CHAPTER 4
Path Dependent Options

The competition pressure for financial products push financial institutions to design and develop more innovative financial derivatives, many of which are tended toward specific needs of customers. Recently, there has been a growing popularity for path dependent options, so called since their payouts are related to the underlying asset price path history during the whole or part of the life of the option. The *barrier option* is the most popular path dependent option that is either nullified, activated or exercised when the underlying asset price breaches a *barrier* during the life of the option. The capped stock-index options, based on Standard and Poor's (S & P) 100 and 500 Indexes, are barrier options now traded in option exchanges (they were launched in the Chicago Board of Exchange in 1991). These capped options will be exercised automatically when the index value exceeds the cap at the close of the day. The payoff of a *lookback option* depends on the minimum or maximum price of the underlying asset attained during certain period of the life of the option, while the payoff of an *average option* (usually called an *Asian option*) depends on the average asset price over some period within the life of the option. An interesting example is the *Russian option*, which is in fact a perpetual American lookback option. The owner of a Russian option on a stock receives the historical maximum value of the asset price when the option is exercised and the option has no pre-set expiration date.

In this chapter, we present details of the product nature of barrier options, lookback options and Asian options, together with the analytic procedures for their valuation. Due to the path dependent nature of these options, the asset price process is monitored over the life of the option contract either for breaching of a barrier level, observation of new extremum value or sampling of asset prices for computing average value. In actual implementation, these monitoring procedures can only be performed at discrete time instants rather than continuously at all times. However, most pricing models of path dependent options assume continuous monitoring of asset price in order to achieve good analytic tractability. We derive analytic price formulas for most common types of continuously monitored barrier and lookback options, and geometric averaged Asian options. For arithmetic averaged Asian options, we manage to obtain analytic approximation valuation formulas. Under the Black-Scholes pricing paradigm, we assume that the uncertainty in the financial market over the time horizon $[0, T]$ is modeled by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, Q)$, where Q is the unique risk neutral (equivalent martingale) probability measure and the filtration $\mathcal{F}_t$ is generated by the standard Brownian process $\{Z(u) : 0 \leq u \leq t\}$. Under Q, the asset price process S_t follows the Geometric Brownian process with riskless interest rate r as the drift rate and constant volatility σ. In addition, all discounted prices of securities are martingales under Q.

When the asset price path is monitored at discrete time instants, the analytic forms of the price formulas become quite daunting since they involve multi-dimensional cumulative normal distribution functions and the dimension is equal to the number of monitoring instants. We discuss briefly some effective analytic approximation techniques for estimating the prices of discretely monitored path dependent options.

4.1 Barrier options

Options with the barrier feature are considered to be the simplest types of path dependent options. The distinctive feature is that the payoff depends not only on the final price of the underlying asset but also whether the asset price has breached (one-touch) some barrier level during the life of the option. An *out-barrier option* (or knock-out option) is one where the option is nullified prior to expiration if the underlying asset price touches the barrier. The holder of the option may be compensated by a rebate payment for the cancellation of the option. An *in-barrier option* (or knock-in option) is one where the option only comes in existence if the asset price crosses the in-barrier, though the holder has paid the option premium up front. When the barrier is upstream with respect to the asset price, the barrier option is called an *up-option*; otherwise, it is called a *down-option*. One can identify eight types of European barrier options, such as down-and-out calls, up-and-out calls, down-and-in puts, down-and-out puts, etc. Also, we may have two-sided barrier options that have both upside and downside barriers. Nullification or activation of the contract occurs either when one of the barriers is touched or only when the two barriers are breached in a pre-specified sequential order. The later type of options are called *sequential barrier options*. Suppose the knock-in or knock-out feature is activated only when the asset price breaches the barrier for a pre-specified length of time (rather than one touch of the barrier), we name this special type of barrier options as *Parisian options* (Chesney *et al.*, 1997).

Why barrier options are popular? From the perspective of the buyer of an option contract, he can achieve *option premium reduction* through the barrier provision by not paying premium to cover those unlikely scenarios as viewed by himself. For example, the buyer of a down-and-out call believes that the asset price would never fall below some floor value, so he can reduce option premium by allowing the option to be nullified when the asset price does fall below the perceived floor value. How both buyer and writer may benefit from the up-and-out call? With an appropriate rebate paid upon breaching the upside barrier, this type of barrier options provide the upside exposure for option buyer but at a lower cost. On the other hand, the option writer is not exposed to unlimited liabilities when the asset price rises acutely. In general, barrier options are attractive since they give investors more flexibility to express their view on the asset price movement in the option contract design.

The very nature of discontinuity at the barrier (circuit breaker effect upon knock-out) creates hedging problems with the barrier options. It is extremely difficult for option writers to hedge barrier options when the asset price is around the barrier level. Pitched battles often erupt around popular knock-out barriers in currency barrier options and these add much unwanted volatility to the markets. George Soros once said "knock-out options relate to ordinary options the way crack relates to cocaine." More details on the discussion of hedging problems of barrier options can be found in Linetsky's paper (1999). Also, Hsu (1997) discusses the difficulty of market implementation of criteria for determining barrier events. In order to avoid unpleasant surprises of being knocked out, the criteria used should be impartial, objective and consistent.

Consider a portfolio of one European in-option and one European out-option, both have the same barrier, strike price and date of expiration. The sum of their values is simply the same as that of a corresponding European option with the same strike price and date of expiration. From financial intuition, this is obvious since only one of the two barrier options survives at expiry and either payoff is the same as that of the European option. Hence, provided there is no rebate payment, we have

$$c_{\text{ordinary}} = c_{\text{down-and-out}} + c_{\text{down-and-in}} \tag{4.1.1a}$$

$$p_{\text{ordinary}} = p_{\text{up-and-out}} + p_{\text{up-and-in}} \tag{4.1.1b}$$

where c and p denote call and put values, respectively. That is, the value of an out-option can be found easily once the value of the corresponding in-option is available, or vice versa.

In this section, we derive analytic price formulas for European options with either one-sided barrier or two-sided barriers based on continuous monitoring of the asset price process. Under the Black-Scholes pricing paradigm, we can solve the pricing models using both the partial differential equation approach and martingale pricing approach. We derive the Green function (fundamental solution) of the governing Black-Scholes equation in a restricted domain using the *method of images*. When the martingale approach is used, we obtain the transition density function using the *reflection principle* in the Brownian motion literature. To compute the expected present value of the rebate payment, we derive the density function of the first passage time to the barrier. We also extend our derivation of price formulas to options with double barriers. The effects of discrete monitoring of the barrier on option prices will be discussed at the end of the section. Barrier option models can also be used to model the nullifying effect on a bond when its issuer defaults. We model default occurrence of a bond issuer by the fall of the issuer's firm value to some liability level. In Sec. 9.1, we show how to apply the barrier option models to devise the pricing models of defaultable bonds.

4.1.1 European down-and-out call options

The down-and-out call options have been available in the U.S. market since 1967 and the analytic price formula first appears in the pioneering paper by Merton (1973; Chap. 1). A down-and-out call has similar features as an ordinary call option, except that it becomes nullified when the asset price S falls below the knock-out level B.

Partial differential equation formulation

Let B denote the constant down-and-out barrier. The domain of definition for the barrier option model now becomes $[B, \infty) \times [0, T]$ in the $S - \tau$ plane. Let $R(\tau)$ denote the time-dependent rebate paid to the holder when the barrier is hit. Taking the usual Black-Scholes assumptions (frictionless market, continuous trading, etc.), the governing equation of the down-and-out barrier call option is given by

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + rS \frac{\partial c}{\partial S} - rc, \quad S > B \text{ and } \tau \in (0, T], \tag{4.1.2}$$

subject to

$$\begin{aligned} &\text{knock-out condition:} \quad && c(B, \tau) = R(\tau) && (4.1.3a) \\ &\text{terminal payoff:} \quad && c(S, 0) = \max(S - X, 0), && (4.1.3b) \end{aligned}$$

where $c = c(S, \tau)$ is the barrier option value, r and σ are the constant riskless interest rate and volatility, respectively. The down-barrier is normally set to be below the strike price X, otherwise the down-and-out call may be knocked out even it expires in-the-money. The partial differential equation formulation implies that knock-out occurs when the barrier is breached at any time during the life of the option.

Suppose we apply the transformation of the independent variable: $y = \ln S$, the barrier becomes the line $y = \ln B$. Now, the Black-Scholes equation (4.1.2) is reduced to the following constant coefficient equation for $c(y, \tau)$

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 c}{\partial y^2} + \left(r - \frac{\sigma^2}{2} \right) \frac{\partial c}{\partial y} - rc \tag{4.1.4a}$$

defined in the semi-infinite domain: $y > \ln B$ and $\tau \in (0, T]$. The auxiliary conditions become

$$c(\ln B, \tau) = R(\tau) \text{ and } c(y, 0) = \max(e^y - X, 0). \tag{4.1.4b}$$

Write $\mu = r - \dfrac{\sigma^2}{2}$ and recall that the Green function of Eq. (4.1.4a) in the infinite domain: $-\infty < y < \infty$ is given by [see Eq. (3.3.5)]

$$G_0(y, \tau; \xi) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{(y + \mu\tau - \xi)^2}{2\sigma^2\tau}\right), \tag{4.1.5}$$

where $G_0(y, \tau; \xi)$ satisfies the initial condition: $\lim_{\tau \to 0^+} G_0(y, \tau; \xi) = \delta(y - \xi)$.

Method of images

We would like to seek the Green function of Eqs. (4.1.4a,b) in the semi-infinite domain: $\ln B < y < \infty$ with zero Dirichlet boundary condition at $y = \ln B$. Assuming that the Green function takes the form

$$G(y, \tau; \xi) = G_0(y, \tau; \xi) - H(\xi) G_0(y, \tau; \eta), \tag{4.1.6}$$

we are required to determine $H(\xi)$ and η (in terms of ξ) such that the zero Dirichlet boundary condition $G(\ln B, \tau; \xi) = 0$ is satisfied. Note that $G(y, \tau; \xi)$ satisfies Eq. (4.1.4a) since both $G_0(y, \tau; \xi)$ and $H(\xi) G_0(y, \tau; \eta)$ satisfy the differential equation. Also, provided that $\eta \notin (\ln B, \infty)$, then $\lim_{\tau \to 0^+} G_0(y, \tau; \eta) = 0$ for all $y > \ln B$. By imposing the boundary condition, $H(\xi)$ has to satisfy

$$H(\xi) = \frac{G_0(\ln B, \tau; \xi)}{G_0(\ln B, \tau; \eta)} = \exp\left(\frac{(\xi - \eta)[2(\ln B + \mu\tau) - (\xi + \eta)]}{2\sigma^2\tau}\right). \tag{4.1.7}$$

The assumed form of $G(y, \tau; \xi)$ is feasible only if the right hand side of Eq. (4.1.7) becomes a function of ξ only. This can be achieved by the judicious choice of

$$\eta = 2 \ln B - \xi, \tag{4.1.8a}$$

so that

$$H(\xi) = \exp\left(\frac{2\mu}{\sigma^2}(\xi - \ln B)\right). \tag{4.1.8b}$$

The parameter η can be visualized as the *mirror image* of ξ with respect to the barrier $y = \ln B$. This is how the name of this method is derived (see Fig. 4.1). Now, by grouping the terms involving exponentials, the second term in Eq. (4.1.6) can be expressed as

$$\begin{aligned} &H(\xi) G_0(y, \tau; \eta) \\ &= \exp\left(\frac{2\mu}{\sigma^2}(\xi - \ln B)\right) \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{[y + \mu\tau - (2\ln B - \xi)]^2}{2\sigma^2\tau}\right) \\ &= \left(\frac{B}{S}\right)^{2\mu/\sigma^2} \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{[(y - \xi) + \mu\tau - 2(y - \ln B)]^2}{2\sigma^2\tau}\right). \end{aligned} \tag{4.1.9}$$

Collecting all the terms together, the Green function in the specified semi-infinite domain: $\ln B < y < \infty$ becomes

$$G(y,\tau;\xi) = \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \left\{ \exp\left(-\frac{(u-\mu\tau)^2}{2\sigma^2\tau}\right) - \left(\frac{B}{S}\right)^{2\mu/\sigma^2} \exp\left(-\frac{(u-2\beta-\mu\tau)^2}{2\sigma^2\tau}\right) \right\}, \tag{4.1.10}$$

where $u = \xi - y$ and $\beta = \ln B - y = \ln \dfrac{B}{S}$.

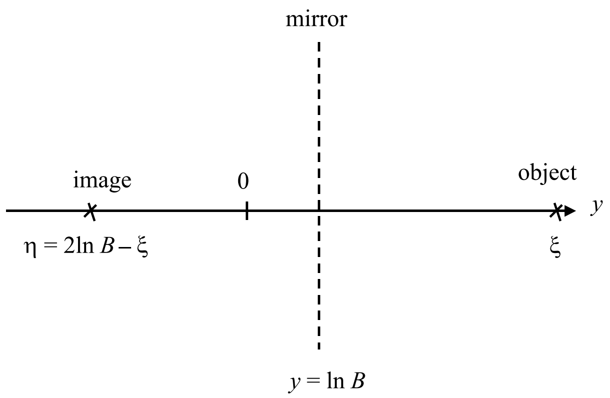

Fig. 4.1 Pictorial representation of the method of image. The mirror is placed at $y = \ln B$.

We consider the barrier option with zero rebate, where $R(\tau) = 0$, and let $K = \max(B, X)$. The price of the zero-rebate European down-and-out call can be expressed as

$$\begin{aligned} c_{do}(y,\tau) &= \int_{\ln B}^{\infty} \max(e^{\xi} - X, 0) G(y,\tau;\xi)\, d\xi \\ &= \int_{\ln K}^{\infty} (e^{\xi} - X) G(y,\tau;\xi)\, d\xi \\ &= \frac{e^{-r\tau}}{\sigma\sqrt{2\pi\tau}} \int_{\ln K/S}^{\infty} (Se^{u} - X) \left[\exp\left(-\frac{(u-\mu\tau)^2}{2\sigma^2\tau}\right) \right. \\ &\quad \left. - \left(\frac{B}{S}\right)^{2\mu/\sigma^2} \exp\left(-\frac{(u-2\beta-\mu\tau)^2}{2\sigma^2\tau}\right) \right] du. \end{aligned} \tag{4.1.11}$$

The direct evaluation of the integral gives

$$c_{do}(S,\tau) = S\left[N(d_1) - \left(\frac{B}{S}\right)^{\delta+1} N(d_3)\right] - Xe^{-r\tau}\left[N(d_2) - \left(\frac{B}{S}\right)^{\delta-1} N(d_4)\right], \quad (4.1.12a)$$

where

$$d_1 = \frac{\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau},$$
$$d_3 = d_1 + \frac{2}{\sigma\sqrt{\tau}}\ln\frac{B}{S}, \quad d_4 = d_2 + \frac{2}{\sigma\sqrt{\tau}}\ln\frac{B}{S}, \quad \delta = \frac{2r}{\sigma^2}. \quad (4.1.12b)$$

Suppose we define

$$\widetilde{c}_E(S,\tau;X,K) = SN(d_1) - Xe^{-r\tau}N(d_2), \quad (4.1.13a)$$

then $c_{do}(S,\tau;X,B)$ can be expressed in the following succinct form

$$c_{do}(S,\tau;X,B) = \widetilde{c}_E(S,\tau;X,K) - \left(\frac{B}{S}\right)^{\delta-1}\widetilde{c}_E\left(\frac{B^2}{S},\tau;X,K\right). \quad (4.1.13b)$$

One can show by direct calculation that the function $\left(\frac{B}{S}\right)^{\delta-1}\widetilde{c}_E\left(\frac{B^2}{S},\tau\right)$ satisfies the Black-Scholes equation identically (see Problem 4.1). The above form allows us to observe readily the satisfaction of the boundary condition: $c_{do}(B,\tau) = 0$, and terminal payoff condition.

The barrier option price formula (4.1.13b) indicates that $c_{do}(S,\tau;X,B) < \widetilde{c}_E(S,\tau;X,K) < c_E(S,\tau;X)$, that is, a down-and-out call is less expensive than the corresponding vanilla European call. This is obvious since the inherent nullifying property lowers the option premium.

Remarks

1. Closed form analytic price formulas for barrier options with exponential time dependent barrier, $B(\tau) = Be^{-\gamma\tau}$, can also be derived [see Problem 4.2]. However, when the barrier level has arbitrary time dependence, the search for analytic price formula for the barrier option fails. Roberts and Shortland (1997) show how to derive the analytic approximation formula by estimating diffusion process boundary hitting times via the Brownian bridge technique.
2. Closed form price formulas for barrier options can also be obtained for other types of diffusion processes followed by the underlying asset price. Lo *et al*. (2002) derive barrier option price formulas under the square root constant elasticity of variance process while Sepp (2004) uses Laplace transform method to obtain price formulas under the double exponential jump diffusion process.

3. By the relation (4.1.1a) and assuming $B < X$, the price of a down-and-in call option can be deduced to be

$$c_{di}(S, \tau; X, B) = \left(\frac{B}{S}\right)^{\delta - 1} c_E\left(\frac{B^2}{S}, \tau; X\right). \tag{4.1.14}$$

4. The barrier option with rebate payment will be considered later when the density function of the first passage time is available. Alternatively, one can use the known solution for the diffusion equation in the semi-infinite domain with time dependent boundary condition to derive the additional option premium due to the rebate (see Problem 4.3).
5. The method of images can be extended to derive the density functions of restricted multi-state diffusion processes where barrier occurs in only one of the state variables (Wong and Kwok, 2003a). The pricing of a two-asset down-and-out call is illustrated in Problem 4.6.
6. The monitoring period for breaching of the barrier may be limited to only part of the life of the option. The price formulas for these *partial barrier options* have been obtained by Heynen and Kat (1994a) (see Problem 4.7).

4.1.2 Transition density function and first passage time density

We may formulate the pricing models of barrier options using the martingale pricing approach and derive the price formulas by computing the discounted expectation of the terminal payoff (subject to knock-out or knock-in provision) under the risk neutral measure Q. Let the time interval $[0, T]$ denote the life of the barrier option, that is, the option is initiated at time zero and will expire at time T. The realized maximum and minimum value of the asset price from time zero to time t (under continuous monitoring) are defined by

$$\begin{aligned} m_0^t &= \min_{0 \le u \le t} S_u \\ M_0^t &= \max_{0 \le u \le t} S_u, \end{aligned} \tag{4.1.15}$$

respectively. The terminal payoffs of various types of barrier options can be expressed in terms of m_0^T and M_0^T. For example, consider the down-and-out call and up-and-out put, their respective terminal payoff can be expressed as

$$\begin{aligned} c_{do}(S_T, T; X, B) &= \max(S_T - X, 0)\mathbf{1}_{\{m_0^T > B\}} \\ p_{uo}(S_T, T; X, B) &= \max(X - S_T, 0)\mathbf{1}_{\{M_0^T < B\}}, \end{aligned} \tag{4.1.16}$$

respectively. Suppose B is the down-barrier, we define τ_B to be the stopping time at which the underlying asset price crosses the barrier for the first time:

$$\tau_B = \inf\{t | S_t \le B\}, \quad S_0 = S. \tag{4.1.17a}$$

Assume $S > B$ and due to path continuity, we may express τ_B (commonly called the *first passage time*) as

$$\tau_B = \inf\{t|S_t = B\}. \tag{4.1.17b}$$

In a similar manner, if B is the up-barrier and $S < B$, we have

$$\tau_B = \inf\{t|S_t \geq B\} = \inf\{t|S_t = B\}. \tag{4.1.17c}$$

It is easily seen that $\{\tau_B > T\}$ and $\{m_0^T > B\}$ are equivalent events if B is a down-barrier. By virtue of the risk neutral valuation principle, the price of a down-and-out call at time zero is given by

$$\begin{aligned} c_{do}(S, 0; X, B) &= e^{-rT} E_Q[\max(S_T - X, 0)\mathbf{1}_{\{m_0^T > B\}}] \\ &= e^{-rT} E_Q[(S_T - X)\mathbf{1}_{\{S_T > \max(X,B)\}}\mathbf{1}_{\{\tau_B > T\}}]. \end{aligned} \tag{4.1.18}$$

The determination of the price function $c_{d_0}(S, 0; X, B)$ requires the determination of the joint distribution function of S_T and m_0^T. We illustrate how to obtain the joint distribution function step by step via the reflection principle.

Reflection principle

We illustrate how to apply the reflection principle to derive the joint law of the minimum value over $[0, T]$ and terminal value of a Brownian motion. Let W_t^0 (W_t^μ) denote the Brownian motion that starts at zero, with volatility σ and zero drift rate (drift rate μ). We would like to find $P[m_0^T < m, W_T^\mu > x]$, where $x \geq m$ and $m < 0$. First, we consider the zero-drift Brownian motion W_t^0. Given that the minimum value m_0^T falls below m, then there exists some time instant $\xi, 0 < \xi < T$, such that ξ is the first time that W_ξ^0 equals m. As Brownian paths are continuous, there exist some times during which $W_t^0 < m$. In other words, W_t^0 decreases at least below m and then increases at least up to level x (higher than m) at time T. Suppose we define a random process

$$\widetilde{W}_t^0 = \begin{cases} W_t^0 & \text{for } t < \xi \\ 2m - W_t^0 & \text{for } \xi \leq t \leq T, \end{cases} \tag{4.1.19}$$

that is, $\widetilde{W}_t^0$ is the mirror reflection of W_t^0 at the level m within the time interval between ξ and T (see Fig. 4.2). It is then obvious that $\{W_T^0 > x\}$ is equivalent to $\{\widetilde{W}_T^0 < 2m - x\}$. Also, the reflection of the Brownian path dictates that

$$\widetilde{W}_{\xi+u}^0 - \widetilde{W}_\xi^0 = -(W_{\xi+u}^0 - W_\xi^0), \quad u > 0. \tag{4.1.20}$$

The stopping time ξ only depends on the path history $\{W_t^0 : 0 \leq t \leq \xi\}$ and it will not affect the Brownian motion at later times. By the strong Markov property of Brownian motions, we argue that the two Brownian increments in Eq. (4.1.20) have the same distribution, and the distribution has zero mean and variance $\sigma^2 u$. For every Brownian path that starts at 0, travels at least m units (downward, $m < 0$) before T and later travels at least $x - m$ units (upward, $x \geq m$), there is an equally likely path that starts at 0, travels m units (downward, $m < 0$) some time before T and travels at least $m - x$ units (further downward, $m \leq x$). Suppose $W_T^0 > x$, then $\widetilde{W}_0^T < 2m - x$, and together with relation (4.1.20), we obtain

$$P[W_T^0 > x, m_0^T < m] = P[\widetilde{W}_T^0 < 2m - x] = P[W_T^0 < 2m - x]$$
$$= N\left(\frac{2m - x}{\sigma\sqrt{T}}\right), \quad m \leq \min(x, 0). \qquad (4.1.21)$$

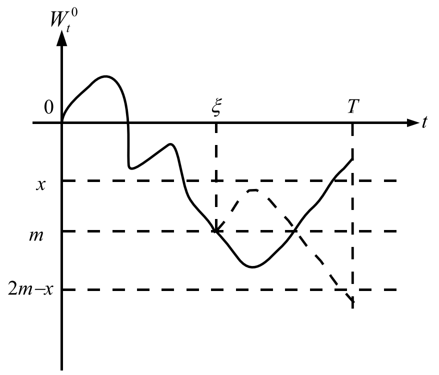

Fig. 4.2 Pictorial representation of the reflection principle of the Brownian motion W_t^0. The dotted path after time ξ is the mirror reflection of the Brownian path at the level m. Suppose W_T^0 ends up at a value higher than x, then the reflected path at time T has a value lower than $2m - x$.

Next, we apply the Girsanov Theorem to effect the change of measure for finding the above joint distribution when the Brownian motion has non-zero drift. Suppose under the measure Q, W_t^μ is a Brownian motion with drift rate μ. We change the measure from Q to $\widetilde{Q}$ such that W_t^μ becomes a Brownian motion with zero drift under $\widetilde{Q}$. Consider the following joint distribution

$$\begin{aligned} &P[W_T^\mu > x, m_0^T < m] \\ &= E_Q[\mathbf{1}_{\{W_T^\mu > x\}}\mathbf{1}_{\{m_0^T < m\}}] \\ &= E_{\widetilde{Q}}\left[\mathbf{1}_{\{W_T^\mu > x\}}\mathbf{1}_{\{m_0^T < m\}} \exp\left(\frac{\mu W_T^\mu}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}\right)\right], \end{aligned} \qquad (4.1.22a)$$

where the factor $\exp\left(\frac{\mu W_T^\mu}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}\right)$ is appended due to the Girsanov Theorem [see Eq. (2.4.40)]. Next, by applying the reflection principle and observing that W_T^μ is a zero-drift Brownian motion under $\widetilde{Q}$, we obtain

$$\begin{aligned}
&P[W_T^\mu > x, m_0^T < m] \\
&= E_{\widetilde{Q}}\left[\mathbf{1}_{\{2m - W_T^\mu > x\}} \exp\left(\frac{\mu}{\sigma^2}(2m - W_T^\mu) - \frac{\mu^2 T}{2\sigma^2}\right)\right] \\
&= e^{\frac{2\mu m}{\sigma^2}} E_{\widetilde{Q}}\left[\mathbf{1}_{\{W_T^\mu < 2m - x\}} \exp\left(-\frac{\mu}{\sigma^2} W_T^\mu - \frac{\mu^2 T}{2\sigma^2}\right)\right] \\
&= e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{2m-x} e^{-\frac{z^2}{2\sigma^2 T}} e^{-\frac{\mu z}{\sigma^2} - \frac{\mu^2 T}{2\sigma^2}}\, dz \\
&= e^{\frac{2\mu m}{\sigma^2}} \int_{-\infty}^{2m-x} \frac{1}{\sqrt{2\pi\sigma^2 T}} \exp\left(-\frac{(z + \mu T)^2}{2\sigma^2 T}\right) dz \\
&= e^{\frac{2\mu m}{\sigma^2}} N\left(\frac{2m - x + \mu T}{\sigma\sqrt{T}}\right), \quad m \le \min(x, 0). \qquad (4.1.22b)
\end{aligned}$$

Suppose the Brownian motion W_t^μ has a downstream barrier m over the period $[0, T]$ so that $m_0^T > m$, we would like to derive the joint distribution

$$P[W_T^\mu > x, m_0^T > m], \quad \text{where} \quad m \le \min(x, 0).$$

By applying the law of total probabilities, we obtain

$$\begin{aligned}
&P[W_T^\mu > x, m_0^T > m] \\
&= P[W_T^\mu > x] - P[W_T^\mu > x, m_0^T < m] \\
&= N\left(\frac{-x + \mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu m}{\sigma^2}} N\left(\frac{2m - x + \mu T}{\sigma\sqrt{T}}\right), \quad m \le \min(x, 0). \quad (4.1.23a)
\end{aligned}$$

Under the special case $m = x$, we have

$$P[m_0^T > m] = N\left(\frac{-m + \mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu m}{\sigma^2}} N\left(\frac{m + \mu T}{\sigma\sqrt{T}}\right). \qquad (4.1.23b)$$

When the Brownian motion W_t^μ has an upstream barrier M over the period $[0, T]$ so that $M_0^T < M$, the joint distribution function of W_T^μ and M_0^T can be deduced using the following relation between M_0^T and m_0^T:

$$M_0^T = \max_{0 \le t \le T} (\sigma Z_t + \mu t) = -\min_{0 \le t \le T} (-\sigma Z_t - \mu t), \qquad (4.1.24)$$

where Z_t is the standard Brownian motion. Since $-Z_t$ has the same distribution as Z_t, the distribution of the maximum value of W_t^μ is the same as that of the negative of the minimum value of $W_t^{-\mu}$. By swapping $-\mu$ for μ, $-M$ for m and $-y$ for x in Eq. (4.1.22b), we obtain

$$P[W_T^\mu < y, M_0^T > M] = e^{\frac{2\mu M}{\sigma^2}} N\left(\frac{y - 2M - \mu T}{\sigma\sqrt{T}}\right), M \ge \max(y, 0). \ (4.1.25)$$

In a similar manner, we obtain

$$\begin{aligned}&P[W_T^\mu < y, M_0^T < M]\\&= P[W_T^\mu < y] - P[W_T^\mu < y, M_0^T > M]\\&= N\left(\frac{y-\mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu M}{\sigma^2}} N\left(\frac{y-2M-\mu T}{\sigma\sqrt{T}}\right), \quad M \geq \max(y,0), \quad (4.1.26a)\end{aligned}$$

and by setting $y = M$, we obtain

$$P[M_0^T < M] = N\left(\frac{M-\mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu M}{\sigma^2}} N\left(-\frac{M+\mu T}{\sigma\sqrt{T}}\right). \quad (4.1.26b)$$

We define $f_{down}(x, m, T)$ to be the density function of W_T^μ with the downstream barrier m, where $m \leq \min(x, 0)$, that is,

$$f_{down}(x, m, T)\,dx = P[W_T^\mu \in dx, m_0^T > m]. \quad (4.1.27a)$$

By differentiating Eq. (4.1.23a) with respect to x and swapping the sign, we obtain

$$\begin{aligned}&f_{down}(x, m, T)\\&= \frac{1}{\sigma\sqrt{T}}\left[n\left(\frac{x-\mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu m}{\sigma^2}} n\left(\frac{x-2m-\mu T}{\sigma\sqrt{T}}\right)\right]\\&\quad \mathbf{1}_{\{m\leq\min(x,0)\}}. \quad (4.1.27b)\end{aligned}$$

In a similar manner, we define $f_{up}(x, M, T)$ be the density function of W_T^μ with the upstream barrier M, where $M > \max(y, 0)$, then

$$\begin{aligned}&P[W_T^\mu \in dy, M_0^T < M]\\&= f_{up}(y, M, T)\,dy\\&= \frac{1}{\sigma\sqrt{T}}\left[n\left(\frac{y-\mu T}{\sigma\sqrt{T}}\right) - e^{\frac{2\mu M}{\sigma^2}} n\left(\frac{y-2M-\mu T}{\sigma\sqrt{T}}\right)\right] dy\\&\quad \mathbf{1}_{\{M>\max(y,0)\}}. \quad (4.1.27c)\end{aligned}$$

Suppose the asset price S_t follows the lognormal process under the risk neutral measure such that $\ln\frac{S_t}{S} = W_t^\mu$, where S is the asset price at time zero and the drift rate $\mu = r - \frac{\sigma^2}{2}$. Let $\psi(S_T; S, B)$ denote the transition density of the asset price S_T at time T given the asset price S at time zero and conditional on $S_t > B$ for $0 \leq t \leq T$. Here, B is the downstream barrier. By Eq. (4.1.27b), we deduce that $\psi(S_T; S, B)$ is given by

$$\begin{aligned}\psi(S_T; S, B) = \frac{1}{\sigma\sqrt{T}S_T}&\left[n\left(\frac{\ln\frac{S_T}{S} - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right.\\&\left.- \left(\frac{B}{S}\right)^{\frac{2r}{\sigma^2}-1} n\left(\frac{\ln\frac{S_T}{S} - 2\ln\frac{B}{S} - \left(r-\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right)\right]. \quad (4.1.28)\end{aligned}$$

First passage time density functions

Let $Q(u;m)$ denote the density function of the first passage time at which the downstream barrier m is first bit by the Brownian process W_t^μ, that is, $Q(u;m)\,du = P[\tau_m \in du]$. First, we determine the distribution function $P[\tau_m > u]$ by observing that $\{\tau_m > u\}$ and $\{m_0^u > m\}$ are equivalent events. By Eq. (4.1.23b), we obtain

$$\begin{aligned} P[\tau_m > u] = &\ P[m_0^u > m] \\ = &\ N\left(\frac{-m+\mu u}{\sigma\sqrt{u}}\right) - e^{\frac{2\mu m}{\sigma^2}} N\left(\frac{m+\mu u}{\sigma\sqrt{u}}\right). \end{aligned} \tag{4.1.29}$$

The density function $Q(u;m)$ is then given by

$$\begin{aligned} Q(u;m)\,du = &\ P[\tau_m \in du] \\ = &\ -\frac{\partial}{\partial u}\left[N\left(\frac{-m+\mu u}{\sigma\sqrt{u}}\right) - e^{\frac{2\mu m}{\sigma^2}} N\left(\frac{m+\mu u}{\sigma\sqrt{u}}\right)\right] du\, \mathbf{1}_{\{m<0\}} \\ = &\ \frac{-m}{\sqrt{2\pi\sigma^2 u^3}} \exp\left(-\frac{(m-\mu u)^2}{2\sigma^2 u}\right) du\, \mathbf{1}_{\{m<0\}}. \end{aligned} \tag{4.1.30a}$$

In a similar manner, let $Q(u;M)$ denote the first passage time density associated with the upstream barrier M. Using the result in Eq. (4.1.26b), we have

$$\begin{aligned} Q(u;M) = &\ -\frac{\partial}{\partial u}\left[N\left(\frac{M-\mu u}{\sigma\sqrt{u}}\right) - e^{\frac{2\mu M}{\sigma^2}} N\left(-\frac{M+\mu u}{\sigma\sqrt{u}}\right)\right] \mathbf{1}_{\{M>0\}} \\ = &\ \frac{M}{\sqrt{2\pi\sigma^2 u^3}} \exp\left(-\frac{(M-\mu u)^2}{2\sigma^2 u}\right) \mathbf{1}_{\{M>0\}}. \end{aligned} \tag{4.1.30b}$$

We write B as the barrier, either upstream or downstream. When the barrier is downstream (upstream), we have $\ln\frac{B}{S} < 0 \left(\ln\frac{B}{S} > 0\right)$. We may combine Eqs. (4.1.30a,b) into one equation as follows

$$Q(u;B) = \frac{\left|\ln\frac{B}{S}\right|}{\sqrt{2\pi\sigma^2 u^3}} \exp\left(-\frac{\left[\ln\frac{B}{S} - \left(r - \frac{\sigma^2}{2}\right)u\right]^2}{2\sigma^2 u}\right). \tag{4.1.31}$$

Suppose a rebate $R(t)$ is paid to the option holder upon breaching the barrier at level B at time t. Since the expected rebate payment over the time interval $[u, u+du]$ is given by $R(u)Q(u;B)\,du$, then the expected present value of the rebate is given by

$$\text{rebate value } = \int_0^T e^{-ru} R(u)Q(u;B)\,du. \tag{4.1.32}$$

When $R(t) = R_0$, a constant value, direct integration of the above integral gives

$$\text{rebate value} = R_0\left[\left(\frac{B}{S}\right)^{\alpha_+} N\left(\delta\frac{\ln\frac{B}{S}+\beta T}{\sigma\sqrt{T}}\right) + \left(\frac{B}{S}\right)^{\alpha_-} N\left(\delta\frac{\ln\frac{B}{S}-\beta T}{\sigma\sqrt{T}}\right)\right], \qquad (4.1.33a)$$

where

$$\beta = \sqrt{\left(r-\frac{\sigma^2}{2}\right)^2 + 2r\sigma^2}, \quad \alpha_\pm = \frac{r-\frac{\sigma^2}{2}\pm\beta}{\sigma^2}$$
$$\text{and} \quad \delta = \text{sign}\left(\ln\frac{S}{B}\right). \qquad (4.1.33b)$$

Here, δ is a binary variable indicating whether the barrier is downstream ($\delta = 1$) or upstream ($\delta = -1$).

Fokker-Planck equation formulation

We would like to find the partial differential equation formulation of the transition density function $\psi_B(x,t;x_0,t_0)$ for the restricted Brownian process with upstream absorbing barrier B. The absorbing condition resembles the knock-out feature in barrier options. The appropriate boundary condition for an absorbing barrier is given by (Cox and Miller, 1965)

$$\psi_B(x,t;x_0,t_0)\Big|_{x=B} = 0. \qquad (4.1.34)$$

The forward Fokker-Planck equation that governs ψ_B is known to be [see Eq. (2.3.12)]

$$\frac{\partial\psi_B}{\partial t} = -\mu\frac{\partial\psi_B}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2\psi_B}{\partial x^2}, \quad -\infty < x < B, t > t_0, \qquad (4.1.35)$$

with boundary condition: $\psi_B(B,t) = 0$. Since $x \to x_0$ as $t \to t_0$ so that

$$\lim_{t\to t_0}\psi_B(x,t;x_0,t_0) = \delta(x-x_0). \qquad (4.1.36)$$

As deduced from the density function in Eq. (4.1.27c), ψ_B is found to be

$$\psi_B(x,t;x_0,t_0) = \frac{1}{\sigma\sqrt{t-t_0}}\left[n\left(\frac{x-x_0-\mu(t-t_0)}{\sigma\sqrt{t-t_0}}\right) - e^{\frac{2\mu(B-x_0)}{\sigma^2}} n\left(\frac{(x-x_0)-2(B-x_0)-\mu(t-t_0)}{\sigma\sqrt{t-t_0}}\right)\right],$$
$$x < B, t > t_0, x_0 < B. \qquad (4.1.37)$$

The probability that W_t^μ never crosses the barrier over $[t_0, t]$ is given by

$$P[\tau_B > t] = P\left[W_t^\mu \le B, M_{t_0}^t \le B \middle| W_{t_0}^\mu = x_0\right]$$
$$= \int_{-\infty}^{B} \psi_B(x, t; x_0, t_0)\, dx. \tag{4.1.38}$$

Price formula for European up-and-out call

Consider a European up-and-out call with strike X and upstream barrier B. Since the writer uses the knock-out feature to cap the upside liability, the option structure makes sense only if we choose $X < B$. As the option is always in-the-money upon knock-out, some form of rebate (say, $B - X$) should be paid upon breaching the barrier. Suppose $X \ge B$, the up-and-out call is knocked out before it becomes in-the-money. Hence, the up-and-out call is indeed worthless.

We would like to compute the non-rebate portion of the value of the up-and-out call. By the risk neutral valuation principle, the non-rebate call value is given by

$$e^{-rT} E_Q\left[(S_T - X)\mathbf{1}_{\{X<S_T<B\}}\mathbf{1}_{\{M_0^T<B\}}\right]$$
$$= e^{-rT}\int_{\ln X/S}^{\ln B/S} (Se^y - X) f_{up}(y, B, T)\, dy. \tag{4.1.39}$$

Recall that the value of the down-and-out call ($B < X$) is given by

$$e^{-rT}\int_{\ln X/S}^{\infty} (Se^x - X) f_{down}(x, B, T)\, dx$$
$$= c_E(S, T; X) - \left(\frac{B}{S}\right)^{\delta-1} c_E\left(\frac{B^2}{S}, T; X\right), \tag{4.1.40}$$

where $c_E(S, \tau; X)$ is the price function of a European vanilla call option with time to expiry τ. Since $f_{down}(x, B, T)$ and $f_{up}(y, B, T)$ have the same analytic form, the integral in Eq. (4.1.39) can be related to the integral in Eq. (4.1.40). Hence, we deduce that

$$\text{up-and-out call value}$$
$$= \left[c_E(S, T; X) - \left(\frac{B}{S}\right)^{\delta-1} c_E\left(\frac{B^2}{S}, T; X\right)\right]$$
$$- \left[c_E(S, T; B) - \left(\frac{B}{S}\right)^{\delta-1} c_E\left(\frac{B^2}{S}, T; B\right)\right]. \tag{4.1.41}$$

The corresponding up-and-in call value is given by

$$\text{up-and-in call value} = \left(\frac{B}{S}\right)^{\delta-1} c_E\left(\frac{B^2}{S}, T; X\right) + c_E(S, T; B) - \left(\frac{B}{S}\right)^{\delta-1} c_E\left(\frac{B^2}{S}, T; B\right). \tag{4.1.42}$$

All other price formulas for European out/in put options with either upstream or downstream barrier can be derived in a similar manner (see Problem 4.5). The paper by Rich (1994) contains a comprehensive list of price formulas of European options with one-sided barrier.

4.1.3 Options with double barriers

A double barrier option has two barriers, upstream barrier at U and downstream barrier at L. In its simplest form, when the asset price reaches one of the two barriers, either nullification (knock-out) or activation (knock-in) is triggered. More complicated payoff structure may include rebate payment upon hitting either one of the knock-out barriers as a compensation to the option holder. In sequential barrier options, knock-out is triggered only when the two barriers are hit at a pre-specified sequential order (see Problem 4.12). The papers by Luo (2001) and Kolkiewicz (2002) contain comprehensive discussions on various types of double barrier options that can be structured.

With the presence of two barriers, the following first passage times of the asset price process S_t can be defined

$$\tau_U = \inf\{t|S_t = U\} \quad \text{and} \quad \tau_L = \inf\{t|S_t = L\}. \tag{4.1.43}$$

During the life of the option $[0, T]$, we can distinguish the following three mutually exclusive events: (i) the upper barrier is first reached, (ii) the lower barrier is first reached, (iii) neither of the two barriers is reached.

We take the usual assumption that S_t is a Geometric Brownian process with volatility σ under the risk neutral measure Q. Let $X_t = \ln \frac{S_t}{S_0}$ so that $X_0 = 0$ and X_t is a Brownian process with drift rate $r - \frac{\sigma^2}{2}$ and variance rate σ^2. We are interested to find the following density functions:

$$g(x, T)\,dx = P[X_T \in dx, \min(\tau_L, \tau_U) > T] \tag{4.1.44a}$$

$$g^+(x, T)\,dx = P[X_T \in dx, \min(\tau_L, \tau_U) \leq T, \tau_U < \tau_L] \tag{4.1.44b}$$

$$g^-(x, T)\,dx = P[X_T \in dx, \min(\tau_L, \tau_U) \leq T, \tau_L < \tau_U]. \tag{4.1.44c}$$

Once the above densities are available, most double barrier options can be priced easily. Some examples are listed below.

1. Double knock-out call option (call payoff received at maturity if none of the barriers is breached)

$$\begin{aligned} c_{LU}^o &= e^{-rT} E\left[(S_T - X)\mathbf{1}_{\{S_T > X\}}\mathbf{1}_{\{\min(\tau_L,\tau_U) > T\}}\right] \\ &= e^{-rT}\int_{\ln X/S}^{\ln U/S} (Se^x - X) g(x,T)\,dx, \quad X \in (L,U), \end{aligned} \tag{4.1.45}$$

where $S_0 = S$ and the expectation E is taken under the risk neutral measure Q.

2. Upper-barrier knock-in call option (a vanilla call comes into life if the upper barrier is breached before the lower barrier is breached during the option life, that is, $\tau_U < \tau_L$ and $\tau_U \leq T$)

$$\begin{aligned} c_U^i &= e^{-rT} E\left[(S_T - X)\mathbf{1}_{\{S_T > X\}}\mathbf{1}_{\{\tau_U < \tau_L\}}\mathbf{1}_{\{\min(\tau_L,\tau_U) \leq T\}}\right] \\ &= e^{-rT}\int_{\ln X/S}^{\ln U/S} (Se^x - X) g^+(x,T)\,dx, \quad X \in (L,U). \end{aligned} \tag{4.1.46}$$

The analytic form for $g^+(x,T)$ can be found in Problem 4.11.

3. Lower-barrier knock-out call option (call payoff received at maturity if the lower barrier is never breached or the upper barrier is breached before the lower barrier, that is, $\tau_L > T$ or $\tau_U < \tau_L$).

Since the sum of a lower-barrier knock-out call and a lower-barrier knock-in call equals a vanilla call, we have

$$\begin{aligned} c_L^o &= c_E - c_L^i \\ &= c_E - e^{-rT}\int_{\ln X/S}^{\ln U/S} (Se^x - X) g^-(x,T)\,dx, \quad X \in (L,U), \end{aligned} \tag{4.1.47}$$

where c_E is the price of a European vanilla call option.

Density functions of Brownian processes with two-sided barriers

All the density functions defined in Eqs. (4.1.44a,b,c) satisfy the forward Fokker-Planck equation. Their complete partial differential equation formulations require the prescription of appropriate auxiliary conditions.

We take the initial position $X(0) = x_0 = 0$. Let $g(x,t;\ell,u)$ denote the density function of the restricted Brownian process X_t with two-sided absorbing barriers at $x = \ell$ and $x = u$, where the barriers are positioned such that $\ell < 0 < u$. Recall that $X_t = \ln \dfrac{S_t}{S_0}$, and if L and U are the absorbing barriers of the asset price process S_t, respectively, then $\ell = \ln \dfrac{L}{S}$ and $u = \ln \dfrac{U}{S}$ with $S_0 = S$. The partial differential equation formulation for $g(x,t;\ell,u)$ is given by

$$\frac{\partial g}{\partial t} = -\mu \frac{\partial g}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 g}{\partial x^2}, \quad \ell < x < u, \quad t > 0, \tag{4.1.48a}$$

with auxiliary conditions:

$$g(\ell, t) = g(u, t) = 0 \quad \text{and} \quad g(x, 0^+) = \delta(x). \tag{4.1.48b}$$

By defining the transformation

$$g(x, t) = e^{\frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}} \widehat{g}(x, t), \tag{4.1.49}$$

then $\widehat{g}(x, t)$ satisfies the forward Fokker-Planck equation with zero drift:

$$\frac{\partial \widehat{g}}{\partial t}(x, t) = \frac{\sigma^2}{2} \frac{\partial^2 \widehat{g}}{\partial x^2}(x, t). \tag{4.1.50}$$

The auxiliary conditions for $\widehat{g}(x, t)$ remain the same as those given in Eq. (4.1.48b). Without the barriers, the infinite-domain fundamental solution to Eq. (4.1.50) is known to be

$$\phi(x, t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{x^2}{2\sigma^2 t}\right). \tag{4.1.51}$$

Like the one-sided barrier case, we try to add extra terms to the above solution such that the homogeneous boundary conditions at $x = \ell$ and $x = u$ are satisfied. The following procedure is an extension of the method of images to two-sided barriers. First, we attempt to add the pair of negative terms $-\phi(x - 2\ell, t)$ and $-\phi(x - 2u, t)$ whereby

$$[\phi(x, t) - \phi(x - 2\ell, t)]\Big|_{x=\ell} = 0 \tag{4.1.52a}$$

and

$$[\phi(x, t) - \phi(x - 2u, t)]\Big|_{x=u} = 0. \tag{4.1.52b}$$

Note that $\phi(x-2\ell, t)$ and $\phi(x-2u, t)$ correspond to the fundamental soluton with initial condition $\delta(x-2\ell)$ and $\delta(x-2u)$, respectively. Writing the above partial sum with three terms as

$$\widehat{g}_3(x, t) = \phi(x, t) - \phi(x - 2\ell, t) - \phi(x - 2u, t), \tag{4.1.53}$$

we observe that the homogeneous boundary conditions are not yet satisfied since

$$\widehat{g}_3(\ell, t) = -\phi(x - 2u, t)\Big|_{x=\ell} \neq 0 \tag{4.1.54a}$$

$$\widehat{g}_3(u, t) = -\phi(2 - 2\ell, t)\Big|_{x=u} \neq 0. \tag{4.1.54b}$$

To nullify the non-zero value of $-\phi(x - 2u, t)\Big|_{x=\ell}$ and $-\phi(x - 2\ell, t)\Big|_{x=u}$, we add a new pair of positive terms $\phi(x - 2(u - \ell), t)$ and $\phi(x + 2(u - \ell), t)$. Similarly, we write the partial sum with five terms as

$$\widehat{g}_5(x,t) = \widehat{g}_3(x,t) + \phi(x - 2(u-\ell), t) + \phi(x + 2(u-\ell), t), \quad (4.1.55)$$

and observe that

$$\widehat{g}_5(\ell,t) = \phi(x - 2(u-\ell), t)\Big|_{x=\ell} \neq 0 \quad (4.1.56a)$$

$$\widehat{g}_5(u,t) = \phi(x + 2(u-\ell), t)\Big|_{x=u} \neq 0. \quad (4.1.56b)$$

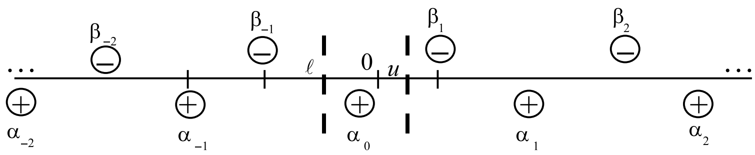

Fig. 4.3 A pictorial representation of the infinite number of sources and sinks due to a pair of absorbing barriers (mirrors) with the object placed at the origin. The positions of the sources and sinks are $\alpha_j = 2(u-\ell)j, j = 0, \pm1, \pm2, \cdots; \beta_j = 2u + 2(u-\ell)(j-1)$ if $j > 0$ and $\beta_j = 2\ell + 2(u-\ell)(j-1)$ if $j < 0$.

Whenever a new pair of positive terms or negative terms are added, the value of the partial sum at $x = \ell$ and $x = u$ becomes closer to zero. In a recursive manner, we add successive pairs of positive and negative terms so as to come closer to the satisfaction of the homogeneous boundary conditions at $x = \ell$ and $x = u$. Apparently, the two absorbing barriers may be visualized as a pair of mirrors with the object placed at the origin (see Fig. 4.3 for a pictorial representation). The source at the origin generates a sink at $x = 2\ell$ due to the mirror at $x = \ell$ and another sink at $x = 2u$ due to the mirror at $x = u$. To continue, the sink at $x = 2\ell$ ($x = 2u$) generates a source at $x = 2(u-\ell)$ $[x = 2(\ell - u)]$ due to the mirror at $x = u$ ($x = \ell$). As the procedure continues, this leads to the sum of an infinite number of positive and negative terms. The solution to $g(x,t)$ is deduced to be

$$\begin{aligned} g(x,t) &= e^{\frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}} \widehat{g}(x,t) \\ &= e^{\frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}} \sum_{n=-\infty}^{\infty} [\phi(x - 2n(u-\ell,t),t) - \phi(x - 2\ell - 2n(u-\ell),t)] \\ &= \frac{e^{\frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}}}{\sqrt{2\pi\sigma^2 t}} \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{[x - 2n(u-\ell)]^2}{2\sigma^2 t}\right) \right. \\ &\qquad \left. - \exp\left(-\frac{[(x-2\ell) - 2n(u-\ell)]^2}{2\sigma^2 t}\right) \right]. \end{aligned} \quad (4.1.57)$$

The double-mirror analogy provides the intuitive argument showing why $g(x,t)$ involves an infinite number of terms [see the text by Karatzas and Shreve (1991) for a rigorous proof]. Once $g(x,t)$ is known, it becomes quite straightforward to derive the price formula of c_{LU}^o (see Problem 4.8).

Next, we would like to derive the density function of the first passage time to either barrier, which is defined by

$$q(t;\ell,u)\,dt = P[\min(\tau_\ell,\tau_u) \in dt], \tag{4.1.58}$$

where $\tau_\ell = \inf\{t|X_t = \ell\}$ and $\tau_u = \inf\{t|X_t = u\}$. First, we consider the corresponding distribution function

$$\begin{aligned} P[\min(\tau_\ell,\tau_u) \le t] &= 1 - P[\min(\tau_\ell,\tau_u) > t] \\ &= 1 - \int_\ell^u g(x,t)\,dx \qquad (4.1.59a) \end{aligned}$$

so that

$$q(t;\ell,u) = -\frac{\partial}{\partial t}\int_\ell^u g(x,t)\,dx. \qquad (4.1.59b)$$

After some tedious manipulation, we manage to obtain

$$\begin{aligned} q(t;\ell,u) = {} & \frac{1}{\sqrt{2\pi\sigma^2 t^3}} \\ & \sum_{n=-\infty}^{\infty} [2n(u-\ell)-\ell]\exp\left(\frac{\mu\ell}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right)\exp\left(-\frac{(2n(u-\ell)-\ell]^2}{2\sigma^2 t}\right) \\ & + [2n(u-\ell)+u]\exp\left(\frac{\mu u}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right)\exp\left(-\frac{[2n(u-\ell)+u]^2}{2\sigma^2 t}\right). \quad (4.1.60) \end{aligned}$$

We may also be interested to find the exit time to a particular barrier. The density function of the exit time to the lower and upper barriers are defined by

$$q^-(t;\ell,u)\,dt = P[\tau_\ell \in dt, \tau_\ell < \tau_u] \qquad (4.1.61a)$$

$$q^+(t;\ell,u)\,dt = P[\tau_u \in dt, \tau_u < \tau_\ell]. \qquad (4.1.61b)$$

Since $\{\tau_\ell \in dt, \tau_\ell < \tau_u\} \cup \{\tau_u \in dt, \tau_u < \tau_\ell\} = \{\min(\tau_\ell,\tau_u) \in dt\}$, we deduce that

$$q(t;\ell,u) = q^-(t;\ell,u) + q^+(t;\ell,u). \tag{4.1.62}$$

A judicious decomposition of $q(t;\ell,u)$ in Eq. (4.1.60) into its two components would suggest (see also Problem 4.10)

$$q^-(t;\ell,u) = \frac{1}{\sqrt{2\pi\sigma^2 t^3}} \sum_{n=-\infty}^{\infty} [2n(u-\ell)-\ell]$$
$$\exp\left(\frac{\mu\ell}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right) \exp\left(-\frac{[2n(u-\ell)-\ell]^2}{2\sigma^2 t}\right) \quad (4.1.63a)$$
$$q^+(t;\ell,u) = \frac{1}{\sqrt{2\pi\sigma^2 t^3}} \sum_{n=-\infty}^{\infty} [2n(u-\ell)+u]$$
$$\exp\left(\frac{\mu u}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right) \exp\left(-\frac{[2n(u-\ell)+u]^2}{2\sigma^2 t}\right). \quad (4.1.63b)$$

To show the claim, we define the probability flow by

$$J(x,t) = \mu g(x,t) - \frac{\sigma^2}{2}\frac{\partial g}{\partial x}(x,t) \quad (4.1.64)$$

and observe that

$$q(t;\ell,u) = -\frac{\partial}{\partial t}\int_\ell^u g(x,t)\,dx = \int_\ell^u -\frac{\partial g}{\partial t}\,dx. \quad (4.1.65)$$

Since g satisfies the forward Fokker-Planck equation, we have

$$q(t;\ell,u) = \int_\ell^u \left(\mu\frac{\partial g}{\partial x} - \frac{\sigma^2}{2}\frac{\partial^2 g}{\partial x^2}\right) dx = J(u,t) - J(\ell,t). \quad (4.1.66)$$

One may visualize the probability flow across $x = \ell$ and $x = u$ as

$$-J(\ell,t) = P[\tau_\ell \in dt, \tau_\ell < \tau_u] \quad (4.1.67a)$$
$$J(u,t) = P[\tau_u \in dt, \tau_u < \tau_\ell]. \quad (4.1.67b)$$

The exit time densities $q^-(t;\ell,u)$ and $q^+(t;\ell,u)$ are seen to satisfy

$$q^-(t;\ell,u) = -J(\ell,t) = -\left[\mu g(x,t) - \frac{\sigma^2}{2}\frac{\partial g}{\partial x}(x,t)\right]\bigg|_{x=\ell} \quad (4.1.68a)$$
$$q^+(t;\ell,u) = J(u,t) = \mu g(x,t) - \frac{\sigma^2}{2}\frac{\partial g}{\partial x}(x,t)\bigg|_{x=u}. \quad (4.1.68b)$$

An alternative proof to Eqs. (4.1.68a,b) is given by Kolkiewicz (2002). Suppose rebate $R^-(t)\,[R^+(t)]$ is paid when the lower (upper) barrier is first breached during the life of the option, then the value of the rebate portion of the double-barrier option is given by

$$\text{rebate value } = \int_0^T e^{-r\xi}[R^-(\xi)q^-(\xi;\ell,u) + R^+(\xi)q^+(\xi;\ell,u)]\,d\xi. \quad (4.1.69)$$

4.1.4 Discretely monitored barrier options

The barrier option pricing models have good analytic tractability when the barrier is monitored continuously (knock-in or knock-out is presumed to occur if the barrier is breached at any instant). In financial markets, practitioners necessarily have to specify a discrete monitoring frequency. Kat and Verdonk (1995) show that the price differences between discrete and continuous barrier options can be quite substantial, even under daily monitoring of the barrier. We would expect that discrete monitoring would lower the cost of knock-in options but raise the cost of knock-out options, compared to their counterparts with continuous monitoring. The analytic price formulas of discrete monitoring barrier options can be derived, but their numerical valuation would be much tedious. The analytic representation involves the multi-variate normal distribution functions (see Problem 4.13).

Broadie *et al.* (1997) obtain an approximation formula for discretely monitored barrier options, which requires only a simple continuity correction to the continuous barrier option formulas. Let δt denote the uniform time interval between monitoring instants and there are m monitoring instants prior to expiration. The precise statement of their analytic approximation formula is given below.

Correction formula for discretely monitored barrier options
Let $V(B;m)$ be the price of a discretely monitored knock-in or knock-out down call or up put option with constant barrier B. Let $V(B)$ be the price of the corresponding continuously monitored barrier option. We have

$$V(B;m) = V(Be^{\pm\beta\sigma\sqrt{\delta t}}) + o\left(\frac{1}{\sqrt{m}}\right), \tag{4.1.70}$$

where $\beta = -\xi\left(\frac{1}{2}\right) \Big/ \sqrt{2\pi} \approx 0.5826$, ξ is the Riemann zeta function, σ is the volatility. The "+" sign is chosen when $B > S$, while the "−" sign is chosen when $B < S$. One observes that the correction shifts the barrier away from the current underlying asset price by a factor of $e^{\beta\sigma\sqrt{\delta t}}$. The extensive numerical experiments performed by Broadie *et al.* (1997) reveal the remarkably good accuracy of the above approximation formula.

4.2 Lookback options

Lookback options are path dependent options whose payoffs depend on the maximum or the minimum of the underlying asset price attained over a certain period of time (called the *lookback period*). We first consider lookback options where the lookback period is taken to be the whole life of the option. Let T denote the time of expiration of the option and $[T_0, T]$ be the lookback period. We denote the minimum value and maximum value of the asset price realized from T_0 to the current time t ($T_0 \leq t \leq T$) by

$$m_{T_0}^t = \min_{T_0 \leq \xi \leq t} S_\xi \tag{4.2.1a}$$

and

$$M_{T_0}^t = \max_{T_0 \leq \xi \leq t} S_\xi \tag{4.2.1b}$$

respectively. The above formulas implicitly imply continuous monitoring of the asset price, though discrete monitoring for the extremum value is normally adopted in practical implementation. Lookback options can be classified into two types: *fixed strike* and *floating strike*. A floating strike lookback call gives the holder the right to buy at the lowest realized price while a floating strike lookback put allows the holder to sell at the highest realized price over the lookback period. Since $S_T \geq m_{T_0}^T$ and $M_{T_0}^T \geq S_T$ so that the holder of a floating strike lookback option always exercise the option. Hence, the respective terminal payoff of the lookback call and put are given by $S_T - m_{T_0}^T$ and $M_{T_0}^T - S_T$. In a sense, floating strike lookback options are not options. A fixed strike lookback call (put) is a call (put) option on the maximum (minimum) realized price. The respective terminal payoff of the fixed strike lookback call and put are $\max(M_{T_0}^T - X, 0)$ and $\max(X - m_{T_0}^T, 0)$, where X is the strike price. Lookback options guarantee "no-regret" outcome for the holders, and thus the holders are relieved from making the difficult decision on the optimal timing for entry into or exit from the market. Generally speaking, lookback options are most desirable for investors who have confidence in the view of the range of the asset price movement over a certain period. One would expect that the prices of lookback options are much more sensitive to volatility. Also, the writer would charge a much higher premium in view of the favorable payoff to the holder.

In this section, we derive the price formulas for various types of lookback options, including those with exotic forms of lookback specification. The analytic formulas are limited to options which are European style, and *continuous monitoring* of the asset price for the extremum value is assumed. We adopt the usual Black-Scholes pricing framework and the underlying asset price is assumed to follow the Geometric Brownian process. Under the risk neutral measure, the process for the stochastic variable $U_\xi = \ln \dfrac{S_\xi}{S}$ is a Brownian process with drift rate $\mu = r - \dfrac{\sigma^2}{2}$ and variance rate σ^2, where r is the riskless interest rate and S is the asset price at current time t (dropping subscript t for brevity).

We define the following stochastic variables

$$y_T = \ln \frac{m_t^T}{S} = \min\{U_\xi, \xi \in [t, T]\} \tag{4.2.2a}$$

$$Y_T = \ln \frac{M_t^T}{S} = \max\{U_\xi, \xi \in [t, T]\}, \tag{4.2.2b}$$

and write $\tau = T - t$. For $y \leq 0$ and $y \leq u$, we can deduce the following joint distribution function of U_T and y_T from the transition density function of the Brownian process with the presence of a downstream barrier [see Eq. (4.1.23a)]

$$P[U_T \geq u, y_T \geq y] = N\left(\frac{-u+\mu\tau}{\sigma\sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-u+2y+\mu\tau}{\sigma\sqrt{\tau}}\right). \quad (4.2.3a)$$

Here, U_ξ is visualized as a restricted Brownian process with drift rate μ and downstream absorbing barrier y. Similarly, for $y \geq 0$ and $y \geq u$, the corresponding joint distribution function of U_T and Y_T is given by [see Eq. (4.1.26a)]

$$P[U_T \leq u, Y_T \leq y] = N\left(\frac{u-\mu\tau}{\sigma\sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{u-2y-\mu\tau}{\sigma\sqrt{\tau}}\right). \quad (4.2.3b)$$

By taking $y = u$ in the above two joint distribution functions, we obtain the distribution functions for y_T and Y_T

$$P(y_T \geq y) = N\left(\frac{-y+\mu\tau}{\sigma\sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{y+\mu\tau}{\sigma\sqrt{\tau}}\right), \quad y \leq 0, \quad (4.2.4a)$$

$$P(Y_T \leq y) = N\left(\frac{y-\mu\tau}{\sigma\sqrt{\tau}}\right) - e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-y-\mu\tau}{\sigma\sqrt{\tau}}\right), \quad y \geq 0. \quad (4.2.4b)$$

The density functions of y_T and Y_T can be obtained by differentiating the above distribution functions (see Problem 4.14).

4.2.1 European fixed strike lookback options

Consider a European fixed strike lookback call option whose terminal payoff is $\max(M_{T_0}^T - X, 0)$. The value of this lookback call option at the current time t is given by

$$c_{fix}(S, M, t) = e^{-r\tau} E\left[\max(\max(M, M_t^T) - X, 0)\right], \quad (4.2.5)$$

where $S_t = S$, $M_{T_0}^t = M$ and $\tau = T - t$, and the expectation is taken under the risk neutral measure. The payoff function can be simplified into the following forms, depending on $M \leq X$ or $M > X$:

(i) $M \leq X$

$$\max(\max(M, M_t^T) - X, 0) = \max(M_t^T - X, 0) \quad (4.2.6a)$$

(ii) $M > X$

$$\max(\max(M, M_t^T) - X, 0) = (M - X) + \max(M_t^T - M, 0). \quad (4.2.6b)$$

Define the function H by

$$H(S, \tau; K) = e^{-r\tau} E[\max(M_t^T - K, 0)], \tag{4.2.7}$$

where K is a positive constant. Once $H(S, \tau; K)$ is determined, then

$$\begin{aligned} c_{fix}(S, M, \tau) &= \begin{cases} H(S, \tau; X) & \text{if } M \leq X \\ e^{-r\tau}(M - X) + H(S, \tau; M) & \text{if } M > X \end{cases} \\ &= e^{-r\tau} \max(M - X, 0) + H(S, \tau; \max(M, X)). \end{aligned} \tag{4.2.8}$$

Interesting, $c_{fix}(S, M, \tau)$ is independent of M when $M \leq X$. This is obvious because the terminal payoff is independent of M when $M \leq X$. On the other hand, when $M > X$, the terminal payoff is guaranteed to have the floor value $M - X$. If we subtract the present value of this guaranteed floor value, then the remaining value of the fixed strike call option is equal to a new fixed strike call but with the strike being increased from X to M.

Since $\max(M_t^T - K, 0)$ is a non-negative random variable, its expected value is given by the integral of the tail probabilities where

$$\begin{aligned} & e^{-r\tau} E[\max(M_t^T - K, 0)] \\ &= e^{-r\tau} \int_0^\infty P[M_t^T - K \geq x]\, dx \\ &= e^{-r\tau} \int_K^\infty P\left[\ln \frac{M_t^T}{S} \geq \ln \frac{z}{S}\right] dz \qquad z = x + K \\ &= e^{-r\tau} \int_{\ln \frac{K}{S}}^\infty S e^y P[Y_T \geq y]\, dy \qquad y = \ln \frac{z}{S} \\ &= e^{-r\tau} \int_{\ln \frac{K}{S}}^\infty S e^y \left[N\left(\frac{-y + \mu\tau}{\sigma\sqrt{\tau}}\right) + e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-y - \mu\tau}{\sigma\sqrt{\tau}}\right)\right] dy, \end{aligned} \tag{4.2.9}$$

where the last integral is obtained by using the distribution function in Eq. (4.2.4b). By performing straightforward integration, we obtain

$$\begin{aligned} H(S, \tau; K) &= SN(d) - e^{-r\tau} K N(d - \sigma\sqrt{\tau}) \\ &\quad + e^{-r\tau} \frac{\sigma^2}{2r} S \left[e^{r\tau} N(d) - \left(\frac{S}{K}\right)^{-\frac{2r}{\sigma^2}} N\left(d - \frac{2r}{\sigma}\sqrt{\tau}\right)\right], \end{aligned} \tag{4.2.10a}$$

where

$$d = \frac{\ln \frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}. \tag{4.2.10b}$$

The European fixed strike lookback put option with terminal payoff $\max(X - m_{T_0}^T, 0)$ can be priced in a similar manner. Write $m = m_{T_0}^t$ and define the function

$$h(S, \tau; K) = e^{-r\tau} E[\max(K - m_t^T, 0)]. \tag{4.2.11}$$

The value of this lookback put can be expressed as

$$p_{fix}(S,m,\tau) = e^{-r\tau}\max(X-m,0) + h(S,\tau;\min(m,X)), \quad (4.2.12)$$

where

$$\begin{aligned}
h(S,\tau;K) =&\ e^{-r\tau}\int_0^\infty P[\max(K-m_t^T,0)\ge x]\,dx \\
=&\ e^{-r\tau}\int_0^K P[K-m_t^T\ge x]\,dx \quad \text{since } 0\le\max(K-m_t^T,0)\le K \\
=&\ e^{-r\tau}\int_0^K P[m_t^T\le z]\,dz \qquad z=K-x \\
=&\ e^{-r\tau}\int_0^{\ln\frac{K}{S}} Se^y P[y_T\le y]\,dy \qquad y=\ln\frac{y}{S} \\
=&\ e^{-r\tau}\int_0^{\ln\frac{K}{S}} Se^y\left[N\left(\frac{y-\mu\tau}{\sigma\sqrt{\tau}}\right)+e^{\frac{2\mu y}{\sigma^2}}N\left(\frac{y+\mu\tau}{\sigma\sqrt{\tau}}\right)\right]dy \\
=&\ e^{-r\tau}KN(-d+\sigma\sqrt{\tau}) - SN(-d) + e^{-r\tau}\frac{\sigma^2}{2r}S \\
&\left[\left(\frac{S}{K}\right)^{-2r/\sigma^2}N\left(-d+\frac{2r}{\sigma}\sqrt{\tau}\right)-e^{r\tau}N(-d)\right]. \qquad (4.2.13)
\end{aligned}$$

4.2.2 European floating strike lookback options

By exploring the pricing relations between the fixed and floating lookback options, we can deduce the price functions of floating strike lookback options from those of fixed strike options. Consider a European floating strike lookback call option whose terminal payoff is $S_T - m_{T_0}^T$, the present value of this call option is given by

$$\begin{aligned}
c_{f\ell}(S,m,\tau) =&\ e^{-r\tau}E[S_T-\min(m,m_t^T)] \\
=&\ e^{-r\tau}E[(S_T-m)+\max(m-m_t^T,0)] \\
=&\ S-me^{-r\tau}+h(S,\tau;m) \\
=&\ SN(d_m) - e^{-r\tau}mN(d_m-\sigma\sqrt{\tau}) + e^{-r\tau}\frac{\sigma^2}{2r}S \\
&\left[\left(\frac{S}{m}\right)^{-\frac{2r}{\sigma^2}}N\left(-d_m+\frac{2r}{\sigma}\sqrt{\tau}\right) - e^{r\tau}N(-d_m)\right], \qquad (4.2.14a)
\end{aligned}$$

where

$$d_m = \frac{\ln\frac{S}{m}+\left(r+\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}. \qquad (4.2.14b)$$

In a similar manner, consider a European floating strike lookback put option whose terminal payoff is $M_{T_0}^T - S_T$, the present value of this put option is given by

$$\begin{aligned} p_{f\ell}(S, M, \tau) &= e^{-r\tau} E[\max(M, M_t^T) - S_T] \\ &= e^{-r\tau} E[\max(M_t^T - M, 0) - (S_T - M)] \\ &= H(S, \tau; M) - (S - Me^{-r\tau}) \\ &= e^{-r\tau} MN(-d_M + \sigma\sqrt{\tau}) - SN(-d_M) + e^{-r\tau} \frac{\sigma^2}{2r} S \\ &\left[e^{r\tau} N(d_M) - \left(\frac{S}{M}\right)^{-\frac{2r}{\sigma^2}} N\left(d_M - \frac{2r}{\sigma}\sqrt{\tau}\right)\right], \end{aligned} \quad (4.2.15a)$$

where

$$d_M = \frac{\ln \frac{S}{M} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}. \quad (4.2.15b)$$

Boundary condition at $S = m$

Consider the particular situation when $S = m$, that is, the current asset price happens to be at the minimum value realized so far. The probability that the current minimum value remains to be the realized minimum value at expiration is expected to be zero (see Problem 4.16). Now, we can argue that the value of the floating strike lookback call should be insensitive to infinitesimal changes in m since the change in option value with respect to marginal changes in m is proportional to the probability that m will be the realized minimum at expiry (Goldman *et al.*, 1979). Mathematically, this is represented by

$$\left.\frac{\partial c_{f\ell}}{\partial m}(S, m, \tau)\right|_{S=m} = 0. \quad (4.2.16)$$

The above property can be verified by direct differentiation of the call price formula (4.2.14a).

Rollover strategy and strike bonus premium

The sum of the first two terms in $c_{f\ell}$ can be seen as the price function of a European vanilla call with strike price m, while the third term can be interpreted as the strike bonus premium (Garman, 1992). To interpret the strike bonus premium, we consider the hedging of the floating strike lookback call by the following rollover strategy. At any time, we hold a European vanilla call with the strike price set at the current realized minimum asset value. In order to replicate the payoff of the floating strike lookback call at expiry, whenever a new realized minimum value of the asset price is established at a later time, one should sell the original call option and buy a new call with the same expiration date but with the strike price set equal to the newly established minimum value. Since the call with a lower strike is always more expensive, an extra premium is required to adopt the rollover strategy. The sum of these expected costs of rollover is termed the strike bonus premium.

We would like to show how the strike bonus premium can be obtained by integrating a joint probability distribution function involving m_t^T and S_T. First, we observe that

$$\begin{aligned}\text{strike bonus premium } &= h(S,\tau;m) + S - me^{-r\tau} - c_E(S,\tau;m)\\ &= h(S,\tau;m) - p_E(S,\tau;m),\end{aligned}\qquad (4.2.17)$$

where $c_E(S,\tau,m)$ and $p_E(S,\tau;m)$ are the price functions of European vanilla call and put, respectively. The last result is due to put-call parity relation. Recall from Eq. (4.2.13) that

$$h(S,\tau;m) = e^{-r\tau}\int_0^m P[m_t^T \leq \xi]\, d\xi \qquad (4.2.18a)$$

and in a similar manner

$$\begin{aligned}p_E(S,\tau;m) =&\ e^{-r\tau}\int_0^\infty P[\max(m - S_T, 0) \geq x]\, dx\\ =&\ e^{-r\tau}\int_0^m P[S_T \leq \xi]\, d\xi.\end{aligned}\qquad (4.2.18b)$$

Since the two stochastic state variables always satisfies $0 \leq m_t^T \leq S_T$, we have

$$P[m_t^T \leq \xi] - P[S_T \leq \xi] = P[m_t^T \leq \xi < S_T] \qquad (4.2.19a)$$

so that (Wong and Kwok, 2003b)

$$\text{strike bonus premium } = e^{-r\tau}\int_0^m P[m_t^T \leq \xi \leq S_T]\, d\xi. \quad (4.2.19b)$$

4.2.3 More exotic forms of European lookback options

The lookback options discussed above are the most basic types where the payoff functions at expiry are of standard forms and the lookback period spans the whole life of the option. In the financial market, a wide variety of options have been structured with properties similar to the above prototype lookback options but being less expensive. Some examples are: partial lookback call option with terminal payoff $\max(S_T - \lambda m_{T_0}^T, 0), \lambda > 1$, partial lookback put option with terminal payoff $\max(\lambda M_{T_0}^T - S_T, 0), 0 < \lambda < 1$ (see Problem 4.19). Also, the price of a lookback option will be lowered if the lookback period is spanning only part of the life of the option. When an investor is faced with the problem of deciding the optimal timing for market entry (market exit), he may be interested to purchase an option whose lookback period covers the early part (period near expiration) of the life of the option. In what follows, we discuss the properties of a "limited period" floating strike lookback call option which is designed for optimal market entry.

The discussion of the corresponding fixed strike lookback options for optimal market exit is relegated to an exercise Problem 4.21.

Lookback options for market entry

Suppose an investor has a view that the asset price will rise substantially in the next 12 months and he buys a call option on the asset with the strike price equal to the current asset price. Suppose the asset price drops a few percent within a few weeks after the purchase, though it does rise up strongly to a high level at expiration, the investor should have a better return if he had bought the option a few weeks later. The optimal timing for market entry is always difficult to be decided. The investor could have avoided the above difficulty if he has purchased a "limited period" floating strike lookback call option whose lookback period only covers the early part of the option's life. It would cause the investor too much if a full period floating strike lookback call were purchased instead.

Let $[T_0, T_1]$ denote the lookback period where $T_1 < T$, T is the expiration time, and let the current time $t \in [T_0, T_1]$. The terminal payoff function of the "limited period" lookback call is $\max(S_T - m_{T_0}^{T_1}, 0)$. We write $S_t = S, m_{T_0}^t = m$ and $\tau = T - t$. The value of this lookback call is given by

$$\begin{aligned} c(S, m, \tau) &= e^{-r\tau} E[\max(S_T - m_{T_0}^{T_1}, 0)] \\ &= e^{-r\tau} E[\max(S_T - m, 0)\mathbf{1}_{\{m \le m_t^{T_1}\}}] \\ &\quad + e^{-r\tau} E[\max(S_T - m_t^{T_1}, 0)\mathbf{1}_{\{m > m_t^{T_1}\}}] \\ &= e^{-r\tau} E[S_T \mathbf{1}_{\{S_T > m, m \le m_t^{T_1}\}}] \\ &\quad - e^{-r\tau} m E\left[\mathbf{1}_{\{S_T > m, m \le m_t^{T_1}\}}\right] \\ &\quad + e^{-r\tau} E[S_T \mathbf{1}_{\{S_T > m_t^{T_1}, m > m_t^{T_1}\}}] \\ &\quad - e^{-r\tau} E[m_t^{T_1} \mathbf{1}_{\{S_T > m_t^{T_1}, m > m_t^{T_1}\}}], \quad t < T_1, \end{aligned} \tag{4.2.20}$$

where E is the expectation under the risk neutral measure. The solution for the call value requires the derivation of the appropriate distribution functions. For the first term, the expectation can be expressed as

$$E[S_T \mathbf{1}_{\{S_T > m, m \le m_t^{T_1}\}}] = \int_{\ln \frac{m}{S}}^{\infty} \int_{y}^{\infty} \int_{\ln \frac{m}{S} - x}^{\infty} S e^{xz} k(z) h(x, y)\, dz\, dx\, dy, \tag{4.2.21a}$$

where $k(z)$ is the density function for $z = \ln \dfrac{S_T}{S_{T_1}}$ and $h(x, y)$ is the bivariate density function for $x = \ln \dfrac{S_{T_1}}{S}$ and $y = \ln \dfrac{m_t^{T_1}}{S}$ [see Eq. (4.2.3b)]. Similarly, the third and fourth terms can be expressed as

$$E[S_T \mathbf{1}_{\{S_T > m_t^{T_1}, m > m_t^{T_1}\}}] = \int_{-\infty}^{\ln \frac{m}{S}} \int_y^\infty \int_{y-x}^\infty S e^{xz} k(z) h(x,y)\, dz\, dx\, dy \tag{4.2.21b}$$

and

$$E[m_t^{T_1} \mathbf{1}_{\{S_T > m_t^{T_1}, m > m_t^{T_1}\}}] = \int_{-\infty}^{\ln \frac{m}{S}} \int_y^\infty \int_{y-x}^\infty S e^{y} k(z) h(x,y)\, dz dx dy. \tag{4.2.21c}$$

After performing tedious integration procedures in the above discounted expectation calculations, the price formula of the present "limited-period" lookback call is found to be (Heynen and Kat, 1994b)

$$\begin{aligned}
& c(S, m, \tau) \\
&= SN(d_1) - me^{-r\tau} N(d_2) + SN_2\left(-d_1, e_1; -\sqrt{\frac{T-T_1}{T-t}}\right) \\
&\quad + e^{-r\tau} m N_2\left(-f_2, d_2; -\sqrt{\frac{T_1 - t}{T-t}}\right)
\end{aligned}$$

$$\begin{aligned}
&+ e^{-r\tau} \frac{\sigma^2}{2r} S \left[\left(\frac{S}{m}\right)^{-\frac{2r}{\sigma^2}} N_2\left(-f_1 + \frac{2r}{\sigma}\sqrt{T_1 - t}, -d_1 + \frac{2r}{\sigma}\sqrt{\tau}; \sqrt{\frac{T_1 - t}{T-t}}\right) \right. \\
&\qquad\qquad \left. - e^{r\tau} N_2\left(-d_1, e_1; -\sqrt{\frac{T-T_1}{T-t}}\right)\right] \\
&+ e^{-r(T-T_1)}\left(1 + \frac{\sigma^2}{2r}\right) SN(e_2) N(-f_1), \quad t < T_1,
\end{aligned} \tag{4.2.22}$$

where

$$\begin{aligned}
d_1 &= \frac{\ln \frac{S}{m} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}, \\
e_1 &= \frac{\left(r + \frac{\sigma^2}{2}\right)(T - T_1)}{\sigma\sqrt{T - T_1}}, \quad e_2 = e_1 - \sigma\sqrt{T - T_1}, \\
f_1 &= \frac{\ln \frac{S}{m} + \left(r + \frac{\sigma^2}{2}\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}}, \quad f_2 = f_1 - \sigma\sqrt{T_1 - t}.
\end{aligned} \tag{4.2.23}$$

One can check easily that when $T_1 = T$ (full lookback period), the above price formula reduces to price formula (4.2.14a). Suppose the current time passes beyond the lookback period, $t > T_1$, the realized minimum value $m_{T_0}^{T_1}$ is now a known quantity. This "limited period" lookback call option then becomes identical to a vanilla call option with the known strike price $m_{T_0}^{T_1}$.

4.2.4 Differential equation formulation

We would like to illustrate how to derive the governing partial differential equation and the associated auxiliary conditions for the European floating strike lookback put option. First, we define the quantity

$$M_n = \left[\int_{T_0}^{t} (S_\xi)^n \, d\xi\right]^{1/n}, \quad t > T_0, \tag{4.2.24}$$

the derivative of which is given by

$$dM_n = \frac{1}{n}\frac{S^n}{(M_n)^{n-1}}dt \tag{4.2.25}$$

so that dM_n is deterministic. Taking the limit $n \to \infty$, we obtain

$$M = \lim_{n\to\infty} M_n = \max_{T_0 \le \xi \le t} S_\xi, \tag{4.2.26}$$

giving the realized maximum value of the asset price process over the lookback period $[T_0, t]$. We attempt to construct a hedged portfolio which contains one unit of a put option whose payoff depends on M_n and $-\triangle$ units of the underlying asset. Again, we choose $\triangle$ so that the stochastic components associated with the option and the underlying asset cancel. Let $p(S, M_n, t)$ denote the value of the lookback put option and let Π denote the value of the above portfolio. We then have

$$\Pi = p(S, M_n, t) - \triangle S. \tag{4.2.27}$$

The dynamics of the portfolio value is given by

$$d\Pi = \frac{\partial p}{\partial t}dt + \frac{1}{n}\frac{S^n}{(M_n)^{n-1}}\frac{\partial p}{\partial M_n}dt + \frac{\partial p}{\partial S}dS + \frac{\sigma^2}{2}S^2\frac{\partial^2 p}{\partial S^2}dt - \triangle dS \tag{4.2.28}$$

by virtue of Ito's lemma. Again, we choose $\triangle = \dfrac{\partial p}{\partial S}$ so that the stochastic terms cancel. Using usual no-arbitrage argument, the non-stochastic portfolio should earn the riskless interest rate so that

$$d\Pi = r\Pi \, dt, \tag{4.2.29}$$

where r is the riskless interest rate. Putting all equations together, we obtain

$$\frac{\partial p}{\partial t} + \frac{1}{n}\frac{S^n}{(M_n)^{n-1}}\frac{\partial p}{\partial M_n} + \frac{\sigma^2}{2}S^2\frac{\partial^2 p}{\partial S^2} + rS\frac{\partial p}{\partial S} - rp = 0. \tag{4.2.30}$$

Now, we take the limit $n \to \infty$ and note that $S \le M$. When $S < M$, $\lim_{n\to\infty} \dfrac{1}{n}\dfrac{S^n}{(M_n)^{n-1}} = 0$; and when $S = M$, $\dfrac{\partial p}{\partial M} = 0$. Hence, the second term in

Eq. (4.2.30) becomes zero as $n \to \infty$. In conclusion, the governing equation for the floating strike lookback put is given by

$$\frac{\partial p}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 p}{\partial S^2} + rS \frac{\partial p}{\partial S} - rp = 0, \quad 0 < S < M, t > T_0, \tag{4.2.31}$$

which is identical to the usual Black-Scholes equation (Goldman *et al.*, 1979), except that the domain of the pricing model has an upper bound M on S. It is interesting to observe that the variable M does not appear in the equation, though M appears as a parameter in the auxiliary conditions. The final condition is the terminal payoff function, namely,

$$p(S, M, T) = M - S. \tag{4.2.32a}$$

In this European floating strike lookback put option, the boundary conditions are applied at $S = 0$ and $S = M$. Once S becomes zero, it stays at the zero value at all subsequent times and the payoff at expiry is certain to be M. Discounting at the riskless interest rate, the lookback put value at the current time t is

$$p(0, M, t) = e^{-r(T-t)} M. \tag{4.2.32b}$$

The boundary condition at the other end $S = M$ is given by

$$\frac{\partial p}{\partial M} = 0 \quad \text{at} \quad S = M. \tag{4.2.32c}$$

Remarks

1. The satisfaction of the differential equation by the price function implies implicitly that the lookback option is hedgeable.
2. One can show by direct differentiation that the put price formula given in Eq. (4.2.15a,b) satisfies Eq. (4.2.31) and the auxiliary conditions (4.2.32a,b,c).
3. When the terminal payoff assumes the more general form $f(S_T, M_{T_0}^T)$, Xu and Kwok (2005) manage to derive an integral representation of the lookback option price formula using the partial differential equation approach.

4.2.5 Discretely monitored lookback options

In real market situations, practitioners necessarily specify a discrete monitoring frequency since continuous monitoring of the asset price movement is almost impractical. We would expect discrete monitoring causes the price of the lookback options to go lower since a new extremum value may be missed out in discrete monitoring. Heynen and Kat (1995) show in their numerical experiments that monitoring the asset price discretely instead of continuously may have a significant effect on the values of lookback options.

The analytic price formulas for lookback options with discrete monitoring (Heynen and Kat, 1995) involve the n-variate normal distribution functions, where n is the number of monitoring instants within the remaining life of the option. Levy and Mantion (1997) propose a simple but effective analytic approximation method to price discretely monitored lookback options. The method assumes a second order Taylor expansion of the option value in powers of $\sqrt{\delta t}$, where δt is the time between successive monitoring instants (assumed to be at regular intervals). The two coefficients of the Taylor expansion are determined by fitting the approximate formula with option values corresponding to $\delta t = \tau$ and $\delta t = \tau/2$, where τ is the time to expiry. The construction and implementation of the method are quite straightforward, the details of which are presented in Problem 4.23. As demonstrated by their numerical experiments, the accuracy of this analytic approximation method is quite remarkable.

Similar to the discretely monitored barrier options, Broadie *et al.* (1999) derive analytic approximation formulas for the price functions of discretely monitored lookback options on a monitoring date in terms of the price functions of the continuously monitored counterparts.

4.3 Asian options

Asian options are averaging options whose terminal payoff depends on some form of averaging of the price of the underlying asset over a part or the whole of option's life. There are frequent situations where traders may be interested to hedge against the average price of a commodity over a period rather than, say, end-of-period price. For example, suppose a manufacturer expects to make a stream of copper purchases for his factory over some fixed time horizon. He would be interested to acquire price protection which is linked to the average price over the period. The hedging of risk against the average price may be achieved through the purchase of an appropriate averaging option. Averaging options are particularly useful for business involved in trading on thinly-traded commodities. The use of such financial instruments may avoid the price manipulation near the end of the period.

Most Asian options are of European-type since an Asian option of American-type may be redeemed as early as the start of the averaging period and lose the intent of protection from averaging. There are two main classes of Asian options, the *fixed strike (average rate)* and the *floating strike (average strike) options*, whose terminal call payoff are $\max(A_T - X, 0)$ and $\max(S_T - A_T, 0)$, respectively. Here, S_T is the asset price at expiry, X is the strike price and A_T denotes some form of average of the price of the underlying asset over the averaging period $[0, T]$. The value of A_T depends on the realization of the asset price path. The most common averaging procedures are the discrete arithmetic averaging defined by

$$A_T = \frac{1}{n}\sum_{i=1}^{n} S_{t_i} \tag{4.3.1a}$$

and the discrete geometric averaging defined by

$$A_T = \left[\prod_{i=1}^{n} S_{t_i} \right]^{1/n}. \tag{4.3.1b}$$

Here, S_{t_i} is the asset price at discrete time $t_i, i = 1, 2, \cdots, n$. In the limit $n \to \infty$, the discrete sampled averages become the continuous sampled averages. The continuous arithmetic average is given by

$$A_T = \frac{1}{T} \int_0^T S_t \, dt, \tag{4.3.2a}$$

while the continuous geometric average is defined to be

$$A_T = \exp \left(\frac{1}{T} \int_0^T \ln S_t \, dt \right). \tag{4.3.2b}$$

A wide variety of averaging options have been proposed. The comprehensive summaries of them can be found in Boyle's (1993) and Zhang's (1994) papers. The most commonly used sampled average is the discrete arithmetic average. If the Geometric Brownian motion is assumed for the underlying asset price process, the pricing of this type of Asian option is in general analytically intractable since the sum of lognormal densities has no explicit representation. On the other hand, the analytic derivation of the price formula of a European Asian option with geometric averaging is feasible since the product of lognormal prices remains lognormal.

In this section, we first derive the general partial differential equation formulation for pricing Asian options. We then consider the pricing of continuously monitored Asian options with geometric or arithmetic averaging. Put-call parity relations and fixed-floating symmetry relations between the prices of continuously monitored Asian options are deduced. For discretely monitored Asian options, we derive closed form price formulas for geometric averaging options and deduce analytic approximation formulas for arithmetic averaging options using the Edgeworth expansion technique.

4.3.1 Partial differential equation formulation

Suppose we write the average of the asset price as

$$A = \int_0^t f(S, u) \, du, \tag{4.3.3}$$

where $f(S, t)$ is chosen according to the type of average adopted in the Asian option. For example, $f(S, t) = \frac{1}{t} S$ corresponds to continuous arithmetic average, $f(S, t) = \exp \left(\frac{1}{n} \sum_{i=1}^{n} \delta(t - t_i) \ln S \right)$ corresponds to discrete geometric average, etc. The price of an Asian option is a function of time to expiry and the two state variables, S and A. Suppose $f(S,t)$ is a continuous time function, then by the mean value theorem

$$dA = \lim_{\Delta t \to 0} \int_t^{t+\Delta t} f(S,u)\,du = \lim_{\Delta t \to 0} f(S,u^*)\,dt = f(S,t)\,dt, \quad t < u^* < t + \Delta t, \tag{4.3.4}$$

so dA is deterministic. Hence, a riskless hedge for the Asian option requires only eliminating the asset-induced risk.

Consider a portfolio which contains one unit of the Asian option and $-\Delta$ units of the underlying asset. We then choose Δ such that the stochastic components associated with the option and the underlying asset cancel off each other. Assume the asset price dynamics to be given by

$$dS = [\mu S - D(S,t)]\,dt + \sigma S\,dZ, \tag{4.3.5}$$

where Z is the standard Wiener process, $D(S,t)$ is the dividend yield on the asset, μ and σ are the expected rate of return and volatility of the asset price, respectively. Let $V(S,A,t)$ denote the value of the Asian option and let Π denote the value of the above portfolio. The portfolio value is given by

$$\Pi = V(S,A,t) - \Delta S, \tag{4.3.6}$$

and its differential is found to be

$$d\Pi = \frac{\partial V}{\partial t}\,dt + f(S,t)\frac{\partial V}{\partial A}\,dt + \frac{\partial V}{\partial S}\,dS + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2}\,dt - \Delta\,dS - \Delta D(S,t)\,dt. \tag{4.3.7}$$

The last term in the above equation corresponds to the contribution of the dividend from the asset to the portfolio's value. As usual, we choose $\Delta = \dfrac{\partial V}{\partial S}$ so that the stochastic terms containing dS cancel. The absence of arbitrage dictates

$$d\Pi = r\Pi\ dt, \tag{4.3.8}$$

where r is the riskless interest rate. Putting Eqs. (4.3.7) and (4.3.8) together, we obtain the following governing differential equation for $V(S,A,t)$

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 V}{\partial S^2} + [rS - D(S,t)]\frac{\partial V}{\partial S} + f(S,t)\frac{\partial V}{\partial A} - rV = 0. \tag{4.3.9}$$

The equation is a degenerate diffusion equation since it contains diffusion term corresponding to S only but not for A. The auxiliary conditions in the pricing model depend on the specific details of the Asian option contract.

4.3.2 Continuously monitored geometric averaging options

We would like to derive analytic price formulas for the European Asian options whose terminal payoff depends on the continuously monitored geometric averaging of the underlying asset price. We take time zero to be the time of initiation of the averaging period, t is the current time and T denotes the expiration time. We define the continuously monitored geometric averaging of the asset price S_u over the time period $[0, t]$ by

$$G_t = \exp\left(\frac{1}{t}\int_0^t \ln S_u\, du\right). \tag{4.3.10}$$

The terminal payoffs of the fixed strike call option and floating strike call option are respectively given by

$$\begin{aligned} c_{fix}(S_T, G_T, T; X) &= \max(G_T - X, 0) \\ c_{f\ell}(S_T, G_T, T) &= \max(S_T - G_T, 0), \end{aligned} \tag{4.3.11}$$

where X is the fixed strike price. We illustrate how to use the risk neutral expectation approach to derive the price formula of the European fixed strike Asian call option. On the other hand, the partial differential equation method is used to derive the price formula of the floating strike counterpart.

European fixed strike Asian call option

We assume the existence of a risk neutral pricing measure Q under which discounted asset prices are martingales, implying the absence of arbitrage. Under the measure Q, the asset price follows

$$\frac{dS_t}{S_t} = (r - q)\, dt + \sigma\, dZ_t, \tag{4.3.12}$$

where Z_t is a Q-Brownian motion. For $0 < t < T$, the solution of the above stochastic differential equation is given by [see Eqs. (2.4.18a,b)]

$$\ln S_T = \ln S_t + \left(r - q - \frac{\sigma^2}{2}\right)(T - t) + \sigma(Z_T - Z_t). \tag{4.3.13}$$

By integrating $\ln S_t$ as required by Eq. (4.3.10), we obtain

$$\begin{aligned} \ln G_T = \frac{t}{T}\ln G_t + \frac{1}{T}\left[(T - t)\ln S_t + \left(r - q - \frac{\sigma^2}{2}\right)\frac{(T-t)^2}{2}\right] \\ + \frac{\sigma}{T}\int_t^T (Z_u - Z_t)\, du. \end{aligned} \tag{4.3.14}$$

The stochastic term $\frac{\sigma}{T}\int_t^T (Z_u - Z_t)\, du$ can be shown to be Gaussian with zero mean and variance $\frac{\sigma^2}{T^2}\frac{(T-t)^3}{3}$ (see Problem 2.32). By the risk neutral valuation principle, the value of the European fixed strike Asian call option is given by

$$c_{fix}(S_t, G_t, t) = e^{-r(T-t)} E[\max(G_T - X, 0)], \qquad (4.3.15)$$

where E is the expectation under Q conditional on the filtration generated by the Q-Brownian motion. We assume the current time t to be within the averaging period. By defining

$$\overline{\mu} = \left(r - q - \frac{\sigma^2}{2}\right) \frac{(T-t)^2}{2T} \quad \text{and} \quad \overline{\sigma} = \frac{\sigma}{T}\sqrt{\frac{(T-t)^3}{3}}, \qquad (4.3.16a)$$

G_T can be written as

$$G_T = G_t^{t/T} S_t^{(T-t)/T} \exp(\overline{\mu} + \overline{\sigma}\widehat{Z}), \qquad (4.3.16b)$$

where $\widehat{Z}$ is the standard normal random variable. Recall from our usual expectation calculations with call payoff that

$$\begin{aligned} &E[\max(F\exp(\overline{\mu} + \overline{\sigma}\widehat{Z}) - X, 0] \\ &= Fe^{\overline{\mu}+\overline{\sigma}^2/2} N\left(\frac{\ln\frac{F}{X} + \overline{\mu} + \overline{\sigma}^2}{\overline{\sigma}}\right) - XN\left(\frac{\ln\frac{F}{X} + \overline{\mu}}{\overline{\sigma}}\right), \end{aligned} \qquad (4.3.17)$$

we then deduce that

$$c_{fix}(S_t, G_t, t) = e^{-r(T-t)} \left[G_t^{t/T} S_t^{(T-t)/T} e^{\overline{\mu}+\overline{\sigma}^2/2} N(d_1) - XN(d_2)\right], \qquad (4.3.18)$$

where

$$\begin{aligned} d_2 &= \left(\frac{t}{T}\ln G_t + \frac{T-t}{T}\ln S_t + \overline{\mu} - \ln X\right) \Big/ \overline{\sigma}, \\ d_1 &= d_1 + \overline{\sigma}. \end{aligned} \qquad (4.3.19)$$

European floating strike Asian call option

Since the terminal payoff of the floating strike Asian call option involves S_T and G_T, pricing by the risk neutral expectation approach would require the joint distribution of S_T and G_T. For floating strike Asian options, the partial differential equation method provides the more effective approach to derive the price formula for $c_{f\ell}(S, G, t)$. This is because the similarity reduction technique can be applied to reduce the dimension of the differential equation.

When continuously monitored geometric averaging is adopted, the governing equation for $c_{f\ell}(S, G, t)$ can be expressed as

$$\frac{\partial c_{f\ell}}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 c_{f\ell}}{\partial S^2} + (r-q)S\frac{\partial c_{f\ell}}{\partial S} + \frac{G}{t}\ln\frac{S}{G}\frac{\partial c_{f\ell}}{\partial G} - rc_{f\ell} = 0, \quad 0 < t < T. \qquad (4.3.20)$$

Next, we define the similarity variables

$$y = t \ln \frac{G}{S} \quad \text{and} \quad W(y,t) = \frac{c_{f\ell}(S,G,t)}{S}. \tag{4.3.21}$$

This is equivalent to choose S as the numeraire. In terms of the similarity variables, the governing equation for $c_{f\ell}(S,G,t)$ becomes

$$\frac{\partial W}{\partial t} + \frac{\sigma^2 t^2}{2}\frac{\partial^2 W}{\partial y^2} - \left(r - q + \frac{\sigma^2}{2}\right) t \frac{\partial W}{\partial y} - qW = 0, \quad 0 < t < T, \tag{4.3.22}$$

with terminal condition: $W(y,T) = \max(1 - e^{y/T}, 0)$.

We write $\tau = T - t$ and let $F(y, \tau; \eta)$ denote the fundamental solution to the following parabolic equation with time dependent coefficients

$$\frac{\partial F}{\partial \tau} = \frac{\sigma^2 (T-\tau)^2}{2}\frac{\partial^2 F}{\partial y^2} - \left(r - q + \frac{\sigma^2}{2}\right)(T-\tau)\frac{\partial F}{\partial y}, \quad \tau > 0, \tag{4.3.23a}$$

with initial condition at $\tau = 0$ (corresponding to $t = T$) given as

$$F(y, 0; \eta) = \delta(y - \eta). \tag{4.3.23b}$$

Though the differential equation has time dependent coefficients, the fundamental solution is readily found to be [see Eq. (3.3.15)]

$$F(y,\tau;\eta) = n\left(\frac{y - \eta - \left(r - q + \frac{\sigma^2}{2}\right)\int_0^\tau (T-u)\,du}{\sigma\sqrt{\int_0^\tau (T-u)^2\,du}}\right). \tag{4.2.24}$$

The solution to $W(y, \tau)$ is then given by

$$W(y,\tau) = e^{-q\tau}\int_{-\infty}^{\infty} \max(1 - e^{\eta/T}, 0) F(y,\tau;\eta)\,d\eta. \tag{4.3.25}$$

The direct integration of the above integral gives (Wu *et al.*, 1999)

$$c_{f\ell}(S,G,t) = Se^{-q(T-t)}N(\widehat{d}_1) - G^{t/T}S^{(T-t)/T}e^{-q(T-t)}e^{-\widehat{Q}}N(\widehat{d}_2), \tag{4.3.26}$$

where

$$\widehat{d}_1 = \frac{t\ln\frac{S}{G} + \left(r - q + \frac{\sigma^2}{2}\right)\frac{T^2 - t^2}{2}}{\sigma\sqrt{\frac{T^3 - t^3}{3}}}, \quad \widehat{d}_2 = \widehat{d}_1 - \frac{\sigma}{T}\sqrt{\frac{T^3 - t^3}{3}},$$

$$\widehat{Q} = \frac{r - q + \frac{\sigma^2}{2}}{2}\frac{T^2 - t^2}{T} - \frac{\sigma^2}{6}\frac{T^3 - t^3}{T^2}. \tag{4.3.27}$$

4.3.3 Continuously monitored arithmetic averaging options

We consider a European fixed strike European Asian call based on continuously monitored arithmetic averaging. The terminal payoff is defined by

$$c_{fix}(S_T, A_T, T; X) = \max(A_T - X, 0). \quad (4.3.28)$$

He and Takahashi (2000) propose a variable reduction method which reduces the dimension of the governing differential equation by one. To motivate the choice of variable transformation, we consider the following expectation representation of the price of the Asian call at time t

$$\begin{aligned} c_{fix}(S_t, A_t, t) &= e^{-r(T-t)} E[\max(A_T - X, 0)] \\ &= e^{-r(T-t)} E\left[\max\left(\frac{1}{T}\int_0^t S_u\,du - X + \frac{1}{T}\int_t^T S_u\,du, 0\right)\right] \\ &= \frac{S_t}{T} e^{-r(T-t)} E\left[\max\left(x_t + \int_t^T \frac{S_u}{S_t}\,du, 0\right)\right], \end{aligned} \quad (4.3.29)$$

where the state variable x_t is defined by

$$x_t = \frac{1}{S_t}(I_t - XT), \quad I_t = \int_0^t S_u\,du = tA_t. \quad (4.3.30)$$

In subsequent exposition, it is more convenient to use I_t instead of A_t as the averaging state variable. Since $S_u/S_t, u > t$, is independent of the history of the asset price up to time t, one argues that the conditional expectation in Eq. (4.3.29) is a function of x_t only. We then deduce that

$$c_{fix}(S_t, I_t, t) = S_t f(x_t, t) \quad (4.3.31)$$

for some function of f. In other words, $f(x_t, t)$ is given by

$$f(x_t, t) = \frac{e^{-r(T-t)}}{T} E\left[\max\left(x_t + \int_t^T \frac{S_u}{S_t}\,du, 0\right)\right]. \quad (4.3.32)$$

If we write the price function of the fixed strike call as $c_{fix}(S, I, t)$, then the governing equation for $c_{fix}(S, I, t)$ is given by

$$\frac{\partial c_{fix}}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c_{fix}}{\partial S^2} + (r - q) S \frac{\partial c_{fix}}{\partial S} + S \frac{\partial c_{fix}}{\partial I} - r c_{fix} = 0. \quad (4.3.33)$$

Suppose we define the following transformation of variables:

$$x = \frac{1}{S}(I - XT) \quad \text{and} \quad f(x, t) = \frac{c_{fix}(S, I, t)}{S}, \quad (4.3.34)$$

then the governing differential equation for $f(x, t)$ becomes

$$\frac{\partial f}{\partial t}+\frac{\sigma^2}{2}x^2\frac{\partial^2 f}{\partial x^2}+[1-(r-q)x]\frac{\partial f}{\partial x}-qf=0, \quad -\infty<x<\infty, t>0. \tag{4.3.35}$$

The terminal condition is given by

$$f(x,T)=\frac{1}{T}\max(x,0). \tag{4.3.36}$$

When $x_t \geq 0$, which corresponds to $\frac{1}{T}\int_0^t S_u\,du \geq X$, it is possible to find closed form analytic solution to $f(x,t)$. Since x_t is an increasing function of t so that $x_T \geq 0$, the terminal condition $f(x,T)$ reduces to x/T. In this case, $f(x,t)$ admits solution of the form

$$f(x,t)=a(t)x+b(t). \tag{4.3.37}$$

By substituting the assumed form of solution into Eq. (4.3.35), we obtain the following pair of governing equations for $a(t)$ and $b(t)$:

$$\begin{aligned} &\frac{da(t)}{dt}-ra(t)=0, && a(T)=\frac{1}{T},\\ &\frac{db(t)}{dt}-a(t)-qb(t)=0, && b(T)=0. \end{aligned} \tag{4.3.38a}$$

When $r \neq q$, $a(t)$ and $b(t)$ are found to be

$$a(t)=\frac{e^{-r(T-t)}}{T} \quad \text{and} \quad b(t)=\frac{e^{-q(T-t)}-e^{-r(T-t)}}{T(r-q)}. \tag{4.3.38b}$$

Hence, the option value for $I \geq XT$ is given by

$$c_{fix}(S,I,t)=\left(\frac{I}{T}-X\right)e^{-r(T-t)}+\frac{e^{-q(T-t)}-e^{-r(T-t)}}{T(r-q)}S. \tag{4.3.39}$$

Though the volatility σ does not appear explicitly in the above price formula, it appears implicitly in S and A. The gamma is easily seen to be zero while the delta is a function of t and $T-t$ but not S or A.

For $I < XT$, there is no closed form analytic solution available. Curran (1994) and Rogers and Shi (1995) propose the conditioning method to find a *lower bound* on the Asian option price. They both use the approach of projecting the averaging state variable A_T on a $\mathcal{F}_T$-measurable Gaussian random variable Z, where

$$\begin{aligned} E[\max(A_T-X,0)] &= E\left[E\left[\max(A_T-X,0)|Z\right]\right]\\ &\geq E\left[\max\left(E[A_T-X|Z],0\right)\right]. \end{aligned} \tag{4.3.40}$$

The resulting expectation involving Z may be solvable in closed form. The choice of Z by Curran is the geometric average while that by Rogers and Shi is the logarithm of the geometric average. The approximation error would be small since the correlation coefficient between the geometric average and arithmetic average is close to one.

The analytic approximation approach has also been applied to continuously monitored floating strike Asian options. Bouaziz *et al.* (1994) use the linear approximation technique to approximate the law of $\{A_T, S_T\}$ by a joint lognormal distribution [see also the extension to quadratic approximation by Chung *et al.* (2003)]. Several other analytic approximation methods for pricing Asian options can be found in the papers by Henderson *et al.* (2004), Milevsky and Posner (1998), Nielsen and Sandmann (2003) and Tsao *et al.* (2003).

4.3.4 Put-call parity and fixed-floating symmetry relations

It is well known that the difference of the prices of European vanilla call and put options is equal to a European forward. Do we have similar put-call parity relations for European Asian options? Also, can we establish symmetry relations between the prices of fixed strike and floating strike Asian options, like those of lookback options? In this section, we derive these parity and symmetry relations for continuously monitored Asian options under the Black-Scholes framework. Some of these relations can be extended to more general stochastic price dynamics (Hoogland and Neumann, 2000).

Put-call parity relation

Let $c_{fix}(S, I, t)$ and $p_{fix}(S, I, t)$ denote the price function of the fixed strike arithmetic averaging Asian call option and put option, respectively. Their terminal payoff functions are given by

$$c_{fix}(S, I, T) = \max\left(\frac{I}{T} - X, 0\right) \qquad (4.3.41a)$$

$$p_{fix}(S, I, T) = \max\left(X - \frac{I}{T}, 0\right). \qquad (4.3.41b)$$

Let $D(S, I, t)$ denote the difference of c_{fix} and p_{fix}. Since both c_{fix} and p_{fix} are governed by the same equation [see Eq. (4.3.33)], so does $D(S, I, t)$. The terminal condition of $D(S, I, t)$ is given by

$$D(S, I, T) = \max\left(\frac{I}{T} - X, 0\right) - \max\left(X - \frac{I}{T}, 0\right) = \frac{I}{T} - X. \qquad (4.3.42)$$

The terminal condition $D(S, I, T)$ is the same as that of the continuously monitored arithmetic averaging option with $I \geq XT$. Hence, when $r \neq q$, the put-call parity relation between the prices of fixed strike Asian options under continuously monitored arithmetic averaging is given by [see also Eq. (4.3.39)]

$$\begin{aligned} & c_{fix}(S,I,t) - p_{fix}(S,I,t) \\ &= \left(\frac{I}{T} - X\right) e^{-r(T-t)} + \frac{e^{-q(T-t)} - e^{-r(T-t)}}{T(r-q)} S. \end{aligned} \tag{4.3.43}$$

Similar techniques can be used to derive the put-call parity relations between other types of Asian options (floating/fixed strike and geometric/arithmetic averaging) (see Problems 4.28 and 4.29).

Fixed-floating symmetry relations

By applying a change of measure and identifying a time-reversal of a Brownian motion (Henderson and Wojakowski, 2002), it is possible to establish the symmetry relations between the prices of floating strike and fixed strike arithmetic averaging Asian options at the start of the averaging period.

Suppose we write the price functions of various continuously monitored arithmetic averaging option at the start of the averaging period (time zero) as

$$c_{f\ell}(S_0, \lambda, r, q, T) = e^{-rT} E\left[\max\left(\lambda S_T - A_T, 0\right)\right] \tag{4.3.44a}$$
$$p_{f\ell}(S_0, \lambda, r, q, T) = e^{-rT} E\left[\max\left(A_T - \lambda S_T, 0\right)\right] \tag{4.3.44b}$$
$$c_{fix}(X, S_0, r, q, T) = e^{-rT} E\left[\max\left(A_T - X, 0\right)\right] \tag{4.3.44c}$$
$$p_{fix}(X, S_0, r, q, T) = e^{-rT} E\left[\max\left(X - A_T, 0\right)\right]. \tag{4.3.44d}$$

Assume that the asset price S_t follows the Geometric Brownian process under the risk neutral measure Q, where

$$\frac{dS_t}{S_t} = (r-q)\,dt + \sigma\,dZ_t. \tag{4.3.45}$$

Here, Z_t is a Q-Brownian process. Suppose the asset price is used as the numeraire, then

$$\begin{aligned} c^*_{f\ell} = \frac{c_{f\ell}}{S_0} &= \frac{e^{-rT}}{S_0} E\left[\max\left(\lambda S_T - A_T, 0\right)\right] \\ &= E\left[\frac{S_T e^{-rT}}{S_0} \frac{\max\left(\lambda S_T - A_T, 0\right)}{S_T}\right]. \end{aligned} \tag{4.3.46}$$

To effect the change of numeraire, we define the measure Q^* by

$$\frac{dQ^*}{dQ} = e^{-\frac{\sigma^2}{2}T + \sigma Z_T} = \frac{S_T e^{-rT}}{S_0 e^{-qT}}. \tag{4.3.47}$$

By virtue of the Girsanov Theorem, $Z^*_t = Z_t - \sigma t$ is a Q^*-Brownian process [see Problem 3.9]. If we write $A^*_T = A_T/S_T$, then

$$c^*_{f\ell} = e^{-qT} E^*\left[\max\left(\lambda - A^*_T, 0\right)\right], \tag{4.3.48}$$

where E^* denotes the expectation under Q^*. Now, we consider

$$A_T^* = \frac{1}{T}\int_0^T \frac{S_u}{S_T}\,du = \frac{1}{T}\int_0^T S_u^*(T)\,du, \qquad (4.3.49a)$$

where

$$S_u^*(T) = \exp\left(-\left(r - q - \frac{\sigma^2}{2}\right)(T-u) - \sigma(Z_T - Z_u)\right). \quad (4.3.49b)$$

In terms of the Q^*-Brownian process Z_t^*, where $Z_T - Z_u = \sigma(T-u) + Z_T^* - Z_u^*$, we can write

$$S_u^*(T) = \exp\left(\left(r - q + \frac{\sigma^2}{2}\right)(u-T) + \sigma(Z_u^* - Z_T^*)\right). \qquad (4.3.50)$$

Furthermore, we define a reflected Q^*-Brownian process starting at zero by $\widehat{Z}_t$, where $\widehat{Z}_t = -Z_t^*$, then $\widehat{Z}_{T-u}$ equals in law to $Z_u^* - Z_T^*$ due to the stationary increment property of a Brownian process. Hence, we establish

$$A_T^* \overset{\text{law}}{=} \widehat{A}_T = \frac{1}{T}\int_0^T e^{\sigma \widehat{Z}_{T-u} + \left(r - q + \frac{\sigma^2}{2}\right)(u-T)}\,du, \qquad (4.3.51a)$$

and via time-reversal of $\widehat{Z}_{T-u}$, we obtain

$$\widehat{A}_T = \frac{1}{T}\int_0^T e^{\sigma \widehat{Z}_\xi + \left(q - r - \frac{\sigma^2}{2}\right)\xi}\,d\xi. \qquad (4.3.51b)$$

Note that $\widehat{A}_T S_0$ is the arithmetic average of the price process with drift rate $q - r$. Summing the results together, we have

$$c_{f\ell} = S_0 c_{f\ell}^* = e^{-qT} E^*\left[\max\left(\lambda S_0 - \widehat{A}_T S_0, 0\right)\right] \qquad (4.3.52)$$

and from which we deduce the following fixed-floating symmetry relation

$$c_{f\ell}(S_0, \lambda, r, q, T) = p_{fix}(\lambda S_0, S_0, q, r, T). \qquad (4.3.53)$$

By combining the put-call parity relations for floating and fixed Asian options and the above symmetry relation, we can derive the fixed-floating symmetry relation between c_{fix} and $p_{f\ell}$ (see Problem 4.30)

$$c_{fix}(X, S_0, r, q, T) = p_{f\ell}\left(S_0, \frac{X}{S_0}, q, r, T\right). \qquad (4.3.54)$$

4.3.5 Fixed strike options with discrete geometric averaging

Consider the discrete geometric averaging of the asset prices at evenly distributed discrete times $t_i = i\Delta t, i = 1, 2, \cdots, n$, where Δt is the uniform time interval between fixings and $t_n = T$ is the time of expiration. Define the running geometric averaging by

$$G_k = \left[\prod_{i=1}^{k} S_{t_i}\right]^{1/k}, \quad k = 1, 2, \cdots, n. \tag{4.3.55}$$

The terminal payoff of a European average value call option with discrete geometric averaging is given by $\max(G_n - X, 0)$, where X is the strike price. Suppose the asset price is lognormally distributed, then the asset price ratio $R_i = \dfrac{S_{t_i}}{S_{t_{i-1}}}, i = 1, 2, \cdots, n$ is also lognormally distributed. Assume that under the risk neutral measure

$$\ln R_i \sim N\left(\left(r - \frac{\sigma^2}{2}\right)\Delta t, \sigma^2 \Delta t\right), \quad i = 1, 2, \cdots, n, \tag{4.3.56}$$

where r is the riskless interest rate and $N(\mu, \sigma^2)$ represents a normal distribution with mean μ and variance σ^2.

European fixed strike call option
The price formula of the European fixed strike call depends on whether the current time t is prior to or after time t_0. First, we consider $t < t_0$ and write

$$\frac{G_n}{S_t} = \frac{S_{t_0}}{S_t}\left\{\frac{S_{t_n}}{S_{t_{n-1}}}\left[\frac{S_{t_{n-1}}}{S_{t_{n-2}}}\right]^2 \cdots \left[\frac{S_{t_1}}{S_{t_0}}\right]^n\right\}^{1/n}, \tag{4.3.57a}$$

so that

$$\ln \frac{G_n}{S_t} = \ln \frac{S_{t_0}}{S_t} + \frac{1}{n}\left[\ln R_n + 2\ln R_{n-1} + \cdots + n \ln R_1\right], \quad t < t_0. \tag{4.3.57b}$$

Since $\ln R_i, i = 1, 2, \cdots n$ and $\ln \dfrac{S_{t_0}}{S_t}$ represent independent Brownian increments over non-overlapping time intervals, they are normally distributed and independent. Observe that $\ln \dfrac{G_n}{S_t}$ is a linear combination of these independent Brownian increments, so it remains to be normally distributed with mean

$$\begin{aligned}
&\left(r - \frac{\sigma^2}{2}\right)(t_0 - t) + \frac{1}{n}\left(r - \frac{\sigma^2}{2}\right)\Delta t \sum_{i=1}^{n} i \\
&= \left(r - \frac{\sigma^2}{2}\right)\left[(t_0 - t) + \frac{n+1}{2n}(T - t_0)\right],
\end{aligned} \tag{4.3.58a}$$

and variance

$$\sigma^2(t_0 - t) + \frac{1}{n^2}\sigma^2 \Delta t \sum_{i=1}^{n} i^2 = \sigma^2 \left[(t_0 - t) + \frac{(n+1)(2n+1)}{6n^2}(T - t_0)\right]. \tag{4.3.58b}$$

Let $\tau = T - t$, where τ is the time to expiry. Suppose we write

$$\sigma_G^2 \tau = \sigma^2 \left\{\tau - \left[1 - \frac{(n+1)(2n+1)}{6n^2}\right](T - t_0)\right\} \tag{4.3.59a}$$

$$\left(\mu_G - \frac{\sigma_G^2}{2}\right)\tau = \left(r - \frac{\sigma^2}{2}\right)\left[\tau - \frac{n-1}{2n}(T - t_0)\right], \tag{4.3.59b}$$

then the transition density function of G_n at time T, given the asset price S_t at an earlier time $t < t_0$, can be expressed as

$$\psi(G_n; S_t) = \frac{1}{G_n\sqrt{2\pi\sigma_G^2\tau}} \exp\left(-\frac{\left\{\ln G_n - \left[\ln S_t + \left(\mu_G - \frac{\sigma_G^2}{2}\right)\tau\right]\right\}^2}{2\sigma_G^2\tau}\right). \tag{4.3.60}$$

By the risk neutral valuation approach, the price of the European fixed strike call with discrete geometric averaging is given by

$$\begin{aligned} c_G(S_t, t) &= e^{-r\tau}E[\max(G_n - X, 0)] \\ &= e^{-r\tau}\left[S_t e^{\mu_G\tau}N(d_1) - XN(d_2)\right], \quad t < t_0 \end{aligned} \tag{4.3.61a}$$

where

$$d_1 = \frac{\ln\frac{S_t}{X} + \left(\mu_G + \frac{\sigma_G^2}{2}\right)\tau}{\sigma_G\sqrt{\tau}}, \quad d_2 = d_1 - \sigma_G\sqrt{\tau}. \tag{4.3.61b}$$

We consider the two extreme cases where $n = 1$ and $n \to \infty$. When $n = 1, \sigma_G^2\tau$ and $\left(\mu_G - \frac{\sigma_G^2}{2}\right)\tau$ reduce to $\sigma^2\tau$ and $\left(r - \frac{\sigma^2}{2}\right)\tau$, respectively, so that the call price reduces to that of a European vanilla call option. We observe that $\sigma_G^2\tau$ is a decreasing function of n, which is consistent with the intuition that the more frequent we take the averaging, the lower the volatility. When $n \to \infty, \sigma_G^2\tau$ and $\left(\mu_G - \frac{\sigma_G^2}{2}\right)\tau$ tend to $\sigma^2\left[\tau - \frac{2}{3}(T - t_0)\right]$ and $\left(r - \frac{\sigma^2}{2}\right)\left(\tau - \frac{T - t_0}{2}\right)$, respectively. Correspondingly, discrete geometric averaging becomes its continuous analog. In particular, the price of a European fixed strike call with continuous geometric averaging at $t = t_0$ is found to be [see also Eq. (4.3.18)]

$$c_G(S_{t_0}, t_0) = S_{t_0}e^{-\frac{1}{2}\left(r + \frac{\sigma^2}{6}\right)(T - t_0)}N(\widehat{d}_1) - Xe^{-r(T - T_0)}N(\widehat{d}_2), \tag{4.3.62a}$$

where

$$\widehat{d}_1 = \frac{\ln \frac{S_{t_0}}{X} + \frac{1}{2}\left(r + \frac{\sigma^2}{6}\right)(T - t_0)}{\sigma\sqrt{\frac{T-t_0}{3}}}, \quad \widehat{d}_2 = \widehat{d}_1 - \sigma\sqrt{\frac{T - t_0}{3}}. \tag{4.3.62b}$$

Next, we consider the in-progress option where the current time t is within the averaging period, that is, $t \geq t_0$. Here, $t = t_k + \xi\Delta t$ for some integer k, $0 \leq k \leq n-1$ and $0 \leq \xi < 1$. Now, $S_{t_1}, S_{t_2} \cdots S_{t_k}, S_t$ are known quantitites; and the price ratios $\dfrac{S_{t_{k+1}}}{S_t}, \dfrac{S_{t_{k+2}}}{S_{t_{k+1}}}, \cdots, \dfrac{S_{t_n}}{S_{t_{n-1}}}$ are independent lognormal random variables. We may write

$$G_n = [S_{t_1} \cdots S_{t_k}]^{1/n} S_t^{(n-k)/n} \left\{\frac{S_{t_n}}{S_{t_{n-1}}}\left[\frac{S_{t_{n-1}}}{S_{t_{n-2}}}\right]^2 \cdots \left[\frac{S_{t_{k+1}}}{S_t}\right]^{n-k}\right\}^{1/n} \tag{4.3.63}$$

so that

$$\ln\frac{G_n}{\widetilde{S}_t} = \frac{1}{n}\left[\ln R_n + 2\ln R_{n-1} + \cdots + (n-k-1)\ln R_{k+2} + (n-k)\ln R_t\right] \tag{4.3.64a}$$

where

$$\widetilde{S}_t = [S_{t_1} \cdots S_{t_k}]^{1/n} S_t^{(n-k)/n} = G_k^{k/n} S_t^{(n-k)/n} \text{ and } R_t = S_{t_{k+1}}/S_t. \tag{4.3.64b}$$

Let the variance and mean of $\ln \dfrac{G_n}{\widetilde{S}_t}$ be denoted by $\widetilde{\sigma}_G^2\tau$ and $\left(\widetilde{\mu}_G - \dfrac{\widetilde{\sigma}_G^2}{2}\right)\tau$, respectively. They are found to be

$$\widetilde{\sigma}_G^2\tau = \sigma^2\Delta t\left[\frac{(n-k)^2}{n^2}(1-\xi) + \frac{(n-k-1)(n-k)(2n-2k-1)}{6n^2}\right] \tag{4.3.65a}$$

and

$$\left(\widetilde{\mu}_G - \frac{\widetilde{\sigma}_G^2}{2}\right)\tau = \left(r - \frac{\sigma^2}{2}\right)\Delta t\left[\frac{n-k}{n}(1-\xi) + \frac{(n-k-1)(n-k)}{2n}\right]. \tag{4.3.65b}$$

Similar to formula (4.3.61a), the price formula of the in-progress European fixed strike call option takes the form

$$c_G(S_t, \tau) = e^{-r\tau}\left[\widetilde{S}_t e^{\widetilde{\mu}_G\tau} N(\widetilde{d}_1) - XN(\widetilde{d}_2)\right], \quad t \geq t_0, \tag{4.3.66a}$$

where

$$\widetilde{d}_1 = \frac{\ln \frac{\widetilde{S}_t}{X} + \left(\widetilde{\mu}_G + \frac{\widetilde{\sigma}_G^2}{2}\right)\tau}{\widetilde{\sigma}_G\sqrt{\tau}}, \quad \widetilde{d}_2 = \widetilde{d}_1 - \widetilde{\sigma}_G\sqrt{\tau}. \tag{4.3.66b}$$

Again, by taking the limit $n \to \infty$, the limiting values of $\tilde{\sigma}_G^2, \tilde{\mu}_G - \dfrac{\tilde{\sigma}_G^2}{2}$ and $\tilde{S}(t)$ become

$$\lim_{n\to\infty} \widetilde{\sigma}_G^2 = \left(\frac{T-t}{T-t_0}\right)^2 \frac{\sigma^2}{3}, \quad \lim_{n\to\infty} \widetilde{\mu}_G - \frac{\widetilde{\sigma}_G^2}{2} = \left(r - \frac{\sigma^2}{2}\right) \frac{T-t}{2(T-t_0)}, \tag{4.3.67a}$$

and

$$\lim_{n\to\infty} \widetilde{S}_t = S_t^{\frac{T-t}{T-t_0}} \widetilde{G}_t \text{ where } \widetilde{G}_t = \exp\left(\frac{1}{T-t_0} \int_{t_0}^t \ln S_u \, du\right). \tag{4.3.67b}$$

The price of the corresponding continuous geometric averaging call option can be obtained by substituting these limiting values into the price formula (4.3.66a).

European fixed strike put option
Using a similar derivation procedure, the price of the corresponding European fixed strike put option with discrete geometric averaging can be found to be

$$p_G(S,\tau) = \begin{cases} e^{-r\tau}\left[XN(-d_2) - Se^{\mu_G\tau}N(-d_1)\right], & t < t_0 \\ e^{-r\tau}\left[XN(-\widetilde{d}_2) - \widetilde{S}e^{\widetilde{\mu}_G\tau}N(-\widetilde{d}_1)\right], & t \geq t_0, \end{cases} \tag{4.3.68}$$

where d_1 and d_2 are given by Eq. (4.3.61b), and $\tilde{d}_1$ and $\tilde{d}_2$ are given by Eq. (4.3.66b). The put-call parity relation for the European fixed strike options with discrete geometric averaging can be deduced to be

$$c_G(S,\tau) - p_G(S,\tau) = \begin{cases} e^{-r\tau}Se^{\mu_G\tau} - Xe^{-r\tau}, & t < t_0 \\ e^{-r\tau}\widetilde{S}e^{\widetilde{\mu}_G\tau} - Xe^{-r\tau}, & t \geq t_0. \end{cases} \tag{4.3.69}$$

Additional analytic price formulas for Asian options with geometric averaging can be found in Boyle's paper (1993).

4.3.6 Fixed strike options with discrete arithmetic averaging

The most common type of Asian options are the fixed strike options whose terminal payoff is determined by the discrete arithmetic average of past prices. The valuation of these options is made difficult by the choice of Geometric Brownian motion for the underlying asset price since the sum of lognormal components has no closed form representation.

First, we give a formal description of this type of option contracts. Suppose the average of asset prices is calculated over the time interval $[t_0, t_n]$ and at discrete points $t_i = t_0 + i\triangle t, t = 0, 1, \cdots, n, \triangle t = \dfrac{t_n - t_0}{n}$. The running average $A(t)$ is defined for the current time $t, t_m \leq t < t_{m+1}$, by

$$A(t) = \frac{1}{m+1} \sum_{i=0}^{m} S_{t_i}, \quad 0 \leq m \leq n, \tag{4.3.70}$$

and $A(t) = 0$ for $t < t_0$. Let t_n be the time of expiration so that the payoff function at expiry is given by $\max(A(t_n) - X, 0)$ for the fixed strike call, where

X is the strike price. It may be more convenient to consider the terminal payoff in terms of $\tilde{A}(t_n;t)$ as defined by

$$\widetilde{A}(t_n;t) = A(t_n) - \frac{m+1}{n+1}A(t) = \frac{1}{n+1}\sum_{i=m+1}^{n} S_{t_i}, \qquad (4.3.71a)$$

which is the average of the unknown stochastic components beyond the current time. For example, the terminal payoff of the fixed strike call can be rewritten as $\max(\widetilde{A}(t_n;t) - X^*, 0)$, where

$$X^* = X - \frac{m+1}{n+1}A(t) \qquad (4.3.71b)$$

is the effective strike price of the option. It is easily seen that if X^* becomes negative, then the option is surely exercised at expiration.

The probability distribution of either $A(t_n)$ or $\widetilde{A}(t_n;t)$ has no available explicit representation. The best approach for deriving approximate analytic price formulas is to approximate the distribution of $A(t_n)$ [or $\widetilde{A}(t_n;t)$] by an approximate lognormal distribution through the method of *generalized Edgeworth series expansion*. The Edgeworth series expansion is quite similar to the Taylor series expansion for analytic functions in complex function theory. A brief discussion of the Edgeworth series expansion is presented below (Jarrow and Rudd, 1982).

Edgeworth series expansion

Given a probability distribution $F_t(s)$, called the true distribution, we would like to approximate $F_t(s)$ by an approximating distribution $F_a(s)$. The distributions considered are restricted to the class where $\dfrac{dF_t(s)}{ds} = f_t(s)$ and $\dfrac{dF_a(s)}{ds} = f_a(s)$ exist, that is, distributions which have continuous density functions. First, we define the following quantities:

(i) jth moment of distribution F:

$$\alpha_j(F) = \int_{-\infty}^{\infty} s^j f(s)\,ds \qquad (4.3.72a)$$

(ii) jth central moment of distribution F:

$$\mu_j(F) = \int_{-\infty}^{\infty} [s - \alpha_1(F)]^j f(s)\,ds \qquad (4.3.72b)$$

(iii) characteristic function of F:

$$\phi(F,t) = \int_{-\infty}^{\infty} e^{its} f(s)\,ds, \quad i = \sqrt{-1}. \qquad (4.3.72c)$$

Here, it is assumed that the moments $\alpha_j(F)$ exist for $j \le n$. Next, the cumulants $k_j(F)$ are defined by

$$\ln \phi(F,t) = \sum_{j=1}^{n-1} k_j(F)\frac{(it)^j}{j!} + o(t^{n-1}). \tag{4.3.73}$$

It can be shown by theoretical analysis that the first $n-1$ cumulants exist, provided that $\alpha_n(F)$ exists. The first four cumulants are

$$k_1(F) = \alpha_1(F), k_2(F) = \mu_2(F), k_3(F) = \mu_3(F), k_4(F) = \mu_4(F) - 3[\mu_2(F)]^2. \tag{4.3.74}$$

Also, we assume the existence of the derivatives: $\dfrac{d^j F_a(s)}{ds^j}$, $j \leq m$. Let $N = \min(n,m)$, the difference of $\ln\phi(F_t,t)$ and $\ln\phi(F_a,t)$ can be represented by

$$\ln \phi(F_t,t) = \sum_{j=1}^{N-1} [k_j(F_t) - k_j(F_a)]\,\frac{(it)^j}{j!} + \ln\phi(F_a,t) + o(t^{N-1}). \tag{4.3.75}$$

Taking the exponentials of the above equation [note that $e^{o(t^{N-1})} = 1 + o(t^{N-1})$], we obtain

$$\phi(F_t,t) = \exp\left(\sum_{j=1}^{N-1} [k_j(F_t) - k_j(F_a)]\,\frac{(it)^j}{j!}\right)\phi(F_a,t) + o(t^{n-1}). \tag{4.3.76}$$

Suppose the above exponential term is expanded in polynomial in it, we have

$$\exp\left(\sum_{j=1}^{N-1} [k_j(F_t) - k_j(F_a)]\,\frac{(it)^j}{j!}\right) = \sum_{j=0}^{N-1} E_j\frac{(it)^j}{j!} + o(t^{N-1}), \tag{4.3.77}$$

where the first few coefficients are

$$\begin{aligned} &E_0 = 1, E_1 = k_1(F_t) - k_1(F_a), E_2 = [k_2(F_t) - k_2(F_a)] + E_1^2, \\ &E_3 = [k_3(F_t) - k_3(F_a)] + 3E_1[k_2(F_t) - k_2(F_a)] + E_1^3, \text{ etc.} \end{aligned} \tag{4.3.78}$$

In terms of cumulants, Eq. (4.3.75) can be rewritten as

$$\phi(F_t,t) = \left[\sum_{j=0}^{N-1} E_j\frac{(it)^j}{j!}\right]\phi(F_a,t) + o(t^{N-1}). \tag{4.3.79}$$

Lastly, we take the inverse Fourier transform of the above equation. Using the following relations

$$f_t(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-its}\phi(F_t,t)\,dt, \tag{4.3.80a}$$

$$(-1)^j\frac{d^j f_a(s)}{ds^j} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-its}(it)^j\phi(F_a,t)\,dt, \quad j = 0,1,\cdots,N-1, \tag{4.3.80b}$$

we obtain the following representation of the Edgeworth series expansion

$$f_t(s) = f_a(s) + \sum_{j=1}^{N-1} E_j \frac{(-1)^j}{j!} \frac{d^j f_a(s)}{ds^j} + \epsilon(s, N) \qquad (4.3.81a)$$

where

$$\epsilon(s, N) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{its} o(t^{N-1})\, dt. \qquad (4.3.81b)$$

In order that $\epsilon(s, N)$ exists for all s, it is necessary to observe

$$\lim_{N\to\infty} |\epsilon(s, N)| = 0 \quad \text{for all } s. \qquad (4.3.82)$$

Suppose all moments of the true and approximating distributions can be calculated, one may claim theoretically that a given distribution can be approximated by another distribution to any desired level of accuracy.

It is most convenient to use the lognormal distribution as the approximating distribution in valuation problems for arithmetic averaging Asian options. Suppose we choose the parameters of the approximating lognormal distribution such that its first two moments match with the first two moments of the true distribution, that is, $\alpha_1(F_t) = \alpha_1(F_a)$ and $\mu_2(F_t) = \mu_2(F_a)$, then the corresponding two-term Edgeworth series expansion becomes

$$f_t(s) = f_a(s) + \epsilon(s, 3), \qquad (4.3.83)$$

since $E_1 = \alpha_1(F_t) - \alpha_1(F_a) = 0$ and $E_2 = \mu_2(F_t) - \mu_2(F_a) + E_1^2 = 0$.

Fixed strike call option

Consider the fixed strike call option with discrete arithmetic averaging whose terminal payoff is $\max(A(t_n) - X, 0)$. By the risk neutral valuation approach, the price of the fixed strike call is given by

$$\begin{aligned} c(S, A, t) &= e^{-r\tau} E[\max(A(t_n) - X, 0)] \\ &= e^{-r\tau} E[\max(\widetilde{A}(t_n; t) - X^*, 0)] \end{aligned} \qquad (4.3.84)$$

where E is the expectation under the risk neutral measure conditional on $S_t = S$ and $A(t) = A$, and $\widetilde{A}(t_n; t)$ and X^* are defined by Eqs. (4.3.71a,b).

We would like to approximate the distribution of $\widetilde{A}(t_n; t)$ by a lognormal distribution. More specifically, we approximate the distribution of $\ln \widetilde{A}(t_n; t)$ by a normal distribution whose mean and variance are $\mu(t)$ and $\sigma(t)^2$, respectively. The first and the second moments of the approximating lognormal distribution are then given by (see Problem 2.24)

$$\alpha_1(F_a) = \mu(t) + \frac{\sigma(t)^2}{2} \qquad (4.3.85a)$$

$$\alpha_2(F_a) = 2\mu(t) + 2\sigma(t)^2 \qquad (4.3.85b)$$

respectively. Suppose we adopt the two-term Edgeworth approximation for $\widetilde{A}(t_n;t)$, which can be achieved by setting the first two moments of the approximating lognormal distribution and the distribution of $\widetilde{A}(t_n;t)$ to be equal, that is,

$$\mu(t) + \frac{\sigma(t)^2}{2} = \ln E[\widetilde{A}(t_n;t)] \tag{4.3.86a}$$

$$2\mu(t) + 2\sigma(t)^2 = \ln E[\widetilde{A}(t_n;t)^2]. \tag{4.3.86b}$$

Solving for $\mu(t)$ and $\sigma(t)^2$, we obtain

$$\mu(t) = 2\ln E[\widetilde{A}(t_n;t] - \frac{1}{2}\ln E[\widetilde{A}(t_n;t)^2] \tag{4.3.87a}$$

$$\sigma(t)^2 = \ln E[\widetilde{A}(t_n;t)^2] - 2\ln E(\widetilde{A}[t_n;t)]. \tag{4.3.87b}$$

Suppose $\ln \widetilde{A}(t_n;t)$ were normally distributed with mean $\mu(t)$ and variance $\sigma(t)^2$, then the value of the fixed strike call would become

$$c(S,A,t) = e^{-r(T-t)}\{E[\tilde{A}(t_n;t)]N(d_1) - X^*N(d_2)\} \tag{4.3.88}$$

where

$$d_1 = \frac{\mu(t) + \sigma(t)^2 - \ln X^*}{\sigma(t)}, \quad d_2 = d_1 - \sigma(t). \tag{4.3.89}$$

The above price formula provides a good approximation to the true price of the fixed strike call when the approximating lognormal distribution is a good approximation to the true distribution of $\widetilde{A}(t_n;t)$. The remaining procedure amounts to the determination of $E[\widetilde{A}(t_n;t)]$ and $E[\widetilde{A}(t_n;t)^2]$.

Let S_t denote the spot asset price at the current time t, where $t = t_m + \xi\triangle t, 0 \leq \xi < 1$. For $t \geq t_0$, we have

$$E[\tilde{A}(t_n;t)] = \frac{1}{n+1}\sum_{i=m+1}^{n} E[S_{t_i}]. \tag{4.3.90}$$

The asset price dynamics under the risk neutral measure Q is assumed to be

$$\frac{dS_t}{S_t} = r\,dt + \sigma\,dZ \tag{4.3.91}$$

where r is the constant riskless interest rate and σ^2 is the constant variance rate. We then have

$$E[S_{t_i}] = S_t e^{r(i-m-\xi)\triangle t}, \tag{4.3.92}$$

so that

$$E[\tilde{A}(t_n;t)] = \frac{S_t}{n+1}e^{r(1-\xi)\triangle t}\left[\frac{1-e^{r(n-m)\triangle t}}{1-e^{r\triangle t}}\right], \quad t \geq t_0. \tag{4.3.93}$$

For $t < t_0$, it can be shown similarly that

$$E[\tilde{A}(t_n;t)] = \frac{S_t}{n+1} e^{r(t_0-t)} \left[\frac{1-e^{r(n+1)\triangle t}}{1-e^{r\triangle t}} \right], \quad t < t_0. \tag{4.3.94}$$

Next, we consider $E[\widetilde{A}(t_n;t)^2]$. For $t \geq t_0$, we have

$$\begin{aligned} E[\widetilde{A}(t_n;t)^2] &= \frac{1}{(n+1)^2} \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} E[S_{t_i} S_{t_j}] \\ &= \frac{1}{(n+1)^2} \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} e^{r(i+j-2\xi)\triangle t + \sigma^2(\min(i,j)-\xi)\triangle t} S_t^2, \end{aligned} \tag{4.3.95}$$

and after some tedious manipulation, we obtain (Levy, 1992)

$$E[\widetilde{A}(t_n;t)^2] = \frac{S_t^2}{(n+1)^2} e^{-2\xi\left(r+\frac{\sigma^2}{2}\right)\triangle t}(A_1 - A_2 + A_3 - A_4), \quad t \geq t_0, \tag{4.3.96a}$$

where

$$\begin{aligned} A_1 &= \frac{e^{(2r+\sigma^2)\triangle t} - e^{(2r+\sigma^2)(N-m+1)\triangle t}}{(1-e^{r\triangle t})[1-e^{(2r+\sigma^2)\triangle t}]}, \\ A_2 &= \frac{e^{[r(N-m+2)+\sigma^2]\triangle t} - e^{(2r+\sigma^2)(N-m+1)\triangle t}}{(1-e^{r\triangle t})[1-e^{(r+\sigma^2)\triangle t}]}, \\ A_3 &= \frac{e^{(3r+\sigma^2)\triangle t} - e^{[r(N-m+2)+\sigma^2]\triangle t}}{(1-e^{r\triangle t})[1-e^{(r+\sigma^2)\triangle t}]}, \\ A_4 &= \frac{e^{2(2r+\sigma^2)\triangle t} - e^{(2r+\sigma^2)(N-m+1)\triangle t}}{[1-e^{(r+\sigma^2)\triangle t}][1-e^{(2r+\sigma^2)\triangle t}]}. \end{aligned} \tag{4.3.96b}$$

The calculation of $E[\widetilde{A}(t_n;t)^2]$ for $t < t_0$ can be performed similarly [see Problem (4.32)].

4.4 Problems

4.1 Consider the function

$$f(S,\tau) = \left(\frac{S}{B}\right)^{\lambda} c_E\left(\frac{B^2}{S}, \tau\right),$$

where $c_E(S,\tau)$ is the price of a vanilla European call option, λ is some constant parameter. Show that $f(S,\tau)$ satisfies the Black-Scholes equation

$$\frac{\partial f}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf$$

when λ is chosen to be $-2\dfrac{r}{\sigma^2} + 1$.

Hint: Substitution of $f(S,\tau)$ into the Black-Scholes equation gives

$$\frac{\partial f}{\partial \tau} - \left[\frac{\sigma^2}{2}S^2\frac{\partial^2 f}{\partial S^2} + rS\frac{\partial f}{\partial S} - rf\right]$$
$$= \left(\frac{S}{B}\right)^{\lambda}\left[\frac{\partial c_E}{\partial \tau} - \frac{\sigma^2}{2}\xi^2\frac{\partial^2 c_E}{\partial \xi^2}\right.$$
$$\left. + (\lambda-1)\sigma^2\xi\frac{\partial c_E}{\partial \xi} - \lambda(\lambda-1)\frac{\sigma^2}{2}c_E - r\lambda c_E + r\xi\frac{\partial c_E}{\partial \xi} + rc_E\right]$$

where $c_E = c_E(\xi,\tau), \xi = \dfrac{B^2}{S}$.

4.2 Consider the European zero-rebate up-and-out put option with exponential barrier function $B(\tau) = Be^{-\gamma\tau}$, where $B(\tau) > X$ for all τ. Show that the price of this barrier option is given by

$$p(S,\tau) = p_E(S,\tau) - \left[\frac{B(\tau)}{S}\right]^{\delta-1} p_E\left(\frac{B(\tau)^2}{S},\tau\right), \quad \delta = \frac{2(r-\gamma)}{\sigma^2},$$

where $p_E(S,\tau)$ is the price of the corresponding European vanilla put option. Deduce the price of the corresponding European up-and-in put option with the same barrier.

Hint: Let $y = \ln\dfrac{S}{B(\tau)}$, show that $p(y,\tau)$ satisfies

$$\frac{\partial p}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial y^2} + \left(r - \frac{\sigma^2}{2} - \gamma\right)\frac{\partial p}{\partial y} - rp.$$

4.3 By applying the following transformation on the dependent variable in the Black-Scholes equation

$$c = e^{\alpha y + \beta\tau}w,$$

where $\alpha = \dfrac{1}{2} - \dfrac{r}{\sigma^2}, \beta = -\dfrac{\alpha^2\sigma^2}{2} - r$, show that Eq. (4.1.4a) is reduced to the prototype diffusion equation

$$\frac{\partial w}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 w}{\partial y^2},$$

while the auxiliary conditions are transformed to become

$$w(0,\tau) = e^{-\beta\tau}R(\tau) \text{ and } w(y,0) = \max(e^{\alpha y}(e^y - X), 0).$$

Consider the following diffusion equation with semi-infinite domain

$$\frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial x^2}, \quad x > 0 \text{ and } t > 0, \quad a \text{ is a positive constant},$$

with initial condition: $v(x,0) = f(x)$ and boundary condition: $v(0,t) = g(t)$, the solution to the above problem is given by (Kevorkian, 1990)

$$v(x,t) = \frac{1}{2a\sqrt{\pi t}} \int_0^\infty f(\xi)[e^{-x-\xi)^2/4a^2t} - e^{-(x+\xi)^2/4a^2t}]\, d\xi \\ + \frac{x}{2a\sqrt{\pi}} \int_0^t \frac{e^{-x^2/4a^2\omega}}{\omega^{3/2}} g(t-\omega)\, d\omega. \tag{i}$$

Using the form of solution given by Eq. (i), show that the price of the European down-and-out option is given by

$$c(y,\tau) = e^{\alpha y+\beta\tau} \left\{ \frac{1}{\sqrt{2\pi\tau}\sigma} \int_0^\infty \max(e^{-\alpha\xi}(e^\xi - X), 0) \right. \\ \left[e^{-(y-\xi)^2/2\sigma^2\tau} - e^{-(y+\xi)^2/2\sigma^2\tau}\right] d\xi \\ \left. + \frac{y}{\sqrt{2\pi}\sigma} \int_0^\tau \frac{e^{-\beta(\tau-\omega)} e^{-y^2/2\sigma^2\omega}}{\omega^{3/2}} R(\tau-\omega)\, d\omega \right\}.$$

Assuming $B < X$, show that the price of the European down-and-out call option is given by [see Eqs. (4.1.13a,b)]

$$c(S,\tau) = c_E(S,\tau) - \left(\frac{B}{S}\right)^{\delta-1} c_E\left(\frac{B^2}{S}, \tau\right) \\ + \int_0^\tau e^{-r\omega} \frac{\ln\frac{S}{B}}{\sqrt{2\pi}\sigma} \frac{\exp\left(\frac{-\left[\ln\frac{S}{B} + \left(r - \frac{\sigma^2}{2}\right)\omega\right]^2}{2\sigma^2 w}\right)}{\omega^{3/2}} R(\tau-\omega)\, d\omega.$$

The last term represents the additional option premium due to the rebate payment.

4.4 Suppose the asset price follows the lognormal distribution with drift rate r, volatility σ, and barrier B. Find the density function of the asset price S_T at expiration time T, with the current asset price S starting below the barrier then breaching the barrier but ending below the barrier at expiration.

4.5 We define

$$d_1 = \frac{\ln\frac{S}{K} + \left(r + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau},$$
$$d_3 = \frac{2\ln\frac{B}{S}}{\sigma\sqrt{\tau}} + d_1, \quad d_4 = d_3 - \sigma\sqrt{\tau}, \delta = \frac{2r}{\sigma^2},$$

$$V_b(S,\tau;K) = [Xe^{-r\tau}N(-d_2) - SN(-d_1)] \\ - \left(\frac{B}{S}\right)^{\delta-1}\left[Xe^{-r\tau}N(-d_4) - \frac{B^2}{S}N(-d_3)\right].$$

Show that the price functions of various barrier put options are given by

$$p_{uo}(S,\tau;X,B) = V_b(S,\tau;\min(X,B))$$
$$p_{do}(S,\tau;X,B) = \max(V_b(S,\tau;X) - V_b(S,\tau;B), 0)$$
$$p_{ui}(S,\tau;X,B) = p_E(S,\tau;X) - p_{uo}(S,\tau;X,B)$$
$$p_{di}(S,\tau;X,B) = p_E(S,\tau;X) - p_{do}(S,\tau;X,B),$$

where $p_E(S,\tau;X)$ is the price function of the European put option.

4.6 Consider a European two-asset down-and-out call option where the terminal payoff depends on the payoff state variable $S_{1,t}$ and knock-out occurs when the barrier state variable $S_{2,t}$ breaches the downstream barrier B_2. Assume that under the risk neutral measure Q.

$$\frac{dS_{i,t}}{S_{i,t}} = r\,dt + \sigma_i\,dZ_{i,t}, \quad i = 1,2 \quad \text{and} \quad dZ_1\,dZ_2 = \rho\,dt.$$

Let X_1 denote the strike price. Show that the price of this down-and-out call with an *external barrier* is given by (Kwok *et al.*, 1998)

$$\begin{aligned} &\text{call price} \\ &= e^{-r\tau}E_Q[(S_{1,T} - X)\mathbf{1}_{\{S_{1,T}>X\}}\mathbf{1}_{\{m_{2,0}^T>B_2\}}] \\ &= S_{1,0}\left[N_2(d_1,e_1;\rho) - \left(\frac{B_2}{S_{2,0}}\right)^{\delta_2-1+2\gamma_{12}} N(d_1',e_1';\rho)\right] \\ &\quad - e^{-rT}X_1\left[N(d_2,e_2;\rho) - \left(\frac{B_2}{S_{2,0}}\right)^{\delta_2-1} N(d_2',e_2';\rho)\right], \end{aligned}$$

where

$$\begin{aligned} d_1 &= \frac{\ln\frac{S_{1,0}}{X_1} + \left(r + \frac{\sigma_1^2}{2}\right)T}{\sigma_1\sqrt{T}}, & d_2 &= d_1 - \sigma_1\sqrt{T}, \\ d_1' &= d_1 + \frac{2\gamma_{12}\ln\frac{B_2}{S_{2,0}}}{\sigma_1\sqrt{T}}, & d_2' &= d_1' - \sigma_1\sqrt{T}, \\ e_1 &= \frac{\ln\frac{S_{2,0}}{B_2} + \left(r - \frac{\sigma_1^2}{2} + \rho\sigma_1\sigma_2\right)T}{\sigma_2\sqrt{T}}, & e_2 &= e_1 - \rho\sigma_1\sqrt{T}, \\ e_1' &= e_1 + \frac{2\ln\frac{B_2}{S_{2,0}}}{\sigma_2\sqrt{T}}, & e_2' &= e_1' - \rho\sigma_1\sqrt{T}, \\ \delta_2 &= \frac{2r}{\sigma_2^2}, \quad \gamma_{12} = \rho\frac{\sigma_1}{\sigma_2}. \end{aligned}$$

4.7 Consider a European down-and-out *partial barrier* call where the barrier provision is activated only between option starting date (time 0) and t_1. Here, t_1 is some time earlier than the expiration date T so that $0 < t_1 < T$. Let B and X denote the down-barrier and strike, respectively, where $B < X$. Let $\ln \dfrac{S_t}{S_0} = r\,dt + \sigma Z_t$ under the risk neutral measure Q. Assuming $S_0 > B$, show that the down-and-out call price is given by (Heynen and Kat, 1994a)

$$\begin{aligned}
&\text{call price}\\
&= e^{-rT} E_Q\left[(S_T - X)\mathbf{1}_{\{S_T > X\}}\mathbf{1}_{\{m_0^{t_1} > B\}}\right]\\
&= S_0\left[N\left(d_1, e_1; \sqrt{\frac{t_1}{T}}\right) - \left(\frac{B}{S}\right)^{\delta+1} N\left(d_1', e_1'; \sqrt{\frac{t_1}{T}}\right)\right]\\
&\quad - e^{-rT} X\left[N\left(d_2, e_2; \sqrt{\frac{t_1}{T}}\right) - \left(\frac{B}{S}\right)^{\delta-1} N\left(d_2', e_2'; \sqrt{\frac{t_1}{T}}\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
d_1 &= \frac{\ln \frac{S_0}{X} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad & d_2 &= d_1 - \sigma\sqrt{T},\\
d_1' &= d_1 + \frac{2\ln \frac{B}{S_0}}{\sigma\sqrt{T}}, & d_2' &= d_1' - \sigma\sqrt{T},\\
e_1 &= \frac{\ln \frac{S_0}{B} + \left(r + \frac{\sigma^2}{2}\right)t_1}{\sigma\sqrt{t_1}}, & e_2 &= e_1 - \sigma\sqrt{t_1},\\
e_1' &= e_1 + \frac{2\ln \frac{B}{S_0}}{\sigma\sqrt{t_1}}, & e_2' &= e_1' - \sigma\sqrt{t_1},\\
\delta &= \frac{2r}{\sigma^2}.
\end{aligned}$$

Show that the above price formula reduces to that in Eq.(4.1.12a,b) when t_1 is set equal T.

Hint: Modify the price formula in Problem 4.6 by setting $\rho = \sqrt{\dfrac{t_1}{T}}$,

$\sigma_1 = \sigma$ and $\sigma_2 = \sqrt{\dfrac{t_1}{T}}\sigma$ so that $\gamma_{12} = 1$.

4.8 Consider a typical term in $g(x, t)$

$$\frac{\exp\left(\frac{\mu x}{\sigma^2} - \frac{\mu^2 t}{2\sigma^2}\right)}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(x-\xi)^2}{2\sigma^2 t}\right),$$

where ξ can be either $2n(u-\ell)$ or $2\ell + 2n(u-\ell)$. Show that the above term can be rewritten as

$$\frac{e^{\frac{\mu\xi}{\sigma^2}}}{\sqrt{2\pi\sigma^2 t}} \exp\left(\frac{-(x-\xi-\mu t)^2}{2\sigma^2 t}\right).$$

Hence, show that

$$\int_\ell^u g(x,t)\,dx$$
$$= \sum_{n=-\infty}^{\infty} \exp\left(\frac{2\mu n(u-\ell)}{\sigma^2}\right)$$
$$\left[N\left(\frac{u-\mu t-2u(u-\ell)}{\sigma\sqrt{t}}\right) - N\left(\frac{\ell-\mu t-2n(u-\ell)}{\sigma\sqrt{t}}\right)\right]$$
$$-\ \exp\left(\frac{2\mu[\ell+n(u-\ell)]}{\sigma^2}\right)$$
$$\left[N\left(\frac{u-\mu t-2\ell-2n(u-\ell)}{\sigma\sqrt{t}}\right) - N\left(\frac{\ell-\mu t-2\ell-2n(u-\ell)}{\sigma\sqrt{t}}\right)\right].$$

Use the above result to derive the price formula of the double knock-out call option [see Eq. (4.1.45)].

Analytic tractability of the price formula remains even when the barriers are exponential functions in time. Suppose the upper and lower barriers become $Ue^{\delta_1 t}$ and $Le^{\delta_2 t}, t \in [0,T]$. Here, δ_1 and δ_2 are constant and the barriers do not intersect over $[0,T]$. Show that the price formula for the double knock-out call option can be expressed as (Kunitomo and Ikeda, 1992)

$$c^o_{LU} = S\sum_{n=-\infty}^{\infty}\left\{\left(\frac{U^n}{L^n}\right)^{\mu_1}\left(\frac{L}{S}\right)^{\mu_2}[N(d_1)-N(d_2)]\right.$$
$$-\ \left(\frac{L^{n+1}}{U^n S}\right)^{\mu_3}[N(d_3)-N(d_4)]$$
$$-\ Xe^{-rT}\sum_{n=-\infty}^{\infty}\left\{\left(\frac{U^n}{L^n}\right)^{\mu_1-2}\left(\frac{L}{S}\right)^{\mu_2} N(d_1-\sigma\sqrt{T})-N(d_2-\sigma\sqrt{T})\right\}$$
$$\left.-\left(\frac{L^{n+1}}{U^n S}\right)^{\mu_3-2}[N(d_3-\sigma\sqrt{T})-N(d_4-\sigma\sqrt{T})]\right\},$$

where

$$d_1 = \frac{\ln\frac{SU^{2n}}{XL^{2n}} + \left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \qquad d_2 = \frac{\ln\frac{SU^{2n}}{FL^{2n}} + \left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$
$$d_3 = \frac{\ln\frac{L^{2n+2}}{XSU^{2n}} + \left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \qquad d_4 = \frac{\ln\frac{L^{2n+2}}{FSU^{2n}} + \left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}},$$

$$\mu_1 = \frac{2[r - \delta_2 - n(\delta_1 - \delta_2)]}{\sigma^2} + 1, \quad \mu_2 = 2n\frac{\delta_1 - \delta_2}{\gamma^2},$$
$$\mu_3 = \frac{2[r - \delta_2 + n(\delta_1 - \delta_2)]}{\sigma^2} + 1, \quad F = Ue^{\delta_1 T}.$$

4.9 Let $P(x, t; x_0, t_0)$ denote the transition density function of the restricted Brownian process $W_t^\mu = \mu t + \sigma Z_t$ with two absorbing barriers at $x = 0$ and $x = \ell$. Using the method of separation of variables (Kevorkian, 1990), show that the solution to $P(x, t; x_0, t_0)$ admits the following eigen-function expansion [which differs drastically in analytic form from that in Eq. (4.1.57)]

$$P(x, t; x_0, t_0) = e^{\frac{\mu}{\sigma^2}(x - x_0)} \frac{2}{\ell} \sum_{k=1}^{\infty} e^{-\lambda_k (t - t_0)} \sin \frac{k\pi x}{\ell} \sin \frac{k\pi x_0}{\ell}$$

where the eigenvalues are given by

$$\lambda_k = \frac{1}{2}\left(\frac{\mu^2}{\sigma^2} + \frac{k^2 \pi^2 \sigma^2}{\ell^2}\right).$$

Hint: $P(x, t; x_0, t_0)$ satisfies the forward Fokker-Planck equation with auxiliary conditions: $P(0, t) = P(\ell, t) = 0$ and $P(x, t_0^+; x_0, t_0) = \delta(x - x_0)$. Pelsser (2000) derives the above solution by performing the Laplace inversion using Bromwich contour integration.

4.10 Let the exit time density $q^+(t; x_0, t_0)$ have dependence on the initial state $X(t_0) = x_0$. We write $\tau = t - t_0$ so that $q^+(t; x_0, t_0) = q^+(x_0, \tau)$. Show that the partial differential equation formulation is given by

$$\frac{\partial q^+}{\partial \tau} = \mu \frac{\partial q^+}{\partial x_0} + \frac{\sigma^2}{2}\frac{\partial^2 q^+}{\partial x_0^2}, \quad \ell < x_0 < u, \quad \tau > 0,$$

[with auxiliary conditions:

$$q^+(u, \tau) = 0, \quad q^+(\ell, \tau) = 0 \quad \text{and} \quad q^+(x_0, 0) = \delta(x_0).$$

Solve for $q^+(x_0, \tau)$ using the partial differential equation approach and compare the solution with that given in Eq. (4.1.63b). Also, show that

$$\int_{t_0}^{t} q^+(s; x_0, t_0)\, ds + \int_{t_0}^{t} q^-(s; x_0, t_0)\, ds + \int_{\ell}^{u} P(x, t; x_0, t_0)\, dx = 1,$$

where $P(x, t; x_0, t_0)$ is the transition density function defined in Problem 4.9.

4.11 By using the method of path counting, Sidenius (1998) shows that $g^+(x, T)$ defined in Eq. (4.1.44b) has the following analytic solution

$$g^{+}(x,T) = \exp\left(\frac{\mu x}{\sigma^2} - \frac{\mu^2}{\sigma^2}T\right)$$
$$\left[\sum_{n=1}^{\infty} \phi(x;\alpha_n^{+},\sigma\sqrt{T}) - \phi(x,\beta_n^{+},\sigma\sqrt{T})\right]$$

where

$$\phi(x;\lambda,\nu) = \frac{1}{\sqrt{2\pi\nu^2}}\exp\left(-\frac{(x-\lambda)^2}{2\nu^2}\right)$$
$$\alpha_n^{+} = 2n(u-\ell)+2\ell \quad \text{and} \quad \beta_n^{+} = 2n(u-\ell).$$

Find the closed form price formula for the upper-barrier knock-in call option [see Eq. (4.1.46) and Luo's paper (2001)].

4.12 A sequential barrier option is a barrier option with two-sided barriers where nullification occurs only when the barriers are breached at a pre-specified order (say, up then down). Show that the price formula of the sequential up-then-down out call option can be inferred from those of the double knock-out call option (see Problem 4.8) except that the infinite summation over n is replaced by summation over only two terms (Li, 1999).

Hint: Show that the price formula obtained by summing the two specific terms satisfies the condition that the sequential up-then-down out call option becomes the corresponding down-and-out call when the asset price hits the upper barrier.

4.13 Consider a discretely monitored down-and-out call option with strike price X and barrier level B_i at discrete time $t_i, i = 1, 2, \cdots, n$. Show that the price of this barrier call is given by (Heynen and Kat, 1996)

$$\begin{aligned} &c_{d_0}(S_0,T;X,B_1,B_2,\cdots,B_n) \\ &= S_0 N_{n+1}(d_1^1,d_1^2,\cdots,d_1^{n+1};\Gamma) - e^{-rT}N_{n+1}(d_2^1,d_2^2,\cdots,d_2^{n+1};\Gamma) \end{aligned}$$

where

$$d_1^i = \frac{\ln\frac{S_0}{B_{t_i}} + \left(r+\frac{\sigma^2}{2}\right)t_i}{\sigma\sqrt{t_i}}, \quad i = 1,2,\cdots,n, \quad d_2^i = d_1^i - \sigma\sqrt{t_i},$$
$$d_1^{n+1} = \frac{\ln\frac{S_0}{X} + \left(r+\frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \quad d_2^{n+1} = d_1^{n+1} - \sigma\sqrt{T}.$$

Also, Γ is the $(n+1)\times(n+1)$ correlation matrix whose entries are given by

$$\rho_{jk} = \frac{\min(t_j,t_k)}{\sqrt{t_j}\sqrt{t_k}}, \quad 1 \le j,k \le n; \quad \rho_{j,n+1} = \sqrt{\frac{t_j}{T}}, j = 1,2,\cdots,n.$$

4.14 Let $f_{min}(y)$ and $f_{max}(y)$ denote the density function of y_T and Y_T, respectively. Show that

$$\begin{aligned} f_{min}(y) =& \frac{1}{\sigma\sqrt{\tau}} n\left(\frac{-y+\mu\tau}{\sigma\sqrt{\tau}}\right) + \frac{2\mu}{\sigma^2} e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{y+\mu\tau}{\sigma\sqrt{\tau}}\right) \\ &+ e^{\frac{2\mu y}{\sigma^2}} \frac{1}{\sigma\sqrt{\tau}} n\left(\frac{y+\mu\tau}{\sigma\sqrt{\tau}}\right), \\ f_{max}(y) =& \frac{1}{\sigma\sqrt{\tau}} n\left(\frac{y-\mu\tau}{\sigma\sqrt{\tau}}\right) - \frac{2\mu}{\sigma^2} e^{\frac{2\mu y}{\sigma^2}} N\left(\frac{-y-\mu\tau}{\sigma\sqrt{\tau}}\right) \\ &+ e^{\frac{2\mu y}{\sigma^2}} \frac{1}{\sigma\sqrt{\tau}} n\left(\frac{-y-\mu\tau}{\sigma\sqrt{\tau}}\right). \end{aligned}$$

Compare with $f_{down}(x, m, T)$ and $f_{up}(y, M, T)$ in Eqs. (4.1.27b,c).

4.15 As an alternative approach to derive the value of a European floating strike lookback call, we consider

$$\begin{aligned} c_{f\ell}(S, m, \tau) &= e^{-r\tau} E[S_T - \min(m, m_t^T)] \\ &= S - e^{-r\tau} E[\min(m, m_t^T)], \end{aligned}$$

where $S_t = S, m_{T_0}^t = m$ and $\tau = T - t$. We may decompose the above expectation calculation into two terms:

$$E[\min(m, m_t^T)] = mP[m \le m_t^T] + E[m_t^T \mathbf{1}_{\{m > m_t^T\}}].$$

Show that the first term is given by

$$\begin{aligned} & mP\left[\ln \frac{m_t^T}{S} \ge \ln \frac{m}{S}\right] \\ &= m\left[N\left(\frac{-\ln \frac{m}{S} + \mu\tau}{\sigma\sqrt{\tau}}\right) - \left(\frac{S}{m}\right)^{1-\frac{2r}{\sigma^2}} N\left(\frac{\ln \frac{m}{S} + \mu\tau}{\sigma\sqrt{\tau}}\right)\right]. \end{aligned}$$

Now, the second term can be expressed as

$$E[m_t^T \mathbf{1}_{\{m_t^T > m\}}] = \int_{-\infty}^{\ln \frac{m}{S}} S e^y f_{min}(y)\, dy.$$

By performing the tedious integration procedure, show that the same price function for $c_{f\ell}(S, m, \tau)$ [see Eq. (4.2.14a,b)] is obtained.

4.16 From the following form of the distribution function of m_t^T [see Eq. (4.2.4a)]

$$P[m \le m_t^T] = N\left(\frac{-\ln \frac{m}{S} + \mu\tau}{\sigma\sqrt{\tau}}\right) - \left(\frac{S}{m}\right)^{1-\frac{2r}{\sigma^2}} N\left(\frac{\ln \frac{m}{S} + \mu\tau}{\sigma\sqrt{\tau}}\right),$$

Show that $P[m \le m_t^T]$ becomes zero when $S = m$.

4.17 Suppose we use a straddle (combination of call and put with the same strike m) in the rollover strategy for hedging the floating strike lookback call and we write

$$c_{f\ell}(S, m, \tau) = c_E(S, \tau; m) + p_E(S, \tau; m) + \text{ strike bonus premium.}$$

Find an integral representation of the strike bonus premium in terms of distribution functions of S_T and m_t^T.

4.18 Prove the following put-call parity relation between the prices of the fixed strike lookback call and floating strike lookback put:

$$c_{fix}(S, M, \tau; X) = p_{f\ell}(S, \max(M, X), \tau) + S - Xe^{-r\tau}.$$

Deduce that

$$\frac{\partial c_{fix}}{\partial M} = 0 \quad \text{for} \quad M < X.$$

Give a financial interpretation why c_{fix} is insensitive to M when $M < X$ [see also Eq. 4.2.8)].

4.19 Suppose the terminal payoffs of the partial lookback call and put options are $\max(S_T - \lambda m_{T_0}^T, 0)$, $\lambda > 1$ and $\max\left(\lambda M_{T_0}^T - S_T, 0\right)$, $0 < \lambda < 1$, respectively. Show that the price formulas of these lookback options are, respectively (Conze and Viswanathan, 1991)

$$\begin{aligned} c\,(S, m, \tau) = {} & SN\left(d_m - \frac{\ln\lambda}{\sigma\sqrt{\tau}}\right) - \lambda m e^{-r\tau} N\left(d_m - \frac{\ln\lambda}{\sigma\sqrt{\tau}} - \sigma\sqrt{\tau}\right) \\ & + e^{-r\tau}\frac{\sigma^2}{2r}\lambda S\left[\left(\frac{S}{m}\right)^{-2r/\sigma^2} N\left(-d_m + \frac{2r}{\sigma}\sigma - \frac{\ln\lambda}{\sigma\sqrt{\tau}}\right)\right. \\ & \left. - e^{r\tau}\lambda^{2r/\sigma^2} N\left(-d_m - \frac{\ln\lambda}{\sigma\sqrt{\tau}}\right)\right] \\ p\,(S, M, \tau) = {} & -SN\left(-d_M + \frac{\ln\lambda}{\sigma\sqrt{\tau}}\right) \\ & + \lambda M e^{-r\tau} N\left(-d_M + \frac{\ln\lambda}{\sigma\sqrt{\tau}} + \sigma\sqrt{\tau}\right) \\ & - e^{-r\tau}\frac{\sigma^2}{2r}\lambda S\left[\left(\frac{S}{M}\right)^{-2r/\sigma^2} N\left(d_M - \frac{2r}{\sigma}\sqrt{\tau} + \frac{\ln\lambda}{\sigma\sqrt{\tau}}\right)\right. \\ & \left. - e^{r\tau}\lambda^{2r/\sigma^2} N\left(d_M + \frac{\ln\lambda}{\sigma\sqrt{\tau}}\right)\right], \end{aligned}$$

where d_m and d_M are given by Eqs. (4.2.14b) and (4.2.15b), respectively.

4.20 The terminal payoff of the lookback spread option is given by

$$c_{sp}(S, m, M, 0) = \max(M_{T_0}^T - m_{T_0}^T - X, 0).$$

Show that the current value of the European lookback spread option can be expressed as (Wong and Kwok, 2003b)

(i) currently at- or in-the-money, that is, $M - m - X \geq 0$

$$c_{sp}(S, m, M, \tau) = c_{f\ell}(S, m, \tau) + p_{f\ell}(S, M, \tau) - Xe^{-r\tau};$$

(ii) currently out-of-the-money, that is, $M - m - X < 0$

$$\begin{aligned} c_{sp}(S, m, M, \tau) &= c_{f\ell}(S, m, \tau) + p_{f\ell}(S, M, \tau) - Xe^{-r\tau} \\ &\quad + \ e^{-r\tau} \int_M^{m+X} P[M_t^T < \xi \leq m_t^T + X]\, d\xi. \end{aligned}$$

4.21 An investor holding a European in-the-money call option may suffer loss in profits if the asset price drops substantially just before expiration. The "limited period" fixed strike lookback feature may help remedy the above market exit problem. To achieve optimal timing for market exit, it is therefore sensible to choose the lookback period starting some time after the option's starting date and ending at expiration. Let $[T_1, T]$ denote the lookback period and consider the pricing of this fixed strike lookback call at time t before the lookback period, whose terminal payoff is $\max(M_{T_1}^T - X, 0)$. Note that when $S_{T_1} > X$, the call is guaranteed to be in-the-money at expiration since $M_{T_1}^T \geq S_{T_1}$. Show that the value of the "limited-period" lookback call is given by

$$\begin{aligned} c(S, \tau) &= e^{-r\tau} E\left[\max\left(M_{T_1}^T - X, 0\right)\right] \\ &= e^{-r\tau} E\left[\left\{S_{T_1}\left(\frac{M_{T_1}^T}{S_{T_1}}\right) - X\right\} \mathbf{1}_{\{m_{T_1}^T > X, S_{T_1} < X\}}\right] \\ &\quad + e^{-r\tau} E\left[\left\{S_{T_1}\left(\frac{M_{T_1}^T}{S_{T_1}}\right) - X\right\} \mathbf{1}_{\{S_{T_1} > X\}}\right], \quad t < T_1, \end{aligned}$$

where X is the strike price. Assuming the usual lognormal diffusion process for the asset price under the risk neutral measure, show that

$$
\begin{aligned}
c(S,\tau) = {} & SN(d_1) - e^{-r\tau}XN(d_2) - SN_2\left(-e_1, d_1; -\sqrt{\frac{T-T_1}{T-t}}\right) \\
& - e^{-r\tau}XN_2\left(f_2, -d_2; -\sqrt{\frac{T_1-t}{T-t}}\right) + e^{-r\tau}\frac{\sigma^2}{2r}S\left[-\left(\frac{S}{X}\right)^{-\frac{2r}{\sigma^2}}\right. \\
& N_2\left(d_1 - \frac{2r}{\sigma}\sqrt{\tau}, -f_1 + \frac{2r}{\sigma}\sqrt{T_1-t}; -\sqrt{\frac{T_1-t}{T-t}}\right) \\
& \left. + e^{r\tau}N_2\left(e_1, d_1; \sqrt{\frac{T-T_1}{T-t}}\right)\right] \\
& + e^{-r(T-T_1)}\left(1 - \frac{\sigma^2}{2r}\right)SN(f_1)N(-e_2), \quad t < T_1,
\end{aligned}
$$

where $d_1, e_1, f_1, \cdots$, etc. are the same as those defined in Eq. (4.2.23) except that m is replaced by X accordingly (Heynen and Kat, 1994b). Deduce the price formula for the corresponding "limited period" fixed strike lookback put option whose terminal payoff function is $\max(X - m_{T_1}^T, 0)$.

4.22 Use Eq. (4.2.31) to derive the following partial differential equation for the floating strike lookback put option

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial \xi^2} - \left(r + \frac{\sigma^2}{2}\right)\frac{\partial V}{\partial \xi}, \quad 0 < \xi < \infty, \tau > 0,$$

where $V(\xi,\tau) = p_{f\ell}(S,M,t)/S$ and $\tau = T-t$, $\xi = \ln\dfrac{M}{S}$. The auxiliary conditions are

$$V(\xi, 0) = e^{\xi} - 1 \quad \text{and} \quad \frac{\partial V}{\partial \xi}(0,\tau) = 0.$$

Solve the above Neumann boundary value problem and check the result with the put price formula given in Eqs. (4.2.15a,b).

Hint: Define $W = \dfrac{\partial V}{\partial \xi}$ so that W satisfies the same governing differential equation but the boundary condition becomes $W(0,\tau) = 0$. Solve for $W(\xi,\tau)$, then integrate W with respect to ξ to obtain V. Be aware that an arbitrary function $\phi(t)$ is generated upon integration with respect to ξ. Obtain an ordinary differential equation for $\phi(t)$ by substituting the solution for V into the original differential equation.

4.23 Let $p(S,t;\delta t)$ denote the value of a floating strike lookback put option with discrete monitoring of the realized maximum value of the asset price, where δt is the regular interval between monitoring instants. Suppose we assume the following two-term Taylor expansion of $p(S,t;\delta t)$ in powers of $\sqrt{\delta t}$ (Levy and Mantion, 1997)

$$p(S,t;\delta t) \approx p(S,t;0) + \alpha\sqrt{\delta t} + \beta\delta t.$$

With $\delta t = 0$, $p(S,t;0)$ represents the lookback put value corresponding to continuous monitoring. Let τ denote the time to expiry. By setting $\delta t = \tau$ and $\delta t = \tau/2$, we deduce the following pair of linear equations for α and β:

$$\alpha\sqrt{\frac{\tau}{2}} + \beta\left(\frac{\tau}{2}\right) = p\left(S,t;\frac{\tau}{2}\right) - p(S,t;0)$$
$$\alpha\sqrt{\tau} + \beta\tau = p(S,t;\tau) - p(S,t;0).$$

Hence, $p(S,t;\tau)$ is simply the vanilla put value with strike price equal to the current realized maximum asset price M. With only one monitoring instant at the mid-point of the remaining option's life, show that $p\left(S,t;\frac{\tau}{2}\right)$ is given by

$$\begin{aligned} & p\left(S,t;\frac{\tau}{2}\right) \\ &= -\,e^{-q\tau}S + e^{-r\tau}\left[MN_2\left(-d_M\left(\frac{\tau}{2}\right) + \sigma\sqrt{\frac{\tau}{2}}, -d_M(\tau) + \sigma\sqrt{\tau}; \frac{1}{\sqrt{2}}\right)\right. \\ &\qquad + \; e^{(r-q)\tau}SN_2\left(d_M(\tau), d\left(\frac{\tau}{2}\right); \frac{1}{\sqrt{2}}\right) \\ &\qquad \left. + \; e^{(r-q)\frac{\tau}{2}}SN_2\left(d_M\left(\frac{\tau}{2}\right), -d\left(\frac{\tau}{2}\right) + \sigma\sqrt{\frac{\tau}{2}}; 0\right)\right], \end{aligned}$$

where

$$d_M(\tau) = \frac{\ln\frac{S}{M} + \left(r - q + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \quad \text{and } d(\tau) = \frac{\left(r - q + \frac{\sigma^2}{2}\right)\sqrt{\tau}}{\sigma}.$$

Once α and β are determined, we then obtain an approximate price formula of the discretely monitored floating strike lookback put.

4.24 Explain why

$$\text{call}_{\text{European}} \;\geq\; \text{Asian call}_{\text{arithmetic}} \;\geq\; \text{Asian call}_{\text{geometric}}$$

Hint: An average price is less volatile than the series of prices from which it is computed.

4.25 Consider the exchange option which entities the holder the right but not the obligation to exchange risky asset S_2 for another risky asset S_1. Let the price dynamics of S_1 and S_2 under the risk neutral measure be governed by

$$\frac{dS_i}{S_i} = (r - q_i)\,dt + \sigma_i\,dZ_i, \quad i = 1, 2,$$

where $dZ_1\,dZ_2 = \rho\,dt$. Let $V(S_1, S_2, \tau)$ denote the price function of the exchange option, whose terminal payoff takes the form

$$V(S_1, S_2, 0) = \max(S_1 - S_2, 0).$$

Show that the governing equation for $V(S_1, S_2, \tau)$ is given by

$$\begin{aligned}\frac{\partial V}{\partial \tau} &= \frac{\sigma_1^2}{2} S_1 \frac{\partial^2 V}{\partial S_1^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{\sigma_2^2}{2} S_2^2 \frac{\partial^2 V}{\partial S_2^2} \\ &\quad + (r - q_2) S_1 \frac{\partial V}{\partial S_2} + (r - q_2) S_2 \frac{\partial V}{\partial S_2} - rV.\end{aligned}$$

By taking S_2 as the numeraire and defining the similarity variables

$$x = \frac{S_1}{S_2} \quad \text{and} \quad W(x, \tau) = \frac{V(S_1, S_2, \tau)}{S_2},$$

show that the governing equation for $W(x, \tau)$ becomes

$$\frac{\partial W}{\partial \tau} = \frac{\sigma^2}{2} x^2 \frac{\partial^2 W}{\partial x^2} + (q_1 - q_2) x \frac{\partial W}{\partial x}.$$

Verify that the solution to $W(x, \tau)$ is given by

$$W(x, \tau) = e^{-q_1\tau} x N(d_1) - e^{-q_2\tau} N(d_2)$$

or

$$V(S_1, S_2, \tau) = S_1 e^{-q_1\tau} N(d_1) - S_2 e^{-q_2\tau} N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\ln \frac{S_1}{S_2} + \left(q_2 - q_1 + \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}, \quad d_2 = d_1 - \sigma\sqrt{\tau}, \\ \sigma^2 &= \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2. \end{aligned}$$

Apply the above exchange option price formula to price the floating strike Asian call option based on the knowledge of the price formula of the fixed strike Asian call option.

Hint: The covariance between S_T and G_T is equal to $\dfrac{\sigma^2(T-t)^2}{2}$.

4.26 We define the geometric average of the price path of asset $i, i = 1, 2$, during the time interval $[t, t + T]$ by

$$G_i(t + T) = \exp\left(\frac{1}{T}\int_0^T \ln S_i(t + u)\,du\right).$$

Consider an Asian option involving two assets whose terminal payoff is given by $\max(G_1(t+T) - G_2(t+T), 0)$. Show that the price formula of this European Asian option at time t is given by (Boyle, 1993)

$$V(S_1, S_2, t; T) = \widetilde{S}_1 N(d_1) - \widetilde{S}_2 N(d_2)$$

where

$$\widetilde{S}_i = S_i \exp\left(-\left(\frac{r^2}{2} + \frac{\sigma^2}{12}\right) T\right), i = 1, 2, \quad \sigma^2 = \frac{\sigma_1^2}{3} + \frac{\sigma_2^2}{3} - \frac{2}{3}\rho\sigma_1\sigma_2,$$

$$d_1 = \frac{\ln \frac{\widetilde{S}_1}{\widetilde{S}_2} + \frac{\sigma^2}{2}T}{\sigma\sqrt{T}}, \quad d_2 = d_1 - \sigma\sqrt{T}.$$

Hint: Apply the price formula of the exchange option.

4.27 Suppose we define the flexible geometric average $G_F(n)$ of asset prices at n evenly spaced time instants by

$$G_F(n) = \prod_{i=1}^{n} S_i^{\omega_i},$$

where $\omega_i = \dfrac{i^\alpha}{\sum_{i=1}^{n} i^\alpha}$, $i = 1, 2, \cdots, n$ and S_i is the asset price at time t_i. Here, ω_i is the weighting factor associated with S_i with the property that the larger the value of α, the heavier are the weights allocated to the more recent asset price. Under the risk neutral measure, the asset price is assumed to follow the lognormal process

$$\frac{dS}{S} = r\,dt + \sigma\,dZ.$$

We consider the fixed strike Asian option whose terminal payoff function is

$$V(S, G_F, T) = \max(\phi(G_F(n) - X), 0),$$

where X is the strike price, and ϕ is the binary variable which is set to 1 for a call or -1 for a put. Show that the Asian option value is given by (Zhang, 1994)

$$V(S, G_F, t) = \phi\left[SA_j^f N\left(\phi\left(d_{n-j}^f + \sigma\sqrt{T_{n-j}^f}\right)\right) - Xe^{-r\tau} N\left(\phi d_{n-j}^f\right)\right],$$

where

$$A_j^f = \exp\left(-r\left(\tau - T_{\mu,n-j}^f\right) - \frac{\sigma^2}{2}\left(T_{\mu,n-j}^f - T_{n-j}^f\right)\right) B_j^f,$$

$$B_0^f = 1, \quad B_j^f = \prod_{i=1}^{j}\left(\frac{S_{n-i}}{S}\right)^{\omega_i}, \quad 1 \le j \le n,$$

$$d_{n-j}^f = \frac{\ln\frac{S}{X} + \left(r - \frac{\sigma^2}{2}\right) T_{\mu,n-j}^f + \ln B_j^f}{\sigma\sqrt{T_{n-j}^f}},$$

$$T_{\mu,n-j}^f = \sum_{i=j+1}^{n} \omega_i[\tau - (n-i)\Delta t],$$

$$T_{n-j}^f = \sum_{i=j+1}^{n} \omega_i^2[\tau - (n-i)\Delta t] + 2\sum_{i=2}^{n-j}\sum_{k=1}^{i-1} \omega_i\omega_k[\tau - (n-k)\Delta t],$$

n is the number of asset prices taken for averaging, Δt is the time interval between successive observational instants, j is the number of observations already passed, B_j^f can be considered as the weighted average of the returns of those observations that have already passed.

4.28 Deduce the following put-call parity relation between the prices of European fixed strike Asian call and put options under continuously monitored geometric averaging

$$\begin{aligned} &c(S, G, t) - p(S, G, t) \\ &= e^{-r(T-t)}\left\{G^{t/T}S^{(T-t)/T}\exp\left((T-t)\left[\frac{\sigma^2}{6}\left(\frac{T-t}{T}\right)^2 \right.\right.\right. \\ &\qquad \left.\left.\left. + \frac{r - q - \frac{\sigma^2}{2}}{2}\frac{T-t}{T}\right] - X\right)\right\}. \end{aligned}$$

4.29 Suppose continuous arithmetic averaging of the asset price is taken from $t = 0$ to T, T is the expiration time. The terminal payoff functions for the floating strike call and put options are, respectively,

$$\max\left(S_T - \frac{1}{T}\int_0^T S_u\,du, 0\right) \text{ and } \max\left(\frac{1}{T}\int_0^T S_u\,du - S_T, 0\right).$$

Show that the put-call parity relation for the above pair of European floating strike options is given by

$$c - p = Se^{-q(T-t)} + \frac{S}{(r-q)T}[e^{-r(T-t)} - e^{-q(T-t)}] - e^{-r(T-t)}A_t,$$

where

$$A_t = \frac{1}{T}\int_0^t S_u\, du.$$

Suppose continuous geometric averaging of the asset price is taken, show that the corresponding put-call parity relation is given by

$$c - p = Se^{-q(T-t)} - G^{t/T} S^{(T-t)/T}$$
$$\exp\left(\frac{\sigma^2 (T-t)^3}{6T^2} + \frac{\left(r - q - \frac{\sigma^2}{2}\right)(T-t)^2}{2T} - r(T-t)\right),$$

where

$$G_t = \exp\left(\frac{1}{t}\int_0^t \ln S_u\, du\right).$$

4.30 Show that the put-call parity relations between the prices of floating strike and fixed strike Asian options at the start of the averaging period are given by

$$p_{f\ell}(S_0, \lambda, r, q, T) - c_{f\ell}(S_0, \lambda, r, q, T) = \frac{S(e^{-qT} - e^{-rT})}{(r-q)T} - \lambda S_0$$
$$c_{fix}(X, S_0, r, q, T) - p_{fix}(X, S_0, r, q, T) = \frac{S(e^{-qT} - e^{-rT})}{(r-q)T} - e^{-rT}X.$$

By combining the above put-call parity relations with the fixed-floating symmetry relation between $c_{f\ell}$ and p_{fix}, deduce the following symmetry relation between c_{fix} and $p_{f\ell}$

$$c_{fix}(X, S_0, r, q, T) = p_{f\ell}\left(S_0, \frac{X}{S_0}, q, r, T\right).$$

4.31 Consider the European continuously monitored arithmetic average Asian option with terminal payoff $\max(A_T - X_1 S_T - X_2, 0)$, where

$$A_T = \frac{1}{T}\int_0^T S_u\, du.$$

At the current time $t > 0$, the average value over the time period $[0, t]$ has been realized. Let $V(S, \tau; X_1, X_2)$ denote the price function of the Asian option at the start of the averaging period. Show that the value of the in-progress Asian option is given by

$$\frac{T-t}{T} V\left(S, T-t; \frac{X_1 T}{T-t}, \frac{X_2 T}{T-t} - \frac{A_t - t}{T-t}\right).$$

4.32 Consider $E\left[\widetilde{A}(t_n; t)^2\right]$ defined in Eq. (4.3.95). Show that when $t < t_0$, we have (Levy, 1992)

$$E[\widetilde{A}(t_n;t)^2] = \frac{S_t^2}{(n+1)^2} e^{(2r+\sigma)(t_0-t)}(B_1 - B_2 + B_3 - B_4)$$

where

$$B_1 = \frac{1 - e^{(2r+\sigma^2)(n+1)\Delta t}}{(1-e^{r\Delta t})[1-e^{(2r+\sigma^2)\Delta t}]}, \quad B_2 = \frac{e^{r(n+1)\Delta t} - e^{(2r+\sigma^2)(n+1)\Delta t}}{(1-e^{r\Delta t})[1-e^{(r+\sigma^2)\Delta t}]},$$

$$B_3 = \frac{e^{r\Delta t} - e^{r(n+1)\Delta t}}{(1-e^{r\Delta t})[1-e^{(r+\sigma^2)\Delta t}]}, \quad B_4 = \frac{e^{(2r+\sigma^2)\Delta t} - e^{(2r+\sigma^2)(n+1)\Delta t}}{[1-e^{(r+\sigma^2)\Delta t}][1-e^{(2r+\sigma^2)\Delta t}]}.$$