X[1 - e^{-r(T-t_d)}]$, implying that the dividend is sufficiently deep to offset the loss on time value of strike.

Analytic price formula for an one-dividend American call

Since the American call on an asset paying known discrete dividends will be exercised only at instants immediately prior to ex-dividend dates, the

American call can be replicated by a European compound option with the expiration dates of the compound options coinciding with the ex-dividend dates. Such a replication strategy makes possible the derivation of an analytic price formula for an American call on an asset paying discrete dividends.

If the whole asset price S follows the lognormal process, this would imply there exists some non-zero probability that the dividends cannot be paid since the asset price may fall below the dividend payment on a dividend date. The difficulty can be resolved if we modify the assumption on the diffusion process where the asset price net of the present value of the escrowed dividends, denoted by $\widetilde{S}$, follows the lognormal diffusion process. We call $\widetilde{S}$ to be the risky component of the asset price.

Suppose the asset pays single discrete dividend of amount D at time t_d, then the risky component of S is defined by

$$\widetilde{S} = \begin{cases} S & \text{for} \quad t_d^+ \le t \le T \\ S - De^{-r(t_d - t)} & \text{for} \quad t \le t_d^- . \end{cases} \tag{5.1.39}$$

Note that $\widetilde{S}$ is continuous across the dividend date. The Black-Scholes assumption on the asset price movement is modified such that under the risk neutral measure the risky component $\widetilde{S}$ follows the lognormal diffusion process

$$\frac{d\widetilde{S}}{\widetilde{S}} = r\,dt + \sigma\,dZ, \tag{5.1.40}$$

where σ is the volatility of the risky component of the asset price.

Now, we would like to derive the price formula for an American call option on an asset paying single discrete dividend D at time t_d, where $D > X[1 - e^{-r(T-t_d)}]$. Let $C_d(\widetilde{S}, t)$ denote the price of this one-dividend American call and $c(\widetilde{S}, t)$ denote the European call price given by the Black-Scholes formula, where t is the calendar time. Let $\widetilde{S}_d$ denote the risky component of the asset value on the ex-dividend date t_d. Let $\widetilde{S}_d^*$ denote the critical value of the risky component at $t = t_d$, above which it is optimal to exericse. This critical value $\widetilde{S}_d^*$ is the solution to the following equation [see Eq. (5.1.37)]

$$\widetilde{S}_d + D - X = c(\widetilde{S}_d, t_d). \tag{5.1.41}$$

The one-dividend American call option can be replicated by a European compound option with a zero strike price whose first expiration date coincides with the ex-dividend date t_d. The compound option pays at t_d either $\widetilde{S}_d + D - X$ if $\widetilde{S}_d \ge \widetilde{S}_d^*$ or a European call option with strike price X and time to expiry $T - t_d$ if $\widetilde{S}_d < \widetilde{S}_d^*$. Let $\psi(\widetilde{S}_d, \widetilde{S}; t_d, t)$ denote the transition density function under the risk neutral measure of $\widetilde{S}_d$ at time t_d, given the asset price $\widetilde{S}$ at an earlier time $t < t_d$. The one-dividend American call price at time t earlier than t_d is given by (Whaley, 1981)

$$C_d(\widetilde{S},t) = e^{-r(t_d-t)}\left[\int_{\widetilde{S}_d^*}^{\infty} [\widetilde{S}_d - (X-D)]\,\psi(\widetilde{S}_d,\widetilde{S};t_d,t)\,d\widetilde{S}_d \right.$$
$$\left. + \int_0^{\widetilde{S}_d^*} c(\widetilde{S}_d,t_d)\,\psi(\widetilde{S}_d,\widetilde{S};t_d,t)\,d\widetilde{S}_d\right], \quad t < t_d. \tag{5.1.42}$$

The first term may be interpreted as the price of a European call with two different strike prices. The strike price $\widetilde{S}_d^*$ determines the moneyness of the call option at expiry and the other strike price $X - D$ is the amount paid in exchange of the asset at expiry. The second term represents the price of a European put-on-call with strike price $\widetilde{S}_d^*$ at t_d and strike price X at T. The price formula for the one-dividend American call option is obtained as follows:

$$\begin{aligned}
&C_d(\widetilde{S},t) \\
&= \widetilde{S}N(a_1) - (X-D)e^{-r(t_d-t)}N(a_2) - Xe^{-r(T-t)}N_2\left(-a_2, b_2; -\sqrt{\frac{t_d-t}{T-t}}\right) \\
&\quad + \widetilde{S}N_2\left(-a_1, b_1; -\sqrt{\frac{t_d-t}{T-t}}\right) \\
&= \widetilde{S}\left[1 - N_2\left(-a_1, -b_1; \sqrt{\frac{t_d-t}{T-t}}\right)\right] + De^{-r(t_d-t)}N(a_2) \\
&\quad - X\left[e^{-r(t_d-t)}N(a_2) + e^{-r(T-t)}N_2\left(-a_2, b_2; -\sqrt{\frac{t_d-t}{T-t}}\right)\right],
\end{aligned} \tag{5.1.43a}$$

where

$$a_1 = \frac{\ln\frac{\widetilde{S}}{\widetilde{S}_d^*} + (r + \frac{\sigma^2}{2})(t_d - t)}{\sigma\sqrt{t_d - t}}, \qquad a_2 = a_1 - \sigma\sqrt{t_d - t},$$
$$b_1 = \frac{\ln\frac{\widetilde{S}}{X} + (r + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}}, \qquad b_2 = b_1 - \sigma\sqrt{T - t}. \tag{5.1.43b}$$

The generalization of the pricing procedure to the two-dividend American call option model is considered in Problem 5.17.

Black's approximation formula

Black (1975) proposes an approximate pricing formula for the one-dividend American call option model. Let $c(S,\tau)$ denote the price function of a European call, where the temporal variable τ is the time to expiry. The approximate value of the one-dividend American call is given by $\max\{c(\widetilde{S}, T - t; X), c(S, t_d - t; X)\}$. The first term gives the one-dividend American call value when the probability of early exercise is zero while the second term assumes the probability of early exercise to be one. Since both cases represent sub-optimal early exercise policies, it is obvious that

$$C_d(\widetilde{S}, T-t; X) \geq \max\{c(\widetilde{S}, T-t; X), c(S, t_d-t; X)\}, \quad t < t_d. \quad (5.1.44)$$

5.1.6 One-dividend and multi-dividend American put options

Consider an American put on an asset which pays out discrete dividends with certainty during the life of the option, the corresponding optimal exercise policy exhibits more complicated behaviors compared to its call counterpart. Within some short time period prior to a dividend payment date, the put holder may choose not to exercise at any asset price level due to the anticipation of the dividend payment. That is, the holder prefers to defer early exercise until immediately after an ex-dividend date in order to benefit from the receipt of the dividend by holding the asset through the dividend date. From the last dividend date to expiration, the optimal exercise boundary behaves like that of an American put on a non-dividend payment asset, so the optimal exercise price $S^*(t)$ increases monotonically with increasing calendar time t. For times in between the dividend dates and before the first dividend date, $S^*(t)$ may rise or fall with increasing t or even becomes zero (see Figs. 5.5 and 5.6). Due to the complicated nature of the optimal exercise policy, no analytic price formula exists for an American put on an asset paying discrete dividends.

One-dividend American put

First, we would like to consider the early exercise policy for the one-dividend American put model. Let the ex-dividend date be t_d, the expiration date be T and the dividend amount be D. Since the exercise policy at $t > t_d$ is identical to that of American put on the same asset with zero dividend, it suffices to consider the exercise policy at time t before the ex-dividend date. Suppose the American put is exercised at time t, then the interest received from t to t_d arising from time value of the strike price X is $X[e^{r(t_d-t)} - 1]$, where r is the riskless interest rate. When the interest is less than the discrete dividend, that is, $X[e^{r(t_d-t)} - 1] < D$, the early exercise of the American put is never optimal. This is because the benefit from the receipt of the dividend amount D by holding the asset through the dividend date t_d is more attractive than the interest income gained. Therefore, there exists a period prior to t_d such that it is never optimal for the holder to exercise the one-dividend American put.

One observes that the interest income $X[e^{r(t_d-t)} - 1]$ depends on $t_d - t$, and its value increases when $t_d - t$ increases. There exists a critical value t_s such that

$$X[e^{r(t_d-t_s)} - 1] = D. \quad (5.1.45a)$$

Solving for t_s, we obtain

$$t_s = t_d - \frac{1}{r} \ln\left(1 + \frac{D}{X}\right). \tag{5.1.45b}$$

Over the interval $[t_s, t_d]$, it is never optimal to exercise the American put.

When $t < t_s$, we have $X[e^{r(t_d - t)} - 1] > D$. Under such condition, early exercise may become optimal when the asset price is below certain critical value. The optimal exercise price $S^*(t)$ is governed by two offsetting effects, the time value of the strike and the discrete dividend. When t is approaching t_s, the dividend effect is more dominant so that the American put would be exercised only when it is deeper-in-the-money, that is, at a lower optimal exercise price $S^*(t)$. When t is farther away from t_s, the dividend effect diminishes so that the optimal exercise policy behaves more like usual American put on a zero-dividend asset. In this case, $S^*(t)$ assumes a lower value as t stays farther from t_s. As a result, the plot of $S^*(t)$ against t resembles a humped shape curve for the time interval prior to t_s (see Fig. 5.5).

From Eq. (5.1.45b), t_s is seen to increase with increasing r so that the interval of "no-exercise" $[t_s, t_d]$ shrinks with higher interest rate. Since the early exercise of an American put results in gain of time value of strike, a higher interest rate implies a higher opportunity cost of holding an in-the-money American put so that the propensity of early exercise increases.

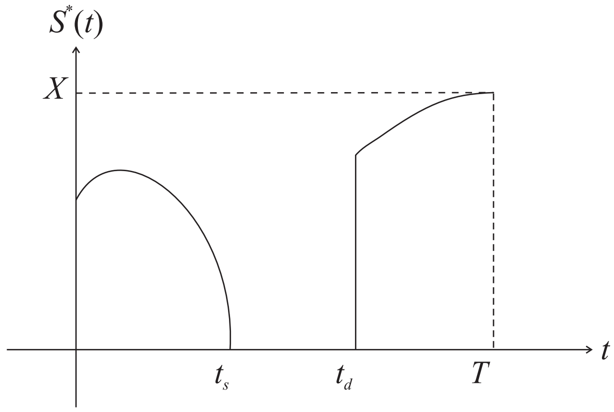


Fig. 5.5 The behaviors of the optimal exercise boundary $S^*(t)$ as a function of t for a one-dividend American put option.

In summary, the optimal exercise boundary $S^*(t)$ of the one-dividend American put model exhibits the following behaviors (see Figure 5.5).

(i) When $t < t_s$, $S^*(t)$ first increases then decreases smoothly with increasing t until it drops to the zero value at t_s.

(ii) $S^*(t)$ stays at the zero value in the interval $[t_s, t_d]$.

(iii) When $t \in (t_d, T], S^*(t)$ is a montonically increasing function of t with $S^*(T) = X$.

Multi-dividend American put

The analysis of the optimal exercise policy for the multi-dividend American put model can be performed in a similar manner. Suppose dividends of amount $D_1, D_2, \cdots, D_n$ are paid on the ex-dividend dates $t_1, t_2, \cdots, t_n$, there is an interval $[t_j^*, t_j]$ before the ex-dividend time $t_j, j = 1, 2, \cdots, n$ such that it is never optimal to exercise the put prematurely. That is, $S^*(t) = 0$ for $t \in [t_j^*, t_j],\ j = 1, 2, \cdots, n.$, where the critical time t_j^* is given by

$$t_j^* = t_j - \frac{1}{r} \ln \left(1 + \frac{D_j}{X} \right), \quad j = 1, 2, \cdots, n. \tag{5.1.46}$$

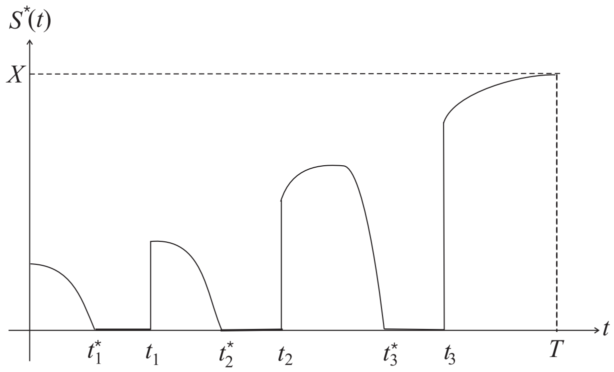


Fig. 5.6 The characterization of the optimal exercise boundary $S^*(t)$ as a function of the calendar time t for a three-dividend American put option model. Observe that $S^*(t)$ is monotonically increasing in (t_3, T) and $S^*(T) = X$. It stays at the zero value in $[t_3^*, t_3]$. Furthermore, $S^*(t)$ can be increasing to some peak value then decreasing as in (t_2, t_3^*), or simply decreasing monotonically as in (t_1, t_2^*).

Here, we follow the calendar time in the description of the optimal exercise boundary. At times falling within the intervals $(t_{j-1}, t_j^*),\ j = 2, \cdots, n$ and $t \le t_1^*$, the optimal exercise price $S^*(t)$ may first increase with time to some peak value, then decreases and eventually drops to the zero value when the time reaches t_j^*. When the dividend is sufficently deep, $S^*(t)$ may decrease monotonically throughout the interval (t_{j-1}, t_j^*) from some peak value to the zero value. When D_j increases further, it may be possible that t_j^* is less than t_{j-1}. As a consequence, $S^*(t) = 0$ for the whole time interval $[t_{j-1}, t_j]$. For the last time interval $(t_n, T]$, the optimal exercise price increases monotonically to X as expiration is approached.

The behaviors of the optimal exercise boundary $S^*(t)$ of a three-dividend American put model as a function of the calendar time t are depicted in Fig. 5.6. Meyer (2001) performed careful numerical studies on the optimal exercise policies of multi-dividend American put options. His results are consistent with the behaviors of $S^*(t)$ described above.

5.2 Analytic formulations of American option pricing models

In this section, we consider two analytic formulations of American option pricing models, namely, the linear complementarity formulation and the formulation as an optimal stopping problem. First, we develop the variational inequalities that are satisfied by the American option price function, and from which we derive the linear complementarity formulation. Alternatively, the American option price can be seen to be the supremum of the discounted expectation of the exercise payoff among all possible stopping times. It can be shown rigorously that the solution to the optimal stopping formulation satisfies the linear complementarity formulation. From the theory of controlled diffusion process, we are able to derive the integral representation of an American price formula in terms of the optimal exercise boundary. We also show how to obtain the integral representation of the early exercise premium using the financial argument of delay exercise compensation. Using the fact that the optimal exercise price is the asset price at which one is indifferent between exercising or non-exercising, we deduce the integral equation for the optimal exercise price. This section is ended with the discussion of two types of American path dependent option models. We consider the pricing of the American barrier option and a special form of perpetual American lookback option coined with the name "Russian option".

5.2.1 Linear complementarity formulation

The valuation of an American option can be formulated as a free boundary value problem, where the free boundary is the optimal exercise boundary which separates the continuation and stopping regions. When the asset price falls into the stopping region where the American call option should be exercised optimally, we have

$$C(S, \tau) = S - X, \quad S \geq S^*(\tau). \tag{5.2.1}$$

The exercise payoff, $C = S - X$, does not satisfy the Black-Scholes equation since

$$\left[\frac{\partial}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2}{\partial S^2} - (r-q)S\frac{\partial}{\partial S} + r\right](S - X) = qS - rX. \quad (5.2.2a)$$

From $S \geq S^*(\tau) > S^*(0^+) = X \max\left(1, \frac{r}{q}\right)$, we deduce that $qS - rX > 0$. We then deduce that in the stopping region, the call value $C(S,\tau)$ observes the following inequality

$$\frac{\partial C}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2} - (r-q)S\frac{\partial C}{\partial S} + rC > 0 \quad \text{for} \quad S \geq S^*(\tau). \qquad (5.2.2b)$$

The above inequality can also be deduced from the following financial argument. Let Π denote the value of the riskless hedging portfolio defined by

$$\Pi = C - \Delta S \text{ where } \Delta = \frac{\partial C}{\partial S}. \qquad (5.2.3a)$$

We argue that optimal early exercise of the American call occurs when the rate of return from the riskless hedging portfolio is less than the riskless interest rate, that is,

$$d\Pi < r\Pi\, dt. \qquad (5.2.3b)$$

By computing $d\Pi$ using Ito's lemma, the above inequality can be shown to be equivalent to Ineq. (5.2.2b).

In the continuation region where the asset price S is less than the optimal exercise price $S^*(\tau)$, the American call value satisfies the Black-Scholes equation. We then conclude that

$$\frac{\partial C}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2} - (r-q)S\frac{\partial C}{\partial S} + rC \geq 0, \quad S > 0 \text{ and } \tau > 0, \qquad (5.2.4)$$

where equality holds when (S,τ) lies in the continuation region. On the other hand, the American call value is always above the intrinsic value $S - X$ when $S < S^*(\tau)$ and equal to the intrinsic value when $S \geq S^*(\tau)$, that is,

$$C(S,\tau) \geq S - X, \quad S > 0 \text{ and } \tau > 0. \qquad (5.2.5)$$

In the above inequality, equality holds when (S,τ) lies in the stopping region. Since (S,τ) is either in the continuation region or stopping region, equality holds in one of the above pair of variational inequalities. We then deduce that

$$\left[\frac{\partial C}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 C}{\partial S^2} - (r-q)S\frac{\partial C}{\partial S} + rC\right][C - (S - X)] = 0, \qquad (5.2.6)$$

for all values of $S > 0$ and $\tau > 0$. To complete the formulation of the model, we have to include the terminal payoff condition in the model formulation

$$C(S,0) = \max(S - X, 0). \qquad (5.2.7)$$

Inequalities (5.2.4–5) and Eq. (5.2.6) together with the auxiliary condition (5.2.7) constitute the *linear complementarity formulation* of the American call option pricing model (Dewynne *et al.*, 1993).

From the above linear complementarity formulation, we can deduce the following two properties for the optimal exercise price $S^*(\tau)$ of an American call.

1. It is the lowest asset price for which the American call value is equal to the exercise payoff.
2. It is the asset price at which one is indifferent between exercising and not exercising the American call.

Bunch and Johnson (2000) presents another interesting property of $S^*(\tau)$. It is the lowest asset price at which the American call value does not depend on the time to expiry, that is,

$$\frac{\partial C}{\partial \tau} = 0 \quad \text{at} \quad S = S^*(\tau). \tag{5.2.8}$$

This agrees with the financial intuition that at the moment when it is optimal to exercise immediately, it does not matter how much time is left to maturity. A simple mathematical proof can be constructed as follows. On the optimal exercise boundary $S^*(\tau)$, we have

$$C(S^*(\tau), \tau) = S^*(\tau) - X. \tag{5.2.9a}$$

Differentiating both sides with respect to τ, we obtain

$$\frac{\partial C}{\partial \tau}(S^*(\tau), \tau) + \frac{\partial C}{\partial S}(S^*(\tau), \tau)\frac{dS^*(\tau)}{d\tau} = \frac{dS^*(\tau)}{\partial \tau}. \tag{5.2.9b}$$

Using the smooth pasting condition: $\frac{\partial C}{\partial S}(S^*(\tau), \tau) = 1$, we then obtain the result in Eq. (5.2.8).

5.2.2 Optimal stopping problem

The pricing of an American option can also be formulated as an *optimal stopping problem*. A stopping time t^* can be considered as a function assuming value over an interval $[0, T]$ such that the decision to "stop at time t^*" is determined by the information on the asset price path $S_u, 0 \leq u \leq t^*$. Consider an American put option, and suppose that it is exercised at time $t^*, t^* < T$, the payoff is $\max(X - S_{t^*}, 0)$. The fair value of the put option with payoff at t^* defined above is given by

$$E_t[e^{-r(t^*-t)} \max(X - S_{t^*}, 0)],$$

where E_t is the expectation under the risk neutral measure conditional on the filtration $\mathcal{F}_t$. This is valid provided that t^* is a stopping time, independent of whether it is deterministic or random.

Since the holder can exercise at any time during the life of the option, we deduce that the American put value is given by (Karatzas, 1988; Jacka, 1991; Myneni, 1992)

$$P(S_t, t) = \sup_{t \le t^* \le T} E_t[e^{-r(t^*-t)} \max(X - S_{t^*}, 0)], \tag{5.2.10}$$

where t is the calendar time and the supremum is taken over all possible stopping times. Recall that $P(S_t, t)$ always stays at or above the payoff and $P(S_t, t)$ equals the payoff at the stopping time t^*. The above supremum is reached at the optimal stopping time (Krylov, 1980) so that

$$t^* = \inf_u \{t \le u \le T : P(S_u, u) = \max(X - S_u, 0)\}, \tag{5.2.11}$$

the first time that the American put value drops to its payoff value.

We would like to verify that the solution to the linear complementarity formulation gives the American put value as stated in Eq. (5.2.10), where the optimal stopping time is determined by Eq. (5.2.11). We recall the renowed *optional stopping theorem* which states that if $(M_t)_{t\ge 0}$ is a continuous martingale with respect to the filtration $(\mathcal{F}_t)_{t\ge 0}$, and if t_1^* and t_2^* are two stopping times, $t_1^* < t_2^*$, then

$$E[M_{t_2^*} | \mathcal{F}_{t_1^*}] = M_{t_1^*}. \tag{5.2.12}$$

For any stopping time $t^*, t < t^* < T$, we apply Ito's formula to the solution $P(S_t, t)$ of the linear complementarity formulation to obtain

$$\begin{aligned} &e^{-rt^*} P(S_{t^*}, t^*) \\ &= e^{-rt} P(S_t, t) \\ &\quad + \int_t^{t^*} e^{-ru} \left[\frac{\partial}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2}{\partial S^2} + (r - q) S \frac{\partial}{\partial S} - r \right] P(S_u, u)\, du \\ &\quad + \int_t^{t^*} e^{-r} \sigma S \frac{\partial P}{\partial S}(S_u, u)\, dZ_u. \end{aligned} \tag{5.2.13}$$

Now, the integrand of the first integral is non-positive as deduced from one of the variational inequalities [see Eq. (5.2.4)]. When we take the expectation of the martingle term in the second integral, the expectation value becomes zero by virtue of the optional sampling theorem. We then have

$$\begin{aligned} P(S_t, t) &\ge E_t[e^{-r(t^*-t)} P(S_{t^*}, t^*)] \\ &= E_t[e^{-r(t^*-t)} \max(X - S_{t^*}, 0)]. \end{aligned} \tag{5.2.14}$$

Lastly, if we choose t^* as defined by Eq. (5.2.11), the above inequality becomes an equality, hence the result in Eq. (5.2.10).

5.2.3 Integral representation of the early exercise premium

From the theory of controlled diffusion process, the American put price is given by [a rigorous proof is presented in Krylov's text (1980)]

$$\begin{aligned} P(S_t, t) = &\, E_t[e^{-r(T-t)} \max(X - S_T, 0)] \\ &+ \int_t^T e^{-r(u-t)} E_u \left[(rX - qS_u) \mathbf{1}_{\{S_u < S^*(u)\}} \right] du. \end{aligned} \quad (5.2.15)$$

The first term represents the usual European put price while the second term represents the early exercise premium. Let $\psi(S_u; S_t)$ denote the transition density function of S_u conditional on S_t. We may rewrite the above put price formula as follows

$$\begin{aligned} P(S_t, t) = &\, e^{-r(T-t)} \int_0^X (X - S_T)\psi(S_T; S_t)\, dS_T \\ &+ \int_t^T e^{-r(u-t)} \int_0^{S^*(u)} (rX - qS_u)\psi(S_u; S_t)\, dS_u\, du. \end{aligned} \quad (5.2.16)$$

The early exercise premium is seen to be positive since

$$rX - qS_u > 0 \quad \text{as} \quad S_u < S^*(u) < \frac{rX}{q}.$$

We would like to provide an intuitive proof to the American put price formula by arguing that the early exercise premium can be interpreted as delay exercise compensation (Jamshidian, 1992).

Delay exercise compensation

In order that the American put option is kept alive for all values of asset price until expiration, the holder needs to be compensated by a continuous cash flow when the put should be exercised optimally. Within the time interval between u and $u + du$ and suppose S_u falls within the stopping region, the amount of compensation paid to the holder of the American put should be $(rX - qS_u)\, du$ in order that the holder agrees not to exercise even when it is optimal for him to do so. This is because the holder would earn interest $rX\, du$ from the cash received and lose dividend $qS_u\, du$ from the short position of the asset if he were to choose to exercise his put. The discounted expectation for the above continuous cash flow compensation is given by

$$e^{-r(u-t)} \int_0^{S^*(u)} (rX - qS_u)\psi(S_u; S_t)\, dS_u.$$

The integration of the above discounted cash flow from $u = t$ to $u = T$ gives the early exercise premium of the American put option, which is precisely the early exercise premium term in Eq. (5.2.16).

Value matching and smooth pasting conditions

Here, we present the financial interpretation of the necessity of the "value matching" and "smooth pasting" conditions, namely, the continuity of P and $\frac{\partial P}{\partial S}$ across the optimal exercise boundary $S^*(u)$. Consider the following *dynamic trading strategy* proposed by Carr *et al.* (1992). After purchasing the American put at the current time t, the investor would instantaneously exercise the put whenever the asset price falls from above to the optimal exercise price $S^*(u)$ and purchase back the put whenever the asset price rises from below to $S^*(u)$. Since the transactions of converting put into holding of cash plus short position in asset and vice versa all occur on the early exercise boundary, we require the "value matching" and "smooth pasting" conditions in order to ensure that these transactions are self-financing, that is, each portfolio revision undertaken is exactly financed by the proceeds from the sale of the previous position.

Analytic representation of American put price function

In the subsequent exposition in this section, we use the time to expiry τ as the temporal variable in optimal exericse boundary $S^*(\tau)$ and write S for S_t. The integrals in Eq. (5.2.16) can be evaluated to give the following representation of the American put price formula

$$\begin{aligned} P(S,\tau) &= Xe^{-r\tau}N(-d_2) - Se^{-q\tau}N(-d_1) \\ &\quad + \int_0^\tau [rXe^{-r\xi}N(-d_{\xi,2}) - qSe^{-q\xi}N(-d_{\xi,1})]\, d\xi, \end{aligned} \tag{5.2.17a}$$

where $\tau = T - t$ and

$$\begin{aligned} d_1 &= \frac{\ln\frac{S}{X} + \left(r - q + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 = d_1 - \sigma\sqrt{\tau}, \\ d_{\xi,1} &= \frac{\ln\frac{S}{S^*(\tau-\xi)} + \left(r - q + \frac{\sigma^2}{2}\right)\xi}{\sigma\sqrt{\xi}}, \qquad d_{\xi,2} = d_{\xi,1} - \sigma\sqrt{\xi}. \end{aligned} \tag{5.2.17b}$$

The dummy time variable ξ can be considered as the time period lapsed from the current time so that $\xi = 0$ and $\xi = \tau$ correspond to the current time and expiration date, respectively.

When the interest rate is zero, $r = 0$, the early exercise premium becomes

$$-\int_0^\tau qSe^{-q\xi}N(-d_{\xi,1})\, d\xi,$$

which is seen to be a non-positive quantity. However, the early exercise premium must be non-negative. These two arguments together lead to

$$\int_0^\tau qSe^{-q\xi}N(-d_{\xi,1})\, d\xi = 0, \tag{5.2.18}$$

which is satisfied only by setting $S^*(\xi) = 0$ for all values of ξ. The zero value of the optimal exercise price infers that the American put is never exercised. In this case, the value of the American put is the same as that of its European counterpart. The same conclusion has been reached by another argument presented earlier in Sec. 5.1.4.

Integral equations for the optimal exercise boundary
If we apply the boundary condition: $P(S^*(\tau), \tau) = X - S^*(\tau)$ to the put price formula (5.2.17a), we obtain the following integral equation for $S^*(\tau)$:

$$\begin{aligned} X - S^*(\tau) &= Xe^{-r\tau}N(-\widehat{d}_2) - S^*(\tau)e^{-q\tau}N(-\widehat{d}_1) \\ &\quad + \int_0^\tau [rXe^{-r\xi}N(-\widehat{d}_{\xi,2}) - qS^*(\tau)e^{-q\xi}N(-\widehat{d}_{\xi,1})]\, d\xi \end{aligned}, \tag{5.2.19}$$

where

$$\begin{aligned} \widehat{d}_1 &= \frac{\ln \frac{S^*(\tau)}{X} + \left(r - q + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \qquad \widehat{d}_2 = \widehat{d}_1 - \sigma\sqrt{\tau} \\ \widehat{d}_{\xi,1} &= \frac{\ln \frac{S^*(\tau)}{S^*(\tau-\xi)} + \left(r - q + \frac{\sigma^2}{2}\right)\xi}{\sigma\sqrt{\xi}}, \qquad \widehat{d}_{\xi,2} = \widehat{d}_{\xi,1} - \sigma\sqrt{\xi}. \end{aligned} \tag{5.2.20}$$

The solution for $S^*(\tau)$ requires the knowledge of $S^*(\tau - \xi)$, $0 < \xi \leq \tau$. The solution procedure starts with $S^*(0)$ and integrates backward in calendar time (that is, increasing τ).

Alternatively, we may use the smooth pasting condition: $\frac{\partial P}{\partial S}(S^*(\tau), \tau) = -1$ along $S^*(\tau)$ to derive another integral equation for $S^*(\tau)$. Taking the partial derivative with respect to S of the terms in Eq. (5.2.19) and setting $S = S^*(\tau)$, we have

$$\begin{aligned} 0 &= 1 + \frac{\partial P}{\partial S}(S^*(\tau), \tau) \\ &= 1 + \frac{\partial p}{\partial S}(S^*(\tau), \tau) \\ &\quad + \int_0^\tau \left[rXe^{-r\xi}\frac{\partial}{\partial S}N(-d_{\xi,2})\bigg|_{S=S^*(\tau)} - qe^{-q\xi}N(-d_{\xi,1})\bigg|_{S=S^*(\tau)} \right. \\ &\qquad \left. - qSe^{-q\xi}\frac{\partial}{\partial S}N(-d_{\xi,1})\bigg|_{S=S^*(\tau)} \right] d\xi \\ &= N(\widehat{d}_1) - \int_0^\tau \left[\frac{(r-q)e^{-q\xi}}{\sigma\sqrt{2\pi\xi}} e^{-\frac{\widehat{d}^2_{\xi,1}}{2}} + qe^{-q\xi}N(-\widehat{d}_{\xi,1}) \right] d\xi. \end{aligned} \tag{5.2.21}$$

Various versions of integral equation for the optimal exercise price can also be derived (Little and Pant, 2000), some of these alternative forms may provide easier analysis of the properties of the optimal exercise boundary. The direct analytic solution to any one of these integral equations is definitely intractable. In Sec. 5.3, we will discuss the recursive integration method for solving the above integral equations.

The integral equation defined in Eq. (5.2.19) may be used to find the optimal exercise price at the limiting case $\tau \to \infty$ [see Eq. (5.1.33a)]. Let $S_P^*(\infty)$ denote $\lim_{\tau\to\infty} S_P^*(\tau)$, which corresponds to the optimal exercise price for the perpetual American put. Taking the limit $\tau \to \infty$ in Eq. (5.2.19), and observing that the value of the perpetual European put is zero, we obtain

$$\begin{aligned} X - S_P^*(\infty) = \int_0^\infty \Bigg[& rXe^{-r\xi} N\left(-\frac{r-q-\frac{\sigma^2}{2}}{\sigma}\sqrt{\xi}\right) \\ & - qS_P^*(\infty)e^{-q\xi} N\left(-\frac{r-q+\frac{\sigma^2}{2}}{\sigma}\sqrt{\xi}\right)\Bigg]\, d\xi. \end{aligned} \tag{5.2.22}$$

The first and second terms in the above integral can be simplifed as follows:

$$\begin{aligned} \int_0^\infty e^{-r\xi} N(-\rho\sqrt{\xi})\, d\xi &= -\frac{e^{-r\xi}}{r} N(-\rho\sqrt{\xi})\Big|_0^\infty - \frac{\rho}{2r}\frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-\rho^2\xi/2}e^{-r\xi}}{\sqrt{\xi}}\, d\xi \\ &= \frac{1}{2r}\left[1 - \frac{\rho}{\sqrt{\rho^2+2r}}\right], \quad \rho = \frac{r-q-\frac{\sigma^2}{2}}{\sigma}; \end{aligned}$$

$$\int_0^\infty e^{-q\xi} N(-\rho'\sqrt{\xi})\, d\xi = \frac{1}{2q}\left[1 - \frac{\rho'}{\sqrt{\rho'^2+2q}}\right], \quad \rho' = \frac{r-q+\frac{\sigma^2}{2}}{\sigma}. \tag{5.2.23}$$

Substituting the above results into Eq. (5.2.22), we obtain

$$X - S_P^*(\infty) = \frac{X}{2}\left[1 - \frac{\rho}{\sqrt{\rho^2+2r}}\right] - \frac{S_P^*(\infty)}{2}\left[1 - \frac{\rho'}{\sqrt{\rho'^2+2q}}\right]. \tag{5.2.24}$$

Rearranging the terms, we have

$$S_P^*(\infty) = \frac{1 + \frac{\rho}{\sqrt{\rho^2+2r}}}{1 + \frac{\rho'}{\sqrt{\rho'^2+2q}}} X = \frac{\mu_-}{\mu_- - 1} X, \tag{5.2.25}$$

where μ_- is defined by Eq. (5.1.33b).

Analytic representation of the American call price function

Similar to the American put price as given in Eq. (5.2.17a), the analytic representation of the American call counterpart is given by

$$\begin{aligned} C(S,\tau) = {} & Se^{-q\tau}N(d_1) - Xe^{-r\tau}N(d_2) \\ & + \int_0^\tau \left[qSe^{-q\xi}N(d_{\xi,1}) - rXe^{-r\xi}N(d_{\xi,2})\right] d\xi. \end{aligned} \tag{5.2.26}$$

The corresponding integral equation for the early exercise boundary $S^*(\tau)$ can be deduced similarly by setting $C(S^*(\tau), \tau) = S^*(\tau) - X$. This gives

$$S^*(\tau) - X = S^*(\tau)e^{-q\tau}N(\widehat{d}_1) - Xe^{-r\tau}N(\widehat{d}_2) + \int_0^{\tau} [qS^*(\tau)e^{-q\xi}N(\hat{d}_{\xi,1}) - rXe^{-r\xi}N(\hat{d}_{\xi,2})]\, d\xi. \tag{5.2.27}$$

Similarly, by taking the limit $\tau \to \infty$ in Eq. (5.2.27), one can also deduce the corresponding asymptotic upper bound of the early exercise boundary of the American call option (see Problem 5.22).

5.2.4 American barrier options

An American barrier option is a barrier option embedded with the early exercise right. For example, an American down-and-out call becomes nullified when the down-barrier is breached by the asset price or prematurely terminated due to the optimal exericse decision of the holder. Like usual American option, the option value of an American out-barrier option can be decomposed into the sum of the value of the European barrier option and the early exercise premium. In this subsection, we derive the price formula of an American down-and-out call and examine some of its pricing behaviors. As a remark, the pricing of an American in-barrier option is much more complicated. This is because the in-trigger region associated with the knock-in feature may intersect with the stopping region of the underlying American option. The pricing models of the American in-barrier options are discussed in Dai and Kwok's paper (2004b). Some interesting results in their paper are presented in Problem 5.26.

In our American down-and-out call option model, we assume that the underlying asset pays a constant dividend yield q and the constant down-barrier B satisfies the condition $B < X$. For simplicity, we assume zero rebate paid upon nullification of the option. The price function $C_B(S, \tau; X, B)$ of the American down-and-out call option is given by

$$C_B(S, \tau; X, B) = e^{-r\tau}E_t\left[\max(S_T - X, 0)\mathbf{1}_{\{m_t^T > B\}}\right] + \int_t^T e^{-ru}E_u\left[(qS_u - rX)\mathbf{1}_{\{m_t^u > B, (S_u, u) \in \mathcal{S}\}}\right] du \tag{5.2.28}$$

where E_t denotes the expectation conditional on the filtration $\mathcal{F}_t$, m_t^u is the realized minimum value of the asset price over the time period $[t, u]$. $\tau = T - t$ and $\mathcal{S}$ denotes the stopping region. The first term gives the value of the European down-and-out barrier option. The second term represents the early exercise premium of the American down-and-out call. The delay exercise compensation is received only when (S_u, u) lies inside the stopping region $\mathcal{S}$ and the barrier option has not been knocked out. To effect the expectation calculations, it is necessary to use the transition density function of the restricted (with down absorbing barrier B) asset price process. After performing the integration procedure, the early exercise premium $e_C(S,\tau;B)$ can be expressed as

$$\begin{aligned} e_C(S,\tau;B) = \int_0^\tau &\{K_C(S,\tau;S^*(\tau-\omega),\omega) \\ &- \left(\frac{S}{B}\right)^{\delta+1} K_C\left(\frac{B^2}{S},\tau;S^*(\tau-\omega),\omega\right)\right\} d\omega, \end{aligned} \tag{5.2.29}$$

where $\delta = 2(q-r)/\sigma^2$ and $S^*(\tau)$ is the optimal exercise price above which the American call option should be exercised. The analytic expression for K_C is given by

$$K_C(S,\tau;S^*(\tau-\omega),\omega) = qSe^{-q\omega}N(d_{\omega,1}) - rXe^{-r\omega}N(d_{\omega,2}), \tag{5.2.30}$$

where

$$d_{\omega,1} = \frac{\ln \frac{S}{S^*(\tau-\omega)} + \left(r - q + \frac{\sigma^2}{2}\right)\omega}{\sigma\sqrt{\omega}}, \quad d_{\omega,2} = d_{\omega,1} - \sigma\sqrt{\omega}. \tag{5.2.31}$$

It can be shown mathematically that

$$K_C(S,\tau;S^*(\tau-\omega),\omega) > \left(\frac{S}{B}\right)^{\delta+1} K_C\left(\frac{B^2}{S},\tau;S^*(\tau-\omega),\omega\right) > 0. \tag{5.2.32}$$

This agrees with the intuition that the early exercise premium is reduced by the presence of the barrier and it always remains positive. Though $e_C(S,\tau)$ apparently becomes negative when $q = 0$, the premium term in fact becomes zero since the early exercise premium must be non-negative. This is made possible by choosing $S^*(\tau-\omega) \to \infty$ for $0 \le \omega \le \tau$. Even with embedded barrier feature, an American call is never exercised when the underlying asset is non-dividend paying.

Next, we explore the effects of the barrier level and rebate on the early exercise policies. Additional pricing properties of the American barrier options can be found in Gao *et al.*'s paper (2000).

Effects of barrier level on early exercise policies

From intuition, it is expected that the optimal exercise price $S^*(\tau;B)$ for an American down-and-out call option decreases with an increasing barrier level B. For an in-the-money American down-and-out call option, the holder should consider to exercise the call at a lower optimal exercise price when the barrier level is higher since the adverse chance of asset price dropping to a level below the barrier is higher.

A semi-rigorous explanation of the above intuition can be argued as follows. Since the price curve of the American barrier call option with a lower barrier level is always above that with higher barrier level, it then intersects tangentially the intrinsic value line $C = S - X$ at a higher optimal exercise price (see Fig. 5.7). Therefore, $S^*(\tau; B)$ is a decreasing function of B.

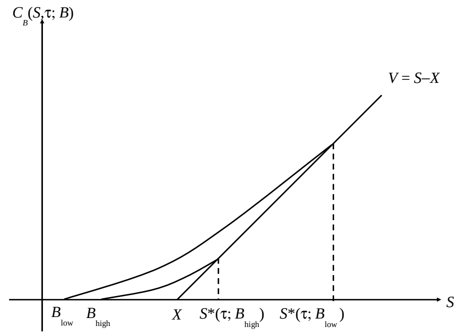


Fig. 5.7 The price curve for an American down-and-out call option with a lower barrier level B_{low} is always above that with a higher barrier level B_{high}.

Effects of rebate on early exercise policies
With the presence of rebate, the holder of an American down-and-out call option will choose to exercise optimally at a higher asset price level since the penalty of adverse movement of asset price dropping below the barrier is lessened. Mathematically, we argue that the price curve of the American down-and-out call option with rebate should be above that without rebate, so it intersects tangentially the intrinsic value line $C = S - X$ at a higher optimal exercise price. Hence, the optimal exercise price is an increasing function of rebate.

5.2.5 American lookback options

The studies of the optimal exercise policies for various types of finite-lived American lookback option remain to be challenging problems. Some of the theoretical results on this topic can be found in a series of papers by Dai and Kwok (2004c, 2005b and 2005c). In this subsection, we consider a special type of a perpetual American option with lookback payoff, coined with the name "Russian option".

The Russian option contract on an asset guarantees the holder of the option to receive the historical maximum value of the asset price path upon exercising the option. Premature exercise of the Russian option can occur at any time chosen by the holder. Let M denote the historical realized maximum of the asset price (the starting date of the lookback period is immaterial) and

S be the asset price, both quantities are taken at the same time. Since it is a perpetual option, the option value is independent of time. Let $V = V(S, M)$ denote the option value and let S^* denote the optimal exercise price at which the Russian option should be exercised. At a sufficiently low asset price, it becomes more attractive to exercise the Russian option and receive the dollar amount M rather than to hold and wait. Therefore, the Russian option is alive when $S^* < S \leq M$ and will be exercised when $S \leq S^*$. The payoff function of the Russian option upon exercising is

$$V(S^*, M) = M, \tag{5.2.35}$$

and the option value stays above M when the option is alive.

Assume that the asset pays a continuous dividend yield q. It will be shown later that the solution to the option model becomes undefined if the underlying asset is non-dividend paying. By dropping the temporal derivative term in the Black-Scholes equation, the governing equation for the Russian option model is given by

$$\frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - q) S \frac{\partial V}{\partial S} - rV = 0, \quad S^* < S < M. \tag{5.2.36}$$

The boundary condition at $S = S^*$ has been given by Eq. (5.2.35). It has been explained in Sec. 4.2 that lookback option value is insensitive to M when $S = M$. Therefore, the other boundary condition at $S = M$ is given by [see Eq. (4.2.20)]

$$\frac{\partial V}{\partial M} = 0 \quad \text{at} \quad S = M. \tag{5.2.37}$$

The optimal exercise price S^* is chosen such that the option value is maximized among all possible values of S^*. The governing equation and boundary conditions can be recast in a more succinct form when the following similarity variables:

$$W = V/M \quad \text{and} \quad \xi = S/M \tag{5.2.38}$$

are employed. In terms of the new similarity variables, the value of the Russian option is governed by

$$\frac{\sigma^2}{2} \xi^2 \frac{d^2 W}{d\xi^2} + (r - q)\xi \frac{dW}{d\xi} - rW = 0, \quad \xi^* < \xi < 1, \tag{5.2.39}$$

where $W = W(\xi)$ and $\xi^* = S^*/M$. The boundary conditions become

$$\frac{dW}{d\xi} = W \quad \text{at} \quad \xi = 1, \tag{5.2.40a}$$

$$W = 1 \quad \text{at} \quad \xi = \xi^*. \tag{5.2.40b}$$

First, we solve for the option value in terms of ξ^*, then determine ξ^* such that the option value is maximized. By substituting the assumed form of the solution $A\xi^\lambda$ into Eq. (5.2.39), we observe that λ should satisfy the following quadratic equation

$$\frac{\sigma^2}{2}\lambda(\lambda-1)+(r-q)\lambda-r=0. \tag{5.2.41}$$

The two roots of the above quadratic equation are

$$\lambda_\pm=\frac{1}{\sigma^2}\left[-(r-q)+\frac{\sigma^2}{2}\pm\sqrt{\left((r-q-\frac{\sigma^2}{2}\right)^2+2\sigma^2r}\right], \tag{5.2.42}$$

where $\lambda_+ > 0$ and $\lambda_- < 0$. The general solution to Eq. (5.2.39) can be expressed as

$$W(\xi)=A_+\xi^{\lambda_+}+A_-\xi^{\lambda_-}, \quad \xi^*<\xi<1, \tag{5.2.43}$$

where A_+ and A_- are arbitrary constants. Applying the boundary conditions (5.2.40a,b), the solution for $W(\xi)$ is found to be

$$W(\xi)=\frac{(1-\lambda_-)\xi^{\lambda_+}-(1-\lambda_+)\xi^{\lambda_-}}{(1-\lambda_-)\xi^{*\lambda_+}-(1-\lambda_+)\xi^{*\lambda_-}}, \quad \xi^*\le\xi\le 1. \tag{5.2.44}$$

The use of calculus reveals that $W(\xi)$ is maximized when ξ^* is chosen to be

$$\xi^*=\left[\frac{\lambda_+(1-\lambda_-)}{\lambda_-(1-\lambda_+)}\right]^{1/(\lambda_--\lambda_+)}. \tag{5.2.45}$$

Besides the above differential equation approach, one may apply the martingale pricing approach to derive the price formula for the Russian option. Interested readers may read the papers by Shepp and Shiryaev (1993) and Gerber and Shiu (1994) for details.

Non-dividend paying underlying asset

How does the price function of the Russian option behave when $q=0$? The two roots then become $\lambda_+=1$ and $\lambda_-=-\frac{2r}{\sigma^2}$. The solution for $W(\xi)$ is reduced to

$$W(\xi)=\frac{\xi}{\xi^*}, \quad \xi^*\le\xi\le 1, \tag{5.2.46}$$

which is maximized when ξ^* is chosen to be zero. The Russian option value becomes infinite when the underlying asset is non-dividend paying. Can you provide a financial argument for the result?

5.3 Analytic approximation methods

Except for a few special cases like the American call on an asset with no dividend or discrete dividends and the perpetual American options, analytic price formulas do not exist for most types of finite-lived American options. In this section, we present three effective analytic approximation methods for finding the American option values and the associated optimal exercise boundaries.

The *compound option approximation method* treats an American option as a compound option by limiting the opportunity set of optimal exercises to be only at a few discrete times rather than at any time during the life of the option. The compound option approach requires the valuation of multi-variate normal integrals in the corresponding approximation formulas, where the dimension of the multi-variate integrals is the same as the number of exercise opportunities allowed. We have seen that one may express the early exercise premium in terms of the optimal exercise boundary in an integral representation, and this naturally leads to an integral equation for the optimal exercise boundary. The *recursive integration method* considers the direct solution of the integral equation for the early exercise boundary by recursive iterations. The iterative algorithm only involves computation of one-dimensional integrals. Even when we take only a few points on the optimal exercise boundary, the numerical accuracy of both compound option method and recursive integration method can be improved quite effectively by extrapolation procedure. The *quadratic approximation method* employs an ingenious transformation of the Black-Scholes equation so that the temporal derivative term can be considered as a quadratic small term and then dropped as an approximation. Once the approximate ordinary differential equation is derived, we only need to determine one optimal exercise point rather than the solution of the whole optimal exercise curve as in the original partial differential equation formulation.

It is commonly observed that most American option values are not too sensitive to the location of the optimal exercise boundary. This may explain why the above analytic approximation methods are quite accurate in calculating the American option values even when only a few points on the optimal exercise boundary are estimated. Evaluation of these analytic approximation formulas normally requires the use of a computer, some of them even require further numerical procedures, like numerical approximation of integrals, iteration and extrapolation. However, they do distinguish from direct numerical methods like the binomial method, finite difference method and Monte Carlo simulation (these numerical methods are discussed in full details in Chapter 6). In analytic approximation methods, the analytic behaviors of the formulation of the American option model are explored to the full extent and ingenious approximations are subsequently applied to reduce the complexity of the problems.

5.3.1 Compound option approximation method

An American option contract normally allows for early exercise at any time prior to expiration. However, by limiting the early exercise privilege to commence only at a few predetermined instants between now and expiration, the American option then resembles a compound option. It then becomes plausible to derive the corresponding analytic price formulas. The approximate price formula will converge to the price formula of the American option in the limit when the number of exercisable instants grows to infinity, since the continuously exercisable property of the American option is then recovered.

First, we derive the formula for a limited exercisable American put option on a non-dividend paying asset where early exercise can only occur at single instant which is halfway to expiration. Let the current time be zero and T be the expiration time. Let $S_{T/2}$ and S_T denote the asset price at times $T/2$ and T, respectively. Between time $T/2$ to the expiration date, the option behaves like an ordinary European option since there is no early exercise privilege. We determine the critical asset price $S^*_{T/2}$ at $T/2$ such that it is indifferent between exercising the put or not at the asset price $S^*_{T/2}$. Accordingly, $S^*_{T/2}$ is obtained by solving the following non-linear algebraic equation

$$p(S^*_{T/2}, T/2; X) = X - S^*_{T/2}, \tag{5.3.1}$$

where X is the strike price of the put. Here, $p(S^*_{T/2}, T/2; X)$ is the Black-Scholes price formula for a European put, where the value assumed by τ is $T/2$.

When $S_{T/2} \leq S^*_{T/2}$, the put option will be exercised with payoff $X - S_{T/2}$. The discounted expectation of $X - S_{T/2}$, conditional on $S_{T/2} \leq S^*_{T/2}$, is found to be

$$\begin{aligned} &e^{-rT/2} \int_0^{S^*_{T/2}} (X - S_{T/2}) \psi(S_{T/2}; S)\ dS_{T/2} \\ &= Xe^{-rT/2} N(-d_2(S, S^*_{T/2}; T/2)) - SN(-d_1(S, S^*_{T/2}; T/2)), \end{aligned} \tag{5.3.2a}$$

where

$$d_1(S_1, S_2; T) = \frac{\ln \frac{S_1}{S_2} + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma\sqrt{T}}, \quad d_2(S_1, S_2; T) = d_1(S_1, S_2; T) - \sigma\sqrt{T}, \tag{5.3.2b}$$

and $\psi(S_{T/2}; S)$ is the transition density function. On the other hand, when $S_{T/2} > S^*_{T/2}$, the put option survives until expiry. At expiry, it will be exercised only when $S_T < X$. The discounted expectation of $X - S_T$, conditional on $S_{T/2} > S^*_{T/2}$ and $S_T < X$, is given by

$$
\begin{aligned}
&\left[e^{-rT}\int_0^X (X-S_T)\psi(S_T;S_{T/2})\,dS_T\right] N(d_2(S,S^*_{T/2};T/2)) \\
&= e^{-rT}\int_{S^*_{T/2}}^{\infty}\int_0^X (X-S_T)\psi(S_T;S_{T/2})\psi(S_{T/2};S)\,dS_T dS_{T/2} \\
&= Xe^{-rT}N_2\left(d_2(S,S^*_{T/2};T/2), -d_2(S,X;T); -1/\sqrt{2}\right) \\
&\quad - SN_2\left(d_1(S,S^*_{T/2};T/2), -d_1(S,X;T); -1/\sqrt{2}\right).
\end{aligned}
\tag{5.3.3}
$$

Note that the correlation coefficient between overlapping Brownian increments over the time intervals $[0, T/2]$ and $[0, T]$ is found to be $1/\sqrt{2}$. The price of the put option with two exercisable instants $T/2$ and T is given by the sum of the above two expectation values. We then have

$$
\begin{aligned}
P_2(S,X;T) &= Xe^{-rT/2}N(-d_2(S,S^*_{T/2};T/2)) - SN(-d_1(S,S^*_{T/2};T/2)) \\
&\quad + Xe^{-rT}N_2\left(d_2(S,S^*_{T/2};T/2), -d_2(S,X;T); -1/\sqrt{2}\right) \\
&\quad - SN_2\left(d_1(S,S^*_{T/2};T/2), -d_1(S,X;T); -1/\sqrt{2}\right).
\end{aligned}
\tag{5.3.4}
$$

The extension to the general case with N exercisable instants (not necessarily equally spaced) can also be derived in a similar manner (see Problem 5.30).

Let P_n denote the value of the put option with n exercisable instants. We expect that the limit of the sequence $P_1, P_2, \cdots, P_n, \cdots$ tends to the American put value. One may apply the acceleration technique to extrapolate the limit based on the first few members of the sequence. Geske and Johnson (1984) propose the following Richardson extrapolation scheme when $n = 3$

$$
P \approx \frac{9P_3 - 8P_2 + P_1}{2}. \tag{5.3.5}
$$

Judging from their numerical experiments, reasonable accuracy is achieved for most cases based on extrapolation formula (5.3.5). Improved accuracy can be achieved by relaxing the requirement of equally spaced exercisable instants and seeking for appropriate exercisable instants such that the approximate put value is maximized (Bunch and Johnson, 1992).

5.3.2 Numerical solution of the integral equation

Recall that Eq. (5.2.19) provides an integral equation for the optimal exercise boundary for an American put option. In the integral equation, the variable τ appears both in the integrand and the upper limit of the integral. A recursive scheme can be derived to solve the integral equation for a given value of τ. In the numerical procedure, all integrals are approximated by the trapezoidal rule. First, we divide τ into n equally spaced subintervals with end points

$\tau_i, i = 0, 1, \cdots, n$ where $\tau_0 = 0, \tau_n = \tau$ and $\Delta\tau = \tau/n$. For convenience, we denote the integrand function by

$$f(S^*(\tau), S^*(\tau-\xi); \tau, \xi) = rXe^{-r\xi}N(-\widehat{d}_{\xi,2}) - qS^*(\tau)e^{-q\xi}N(-\widehat{d}_{\xi,1}), \quad (5.3.6a)$$

where

$$\widehat{d}_{\xi,1} = \frac{\ln \frac{S^*(\tau)}{S^*(\tau-\xi)} + \left(r - q + \frac{\sigma^2}{2}\right)\xi}{\sigma\sqrt{\xi}}, \quad \widehat{d}_{\xi,2} = \widehat{d}_{\xi,1} - \sigma\sqrt{\xi}. \quad (5.3.6b)$$

Let S_i^* denote the numerical approximation to $S^*(\tau_i), i = 0, 1, \cdots, n$. Setting $\tau = \tau_1$ in the integral equation and approximating the integral by

$$\begin{aligned}&\int_0^{\tau_1} \left[rXe^{-r\xi}N(-\widehat{d}_{\xi,2}) - qS^*(\tau)e^{-q\xi}N(-\widehat{d}_{\xi,1})\right] d\xi \\ &\approx \frac{\Delta\tau}{2}\left[f(S_1^*, S_1^*; \tau_1, \tau_0) + f(S_1^*, S_0^*; \tau_1, \tau_1)\right],\end{aligned} \quad (5.3.7)$$

we obtain the following non-linear algebraic equation for S_1^*:

$$X - S_1^* = p(S_1^*, \tau_1) + \frac{\Delta\tau}{2}\left[f(S_1^*, S_1^*; \tau_1, \tau_0) + f(S_1^*, S_0^*; \tau_1, \tau_1)\right]. \quad (5.3.8)$$

Since S_0^* is known to be $\min\left(X, \frac{r}{q}X\right)$, one can solve for S_1^* by any root-finding method. Once S_1^* is known, we proceed to set $\tau = \tau_2$ and approximate the integral over the two subintervals: (τ_0, τ_1) and (τ_1, τ_2). The corresponding non-linear algebraic equation for S_2^* is then given by

$$\begin{aligned}X - S_2^* = p(S_2^*, \tau_2) + \frac{\Delta\tau}{2}[&f(S_2^*, S_2^*; \tau_2, \tau_0) + 2f(S_2^*, S_1^*; \tau_2, \tau_1) \\ &+ f(S_2^*, S_0^*; \tau_2, \tau_2)].\end{aligned} \quad (5.3.9)$$

Recursively, the general algebraic equation for $S_k^*, k = 2, 3, \cdots, n$ can be deduced to be (Huang *et al.*, 1996)

$$\begin{aligned}X - S_k^* = p(S_k^*, \tau_k) + \frac{\Delta\tau}{2}\Bigg[&f(S_k^*, S_k^*; \tau_k, \tau_0) + f(S_k^*, S_0^*; \tau_k, \tau_k) \\ &+ 2\sum_{i=1}^{k-1} f(S_k^*, S_{k-i}^*; \tau_k, \tau_i)\Bigg], k = 2, 3, \cdots, n,\end{aligned} \quad (5.3.10)$$

where $S_k^*, k = 1, 2, \cdots, n$ are solved sequentially. By choosing n to be sufficiently large, the optimal exercise boundary $S^*(\tau)$ can be approximated to sufficient accuracy as desired.

Once $S_k^*, k = 1, 2, \cdots, n$, are known, the American put value can be approximated by

$$P(S,\tau) \approx P_n = p(S,\tau) + \frac{\Delta\tau}{2}\left[f\left(S, S_n^*; \tau_n, \tau_0\right) + f(S, S_0^*; \tau_n, \tau_n) \right. \\ \left. + 2\sum_{i=1}^{n-1} f(S, S_{n-i}^*; \tau_n, \tau_i)\right], \tag{5.3.11}$$

where $\tau = \tau_n$. Obviously, the limit of P_n tends to $P(S,\tau)$ as n tends to infinity. Similar to the compound option approximation method, one may apply the following extrapolation scheme

$$P(S,\tau) \approx \frac{9P_3 - 8P_2 + P_1}{2}, \tag{5.3.12}$$

where P_n is defined in Eq. (5.3.11). The numerical procedure of the recursive integration method is seen to be much less tedious compared to the compound option approximation method since only one-dimensional integrals are involved. Various versions of numerical schemes for more effective numerical valuation of American option values have been reported in the literature. For example, Ju (1998) proposes to price an American option by approximating its optimal exercise boundary as a multi-piece exponential function. The method is claimed to have the advantage of easy implementation since closed form formulas can be obtained in terms of the bases and exponents of the multi-piece exponential function.

One advantage of the recursive integration method is that the greeks of the American option values can also be found effectively without much additional efforts. For example, from the following formula for the delta of the American option price (see Problem 5.20):

$$\Delta = \frac{\partial P}{\partial S} = -N(-d_1) - \int_0^\tau \left[\frac{(r-q)e^{-q\xi}}{\sigma\sqrt{2\pi\xi}} e^{-\frac{d_{\xi,1}^2}{2}} + qe^{-q\xi}N(-d_{\xi,1})\right] d\xi, \tag{5.3.13}$$

one can easily deduce the numerical approximation to the delta Δ by approximating the above integral using the trapezoidal rule as follows:

$$\Delta \approx \Delta_n = -N(-d_1) - \frac{\Delta\tau}{2}\left[g(S, S_n^*;\tau_n, \tau_0) + g(S, S_0^*; \tau_n, \tau_n) \right. \\ \left. + 2\sum_{i=1}^{n-1} g(S, S_{n-i}^*; \tau_n, \tau_i)\right], \tag{5.3.14}$$

where

$$g(S, S^*(\tau-\xi); \tau, \xi) = \frac{(r-q)e^{-q\xi}}{\sigma\sqrt{2\pi\xi}} e^{-\frac{d_{\xi,1}^2}{2}} + qe^{-q\xi}N(-d_{\xi,1}) \\ d_{\xi,1} = \frac{\ln\frac{S}{S^*(\tau-\xi)} + \left(r - q + \frac{\sigma^2}{2}\right)\xi}{\sigma\sqrt{\xi}}. \tag{5.3.15}$$

5.3.3 Quadratic approximation method

The quadratic approximation method is first proposed by MacMillan (1986) for non-dividend paying stock options and later extended to commodity options by Barone-Adesi and Whaley (1987). This method has been proven to be quite efficient with reasonably good accuracy for valuation of American options, particularly for shorter lived options.

Recall that the governing equation for the price of a commodity option with a constant cost of carry b and riskless interest rate r is given by

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + bS\frac{\partial V}{\partial S} - rV, \tag{5.3.16}$$

where σ is the constant volatility of the asset price. We consider an American call option written on a commodity and define the early exercise premium by

$$e(S,\tau) = C(S,\tau) - c(S,\tau). \tag{5.3.17}$$

Inside the continuation region, Eq. (5.3.16) holds for both $C(S,\tau)$ and $c(S,\tau)$. Since the differential equation is linear, the same equation holds for $e(S,\tau)$. By writing $k_1 = 2r/\sigma^2$ and $k_2 = 2b/\sigma^2$, and defining

$$e(S,\tau) = K(\tau)f(S,K), \tag{5.3.18}$$

where $K(\tau)$ will be determined. Now, Eq. (5.3.16) can be transformed into the form

$$S^2\frac{\partial^2 f}{\partial S^2} + k_2 S\frac{\partial f}{\partial S} - k_1 f\left[1 + \frac{\frac{dK}{d\tau}}{rK}\left(1 + \frac{K\frac{\partial f}{\partial K}}{f}\right)\right] = 0. \tag{5.3.19}$$

A judicious choice for $K(\tau)$ is

$$K(\tau) = 1 - e^{-r\tau}, \tag{5.3.20}$$

so that Eq. (5.3.19) becomes

$$S^2\frac{\partial^2 f}{\partial S^2} + k_2 S\frac{\partial f}{\partial S} - \frac{k_1}{K}\left[f + (1-K)K\frac{\partial f}{\partial K}\right] = 0. \tag{5.3.21}$$

Note that the last term in the above equation contains the factor $(1-K)K$, and it becomes zero at $\tau = 0$ and in the limit $\tau \to \infty$. Further, it has a maximum value of $1/4$ at $K = 1/2$. Suppose we drop the quadratic term $(1-K)K\dfrac{\partial f}{\partial K}$, Eq. (5.3.21) is then reduced to an ordinary differential equation with the error being controlled by the magnitude of the quadratic term $(1-K)K$. This is how the name of this approximation method is derived. The approximate equation for f now becomes

$$S^2\frac{\partial^2 f}{\partial S^2} + k_2 S\frac{\partial f}{\partial S} - \frac{k_1}{K}f = 0, \tag{5.3.22}$$

where K is assumed to be non-zero. The special case where $K = 0$ can be considered separately (see Problem 5.31).

When K is treated as a parameter, Eq. (5.3.22) becomes an equi-dimensional differential equation. The general solution for $f(S)$ is given by

$$f(S) = c_1 S^{q_1} + c_2 S^{q_2}, \tag{5.3.23}$$

where c_1 and c_2 are arbitrary constants, q_1 and q_2 are roots of the auxiliary equation

$$q^2 + (k_2 - 1)q - \frac{k_1}{K} = 0. \tag{5.3.24}$$

Solving the above quadratic equation, we obtain

$$q_1 = -\frac{1}{2}\left[(k_2 - 1) + \sqrt{(k_2 - 1)^2 + 4\frac{k_1}{K}}\right] < 0, \tag{5.3.25a}$$

$$q_2 = \frac{1}{2}\left[-(k_2 - 1) + \sqrt{(k_2 - 1)^2 + 4\frac{k_1}{K}}\right] > 0. \tag{5.3.25b}$$

The term $c_1 S^{q_1}$ in Eq. (5.3.23) should be discarded since $f(S)$ tends to zero as S approaches 0. The approximate value $\widetilde{C}(S, \tau)$ of the American call option is then given by

$$C(S, \tau) \approx \widetilde{C}(S, \tau) = c(S, \tau) + c_2 K S^{q_2}. \tag{5.3.26}$$

Lastly, the arbitrary constant c_2 is determined by applying the value matching condition at the critical asset value S^*, namely, $\widetilde{C}(S^*, \tau) = S^* - X$. However, S^* itself is not yet known. The additional equation required to determine S^* is provided by the smooth pasting condition: $\dfrac{\partial \widetilde{C}}{\partial S}(S^*, \tau) = 1$ along the optimal exercise boundary. These two conditions together lead to the following pair of equations for c_2 and S^*

$$S^* - X = c(S^*, \tau) + c_2 K S^{*q_2} \tag{5.3.27a}$$

$$1 = e^{(b-r)\tau} N(d_1(S^*)) + c_2 K q_2 S^{*q_2 - 1} \tag{5.3.27b}$$

where

$$d_1(S^*) = \frac{\ln \frac{S^*}{X} + \left(b + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}. \tag{5.3.28}$$

By eliminating c_2 in Eqs. (5.3.27a,b), we obtain the following non-linear algebraic equation for $S^*(\tau)$:

$$S^* - X = c(S^*, \tau) + \left[1 - e^{(b-r)\tau} N(d_1(S^*))\right] \frac{S^*}{q_2}. \tag{5.3.29}$$

In summary, for $b < r$, the approximate value of the American commodity call option can be expressed as

$$\widetilde{C}(S,\tau) = c(S,\tau) + \frac{S^*}{q_2}\left[1 - e^{(b-r)\tau}N(d_1(S^*))\right]\left(\frac{S}{S^*}\right)^{q_2}, \quad S < S^*, \tag{5.3.30}$$

where S^* is obtained by solving Eq. (5.3.29). The last term in Eq. (5.3.30) gives an approximate value for the early exercise premium, which can be shown to be positive for $b < r$. When $b \geq r$, the American call will never be exercised prematurely (see Problem 5.2) so that the American call option value is the same as that of its European counterpart.

5.4 Options with voluntary reset rights

The reset right embedded in a financial derivative refers to the privilege given to the derivative holder to reset certain terms in the contract according to some specified rules. The reset may be done on the strike price or the maturity date of the derivative or both. The number of resets allowed within the life of the contract may be more than once. Usually there are some predetermined conditions that have to be met in order to activate a reset. The reset may be automatic upon the fulfilment of certain conditions or activated voluntarily by the holder. In this section, we confine our discussion to options with strike reset right, the holder of which can choose optimally the reset moment. We would like to analyze the *optimal reset strategies* adopted by the option holder.

We consider the reset-strike put option, where the strike price can be reset to the prevailing asset price at the reset moment. Let X denote the original strike price set at initiation of the option, S_{t^*} and S_T denote the asset price at the reset date t^* and expiration date T, respectively. Suppose there is only one reset right allowed, the terminal payoff of the reset put option is given by $\max(X - S_T, 0)$ if no reset occurs throughout the option's life, and modified to $\max(S_{t^*} - S_T, 0)$ if reset occurs at time $t^* < T$. Upon reset, the reset-strike put option effectively becomes an at-the-money put option.

The shout options are closely related to the reset-strike put options. Consider the shout option with the call payoff with only one shout right. Suppose the holder has chosen to shout at time t^*, then the terminal payoff is guaranteed to have the floor value $S_{t^*} - X$. More precisely, the terminal payoff is given by $\max(S_T - X, S_{t^*} - X)$ if the holder has shouted at t^* prior to maturity, but stays at the usual call payoff $\max(S_T - X, 0)$ if no shout occurs throughout the option's life. It will be shown later that the shout call can be replicated by a reset-strike put and a forward so that the reset-strike put option and its shout call counterpart follow the same optimal stopping policy [see Eq. (5.4.11)].

Another example of reset right is the *shout floor* feature in an index fund with a protective floor. Essentially, the shout floor feature gives the holder the right to shout at any time during the life of the contract to receive an at-the-money put option. In Sec. 5.4.1, we show how to obtain the closed form price formula of the shout floor feature (Dai *et al.*, 2004a). A similar feature of fund value protection can be found in equity-linked annuities. For example, the dynamic fund protection embedded in an investment fund provides a floor level of protection against a reference stock index, where the investor has the right to reset the fund value to that of the reference stock index. The protected fund may allow a finite number of resets throughout the life of the fund, and the reset instants can be chosen optimally by the investor. The fund holder also has the right to withdraw the fund prematurely. Details of pricing of the reset and withdrawal rights in dynamic fund protection can be found in Chu and Kwok's paper (2004).

There exist a wide variety of derivative instruments in the financial markets with embedded reset features. For example, Macquarie Bank in Australia offers a *Geared Equity Investment*. Basically, it is a collateralized loan with a reset put (Gray and Whaley, 1999). Macquarie Bank provides an investor with a loan, the proceeds of which are used to buy some Australian shares. The investor owns the shares and receives the dividends, but the shares are held by MacQuarie Bank as collateral for the loan. In addition, Macquarie Bank insures the investor against any share price decline by providing an optional reset feature that the strike price is automatically reset to the prevailing share price on a specified reset date (chosen by the investor at origination) should the share price exceed the original strike price. As another example, the Canadian segregated funds are mutual fund investments embedded with a long term maturity guarantee. These fund contracts contain multiple reset options that allow the holder to reset the guarantee level and the maturity date during the life of the contract. The optimal reset policies of options with combined reset rights on strike and maturity are analyzed in details by Dai and Kwok (2005b).

5.4.1 Valuation of the shout floor

The shout floor feature in an index fund gives the holder the right to shout at any time during the life of the contract to install a floor on the return of the fund, where the floor value is set at the prevailing index value S_{t^*} at the shouting time t^*. This shout floor feature gives the fund holder the upside potential of the index fund, while provides a guarantee on the return of the index at the floor value. In essence, the holder receives an at-the-money put option at the shout moment. By virtue of the guarantee on the return, the holder has the right to sell the index fund for the floor value at maturity of the contract. If no shout occurs throughout the life of the contract, then the fund value becomes zero. In summary, the terminal payoff of the shout floor is

$$\begin{cases} \max(S_{t^*} - S_T, 0) & \text{if shout has occurred} \\ 0 & \text{if no shout has occurred,} \end{cases}$$

where S_{t^*} and S_T are the index value at the shout moment t^* and maturity date T, respectively.

Formulation as a free boundary value problem
Interestingly, closed form price formula for the shout floor feature under the usual Black-Scholes pricing framework can be obtained. As usual, the stochastic process for the index value S under the risk neutral measure is assumed to follow the lognormal diffusion process

$$\frac{dS}{S} = (r-q)dt + \sigma\, dZ, \tag{5.4.1}$$

where r and q are the constant riskless interest rate and dividend yield, respectively, and σ is the constant volatility.

Let $V(S,\tau)$ denote the value of the shout floor feature. At the shout moment, the shout floor right is transformed into the ownership of an at-the-money European put option. The price function of an at-the-money put option is seen to be linearly homogeneous in S, which we write it as $Sp^*(\tau)$. By setting the strike price be the current asset price in the Black-Scholes put option price formula, we obtain

$$p^*(\tau) = e^{-r\tau}N(-d_2^*) - e^{-q\tau}N(-d_1^*), \tag{5.4.2a}$$

where

$$d_1^* = \frac{r-q+\frac{\sigma^2}{2}}{\sigma}\sqrt{\tau} \quad \text{and} \quad d_2^* = d_1^* - \sigma\sqrt{\tau}. \tag{5.4.2b}$$

The linear complementarity formulation of the free boundary value problem for the shout floor feature takes a similar form as that of an American option. Recalling that the exercise payoff is $Sp^*(\tau)$ and the terminal payoff is zero, we obtain the following linear complementarity formulation for $V(S,\tau)$

$$\begin{aligned}
&\frac{\partial V}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV \geq 0, \qquad V \geq Sp^*(\tau),\\
&\left[\frac{\partial V}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV\right][V - Sp^*(\tau)] = 0,\\
&V(S,0) = 0.
\end{aligned} \tag{5.4.3}$$

Since there is no strike price X appearing in the shout payoff, the pricing function $V(S,\tau)$ then becomes linearly homogeneous in S. We may write $V(S,\tau) = Sg(\tau)$, where $g(\tau)$ is to be determined. By substituting this assumed form of $V(S,\tau)$ into Eq. (5.4.3), we obtain the following set of variational inequalities for $g(\tau)$:

$$\begin{aligned}
&\frac{d}{d\tau}[e^{q\tau}g(\tau)] \geq 0, \quad g(\tau) \geq p^*(\tau),\\
&\frac{d}{d\tau}[e^{q\tau}g(\tau)]\,[g(\tau) - p^*(\tau)] = 0,\\
&g(0) = 0.
\end{aligned} \tag{5.4.4}$$

The form of solution for $g(\tau)$ depends on the analytic properties of the function $e^{q\tau}p^*(\tau)$. The derivative of $e^{q\tau}p^*(\tau)$ observes the following properties (see Fig. 5.8).

(i) If $r \leq q$, then

$$\frac{d}{d\tau}[e^{q\tau}p^*(\tau)] > 0 \quad \text{for} \quad \tau \in (0, \infty). \tag{5.4.5}$$

(ii) If $r > q$, then there exists a unique critical value $\tau^* \in (0, \infty)$ such that

$$\left.\frac{d}{d\tau}[e^{q\tau}p^*(\tau)]\right|_{\tau=\tau^*} = 0, \tag{5.4.6a}$$

and

$$\frac{d}{d\tau}[e^{q\tau}p^*(\tau)] > 0 \quad \text{for} \quad \tau \in (0, \tau^*), \tag{5.4.6b}$$

$$\frac{d}{d\tau}[e^{q\tau}p^*(\tau)] < 0 \quad \text{for} \quad \tau \in (\tau^*, \infty). \tag{5.4.6c}$$

The hints for the proof of these properties are given in Problem 5.34 [also see Dai *et al*. (2004a)].

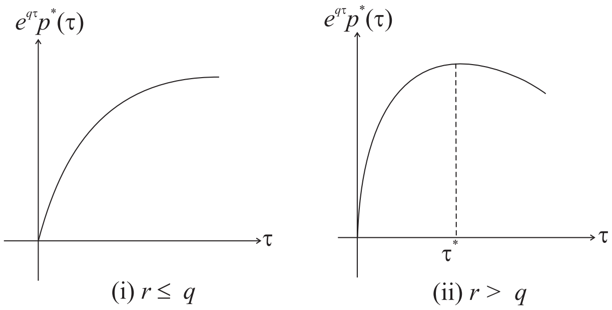


Fig. 5.8 Properties of the function $e^{q\tau}p^*(\tau)$ under (i) $r \leq q$, (ii) $r > q$.

The price function $V(S, \tau)$ of the shout floor takes different forms, depending on whether $r \leq q$ or $r > q$.

(i) $r \leq q$

By Eq. (5.4.5), we observe that $\frac{d}{d\tau}[e^{q\tau}p^*(\tau)]$ is strictly positive for all $\tau > 0$ and $p^*(0) = 0$. One then deduces that pricing formulation (5.4.4) can be satisfied by

$$g(\tau) = p^*(\tau), \qquad \tau \in (0, \infty). \tag{5.4.7}$$

(ii) $r > q$

By Eqs. (5.4.6b,c), we obtain in a similar manner that

$$g(\tau) = p^*(\tau) \quad \text{for} \quad \tau \in (0, \tau^*]. \tag{5.4.8}$$

However, when $\tau > \tau^*$, we cannot have $g(\tau) = p^*(\tau)$ since this would lead to $\frac{d}{d\tau}[e^{q\tau}g(\tau)] = \frac{d}{d\tau}[e^{q\tau}p^*(\tau)] \geq 0$, contradicting the result in Eq. (5.4.6c). By Eq. (5.4.4), we must have $\frac{d}{d\tau}[e^{q\tau}g(\tau)] = 0$ for $\tau \in (\tau^*, \infty)$. Together with the auxiliary condition: $g(\tau^*) = p^*(\tau^*)$, the solution is given by

$$g(\tau) = e^{-q(\tau-\tau^*)}p^*(\tau^*) \quad \text{for} \quad \tau \in (\tau^*, \infty). \tag{5.4.9}$$

In summary, the optimal shouting policy adopted by the holder of the shout floor depends on the relative magnitude of r and q. When $r \leq q$, the holder should shout at once at any time and at any index value level to install the protective floor. When $r > q$, there exists a critical time earlier than which it is never optimal for the holder to shout. However, the holder should shout at once at any index value level once τ falls to the critical value τ^*.

5.4.2 Reset-strike put options

The reset feature embedded in the reset-strike put option allows the holder to reset the original strike price to the prevailing asset price at the reset moment chosen by the holder. The reset-strike put is very similar to the shout floor since the holder receives an at-the-money put option upon reset, except that the reset-strike put has an *initial strike price* X even reset does not occur throughout the life of the contract. Similar to Eq. (5.4.3), the linear complementarity formulation for the price function $U(S, \tau)$ of the reset-strike put option is given by

$$\begin{aligned} &\frac{\partial U}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 U}{\partial S^2} - (r-q)S\frac{\partial U}{\partial S} + rU \geq 0 \quad U \geq Sp^*(\tau), \\ &\left[\frac{\partial U}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 U}{\partial S^2} - (r-q)S\frac{\partial U}{\partial S} + rU\right][U - Sp^*(\tau)] = 0, \\ &\quad U(S, 0) = \max(X - S, 0). \end{aligned} \tag{5.4.10}$$

Unlike the shout floor, the terminal payoff of the reset-strike put option contains the initial strike price X. Now, $U(S, \tau)$ is no longer linear homogeneous in S. From financial intuition, the holder should shout only when the asset price reaches some sufficiently high critical level $S^*(\tau)$ to install a new strike. Obviously, $S^*(\tau)$ must be greater than X. Similar to the American option models, the optimal reset boundary is not known *a prior* but has to be solved in the solution procedure of the above free boundary value problem. A schematic plot of $U(S, \tau)$ against S is shown in Fig. 5.9.

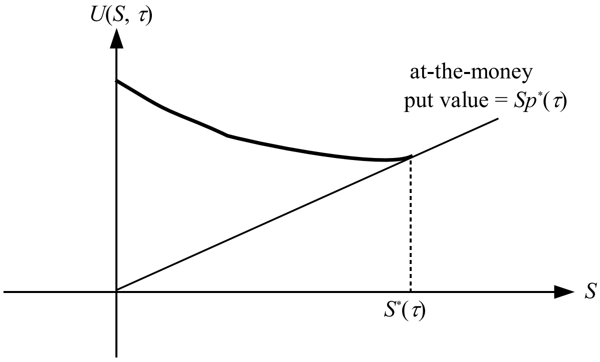


Fig. 5.9 The price curve of the reset-strike put touches tangentially the line representing the at-the-money put value at $S = S^*(\tau)$.

Parity relation between reset-strike put and shout call
Consider the portfolio of holding a reset-strike put and a forward contract. Both derivatives have the same maturity date, and the forward price is taken to be the same as the strike price. The terminal payoff of this portfolio is given by

$$\begin{cases} \max(X - S_T, 0) + S_T - X = \max(S_T - X, 0) & \text{if no reset occurs} \\ \max(S_{t^*} - S_T, 0) + S_T - X = \max(S_T - X, S_{t^*} - X) & \text{if reset occurs} \end{cases}.$$

Here, S_{t^*} is the prevailing asset price at the reset moment t^*. The above payoff structure is identical to that of a shout call. Hence, the shout call can be replicated by a combination of a reset-strike put and a forward. As a consequence, the reset-strike put and shout call should share the same optimal reset/shout policy. Let $W(S,\tau)$ denote the price of the shout call. The parity relation between the prices of reset-strike put and shout call is given by

$$W(S,\tau) = U(S,\tau) + Se^{-q\tau} - Xe^{-r\tau}. \tag{5.4.11}$$

Characterization of the optimal reset policy
We examine the characterization of the optimal reset boundary $S^*(\tau)$ of the strike-reset put option, in particular, the asymptotic behaviors at $\tau \to 0^+$ and $\tau \to \infty$. Since the new strike price upon reset should not be lower than the original strike price, we should have

$$S^*(\tau) \geq X. \tag{5.4.12}$$

Similar to the American call, $S^*(\tau)$ of the reset-strike put is monotonically increasing with respect to τ. Unlike the American call, $S^*(\tau)$ starts at X at $\tau \to 0^+$, independent of r and q. To show the claim, we define

$$D(S,\tau) = U(S,\tau) - Sp^*(\tau) \tag{5.4.13}$$

and note that $D(S,\tau) \geq 0$ for all S and τ. In the continuation region, $D(S,\tau)$ satisfies

$$\frac{\partial D}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 D}{\partial S^2} - (r-q)S\frac{\partial D}{\partial S} + rD = -S[p^{*\prime}(\tau) + qp^*(\tau)], \quad 0 < S < S^*(\tau), \quad \tau > 0. \tag{5.4.14}$$

As $\tau \to 0^+$, we have $-S[p^{*\prime}(\tau) + qp^*(\tau)] \to \infty$. Supposing $S^*(0^+) > X$ and considering $S \in (X, S^*(0^+))$, we have $D(S, 0^+) = 0$ so that

$$\frac{\partial D}{\partial \tau}(S, 0^+) = -S[p^{*\prime}(0^+) + qp(0^+)] < 0. \tag{5.4.15}$$

This would imply $D(S, 0^+) < 0$, a contradiction to $D(S,\tau) \geq 0$ for all τ. Hence, we must have $S^*(0^+) \leq X$. Together with Eq. (5.4.12), we conclude that $S^*(0^+) = X$.

Next, we examine the asymptotic behavior of $S^*(\tau)$ at $\tau \to \infty$. Let $W^\infty(S) = \lim_{\tau\to\infty} e^{r\tau}U(S,\tau)$. The existence of $W^\infty(S)$ requires the existence of $\lim_{\tau\to\infty} e^{r\tau}p^*(\tau)$. It can be shown that when $r \leq q$, we have

$$\lim_{\tau\to\infty} e^{r\tau}p^*(\tau) = 1, \tag{5.4.16}$$

and the limit diverges when $r > q$. The governing differential equation formulation for $W^\infty(S)$ is given by

$$\frac{\sigma^2}{2}S^2\frac{d^2W^\infty}{dS^2} + (r-q)S\frac{dW^\infty}{dS} = 0, \quad 0 < S < S^*_\infty,$$
$$W^\infty(0) = X, \quad W^\infty(S^*_\infty) = S^*_\infty \quad \text{and} \quad \frac{dW^\infty}{dS}(S^*_\infty) = 1. \tag{5.4.17}$$

By following similar procedures as in Sec. 5.1.3, the solution to $W^\infty(S)$ can be found to be (see Problem 5.36)

$$W^\infty(S) = X + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\frac{S^{1+\alpha}}{X^\alpha}, \quad 0 < S < S^*_\infty, \tag{5.4.18}$$

where

$$S^*_\infty = \left(1 + \frac{1}{\alpha}\right)X \quad \text{and} \quad \alpha = \frac{2(q-r)}{\sigma^2}. \tag{5.4.19}$$

When $r < q, S^*(\tau)$ is defined for all $\tau > 0$ with the asymptotic limit $\left(1 + \frac{1}{\alpha}\right)X$ at $\tau \to \infty$. In particular, S^*_∞ becomes infinite when $r = q$.

Now, we consider the case $r > q$. Recall that it is never optimal to exercise the shout floor when $\tau > \tau^*$ [τ^* can be obtained by solving Eq. (5.4.6a)]. Since the reset-strike put is more expensive than the shout floor and their exercise payoffs are the same, so it is also never optimal to exercise the reset-strike put when $\tau > \tau^*$. We write the optimal reset boundary of the reset-strike put as $S^*(\tau; X)$, with dependence on the strike price X. When $X = 0$, it corresponds to the shout floor and $S^*(\tau; 0)$ is known to be zero. When $X \to \infty, S^*(\tau; \infty)$ becomes infinite since it is never optimal to reset at any asset value when the strike price is already at infinite value. One then argues that $S^*(\tau; X)$ is finite when X is finite, $\tau < \tau^*$. When $\tau \to \tau^{*-}, S(\tau; X)$ becomes infinite. In Fig. 5.10, we illustrate the behaviors of $S^*(\tau)$ under the two separate cases: $r < q$ and $r > q$. More detialed discussion of the pricing behaviors of the reset-strike put options can be found in Dai *et al.*'s paper (2004a).

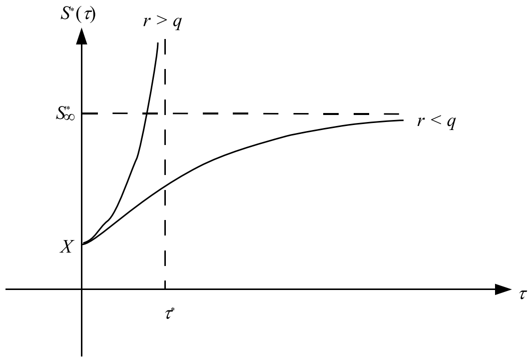


Fig. 5.10 Plot of the optimal reset boundary $S^*(\tau)$ of the reset-strike put against τ. When $r < q, S^*(\tau)$ is defined for all τ and there is a finite asymptotic limit S^*_∞. When $r > q, S^*(\tau)$ is defined only for $\tau \in (0, \tau^*)$.

Multi-reset put options

We consider the pricing formulation of a put option with multiple rights to reset the strike price throughout the option's life. Let $U_n(S, \tau; X)$ denote the price function of the n-reset put option. Upon the j^{th} reset, the reset put becomes an at-the-money $(j-1)$-reset put, where the strike price equals the prevailing asset price at the reset instant. Let t_j denote the time of the j^{th} reset and S_j^* denote the critical asset value at the reset instant t_j^*. The strike price of the reset put with j reset rights remaining is denoted by S_{j+1}^*. For notational convenience, we write $S_{n+1}^* = X$. It is obvious that $S_{j+1}^* < S_j^*, j = 1, 2, \cdots, n$, and $U_{j+1}(S, \tau; X) > U_j(S, \tau; X)$ for all S and τ.

The price function $U_j(S, \tau; X)$ observes linear homogeneity in S and X so that

$$U_j(S, \tau; X) = XU_j\left(\frac{S}{X}, \tau; 1\right). \tag{5.4.20}$$

When the reset put is at-the-money, $S/X = 1$ and this leads to

$$U_j(S,\tau;S) = SU_j(1,\tau;1). \tag{5.4.21}$$

We write $p_j(\tau) = U_j(1,\tau;1), j = 0,1,\cdots,n-1$. The linear complementarity formulation of the pricing model of the n-reset put option is given by

$$\begin{aligned}
&\frac{\partial U_n}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 U_n}{\partial S^2} - (r-q)S\frac{\partial U_n}{\partial S} + rU_n \geq 0, \quad U_n \geq Sp_{n-1}(\tau),\\
&\left[\frac{\partial U_n}{\partial \tau} - \frac{\sigma^2}{2}S^2\frac{\partial^2 U_n}{\partial S^2} - (r-q)S\frac{\partial U_n}{\partial S} + rU_n\right][U_n - Sp_{n-1}(\tau)] = 0,\\
&U_n(S,0) = \max(X-S,0).
\end{aligned} \tag{5.4.22}$$

One has to solve step by step for U_n, starting from $U_1, U_2, \cdots$. For the perpetual n-reset strike put, it is possible to obtain the optimal reset price in closed form when $r < q$. Let $S^*_{n,\infty}$ denote $\lim_{\tau\to\infty} S^*_n(\tau)$. For $r < q$, we have

$$S^*_{n,\infty} = \left(1 + \frac{1}{\alpha}\right)\frac{X}{\beta_n}, \tag{5.4.23a}$$

where $\alpha = \dfrac{2(q-r)}{\sigma^2}$, $\beta_1 = 1$ and

$$\beta_n = 1 + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\beta_{n-1}^{1+\alpha}. \tag{5.4.23b}$$

The hints to derive $S^*_{n,\infty}$ are outlined in Problem 5.36. Taking the limit $n \to \infty$, we obtain

$$\lim_{n\to\infty} \beta_n = 1 + \frac{1}{\alpha}, \tag{5.4.24b}$$

giving

$$\lim_{n\to\infty} S^*_{n,\infty} = X. \tag{5.4.24c}$$

Since $S^*_n(\tau)$ is an increasing function of τ and $S^*_n(\tau) \geq X$, we then deduce that

$$\lim_{n\to\infty} S^*_n(\tau) = X \quad \text{for all} \quad \tau. \tag{5.4.25}$$

How to interpret the above result? When $r < q$, the holder of an infinite-reset put should exercise the reset right whenever the option becomes in-the-money. More precisely, the holder always resets whenever a new maximum value of the asset value is realized. The terminal payoff of the infinite-reset put then becomes $\max(M_0^T - S_T, X - S_T)$, a payoff involving the lookback variable M_0^T, where $M_0^T = \max_{0\leq t\leq T} S_t$. The pricing model of the infinite-reset put is no longer a free boundary value problem. Rather, it becomes a lookback option model [see Dai *et al.*'s paper (2003)].

5.5 Problems

5.1 Find the value of an American vanilla put option when (i) riskless interest rate $r = 0$, (ii) volatility $\sigma = 0$, (iii) strike price $X = 0$, (iv) asset price $S = 0$.

5.2 Find the lower and upper bounds on the difference of the values of the American put and call options on a commodity with cost of carry b.

5.3 Consider an American call option whose underlying asset price follows a Geometric Brownian process. Show that

$$C(\lambda S, \tau) - C(S, \tau) \leq (\lambda - 1)S, \quad \lambda \geq 1.$$

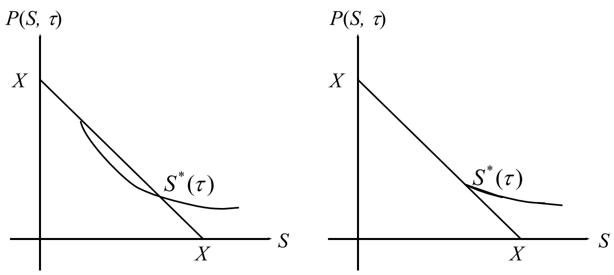
5.4 Explain why an American call (put) futures option is worth more (less) than the corresponding American call (put) option on the underlying asset, when the cost of carry of the underlying asset is positive. Also, why the difference in prices widens when the maturity date of the futures goes beyond the expiration date of the option.

5.5 We would like to show by heuristic arguments that the American price function $P(S, \tau)$ satisfies the smooth pasting condition

$$\left.\frac{\partial P}{\partial S}\right|_{S=S^*(\tau)} = -1$$

at the optimal exercise price $S^*(\tau)$. Consider the behaviors of the American price curve near $S^*(\tau)$ under the following two scenarios:

(i) $\left.\dfrac{\partial P}{\partial S}\right|_{S=S^*(\tau)} < -1$ and (ii) $\left.\dfrac{\partial P}{\partial S}\right|_{S=S^*(\tau)} > -1$.



(a) When $\left.\dfrac{\partial P}{\partial S}\right|_{S=S^*(\tau)} < -1$, the price curve $P(S, \tau)$ at value of S close to but greater than $S^*(\tau)$ falls below the intrinsic value line (see the top left figure).

(b) When $\left.\dfrac{\partial P}{\partial S}\right|_{S=S^*(\tau)} > -1$, argue why the value of the American put option at asset price level close to $S^*(\tau)$ can be increased by choosing a smaller value for $S^*(\tau)$ (see the top right figure).

Explain why both cases do not correspond to the optimal exercise strategy of an American put. Hence, the slope of the American put price curve at $S^*(\tau)$ must satisfy the smooth pasting condition.

5.6 Explain why, when $q \geq r$, an American call on a continuous dividend paying asset which is optimally held to expiration will have zero value at expiration (Kim, 1990).

5.7 Let $P(S, \tau; X, r, q)$ denote the price function of an American put. Show that $P(X, \tau; S, q, r)$ also satisfies the Black-Scholes equation:

$$\frac{\partial P}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 P}{\partial S^2} + (r - q) S \frac{\partial P}{\partial S} - rP$$

together with the auxiliary conditions:

$$\begin{aligned} &P(X, 0; S, q, r) = \max(S - X, 0) \\ &P(X, \tau; S, q, r) \geq \max(S - X, 0) \quad \text{for } \tau > 0. \end{aligned}$$

Note that the auxiliary conditions are identical to those of the price function of the American call. Hence, we can conclude that

$$C(S, \tau; X, r, q) = P(X, \tau; S, q, r).$$

Hint: Write $P(S', \tau) = P\left(\frac{1}{S}, \tau; \frac{1}{X}, q, r\right) = \frac{1}{SX} P(X, \tau; S, q, r)$, show that

$$\begin{aligned} &\frac{\partial}{\partial \tau}[SXP(S', \tau)] - \frac{\sigma^2}{2} S^2 \frac{\partial^2}{\partial S^2}[SXP(S', \tau)] \\ &- (r - q) S \frac{\partial}{\partial S}[SXP(S', \tau)] + rSXP(S', \tau) \\ &= SX \left[\frac{\partial P}{\partial \tau}(S', \tau) - \frac{\sigma^2}{2} S'^2 \frac{\partial^2 P}{\partial S'^2}(S', \tau) \right. \\ &\left. -(q - r) S' \frac{\partial P}{\partial S'}(S', \tau) + qP(S', \tau) \right]. \end{aligned}$$

5.8 From the put-call symmetry relation for the prices of American call and put options derived in Problem 5.7, show that

$$\begin{aligned} \frac{\partial C}{\partial S}(S, \tau; X, r, q) &= \frac{\partial P}{\partial X}(X, \tau; S, q, r) \\ \frac{\partial C}{\partial q}(S, \tau; X, r, q) &= \frac{\partial P}{\partial r}(X, \tau; S, q, r). \end{aligned}$$

Give financial interpretation of the results.

5.9 Consider the pair of American call and put options with the same time to expiry τ and on the same underlying asset. Assume the volatility of the asset price to be at most time dependent. Let S_C and S_P be the spot asset price corresponding to the call and put, respectively (S_C and S_P need not be the same since the calendar times at which we are comparing values need not be the same). Suppose the two options have the same moneyness, that is,

$$\frac{S_C}{X_C} = \frac{X_P}{S_P},$$

where X_C and X_P are the strike price corresponding to the call and put, respectively. Let $C(S_C, \tau; X_C, r, q)$ and $P(S_P, \tau; X_P, r, q)$ denote the price function of the American call and put, respectively. Derive the generalized put-call symmetry relation (Carr and Chesney, 1996)

$$\frac{C(S_C, \tau; X_C, r, q)}{\sqrt{S_C X_C}} = \frac{P(S_P, \tau; X_P, q, r)}{\sqrt{S_P X_P}}.$$

Furthermore, let $S_C^*(\tau; X_C, r, q)$ and $S_P^*(\tau; X_P, r, q)$ denote the optimal exercise price of the American call and put, respectively. Show that

$$S_C^*(\tau; X_C, r, q) S_P^*(\tau; X_P, q, r) = X_C X_P.$$

This relation is a generalization of the result given in Eq. (5.1.23).

5.10 Let H denote the barrier of a perpetual American down-and-out call option. The governing equation for the price of the perpetual American barrier option $C_\infty(S; r, q)$ is given by

$$\frac{\sigma^2}{2} S^2 \frac{d^2 C_\infty}{dS^2} + (r - q) S \frac{dC_\infty}{dS} - r C_\infty = 0, \quad H < S < S_\infty^*,$$

where S_∞^* is the optimal exercise price. Determine S_∞^* and find the option price $C_\infty(S; r, q)$.

Hint: The optimal exercise price is determined by maximizing the solution for the perpetual American call price among all possible exercise prices, that is,

$$C_\infty(S; r, q) = \max_{S_\infty^*} \left\{ \frac{S_\infty^* - X}{H^{\lambda_+} {S_\infty^*}^{\lambda_-} - {S_\infty^*}^{\lambda_+} H^{\lambda_-}} (H^{\lambda_+} S^{\lambda_-} - H^{\lambda_-} S^{\lambda_+}) \right\},$$

where λ_+ and λ_- are roots of the quadratic equation:

$$\frac{\sigma^2}{2} \lambda(\lambda - 1) + (r - q)\lambda - r = 0.$$

5.11 Suppose the continuous dividend paid by an asset is at the constant rate d but not proportional to the asset price S. Show that the American call option on the above asset would not be exercised prematurely if $d < rX$ where r is the riskless interest rate and X is the strike price. Under the above condition, show that the price of the perpetual American call option is given by (Merton, 1973, Chap. 1)

$$C(S,\infty;X) = S - \frac{d}{r}\left[1 - \frac{\left(\frac{2d}{\sigma^2 S}\right)^{\frac{2r}{\sigma^2}}}{\Gamma\left(2+\frac{2r}{\sigma^2}\right)}\, M\left(\frac{2r}{\sigma^2}, 2+\frac{2r}{\sigma^2}, -\frac{2d}{\sigma^2 S}\right)\right],$$

where Γ and M denote the Gamma function and the confluent hypergeometric function, respectively.

5.12 Consider an American call option with a continuously changing strike price $X(\tau)$ where $\frac{dX(\tau)}{d\tau} < 0$. The auxiliary conditions for the American call option model are given by

$$C(S,\tau;X(\tau)) \geq \max(S - X(\tau), 0)$$

and

$$C(S,\tau;X(0)) = \max(S - X(0), 0).$$

Define the following new set of variables:

$$\xi = \frac{S}{X(\tau)} \quad \text{and} \quad F(\xi,\tau) = \frac{C(S,\tau;X(\tau))}{X(\tau)}.$$

Show that the governing equation for the price of the above American call is given by

$$\frac{\partial F}{\partial \tau} = \frac{\sigma^2}{2}\,\xi^2\,\frac{\partial^2 F}{\partial \xi^2} + \eta(\tau)\,\xi\,\frac{\partial F}{\partial \xi} - \eta(\tau)F,$$

where $\eta(\tau) = r + \frac{1}{X}\frac{\partial X}{\partial \tau}$ and r is the riskless interest rate. The auxiliary conditions become

$$F(\xi,0) = \max(\xi - 1, 0) \quad \text{and} \quad F(\xi,\tau) \geq \max(\xi - 1, 0).$$

Show that if $X(\tau) \geq X(0)e^{-r\tau}$, then it is never optimal to exercise the American call prematurely. In such a case, show that the value of the above American call is the same as that of a European call with a fixed strike price $X(0)$ (Merton, 1973, Chap. 1).

Hint: Show that when the time dependent function $\eta(\tau)$ satisfies the condition $\int_0^\tau \eta(s)\, ds \geq 0$, it is then never optimal to exercise the American call prematurely.

5.13 Consider the one-dividend American call option model. Explain why the exercise price S_d^*, which is obtained by solving Eq. (5.1.37), decreases when the dividend amount D increases. Also, show that S_d^* tends to infinity when D falls to the value $X[1 - e^{-r(T-t_d)}]$.

5.14 Give a mathematical proof to the following inequality

$$C_d(\widetilde{S}, T-t; X) \geq \max\{c(\widetilde{S}, T-t; X), c(S, t_d - t; X)\}, \quad t < t_d,$$

which arises from the Black approximation formula for the one-dividend American call (see Sec. 5.1.5). Here, t_d and T are the ex-dividend and expiration dates, respectively; S and $\widetilde{S}$ are the market asset price and the asset price net of the present value of the escrowed dividend, respectively.

5.15 Suppose discrete dividends of amount $D_1, D_2, \cdots, D_n$ are paid at the respective ex-dividend dates $t_1, t_2, \cdots, t_n$ and let t_{n+1} denote the date of expiration T. Show that the risky component is given by

$$\widetilde{S} = S - \sum_{k=j+1}^{n} D_k e^{-r(t_k - t)} \text{ for } t_j^+ \leq t \leq t_{j+1}^-, \quad j = 0, 1, \cdots, n,$$

and $t_0 = 0$.
Hint: Extend the result in Eq. (5.1.39).

5.16 Consider an American call option on an asset which pays discrete dividends at anticipated dates $t_1 < t_2 < \cdots < t_n$. Let the size of the dividends be, respectively, $D_1, D_2, \cdots, D_n$, and $T = t_{n+1}$ be the time of expiration. Show that it is never optimal to exercise the American call at any time prior to expiration if all the discrete dividends are not sufficiently deep, as indicated by the following inequality

$$D_i \leq X[1 - e^{-r(t_{i+1} - t_i)}], \quad i = 1, 2, \cdots, n.$$

5.17 In the two-dividend American call option model, we assume discrete dividends of amount D_1 and D_2 are paid out by the underlying asset at times t_1 and t_2, respectively. Let $\widetilde{S}_t$ denote the asset price at time t, net of the present value of escrowed dividends and $\tilde{S}_{t_1}^*$ ($\tilde{S}_{t_2}^*$) denote the optimal exercise price at time t_1 (t_2) above which the American call should be exercised prematurely. Let r, σ, X and T denote the riskless interest rate, volatility of $\widetilde{S}$, strike price and expiration time, respectively. Let $C(\widetilde{S}_t, t)$ denote the value of the American call at time t. Show that $\tilde{S}_{t_1}^*$ and $\tilde{S}_{t_2}^*$ are given by the solution of the following non-linear algebraic equations

$$\begin{aligned} C(\widetilde{S}_{t_1}^*, t_1) &= \tilde{S}_{t_1}^* \, [1 - N_2(-a_1, -b_1; \rho)] + D_2 e^{-r(t_2 - t_1)} N(a_2) \\ &\quad - X[e^{-r(t_2 - t_1)} N(a_2) + e^{-r(T - t_1)} N_2(-a_2, b_2; -\rho)] \\ C(\widetilde{S}_{t_2}^*, t_2) &= \tilde{S}_{t_2}^* \, N(v_1) - X e^{-r(T - t_2)} N(v_2), \end{aligned}$$

where

$$a_2 = \frac{\ln \frac{\tilde{S}^*_{t_1}}{\tilde{S}^*_{t_2}} + \left(r - \frac{\sigma^2}{2}\right)(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}}, \qquad a_1 = a_2 + \sigma\sqrt{t_2 - t_1},$$

$$b_2 = \frac{\ln \frac{\tilde{S}^*_{t_1}}{X} + \left(r - \frac{\sigma^2}{2}\right)(T - t_1)}{\sigma\sqrt{T - t_1}}, \qquad b_1 = b_2 + \sigma\sqrt{T - t_1},$$

$$v_2 = \frac{\ln \frac{\tilde{S}^*_{t_2}}{X} + \left(r - \frac{\sigma^2}{2}\right)(T - t_2)}{\sigma\sqrt{T - t_2}}, \qquad v_1 = v_2 + \sigma\sqrt{T - t_2}.$$

The American call price is given by (Welch and Chen, 1988)

$$\begin{aligned} C(\widetilde{S}_t, t) = \widetilde{S}_t\, &[1 - N_3(-f_1, -g_1, -h_1; \rho_{12}, \rho_{13}, \rho_{23})] \\ &- X[e^{-r(t_1-t)}N(f_2) + e^{-r(t_2-t)}N_2(-f_2, g_2; -\rho_{12}) \\ &+ e^{-r(T-t)}N_3(-f_2, -g_2, h_2; \rho_{12}, -\rho_{13}, -\rho_{23})] \\ &+ D_1 e^{-r(t_1-t)}N(f_2) \\ &+ D_2 e^{-r(t_2-t)}[N(f_2) + N_2(-f_2, g_2; -\rho_{12})], \end{aligned}$$

where

$$\rho_{12} = \sqrt{\frac{t_1 - t}{t_2 - t}}, \quad \rho_{13} = \sqrt{\frac{t_1 - t}{T - t}}, \quad \rho_{23} = \sqrt{\frac{t_2 - t}{T - t}},$$

$$f_2 = \frac{\ln \frac{\widetilde{S}_t}{\tilde{S}^*_{t_1}} + \left(r - \frac{\sigma^2}{2}\right)(t_1 - t)}{\sigma\sqrt{t_1 - t}}, \qquad f_1 = f_2 + \sigma\sqrt{t_1 - t},$$

$$g_2 = \frac{\ln \frac{\widetilde{S}_t}{\tilde{S}^*_{t_2}} + \left(r - \frac{\sigma^2}{2}\right)(t_2 - t)}{\sigma\sqrt{t_2 - t}}, \qquad g_1 = g_2 + \sigma\sqrt{t_2 - t},$$

$$h_2 = \frac{\ln \frac{\widetilde{S}_t}{X} + \left(r - \frac{\sigma^2}{2}\right)(T - t)}{\sigma\sqrt{T - t}}, \qquad h_1 = h_2 + \sigma\sqrt{T - t}.$$

5.18 Consider the one-dividend American put option model where the discrete dividend at time t_d is paid at the known rate λ, that is, the dividend payment is λS_{t_d}. Show that the slope of the optimal exercise boundary of the American put at time right before t_d is given by (Meyer, 2001)

$$\lim_{t \to t_d^-} \frac{dS^*(t)}{dt} = \frac{r}{\lambda}X,$$

where r is the riskless interest rate.

Hint: Consider the balance of the gain in interest income from the strike price and the loss in dividend over the differential time interval $\triangle t$ right before t_d.

5.19 Bunch and Johnson (2000) give the following three different definitions of the optimal exercise price of an American put.

1. It is the value of the stock price at which one is indifferent between exercising and not exercising the put.
2. It is the highest value of the stock price for which the value of the put is equal to the exercise price less the stock price.
3. It is the highest value of the stock price at which the put value does not depend on time to maturity.

Give financial interpretation to the above three definitions.

5.20 Show that the delta of the price of an American put option on an asset which pays a continuous dividend yield at the rate q is given by

$$\frac{\partial P}{\partial S} = -N(-d_1) - \int_0^\tau \left[\frac{(r-q)e^{-q\xi}}{\sigma\sqrt{2\pi\xi}} e^{-\frac{d_{\xi,1}^2}{2}} + qe^{-q\xi}N(-d_{\xi,1})\right] d\xi,$$

where

$$d_1 = \frac{\ln\frac{S}{X} + \left(r - q + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}},$$

$$d_{\xi,1} = \frac{\ln\frac{S}{S^*(\tau-\xi)} + \left(r - q + \frac{\sigma^2}{2}\right)\xi}{\sigma\sqrt{\xi}}, \qquad d_{\xi,2} = d_{\xi,1} - \sigma\sqrt{\xi}.$$

Examine the sign of the delta of the early exercise premium when $r \geq q$ and $r < q$. Give financial interpretation of the sign behaviors of the above delta. Furthermore, show that

$$\begin{aligned}\frac{\partial^2 P}{\partial S^2} &= \frac{1}{S\sigma\sqrt{2\pi\tau}} e^{-d_1^2/2} \\ &\quad + \int_0^\tau \left[\frac{(r-q)e^{-q\xi}}{S\sigma^2\xi\sqrt{2\pi}} d_{\xi,1} e^{-\frac{d_{\xi,1}^2}{2}} + \frac{qe^{-q\xi}}{S\sigma\sqrt{2\pi\xi}} e^{-\frac{d_{\xi,1}^2}{2}}\right] d\xi.\end{aligned}$$

Find similar expressions for $\frac{\partial P}{\partial \sigma}, \frac{\partial P}{\partial r}$ and $\frac{\partial P}{\partial X}$ (Huang *et al.*, 1996).

5.21 Consider an American put option on an asset which pays no dividend. Show that the early exercise premium $e(S, \tau; X)$ is bounded by

$$rX \int_0^\tau e^{-r\xi} N(-\widetilde{d_\xi})\, d\xi \leq e(S, \tau; X) \leq rX \int_0^\tau e^{-r\xi} N(-\widehat{d_\xi})\, d\xi,$$

where

$$\widetilde{d_\xi} = \frac{\ln\frac{S^*(\tau)}{S^*(0)} + \left(r - \frac{\sigma^2}{2}\right)\xi}{\sigma\sqrt{\xi}}, \qquad \widehat{d_\xi} = \frac{\ln\frac{S^*(\tau)}{S^*(\infty)} + \left(r - \frac{\sigma^2}{2}\right)\xi}{\sigma\sqrt{\xi}},$$

$$S^*(\infty) = \frac{X}{1 + \frac{\sigma^2}{2r}}, \qquad S^*(0) = X.$$

5.22 Let $S_C^*(\infty)$ denote $\lim\limits_{\tau\to\infty} S_C^*(\tau)$, where $S_C^*(\tau)$ is the solution to the integral equation defined in Eq. (5.2.27). By taking the limit $\tau \to \infty$ of the above integral equation, solve for $S_C^*(\infty)$. Compare the result given in Eq. (5.1.19).

5.23 By considering the corresponding integral representation of the early exercise premium of an American commodity option with cost of carry b, show that

(a) when $b \geq r$, r is the riskless interest rate, there is no advantage of early exercise for the American commodity call option;

(b) advantage of early exercise always exists for the American commodity put option for all values of b.

5.24 Let $C_{do}(S,\tau;X,H,r,q)$ and $P_{uo}(S,\tau;X,H,r,q)$ denote the price function of an American down-and-out barrier call and an American up-and-out barrier put, respectively, both with constant barrier level H. Show that the put-call symmetry relation for the prices of the American barrier call and put options is given by (Gao *et al.*, 2000)

$$C_{do}(S,\tau;X,H,r,q) = P_{uo}(X,\tau;SX/H,q,r).$$

Let $S^*_{do,call}(\tau;X,H,r,q)$ and $S^*_{uo,put}(\tau;X,H,r,q)$ denote the optimal exercise price of the American down-and-out call and American up-and-out put, respectively. Show that

$$S^*_{do,call}(\tau;X,H,r,q) = \frac{X^2}{S^*_{uo,put}(\tau;X,X^2/H,q,r)}.$$

5.25 Consider an American up-and-out put option with barrier level $B(\tau) = B_0 e^{-\alpha\tau}$ and strike price X. Assuming that the underlying asset pays a continuous dividend yield q, find the integral representation of the early exercise premium. What would be the effect on the optimal exercise price $S^*(\tau;B(\tau))$ when B_0 decreases?

5.26 Consider a down-and-in American call $C_{di}(S,\tau;X,B)$, where the down-and-in trigger clause entitles the holder to receive an American call option with strike price X when the asset price S falls below the threshold level B. The underlying asset pays dividend yield q and let r denote the riskless interest rate. Let $C(S,\tau;X)$ and $c(S,\tau;X)$ denote the price function of the American call and European call with strike price X, respectively. Show that when $B \leq \max\left(X, \frac{r}{q}X\right)$

$$\begin{aligned} &C_{di}(S,\tau;X,B) \\ &= \left(\frac{S}{B}\right)^{1-\frac{2(r-q)}{\sigma^2}} \left[C\left(\frac{B^2}{S},\tau;X\right) - c\left(\frac{B^2}{S},\tau;X\right)\right] + c_{di}(S,\tau;X,B), \end{aligned}$$

where $c_{di}(S, \tau; X, B)$ is the price function of the European down-and-in call counterpart. Find the corresponding form of the price function $C_{di}(S, \tau; X, B)$ when (i) $B \geq S^*(\infty)$ and (ii) $S^*(0^+) < B < S^*(\infty)$, where $S^*(\tau)$ is the optimal exercise boundary of the American non-barrier call $C(S, \tau; X)$ (Dai and Kwok, 2004b).

5.27 The exercise payoff of an American capped call with the cap L is given by $\max(\min(S, L\ ) - X, 0)$, $L > X$. Let $S^*_{cap}(\tau)$ and $S^*(\tau)$ denote the early exercise boundary of the American capped call and its non-capped counterpart, respectively. Show that (Broadie and Detemple, 1995)

$$S^*_{cap}(\tau) = \min(S^*(\tau), L).$$

5.28 Consider an American call option with the callable feature, where the issuer has the right to recall throughout the whole life of the option. Upon recall by the issuer, the holder of the American option can choose either to exercise his option or receive the constant cash amount K. Let $S^*_{call}(\tau)$ and $S^*(\tau)$ denote the optimal exercise boundary of the callable American call and its non-callable counterpart, respectively. Show that

$$S^*_{call}(\tau) = \min(S^*(\tau), K + X),$$

where X is the strike price. Furthermore, suppose the holder is given a notice period of length τ_n, where his decision to exercise the option or receive the cash amount K is made at the end of the notice period. Show that the optimal exercise boundary $S^*_{call}(\tau)$ now becomes

$$S^*_{call}(\tau) = \min(S^*(\tau), \widehat{S}^*(\tau_n)),$$

where $\widehat{S}^*(\tau_n)$ is the solution to the algebraic equation

$$\widehat{S}^*(\tau_n) - X - Ke^{-r\tau_n} = c(S, \tau_n; K + X).$$

Here, $c(S, \tau_n; K + X)$ is the price of the European option with time to expiry τ_n and strike price $K + X$ (Kwok and Wu, 2000; Dai and Kwok, 2006a).

Hint: Note that $S^*_{call}(\tau)$ cannot be greater than $K + X$. If otherwise, at asset price level satisfying $K + X < S < S^*_{call}(\tau)$, the intrinsic value of the American call is above K. This represents a non-optimal recall policy of the issuer.

5.29 Unlike usual option contracts, the holder of an *installment option* pays the option premium throughout the life of the option. The installment option is terminated if the holder chooses to discontinue the installment payment. In normal cases, the installments are paid at predetermined time instants within the option's life. In this problem, we consider the two separate cases: continuous payment stream and discrete payments.

First, we let s denote the continuous rate of installment payment so that the amount $s\Delta t$ is paid over the interval Δt. Let $V(S,t)$ denote the value of a European installment call option. Show that $V(S,t)$ is governed by

$$\begin{cases} \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + r\frac{\partial V}{\partial S} - rV - s = 0 & \text{if } S > S^*(t) \\ V = 0 & \text{if } S \le S^*(t) \end{cases}.$$

where $S^*(t)$ is the critical asset price at which the holder discontinues the installment payment optimally. Solve for the analytic price formula when the installment option has infinite time to expiration (perpetual installment option).

Next, suppose that installments of equal amount d are paid at discrete instants $t_j, j = 1, \cdots, n$. Explain the validity of the following jump condition across the payment date t_j

$$V(S, t_j^-) = \max(V(S, t_j^+) - d, 0).$$

Finally, give a sketch of the variation of the option value $V(S,t)$ as a function of the calendar time t at varying values of asset value S under discrete installment payments.

Hint: There is an increase in the option value of amount d right after the installment payment. Also, it is optimal not to pay the installment at time t_j if $V(S, t_j^+) \le d$.

5.30 Suppose an American put option is only allowed to be exercised at N time instants between now and expiration. Let the current time be zero and denote the exercisable instants by the time vector $\boldsymbol{t} = (t_1, t_2, \cdots, t_N)^T$. Let $N_i(\boldsymbol{d}_i; R_i)$ denote the i-dimensional multivariate normal integral with upper limits of integration given by the i-dimensional vector $\boldsymbol{d}_i$ and correlation matrix R_i. Define the diagonal matrix $D_i = \text{diag } (1, \cdots 1, -1)$, and let $\boldsymbol{d}_i^* = D_i \boldsymbol{d}_i$ and $R_i^* = D_i R_i D_i$. Show that the value of the above American put with N exercisable instants is found to be (Bunch and Johnson, 1992)

$$P = X \sum_{i=1}^{N} e^{-rt_i} N_i(\boldsymbol{d}_{i_2}^*; R_i^*) - S \sum_{i=1}^{N} N_i(\boldsymbol{d}_{i_1}^*; R_i^*),$$

where

$$\boldsymbol{d}_{i_1} = (d_{11}, d_{21}, \cdots, d_{i1})^T, \quad \boldsymbol{d}_{i_1}^* = D_i \boldsymbol{d}_{i_1},$$
$$\boldsymbol{d}_{i_2} = \boldsymbol{d}_{i_1} - \sigma(\sqrt{t_1}, \sqrt{t_2}, \cdots, \sqrt{t_i})^T, \quad \boldsymbol{d}_{i_2}^* = D_i \boldsymbol{d}_{i_2},$$
$$d_{j1} = \frac{\ln \frac{S}{S_{t_j}^*} + \left(r + \frac{\sigma^2}{2}\right) t_j}{\sigma\sqrt{t_j}}, \quad j = 1, \cdots, i,$$

and $S^*_{t_j}$ is the optimal exercise price at t_j. Also, find the expression for the correlation matrix R_i.

Hint: When $N = 3$ and the exercisable instants are equally spaced, the correlation matrix R_3 is found to be

$$R_3 = \begin{pmatrix} 1 & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1 & \sqrt{2/3} \\ 1/\sqrt{3} & \sqrt{2/3} & 1 \end{pmatrix}.$$

5.31 The approximate equation for f in the quadratic approximation method becomes undefined when $K(\tau) = 1 - e^{-r\tau} = 0$, which corresponds to $r = 0$. Following a similar derivation procedure as in the quadratic approximation method, solve approximately the American option valuation problem for this special case of zero riskless interest rate.

5.32 Show that the approximate value of the American commodity put option based on the quadratic approximation method is given by

$$\widetilde{P}(S,\tau) = p(S,\tau) - \frac{S^*}{q_1}\left[1 - e^{(b-r)}N(-d_1(S^*))\right]\left(\frac{S}{S^*}\right)^{q_1}, \quad S > S^*.$$

Explain why the formula holds for all values of b.

Hint: Show that

$$\widetilde{P}(S,\tau) = p(S,\tau) + c_1 K S^{q_1}$$

and

$$\frac{\partial p}{\partial S}(S^*,\tau) = e^{(b-r)\tau}N(-d_1(S^*)).$$

5.33 Consider the shout call option discussed in Sec. 5.4.2 (Dai *et al.*, 2004a). Explain why the value of the shout call is bounded above by the fixed strike lookback call option with the same strike X.

5.34 Show that

$$e^{q\tau}p^*(\tau; r, q) = p^*(\tau; r - q, 0),$$

where $p^*(\tau)$ is defined in Eq. (5.4.2a). To prove the results in Eqs. (5.4.6a,b,c), it suffices to consider the sign behavior of

$$\frac{d}{d\tau}p^*(\tau; r, 0) = e^{-r\tau}f(\tau),$$

where

$$f(\tau) = -rN(-d_2) + \frac{\sigma}{2\sqrt{\tau}}n(-d_2),$$

$$d_2 = \alpha\sqrt{\tau}, \quad d_1 = d_2 + \sigma\sqrt{\tau} \quad \text{and} \quad \alpha = \frac{r - \frac{\sigma^2}{2}}{\sigma}.$$

Consider the following two cases (Dai *et al.*, 2004a).

(a) For $r \leq 0$, show that

$$\frac{d}{d\tau}p^*(\tau; r, 0) > 0.$$

(b) For $r > 0$, show that

$$f'(\tau) = \frac{\sigma n(-d_2)}{4\sqrt{\tau}}\left[\alpha(\alpha+\sigma) - \frac{1}{\tau}\right],$$

hence deduce the results in Eqs. (5.4.6b,c).

5.35 For the reset-strike put option, assuming $r \leq q$, show that the early reset premium is given by (Dai *et al.*, 2004a)

$$e(S,\tau) = Se^{-q\tau}\int_0^\tau N(d_{1,\tau-u})\frac{d}{du}[e^{qu}p^*(u)]\,du,$$

where

$$d_{1,\tau-u} = \frac{\ln\frac{S}{S^*(u)} + \left(r - q + \frac{\sigma^2}{2}\right)(\tau - u)}{\sigma\sqrt{\tau - u}}.$$

How to modify the formula when $r > q$?

5.36 Let $W_n^\infty(S;X) = \lim_{\tau\to\infty} e^{r\tau}U_n(S,\tau;X)$, where $U_n(S,\tau;X)$ is the value of the n-reset put option [see Eq. (5.4.22)]. For $r < q$, show that the governing equation for $W_n^\infty(S)$ is given by (Dai *et al.*, 2003)

$$\frac{\sigma^2}{2}S^2\frac{d^2W_n^\infty}{dS^2} + (r-q)S\frac{dW_n^\infty}{dS} = 0, \quad 0 < S < S^*_{n,\infty}.$$

The auxiliary conditions are given by

$$W_n^\infty(S^*_{n,\infty}) = \beta_n S^*_{n,\infty} \quad \text{and} \quad \frac{dW_n^\infty}{dS}(S^*_{n,\infty}) = \beta_n,$$

where $\beta_n = W_{n-1}^\infty(1;1)$. Show that

$$W_n^\infty(S;X) = X + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\frac{\beta_n^{1+\alpha}}{X^\alpha}S^{1+\alpha}$$

and

$$S^*_{n,\infty} = \left(1 + \frac{1}{\alpha}\right)\frac{X}{\beta_n},$$

where $\alpha = 2(q-r)/\sigma^2$. The recurrence relation for β_n is deduced to be

$$\beta_n = 1 + \frac{\alpha^\alpha}{(1+\alpha)^{1+\alpha}}\beta_{n-1}^{1+\alpha}.$$

Show that $\beta_1 = 1$ and $\lim_{n\to\infty} \beta_n = 1 + \frac{1}{\alpha}$. Also, find the first few values of $S^*_{n,\infty}$.

5.37 The reload provision in an employee stock option entitles its holder to receive $\frac{X}{S^*}$ units of fresh "reloaded" at-the-money option from the employer upon the exercise of the stock option. Here, X is the original strike price and S^* is the prevailing stock price at the exercise moment. The "reloaded" option has the same date of expiration as the original option. The exercise payoff is given by $S^* - X + \frac{X}{S}c(S^*, \tau; S^*, r, q)$. By the linear homogeneity property of the call price function, we can express the exercise payoff as $S - X + S\widehat{c}(\tau; r, q)$, where

$$\widehat{c}(\tau; r, q) = e^{-q\tau}N(\widehat{d}_1) - e^{-r\tau}N(\widehat{d}_2)$$

and

$$\widehat{d}_1 = \frac{r - q + \frac{\sigma^2}{2}}{\sigma}\sqrt{\tau} \quad \text{and} \quad \widehat{d}_2 = \frac{r - q - \frac{\sigma^2}{2}}{\sigma}\sqrt{\tau}.$$

Let $S^*(\tau; r, q)$ denote the optimal exercise boundary that separates the stopping and continuation regions. The stopping region and the optimal exercise boundary $S^*(\tau)$ observe the following properties (Dai and Kwok, 2006b).

1. The stopping region is contained inside the region defined by

$$\{(S, \tau) : S \geq X, \quad 0 \leq \tau \leq T\}.$$

2. At time close to expiry, the optimal stock price is given by

$$S^*(0^+; r, q) = X, \quad q \geq 0, r > 0.$$

3. When the stock pays dividend at constant yield $q > 0$, the optimal stock price at infinite time to expiry is given by

$$S^*(\infty; r, q) = \frac{\mu_+}{\mu_+ - 1}X,$$

where μ_+ is the positive root of the equation:

$$\frac{\sigma^2}{2}\mu^2 + \left(r - q - \frac{\sigma^2}{2}\right)\mu - r = 0.$$

4. If the stock pays no dividend, then

(a) for $r \leq \frac{\sigma^2}{2}$, $S^*(\tau; r, 0)$ is defined for all $\tau > 0$ and $S^*(\infty; r, 0) = \infty$;

(b) for $r > \frac{\sigma^2}{2}$, $S^*(\tau; r, 0)$ is defined only for $0 < \tau < \tau^*$, where τ^* is the unique solution to the algebraic equation

$$\frac{\sigma}{2\sqrt{\tau}}n\left(-\frac{r+\frac{\sigma^2}{2}}{\sigma}\sqrt{\tau}\right)-rN\left(\frac{r+\frac{\sigma^2}{2}}{\sigma}\sqrt{\tau}\right)=0.$$

5.38 Consider a landowner holding a piece of land who has the right to build a developed structure on the land or abandon the land. Let S be the value of the developed structure and H be the constant rate of holding costs (which may consist of property taxes, property maintenance costs, etc.). Assuming there is no fixed time horizon beyond which the structure cannot be developed, so the value of the land can be modeled as a perpetual American call, whose value is denoted by $C(S)$. Let σ_S denote the volatility of the lognormal process followed by S and r be the riskless interest rate. Suppose the asset value of the developed structure can be hedged by other tradeable asset, use the riskless hedging principle to show that the governing equation for $C(S)$ is given by

$$\frac{\sigma_S^2}{2}S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rC-H=0.$$

Let Z denote the lower critical value of S below which it is optimal to abandon the land. Let W be the higher critical value of S at which it is optimal to build the structure. Let X be the amount of cash investment required to build the structure. Explain why the auxiliary conditions at $S=Z$ and $S=W$ are prescribed by

$$\begin{cases} C(Z)=0 \quad \text{and} \quad \frac{dC}{dS}(Z)=0 \\ C(W)=W-X \quad \text{and} \quad \frac{dC}{dS}(W)=1 \end{cases}.$$

Show that the solution to the perpetual American call model is given by

$$C(S)=\begin{cases} 0 & \text{if } S<Z \\ \alpha_1 S+\alpha_2 S^\lambda-\frac{H}{r} & \text{if } Z\le S\le W \\ S-X & \text{if } S>W \end{cases},$$

where $\lambda=-\dfrac{2r}{\sigma_S^2}$, $W=\dfrac{\lambda}{\lambda-1}\left(X-\dfrac{H}{r}\right)\dfrac{1}{1-\alpha_1}$,

$$Z=\frac{\lambda}{\lambda-1}\frac{H}{r}\left[1-\left(1-\frac{rX}{H}\right)^{(\lambda-1)/\lambda}\right],$$

$$\alpha_1=\frac{1}{1-\left(1-\frac{rX}{H}\right)^{(\lambda-1)/\lambda}}, \quad \alpha_2=-\frac{\alpha_1}{\lambda Z^{\lambda-1}}.$$

This pricing model has two-sided free boundaries, one is associated with the right to abandon the land and the other with the right to build the structure.