

CHAPTER 3

Pricing Models for One-Asset European Options

The revolution on trading and pricing derivative securities in financial markets and academic communities began in early 1970's. In 1973, the Chicago Board of Options Exchange started the trading of options in exchanges, though options had been regularly traded by financial institutions in the *over-the-counter markets* in earlier years. In the same year, Black and Scholes (1973) and Merton (1973) published their seminal papers on the theory of option pricing. Since then the growth of the field of financial engineering has been phenomenal. The Black-Scholes-Merton risk neutrality formulation of the option pricing theory is attractive since valuation formula of a derivative deduced from their model is a function of directly *observable* parameters (except one, which is the volatility parameter). The derivative can be priced as if the market price of risk of the underlying asset is zero. When judged by its ability to explain the empirical data, the option pricing theory is widely acclaimed to be the most successful theory not only in finance, but in all areas of economics. In recognition of their pioneering and fundamental contributions to the pricing theory of derivatives, Scholes and Merton received the 1997 Nobel Award in Economics.

In the next section, we first show how Black and Scholes applied the riskless hedging principle to derive the differential equation that governs the price of a derivative security. We also discuss Merton's approach of dynamically replicating an option by a portfolio of the riskless asset in the form of money market account and the risky underlying asset. The cost of constructing the replicating portfolio gives the fair price of the option. Furthermore, we present an alternative perspective of the risk neutral valuation approach by showing that tradeable securities should have the same market price of risk if they are hedgeable with each other.

In Sec. 3.2, we discuss the renowned martingale pricing theory of options, which is the continuous analog of the discrete risk neutral valuation models for contingent claims. The price of a derivative is given by the discounted expectation of the terminal payoff under the risk neutral measure, where all discounted security prices are martingales under this measure. The Black-Scholes equation can then be deduced from the Feynman-Kac representation formula. We also illustrate how to apply Girsanov's Theorem to effect the change of probability measure in option pricing calculations.

In Sec. 3.3, we solve the Black-Scholes pricing equation for several types of European style one-asset derivative securities. The contractual specifications are translated into appropriate auxiliary conditions of the corresponding option models. The most popular option price formulas are those for the European vanilla call and put options where the underlying asset price follows the Geometric Brownian motion with constant drift rate and variance rate. The comparative statics of these price formulas with respect to different parameters in the option model are derived and their properties are discussed.

The generalization of the above valuation formulas for pricing other European style derivative securities, like futures options and chooser options are considered in Sec. 3.4. We also consider extensions of the Black-Scholes model, which include the effects of dividends, time dependent interest rate and volatility, etc.

Practitioners using the Black-Scholes model are aware that it is less than perfect. The major criticisms are the assumptions of constant volatility, continuity of asset price process without jump and zero transaction costs on hedging. When we try to equate the Black-Scholes option prices with actual quoted market prices of European calls and puts, we have to use different volatility values (called implied volatilities) for options with different maturities and strike prices. In Sec. 3.5, we consider the phenomena of volatility smiles exhibited in implied volatilities. We derive the Dupire equation, which may be considered as the forward version of the option pricing equation. From the Dupire equation, we can compute the local volatility function that give theoretical Black-Scholes option prices which agree with market option prices. We also consider option pricing models that allow jumps in the underlying price process and include the effects of transaction costs in the sale of the underlying asset. Jumps in asset price occur when there are sudden arrivals of information about the firm or economy as a whole. Transaction costs represent market frictions in the trading of assets. We examine how the effects of transaction costs can be incorporated into the option pricing models.

Though most practitioners are aware of the limitations of the Black-Scholes model, why is it still so popular on the trading floor? One simple reason: the model involves only one parameter that is not directly observable in the market, namely, volatility. That gives an option trader the straight and simple insight: sell when volatility is high and buy when it is low. For a simple model like the Black-Scholes model, traders can understand the underlying assumptions and limitations, and make appropriate adjustments if necessary. Also, pricing methodologies associated with the Black-Scholes model are relatively simple. For many European style derivatives, closed form pricing formulas are readily available. Even for more complicated options where pricing formulas do not exist, there exist an arsenal of effective and efficient numerical schemes to calculate the option values and their comparative statics.

3.1 Black-Scholes-Merton formulation

Black and Scholes (1973) revolutionize the pricing theory of options by showing how to hedge continuously the exposure on the short position of an option. Consider a writer of a European call option on a risky asset, he is exposed to the risk of unlimited liability if the asset price rises above the strike price. To protect his short position in the call option, he should consider purchasing certain amount of the underlying asset so that the loss in the short position in the call option is offset by the long position in the asset. In this way, he is adopting the *hedging procedure*. A hedged position combines an option with its underlying asset so as to achieve the goal that either the asset compensates the option against loss or otherwise. This risk monitoring strategy has been commonly used by practitioners in financial markets. By adjusting the proportion of the asset and option continuously in a portfolio, Black and Scholes demonstrate that investors can create a *riskless hedging portfolio* where the risk exposure associated with the asset price movement is eliminated. In an efficient market with no riskless arbitrage opportunity, a riskless portfolio must earn an expected rate of return equal to the riskless interest rate.

Readers may be interested to read Black's article (1989) which tells the story how Black and Scholes came up with the idea of a riskless hedging portfolio.

3.1.1 Riskless hedging principle

We illustrate how to use the riskless hedging principle to derive the governing partial differential equation for the price of a European call option. In their seminal paper (1973), Black and Scholes make the following assumptions on the financial market.

- (i) Trading takes place continuously in time.
- (ii) The riskless interest rate r is known and constant over time.
- (iii) The asset pays no dividend.
- (iv) There are no transaction costs in buying or selling the asset or the option, and no taxes.
- (v) The assets are perfectly divisible.
- (vi) There are no penalties to short selling and the full use of proceeds is permitted.
- (vii) There are no riskless arbitrage opportunities.

The stochastic process of the asset price S is assumed to follow the Geometric Brownian motion

$$\frac{dS}{S} = \rho dt + \sigma dZ, \quad (3.1.1)$$

where ρ is the expected rate of return, σ is the volatility, and $Z(t)$ is the standard Brownian process. Both ρ and σ are assumed to be constant. Consider a portfolio which involves short selling of one unit of a European call option and long holding of Δ units of the underlying asset. The value of the portfolio Π is given by

$$\Pi = -c + \Delta S, \quad (3.1.2)$$

where $c = c(S, t)$ denotes the call price. Note that ΔS here refers to Δ times S , not infinitesimal change in S . Since both c and Π are random variables, we apply the Ito lemma to compute their stochastic differentials as follows:

$$dc = \frac{\partial c}{\partial t} dt + \frac{\partial c}{\partial S} dS + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} dt \quad (3.1.3a)$$

and

$$\begin{aligned} d\Pi &= -dc + \Delta dS \\ &= \left(-\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} \right) dt + \left(\Delta - \frac{\partial c}{\partial S} \right) dS \\ &= \left[-\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + \left(\Delta - \frac{\partial c}{\partial S} \right) \rho S \right] dt + \left(\Delta - \frac{\partial c}{\partial S} \right) \sigma S dZ. \end{aligned} \quad (3.1.3b)$$

The cautious readers may be puzzled by the non-occurrence of the differential term $S d\Delta$ in $d\Pi$. Here, we follow the derivation in Black-Scholes' paper (1973) where they assume the number of units of the underlying asset held in the portfolio to be instantaneously constant. The justification of such simplification can only be explained when we use the concept of replicating the position of the call by the risky asset and money market account [see Eq. (3.1.12) for more details].

The stochastic component of the portfolio appears in the last term: $\left(\Delta - \frac{\partial c}{\partial S} \right) \sigma S dZ$. If we choose $\Delta = \frac{\partial c}{\partial S}$, then the portfolio becomes an instantaneous riskless hedge (note that $\frac{\partial c}{\partial S}$ changes continuously in time). By virtue of no arbitrage, the hedged portfolio should earn the riskless interest rate. Otherwise, suppose the hedged portfolio earns more than the riskless interest rate, then an arbitrageur can lock in riskless profit by borrowing as much money as possible to purchase the hedged portfolio. By setting $d\Pi = r\Pi dt$, we then have

$$d\Pi = \left(-\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} \right) dt = r\Pi dt = r \left(-c + S \frac{\partial c}{\partial S} \right) dt. \quad (3.1.4)$$

Upon rearranging the terms, we obtain

$$\frac{\partial c}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + rS \frac{\partial c}{\partial S} - rc = 0. \quad (3.1.5)$$

The above parabolic partial differential equation is called the *Black-Scholes equation*. Note that the parameter ρ , which is the expected rate of return of the asset, does not appear in the equation. To complete the formulation of the option pricing model, we need to prescribe the auxiliary (terminal payoff) condition for the European call option. At expiry, the payoff of the European call is given by

$$c(S, T) = \max(S - X, 0), \quad (3.1.6)$$

where T is the time of expiration and X is the strike price. Since both the equation and the auxiliary condition do not contain ρ , one can conclude that the call price does not depend on the actual expected rate of return of the stock price. The option pricing model involves five parameters: S, T, X, r and σ ; all except the volatility σ are directly observable parameters. The independence of the pricing model on ρ is related to the concept of *risk neutrality*. In a risk neutral world, investors do not demand extra returns on average for bearing risks. In this way, the rates of return on the underlying asset and the derivative in the risk neutral world are set equal the riskless interest rate. This risk neutral valuation approach is a major breakthrough in the option pricing theory pioneered by Black and Scholes. We will address the concept of risk neutrality from different perspectives in later sections.

The governing equation for a European put option can be derived similarly and the same Black-Scholes equation is obtained. Let $V(S, t)$ denote the price of a derivative security contingent on S , it can be shown that V is governed by

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (3.1.7)$$

The price of a particular derivative security is obtained by solving Eq. (3.1.7) subject to appropriate auxiliary conditions that model the corresponding contractual specifications in the derivative security.

Remark

Readers are reminded that the ability to construct a perfectly hedged portfolio relies on the assumptions of continuous trading and continuous sample path of the asset price. These two and other assumptions in the Black-Scholes pricing model have been critically examined by later works in derivative pricing theory. For example, the Geometric Brownian motion of the asset price movement may not truly reflect the actual behaviors of the price process. Also, the interest rate is widely recognized to be fluctuating over time in an irregular manner, rather than being constant and deterministic. Moreover, the Black-Scholes pricing approach assumes continuous hedging at all times. In the real world of trading with transaction costs, this would lead to infinite transaction costs in the hedging procedure. Even with all these limitations, the Black-Scholes model is still considered to be the most fundamental in derivative pricing theory. Various forms of modification to this basic model have been proposed to accommodate the above shortcomings. Some of these enhanced versions of pricing models will be addressed in later sections.

3.1.2 Dynamic replication strategy

As an alternative to the riskless hedging approach, Merton (1973) derives the option pricing equation via the construction of a self-financing and dynamically hedged portfolio containing the risky asset, option and riskless asset (in the form of money market account). Let $Q_S(t)$ and $Q_V(t)$ denote the number of units of asset and option in the portfolio, respectively, and $M_S(t)$ and $M_V(t)$ the dollar value of $Q_S(t)$ units of asset and $Q_V(t)$ units of option, respectively. The *self-financing portfolio* is set up with zero initial net investment cost and no additional funds added or withdrawn afterwards. The additional units acquired for one security in the portfolio is completely financed by the sale of another security in the same portfolio. The portfolio is said to be *dynamic* since its composition is allowed to change over time. The zero net investment condition at time t can be expressed as

$$\begin{aligned} \Pi(t) &= M_S(t) + M_V(t) + M(t) \\ &= Q_S(t)S + Q_V(t)V + M(t) = 0, \end{aligned} \quad (3.1.8)$$

where $\Pi(t)$ is the value of the portfolio and $M(t)$ the value of the riskless asset invested in riskless money market account. Suppose the asset price process is governed by the stochastic differential equation (3.1.1), we apply the Ito lemma to obtain the differential of the option value V as follows

$$\begin{aligned} dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} dt \\ &= \left(\frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \sigma S \frac{\partial V}{\partial S} dZ. \end{aligned} \quad (3.1.9)$$

Suppose we formally write the stochastic evolution of V as

$$\frac{dV}{V} = \rho_V dt + \sigma_V dZ, \quad (3.1.10)$$

then ρ_V and σ_V are given by

$$\rho_V = \frac{\frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2}}{V} \quad (3.1.11a)$$

$$\sigma_V = \frac{\sigma S \frac{\partial V}{\partial S}}{V}. \quad (3.1.11b)$$

The instantaneous dollar return $d\Pi(t)$ of the above portfolio is attributed to differential price changes of asset and option and interest accrued, and differential changes in the amount of asset, option and money market account held. From Eq. (3.1.8), we have

$$\begin{aligned} d\Pi(t) &= [Q_S(t) dS + Q_V(t) dV + rM(t) dt] \\ &\quad + [S dQ_S(t) + V dQ_V(t) + dM(t)], \end{aligned} \quad (3.1.12)$$

where $dM(t)$ represents the change in the money market account held due to net dollar gained/lost from the sale of asset and option in the portfolio. Since the portfolio is self-financing, the sum of the last three terms in Eq. (3.1.12) is zero. The instantaneous portfolio return $d\Pi(t)$ can then be expressed as

$$\begin{aligned} d\Pi(t) &= Q_S(t) dS + Q_V(t) dV + rM(t) dt \\ &= M_S(t) \frac{dS}{S} + M_V(t) \frac{dV}{V} + rM(t) dt. \end{aligned} \quad (3.1.13)$$

Eliminating $M(t)$ between Eqs. (3.1.8, 3.1.13) and expressing dS/S and dV/V in terms of dt and dZ , we obtain

$$\begin{aligned} d\Pi(t) &= [(\rho - r)M_S(t) + (\rho_V - r)M_V(t)]dt \\ &\quad + [\sigma M_S(t) + \sigma_V M_V(t)]dZ. \end{aligned} \quad (3.1.14)$$

How to make the above self-financing portfolio to be completely riskless so that its return is non-stochastic? This can be achieved by choosing appropriate proportion of asset and option according to

$$\sigma M_S(t) + \sigma_V M_V(t) = \sigma S Q_S(t) + \frac{\sigma S \frac{\partial V}{\partial S}}{V} V Q_V(t) = 0, \quad (3.1.15)$$

that is, the number of units of asset and option in the self-financing portfolio must be in the ratio

$$\frac{Q_S(t)}{Q_V(t)} = -\frac{\partial V}{\partial S} \quad (3.1.16)$$

at all times. The ratio is time dependent, so continuous readjustment of the portfolio is necessary. We now have a dynamic replicating portfolio that is riskless and requires no net investment, so the non-stochastic portfolio return $d\Pi(t)$ must be zero. Equation (3.1.14) now becomes

$$0 = [(\rho - r)M_S(t) + (\rho_V - r)M_V(t)]dt. \quad (3.1.17)$$

Substituting relation (3.1.16) into the above equation, we obtain

$$(\rho - r)S \frac{\partial V}{\partial S} = (\rho_V - r)V. \quad (3.1.18)$$

Lastly, putting Eq. (3.1.11a) into Eq. (3.1.18), we obtain the same Black-Scholes equation for V :

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (3.1.19)$$

Suppose we take $Q_V(t) = -1$ in the above dynamically hedged self-financing portfolio, that is, the portfolio always shorts one unit of the option. The number of units of risky asset held is then kept at the level of $\frac{\partial V}{\partial S}$ units

[see Eq. (3.1.16)], which is changing continuously over time. The net cash flow resulted in the buying/selling of asset in order to maintain at the level of $\frac{\partial V}{\partial S}$ units is siphoned to the money market account.

Replicating portfolio

With the choice of $Q_V(t) = -1$ and knowing that

$$0 = \Pi(t) = -V + \Delta S + M(t), \quad (3.1.20a)$$

the value of the option is found to be

$$V = \Delta S + M(t), \quad \Delta = \frac{\partial V}{\partial S}. \quad (3.1.20b)$$

The above equation implies that the position of the option can be replicated by a self-financing dynamic trading strategy using the risky asset and the riskless asset (in the form of money market account). The replication is possible by adjusting the portfolio in a self-financing manner using the hedge ratio given by Eq. (3.1.20b).

Since the replicating portfolio is self-financing and replicates the terminal payoff of the option at expiration, by virtue of no-arbitrage argument, the initial cost of setting up this replicating portfolio of risky asset and riskless asset must be equal to the value of the option replicated. Thus, the fair price of an option is the value of this self-financing replicating portfolio.

Alternative perspective on risk neutral valuation

We would like to present an alternative perspective by which the argument of *risk neutrality* can be explained in relation to the concept of market price of risk (Cox and Ross, 1976). Suppose we write the stochastic process followed by the option price $V(S, t)$ formally as that given by Eq. (3.1.10), then ρ_V and σ_V are given by Eqs. (3.1.11a,b). By rearranging Eq. (3.1.11a), we obtain the following form of the governing equation for $V(S, t)$:

$$\frac{\partial V}{\partial t} + \rho S \frac{\partial V}{\partial S} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - \rho_V V = 0. \quad (3.1.21)$$

Unlike the Black-Scholes equation, Eq. (3.1.21) contains the parameters ρ and ρ_V . To solve for the option price, we need to determine ρ and ρ_V , or find some other means to avoid such nuisance. The clue lies in the formation of a riskless hedge.

By forming the riskless dynamically hedged portfolio, we obtain [combining Eqs. (3.1.11b) and (3.1.18)]

$$\frac{\rho_V - r}{\sigma_V} = \frac{\rho - r}{\sigma}. \quad (3.1.22)$$

Equation (3.1.22) has specific financial interpretation. The quantities $\rho_V - r$ and $\rho - r$ represent the extra returns over the riskless interest rate on the

option and the asset, respectively. When each is divided by the respective volatility (a measure of the risk of the associated security), the corresponding ratio is called the *market price of risk*, λ . The market price of risk is interpreted as the rate of extra return above the riskless interest rate per unit of risk. The two tradeable securities, option and asset, are hedgeable with each other. Equation (3.1.22) shows that hedgeable securities should have the same market price of risk. Substituting Eqs. (3.1.11a,b) into Eq. (3.1.22), and rearranging the terms, we obtain

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (3.1.23)$$

This is precisely the Black-Scholes equation, which is identical to the equation obtained by setting $\rho_V = \rho = r$ in Eq. (3.1.21). The relation of $\rho_V = \rho = r$ implies zero market price of risk. In the world of zero market price of risk, investors are said to be risk neutral since they do not demand extra returns on holding risky assets. The Black-Scholes equation demonstrates that option valuation can be performed in the risk neutral world by *artificially* taking the expected rate of returns of both the asset and option to be the riskless interest rate.

The use of risk neutrality to find the price of a derivative *relative* to that of the underlying asset would mean that the mathematical relationship between the prices of the derivative and asset is invariant to the risk preferences of investors. We have shown that risk neutrality argument holds when the option can be replicated dynamically by the underlying asset and riskless asset, or in an alternative sense, the risk of the option can be hedged by that of the underlying asset. Indeed the actual rate of return of the underlying asset would affect the asset price and thus indirectly affects the *absolute* derivative price. We simply use the convenience of risk neutrality to arrive at the derivative price relative to that of the underlying asset but we do not assume risk neutrality of the investors.

3.2 Martingale pricing theory

In Sec. 2.2.3, we have seen that for the discrete multi-period securities models the existence of martingale measure is equivalent to the absence of arbitrage. Also, in an arbitrage free complete market, arbitrage price of a contingent claim is given by the discounted expected value of the terminal payoff under the martingale measure. In this section we extend the discrete securities models to their continuous counterparts and develop the pricing theory of derivative under the framework of martingale measure. The martingale pricing theory is developed in a series of seminar papers by Harrison and Pliska (1981, 1983).

A numeraire defines the units in which security prices are measured. For example, if the traded security S is used as the numeraire, then the price of other securities $S_j, j = 1, 2, \dots, n$, relative to the numeraire S is given by S_j/S . Suppose the money market account is used as the numeraire, then the price of a security relative to the money market account is simply the discounted security price. The martingale pricing theory rests on the assertion that a continuous time financial market consisting of traded securities and trading strategies is arbitrage free and complete if for every choice of numeraire there exists a unique equivalent martingale measure such that all security prices relative to that numeraire are martingales under the measure. When we compute the price of an attainable contingent claim, the prices obtained using different martingale measures coincide. That is, the derivative price is invariant with respect to the choice of martingale pricing measure. Depending on the nature of the derivative, we may choose a particular numeraire for a given pricing problem.

In Sec. 3.2.1, we discuss the notion of absence of arbitrage and equivalent martingale measure, and present the risk neutral valuation formula for an attainable contingent claim in a continuous time securities model. We also consider the change of numeraire technique and demonstrate how to compute the Radon-Nikodym derivative that effects the change of measure.

In Sec. 3.2.2, the Black-Scholes model will be revisited. We show that the martingale pricing theory gives the price of a European option as the discounted expectation of the terminal payoff under the equivalent martingale measure. From the Feynman-Kac representation formula (see Sec. 2.4.2), we again obtain the Black-Scholes equation that governs the option price function.

3.2.1 Equivalent martingale measure and risk neutral valuation

Under the continuous time framework, the investors are allowed to trade continuously in the financial market up to finite time T . Most of the tools and results in continuous time securities models are extensions of those in discrete multi-period models. Uncertainty in the financial market is modeled by the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, P)$, where Ω is a sample space, \mathcal{F} is a σ -algebra on Ω , P is a probability measure on (Ω, \mathcal{F}) , \mathcal{F}_t is the filtration and $\mathcal{F}_T = \mathcal{F}$. In the securities model, there are $M + 1$ securities whose price processes are modeled by adapted stochastic processes $S_m(t), m = 0, 1, \dots, M$. Also, we define $h_m(t)$ to be the number of units of the m^{th} security held in the portfolio. The trading strategy $H(t)$ is the vector stochastic process $(h_0(t) \ h_1(t) \ \dots \ h_M(t))^T$, where $H(t)$ is a $(M + 1)$ -dimensional predictable process since the portfolio composition is determined by the investor based on the information available before time t .

The value process associated with a trading strategy $H(t)$ is defined by

$$V(t) = \sum_{m=0}^M h_m(t) S_m(t), \quad 0 \leq t \leq T, \quad (3.2.1)$$

and the gain process $G(t)$ is given by

$$G(t) = \sum_{m=0}^M \int_0^t h_m(u) dS_m(u), \quad 0 \leq t \leq T. \quad (3.2.2)$$

Similar to that in discrete models, $H(t)$ is self-financing if and only if

$$V(t) = V(0) + G(t). \quad (3.2.3)$$

The above equation indicates that the change in portfolio value associated with a self-financing strategy comes only from the changes in the security prices, that is, no additional cash inflows or outflows occur after initial date $t = 0$. We use $S_0(t)$ to denote the money market account process that grows at the riskless interest rate $r(t)$, that is,

$$dS_0(t) = r(t)S_0(t) dt. \quad (3.2.4)$$

The discounted security price process $S_m^*(t)$ is defined as

$$S_m^*(t) = S_m(t)/S_0(t), \quad m = 1, 2, \dots, M. \quad (3.2.5)$$

The discounted value process $V^*(t)$ is defined by dividing $V(t)$ by $S_0(t)$. The discounted gain process $G^*(t)$ is defined by

$$G^*(t) = V^*(t) - V^*(0). \quad (3.2.6)$$

No-arbitrage principle and equivalent martingale measure

A self-financing trading strategy H represents an arbitrage opportunity if and only if (i) $G^*(T) \geq 0$ and (ii) $E_P G^*(T) > 0$ where P is the actual probability measure of the states of occurrence associated with the securities model. A probability measure Q on the space (Ω, \mathcal{F}) is said to be an equivalent martingale measure if it satisfies

- (i) Q is equivalent to P , that is, both P and Q have the same null set;
- (ii) the discounted security price processes $S_m^*(t)$, $m = 1, 2, \dots, M$ are martingales under Q , that is,

$$E_Q[S_m^*(u)|\mathcal{F}_t] = S_m^*(t), \quad \text{for all } 0 \leq t \leq u \leq T. \quad (3.2.7)$$

In discrete time models, we have seen that the absence of arbitrage is equivalent to the existence of a martingale measure (see Theorem 2.3). In continuous time models, Harrison and Pliska (1981) establish the novel result that the existence of an equivalent martingale measure implies the absence of arbitrage. However, absence of arbitrage no longer implies the existence of an equivalent martingale measure.

Contingent claims are modeled as \mathcal{F}_T -measurable random variables. A contingent claim Y is said to be attainable if there exists some trading strategy H such that $V(T) = Y$. We then say that Y is generated by H . The arbitrage price of Y can be obtained by the *risk neutral valuation approach* as stated in Theorem 3.1.

Theorem 3.1

Let Y be an attainable contingent claim generated by some trading strategy H and assume that an equivalent martingale measure Q exists, then for each time $t, 0 \leq t \leq T$, the arbitrage price of Y is given by

$$V(t; H) = S_0(t)E_Q \left[\frac{Y}{S_0(T)} \middle| \mathcal{F}_t \right]. \quad (3.2.8a)$$

The validity of Theorem 3.1 is readily seen if we consider the discounted value process $V^*(t; H)$ to be a martingale under Q . This leads to

$$V(t; H) = S_0(t)V^*(t; H) = S_0(t)E_Q[V^*(T; H)|\mathcal{F}_t]. \quad (3.2.8b)$$

Furthermore, by observing that $V^*(T; H) = Y/S_0(T)$, so the risk neutral valuation formula (3.2.8a) follows.

Even though there may be two replicating portfolios that generate Y , the above risk neutral valuation formula shows that the arbitrage price is determined uniquely by the discounted expectation calculations, independent of the choice of the replicating portfolio.

Change of numeraire

The risk neutral valuation formula (3.2.8a) uses the riskless asset $S_0(t)$ (money market account) as the numeraire. Geman *et al.* (1995) show that the choice of $S_0(t)$ as the numeraire is not unique in order that the risk neutral valuation formula holds. It will be demonstrated in later chapters that the choice of other numeraire such as the stochastic bond price may be more convenient for certain type of option pricing calculations (see Problem 3.5). The following discussion summarizes the powerful tool developed by Geman *et al.* (1995) on the change of numeraire.

Let $N(t)$ be a numeraire whereby we have the existence of an equivalent probability measure Q_N such that all security prices discounted with respect to $N(t)$ are Q_N -martingale. In addition, if a contingent claim Y is attainable under $(S_0(t), Q)$, then it is also attainable under $(N(t), Q_N)$. The arbitrage price of any security given by the risk neutral valuation formula under both measures should agree. We then have

$$S_0(t)E_Q \left[\frac{Y}{S_0(T)} \middle| \mathcal{F}_t \right] = N(t)E_{Q_N} \left[\frac{Y}{N(T)} \middle| \mathcal{F}_t \right]. \quad (3.2.9)$$

To effect the change of measure from Q_N to Q , we multiply $\frac{Y}{N(T)}$ by the Radon-Nikodym derivative so that

$$S_0(t)E_Q \left[\frac{Y}{S_0(T)} \middle| \mathcal{F}_t \right] = N(t)E_Q \left[\frac{Y}{N(T)} \frac{dQ_N}{dQ} \middle| \mathcal{F}_t \right]. \quad (3.2.10)$$

By comparing like terms, we obtain the following formula for the Radon-Nikodym derivative

$$\frac{dQ_N}{dQ} = \frac{N(T)}{N(t)} \frac{S_0(T)}{S_0(t)}. \quad (3.2.11)$$

3.2.2 Black-Scholes model revisited

We consider the Black-Scholes option model again under the martingale pricing framework. The securities model has two basic tradeable securities, the underlying risky asset S and riskless asset M (in the form of money market account). The price processes of $S(t)$ and $M(t)$ are governed by

$$\frac{dS(t)}{S(t)} = \rho dt + \sigma dZ \quad (3.2.12a)$$

$$dM(t) = rM(t) dt. \quad (3.2.12b)$$

Suppose we take the money market account as the numeraire, and define the price of discounted risky asset by $S^*(t) = S(t)/M(t)$. The price process of $S^*(t)$ becomes

$$\frac{dS^*(t)}{S^*(t)} = (\rho - r)dt + \sigma dZ. \quad (3.2.13)$$

We would like to find the equivalent martingale measure Q such that the discounted asset price S^* is Q -martingale. By the Girsanov Theorem, suppose we choose $\gamma(t)$ in the Radon-Nikodym derivative [see Eqs. (2.4.38a,b)] such that

$$\gamma(t) = \frac{\rho - r}{\sigma}, \quad (3.2.14)$$

then \tilde{Z} is a Brownian motion under the probability measure Q and

$$d\tilde{Z} = dZ + \frac{\rho - r}{\sigma} dt. \quad (3.2.15)$$

Under the Q -measure, the process of $S^*(t)$ now becomes

$$\frac{dS^*(t)}{S^*(t)} = \sigma d\tilde{Z}, \quad (3.2.16)$$

hence $S^*(t)$ is Q -martingale. The asset price $S(t)$ under the Q -measure is governed by

$$\frac{dS(t)}{S(t)} = r dt + \sigma d\tilde{Z}, \quad (3.2.17)$$

where the drift rate equals the riskless interest rate r . When the money market account is used as the numeraire, the corresponding equivalent martingale measure is called the *risk neutral measure* and the drift rate of S under the Q -measure is called the *risk neutral drift rate*.

By the risk neutral valuation formula (3.2.8a), the arbitrage price of a derivative is given by

$$V(S, t) = e^{-r(T-t)} E_Q^{t,S} [h(S_T)] \quad (3.2.18)$$

where $E_Q^{t,S}$ is the expectation under the risk neutral measure Q conditional on the filtration \mathcal{F}_t and $S_t = S$. In future discussions, when there is no ambiguity, we choose to write E_Q instead of the full notation $E_Q^{t,S}$. The terminal payoff of the derivative is some function h of the terminal asset price S_T . By the Feynman-Kac representation formula [see Eq. (2.4.24-30) and Problem 2.35], the governing equation of $V(S, t)$ is given by

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0. \quad (3.2.19)$$

This is the same Black-Scholes equation that has been derived earlier.

As an example, consider the European call option whose terminal payoff is $\max(S_T - X, 0)$. Using Eq. (3.2.18), the call price $c(S, t)$ is given by

$$\begin{aligned} c(S, t) &= e^{-r(T-t)} E_Q[\max(S_T - X, 0)] \\ &= e^{-r(T-t)} \{E_Q[S_T \mathbf{1}_{\{S_T \geq X\}}] - X E_Q[\mathbf{1}_{\{S_T \geq X\}}]\}. \end{aligned} \quad (3.2.20)$$

In the next section, we show how to derive the call price formula by computing the above expectations [see Eqs. (3.3.16a,b)].

Exchange rate process under domestic risk neutral measure

Consider a foreign currency option whose payoff function depends on the exchange rate F , which is defined to be the domestic currency price of one unit of foreign currency. Let M_d and M_f denote the money market account process in the domestic market and foreign market, respectively. Suppose the processes of $M_d(t)$, $M_f(t)$ and $F(t)$ are governed by

$$dM_d(t) = rM_d(t) dt, \quad dM_f(t) = r_f M_f(t) dt, \quad \frac{dF(t)}{F(t)} = \mu dt + \sigma dZ_F, \quad (3.2.21)$$

where r and r_f denote the riskless domestic and foreign interest rates, respectively. We would like to find the risk neutral drift rate of the exchange rate process F under the domestic equivalent martingale measure.

We may treat the domestic money market account and the foreign money market account in domestic dollars (whose value is given by FM_f) as traded securities in the domestic currency world. With reference to the domestic equivalent martingale measure, M_d is used as the numeraire. By Ito's lemma, the relative price process $X(t) = F(t)M_f(t)/M_d(t)$ is governed by

$$\frac{dX(t)}{X(t)} = (r_f - r + \mu) dt + \sigma dZ_F. \quad (3.2.22)$$

With the choice of $\gamma = (r_f - r + \mu)/\sigma$ in the Girsanov Theorem, we define

$$dZ_d = dZ_F + \gamma dt, \quad (3.2.23)$$

where Z_d is a Brownian process under Q_d . Now, under the domestic equivalent martingale measure Q_d , the process of X now becomes

$$\frac{dX(t)}{X(t)} = \sigma dZ_d \quad (3.2.24)$$

so that X is Q_d -martingale. The exchange rate process F under the Q_d -measure is given by

$$\frac{dF(t)}{F(t)} = (r - r_f) dt + \sigma dZ_d. \quad (3.2.25)$$

Hence, the risk neutral drift rate of F under Q_d is found to be $r - r_f$.

3.3 Black-Scholes pricing formulas and their properties

In this section, we first derive the Black-Scholes price formula for a European call option by solving directly the Black-Scholes equation augmented with appropriate auxiliary conditions. The price formula of the corresponding European put option can be obtained easily from the put-call parity relation once the corresponding European call price formula is known. We also derive the call price function using the risk neutral valuation formula by computing the discounted expectation of the terminal payoff. By substituting the expectation representation of the call price function into the Black-Scholes equation, we deduce the backward Fokker-Planck equation for the transition density function of the asset price under the risk neutral measure.

For hedging and other trading purposes, it is important to estimate the rate of change of option price with respect to the price of the underlying asset and other option parameters, like the strike price, volatility etc. The formulas for these comparative statics (commonly called the *greeks* of the option formulas) are derived and their characteristics are analyzed.

3.3.1 Pricing formulas for European options

Recall that the Black-Scholes equation for a European vanilla call option takes the form

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + rS \frac{\partial c}{\partial S} - rc, \quad 0 < S < \infty, \tau > 0, \quad (3.3.1)$$

where $c = c(S, \tau)$ is the European call value, S and $\tau = T - t$ are the asset price and time to expiry, respectively. We use the time to expiry τ instead of the calendar time t as the time variable in order that the Black-Scholes equation becomes the usual parabolic type differential equation. The auxiliary conditions of the option pricing model are prescribed as follows:

Initial condition (payoff at expiry)

$$c(S, 0) = \max(S - X, 0), \quad X \text{ is the strike price.} \quad (3.3.2a)$$

Solution behaviors at the boundaries

(i) When $S = 0$ for some $t < T$, S will stay at zero at all subsequent times so that the option is sure to expire out-of-the-money. Hence, the option has zero value, that is,

$$c(0, \tau) = 0. \quad (3.3.2b)$$

(ii) When S is sufficiently large, it becomes almost certain that the call will be exercised. Since the present value of the strike price is $Xe^{-r\tau}$, we have (see Problem 1.10)

$$c(S, \tau) \sim S - Xe^{-r\tau} \quad \text{as } S \rightarrow \infty. \quad (3.3.2c)$$

We illustrate how to apply the Green function technique to determine the solution to $c(S, \tau)$. Using the transformation: $y = \ln S$ and $c(y, \tau) = e^{-r\tau} w(y, \tau)$, the Black-Scholes equation is transformed into the following infinite-domain constant-coefficient parabolic equation

$$\frac{\partial w}{\partial \tau} = \frac{\sigma^2}{2} \frac{\partial^2 w}{\partial y^2} + \left(r - \frac{\sigma^2}{2} \right) \frac{\partial w}{\partial y}, \quad -\infty < y < \infty, \quad \tau > 0. \quad (3.3.3)$$

The initial condition (3.3.2a) for the model now becomes

$$w(y, 0) = \max(e^y - X, 0). \quad (3.3.4)$$

Since the domain of the pricing model is infinite, the differential equation together with the initial condition are sufficient to determine the call price function. Once the price function has been obtained, we can check whether the solution behaviors at the boundaries agree with those stated in Eqs. (3.3.2b,c).

Green function approach

The infinite domain Green function of Eq. (3.3.3) is known to be [see Eqs. (2.3.13-14)]

$$\phi(y, \tau) = \frac{1}{\sigma \sqrt{2\pi\tau}} \exp\left(-\frac{[y + (r - \frac{\sigma^2}{2})\tau]^2}{2\sigma^2\tau}\right). \quad (3.3.5)$$

Here, $\phi(y, \tau)$ satisfies the initial condition:

$$\lim_{\tau \rightarrow 0^+} \phi(y, \tau) = \delta(y), \quad (3.3.6)$$

where $\delta(y)$ is the Dirac function representing a unit impulse at the origin. The Green function $\phi(y, \tau)$ can be considered as the response in the position y and at time to expiry τ due to a unit impulse placed at the origin initially. On the other hand, from the property of the Dirac function, the initial condition can be expressed as

$$w(y, 0) = \int_{-\infty}^{\infty} w(\xi, 0) \delta(y - \xi) d\xi, \quad (3.3.7)$$

so that $w(y, 0)$ can be considered as the superposition of impulses with varying magnitude $w(\xi, 0)$ ranging from $\xi \rightarrow -\infty$ to $\xi \rightarrow \infty$. Since Eq. (3.3.3) is linear, the response in position y and at time to expiry τ due to an impulse of magnitude $w(\xi, 0)$ in position ξ at $\tau = 0$ is given by $w(\xi, 0)\phi(y - \xi, \tau)$. From the principle of superposition for a linear differential equation, the solution to the initial value problem posed in Eqs. (3.3.3–4) is obtained by summing up the responses due to these impulses. This amounts to integration from $\xi \rightarrow -\infty$ to $\xi \rightarrow \infty$. Hence, the solution to $c(y, \tau)$ is given by

$$\begin{aligned} c(y, \tau) &= e^{-r\tau} w(y, \tau) \\ &= e^{-r\tau} \int_{-\infty}^{\infty} w(\xi, 0) \phi(y - \xi, \tau) d\xi \\ &= e^{-r\tau} \int_{\ln X}^{\infty} (e^\xi - X) \frac{1}{\sigma \sqrt{2\pi\tau}} \\ &\quad \exp\left(-\frac{[y + (r - \frac{\sigma^2}{2})\tau - \xi]^2}{2\sigma^2\tau}\right) d\xi. \end{aligned} \quad (3.3.8)$$

Alternatively, $\phi(y - \xi, \tau)$ can be interpreted as the kernel of the integral transform that transforms the initial value $w(\xi, 0)$ to the solution $w(y, \tau)$ at time τ . Consider the following integral, by completing square in the exponential expression, we obtain

$$\begin{aligned} &\int_{\ln X}^{\infty} e^\xi \frac{1}{\sigma \sqrt{2\pi\tau}} \exp\left(-\frac{[y + (r - \frac{\sigma^2}{2})\tau - \xi]^2}{2\sigma^2\tau}\right) d\xi \\ &= \exp(y + r\tau) \int_{\ln X}^{\infty} \frac{1}{\sigma \sqrt{2\pi\tau}} \exp\left(-\frac{[y + (r + \frac{\sigma^2}{2})\tau - \xi]^2}{2\sigma^2\tau}\right) d\xi \quad (3.3.9a) \\ &= e^{r\tau} SN\left(\frac{\ln \frac{S}{X} + (r + \frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}}\right), \quad y = \ln S. \end{aligned}$$

It is even easier to obtain

$$\int_{\ln X}^{\infty} \frac{1}{\sigma \sqrt{2\pi\tau}} \exp\left(-\frac{[y + (r - \frac{\sigma^2}{2})\tau - \xi]^2}{2\sigma^2\tau}\right) d\xi$$

$$= N\left(\frac{y + (r - \frac{\sigma^2}{2})\tau - \ln X}{\sigma \sqrt{\tau}}\right) = N\left(\frac{\ln \frac{S}{X} + (r - \frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}}\right), \quad y = \ln S. \quad (3.3.9b)$$

Hence, the price formula of the European call option is found to be

$$c(S, \tau) = SN(d_1) - Xe^{-r\tau}N(d_2), \quad (3.3.10a)$$

where

$$d_1 = \frac{\ln \frac{S}{X} + (r + \frac{\sigma^2}{2})\tau}{\sigma \sqrt{\tau}}, \quad d_2 = d_1 - \sigma \sqrt{\tau}. \quad (3.3.10b)$$

The initial condition is seen to be satisfied by observing that the limits of both d_1 and d_2 tend to either one or zero depending on $S > X$ or $S < X$ respectively, as $\tau \rightarrow 0^+$. One can easily check that boundary conditions (3.3.2b,c) are satisfied by the analytic call price formula (3.3.10a,b). The European call price formula also gives the price of an American call on a non-dividend paying asset since the American call will not be optimally exercised prior to expiry.

The call value given by formula (3.3.10a) can be shown to lie within the bounds

$$\max(S - Xe^{-r\tau}, 0) \leq c(S, \tau) \leq S, \quad S \geq 0, \tau \geq 0, \quad (3.3.11)$$

which agrees with the distribution free results on the bounds of call price (see Sec. 1.2). In Sec. 3.3.2, we will show that the call price function $c(S, \tau)$ is an increasing convex function of S . A plot of $c(S, \tau)$ against S is shown in Fig. 3.1.

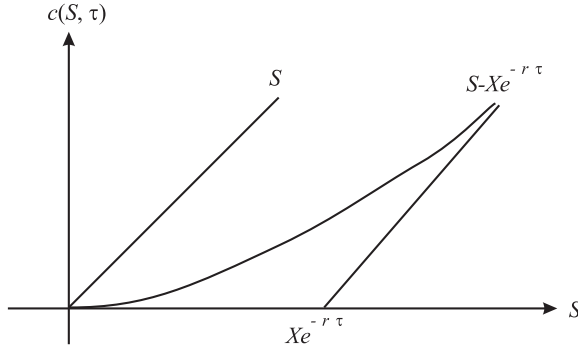


Fig. 3.1 A plot of $c(S, \tau)$ against S at a given time to expiry τ . The European call value is bounded between S and $\max(S - Xe^{-r\tau}, 0)$.

Discounted expectation under the risk neutral measure

We illustrate another approach to obtain the European call value by computing the discounted expectation of the terminal payoff under the risk neutral measure. Let $\psi(S_T, T; S, t)$ denote the transition density function under the risk neutral measure of the terminal asset price S_T at time T , given asset price S at an earlier time t . According to Eq. (3.2.20), the call value $c(S, \tau)$ can be written as

$$\begin{aligned} c(S, \tau) &= e^{-r\tau} E_Q[(S_T - X)\mathbf{1}_{\{S_T \geq X\}}] \\ &= e^{-r\tau} \int_0^\infty \max(S_T - X, 0)\psi(S_T, T; S, t) dS_T. \end{aligned} \quad (3.3.12)$$

Under the risk neutral measure, the asset price follows the Geometric Brownian motion with drift rate r and variance rate σ^2 as governed by Eq. (3.2.17). By observing the results in Eqs. (2.4.17, 20b), we deduce that

$$\ln \frac{S_T}{S} = \left(r - \frac{\sigma^2}{2}\right)\tau + \sigma\tilde{Z}(\tau) \quad (3.3.13)$$

so that $\ln \frac{S_T}{S}$ is normally distributed with mean $\left(r - \frac{\sigma^2}{2}\right)\tau$ and variance $\sigma^2\tau$, $\tau = T - t$. From the density function of a normal random variable [see Eq. (2.3.13)], we deduce that the transition density function is given by

$$\psi(S_T, T; S, t) = \frac{1}{S_T\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{\left[\ln \frac{S_T}{S} - \left(r - \frac{\sigma^2}{2}\right)\tau\right]^2}{2\sigma^2\tau}\right). \quad (3.3.14)$$

Suppose we set $\xi = \ln S_T$ and $y = \ln S$ so that $\ln \frac{S_T}{S} = \xi - y$ and $d\xi = \frac{dS_T}{S_T}$. Substituting the density function into Eq. (3.3.12), the European call value can be expressed as

$$\begin{aligned} c(S, \tau) &= e^{-r\tau} \int_{-\infty}^\infty \max(e^\xi - X, 0) \\ &\quad \frac{1}{\sigma\sqrt{2\pi\tau}} \exp\left(-\frac{\left[y + \left(r - \frac{\sigma^2}{2}\right)\tau - \xi\right]^2}{2\sigma^2\tau}\right) d\xi, \end{aligned} \quad (3.3.15)$$

which is consistent with the result shown in Eq. (3.3.8).

If we compare the price formula (3.3.10a) with the expectation representation in Eq. (3.2.20), we deduce that

$$N(d_2) = E_Q[\mathbf{1}_{\{S_T \geq X\}}] = Q[S_T \geq X] \quad (3.3.16a)$$

$$SN(d_1) = e^{-r\tau} E_Q[S_T \mathbf{1}_{\{S_T \geq X\}}]. \quad (3.3.16b)$$

Hence, $N(d_2)$ is recognized as the probability under the risk neutral measure Q that the call expires in-the-money, so $Xe^{-r\tau}N(d_2)$ represents the present value of the risk neutral expectation of payment paid by the option holder at expiry. Also, $SN(d_1)$ is the discounted risk neutral expectation of the terminal asset price conditional on the call being in-the-money at expiry (see Problem 3.9).

Fokker-Planck equations

What would be the governing equation for the transition density function $\psi(S_T, T; S, t)$? Suppose we substitute the integral in Eq. (3.3.12) into the Black-Scholes equation, we obtain

$$0 = e^{-r(T-t)} \int_0^\infty \max(S_T - X, 0) \left(\frac{\partial \psi}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 \psi}{\partial S^2} + rS \frac{\partial \psi}{\partial S} \right) dS_T. \quad (3.3.17)$$

The integrand function must vanish and thus leads to the following governing equation for $\psi(S_T, T; S, t)$:

$$\frac{\partial \psi}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 \psi}{\partial S^2} + rS \frac{\partial \psi}{\partial S} = 0. \quad (3.3.18a)$$

This is called the backward Fokker-Planck equation since the dependent variables S and t are backward variables. In terms of the forward variables T and S_T , one can show that $\psi(S_T, T; S, t)$ satisfies the following forward Fokker-Planck equation (see Problem 3.7)

$$\frac{\partial \psi}{\partial T} - \frac{\sigma^2}{2} S_T^2 \frac{\partial^2 \psi}{\partial S_T^2} + rS_T \frac{\partial \psi}{\partial S_T} = 0. \quad (3.3.18b)$$

The forward equation reduces to the same form as that in Eq. (2.3.14) if we set $x = \ln S_T$, $\mu = r - \frac{\sigma^2}{2}$ and visualize the time variable t in Eq. (2.3.14) as the forward time variable T in Eq. (3.3.18b).

Put price function

Using the put-call parity relation [see Eq. (1.2.18)], the price of a European put option is given by

$$\begin{aligned} p(S, \tau) &= c(S, \tau) + Xe^{-r\tau} - S \\ &= S[N(d_1) - 1] + Xe^{-r\tau}[1 - N(d_2)] \\ &= Xe^{-r\tau}N(-d_2) - SN(-d_1). \end{aligned} \quad (3.3.19)$$

At sufficiently low asset value, we see that $N(-d_2) \rightarrow 1$ and $SN(-d_1) \rightarrow 0$ so that

$$p(S, \tau) \sim Xe^{-r\tau} \quad \text{as } S \rightarrow 0^+. \quad (3.3.20)$$

The put value can be below its intrinsic value, $X - S$. On the other hand, though $SN(-d_1)$ is of the indeterminate form $\infty \cdot 0$ as $S \rightarrow \infty$, one can show that $SN(-d_1) \rightarrow 0$ as $S \rightarrow \infty$. Hence, we obtain

$$\lim_{S \rightarrow \infty} p(S, \tau) = 0. \quad (3.3.21a)$$

This is not surprising since the European put is certain to expire out-of-the-money as $S \rightarrow \infty$, so it has zero present value. The put price function is a decreasing convex function of S and bounded above by the strike price X . A plot of $p(S, \tau)$ against S is shown in Fig. 3.2.

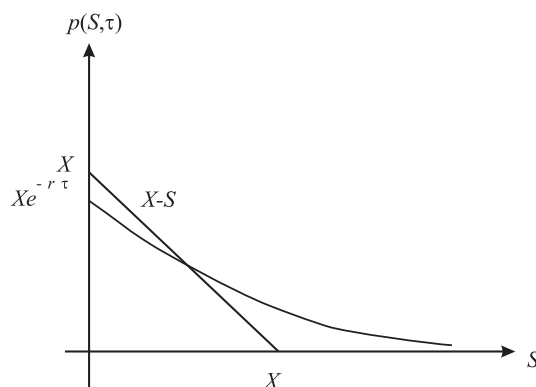


Fig. 3.2 A plot of $p(S, \tau)$ against S at a given time to expiry τ . The European put value may be below the intrinsic value $X - S$ at sufficiently low asset value S .

For a perpetual European put option with infinite time to expiry, since both $N(-d_1) \rightarrow 0$ and $N(-d_2) \rightarrow 0$ as $\tau \rightarrow \infty$, we obtain

$$\lim_{\tau \rightarrow \infty} p(S, \tau) = 0. \quad (3.3.21b)$$

The value of a perpetual European put is zero since the present value of the strike price received at infinite time later is zero.

3.3.2 Comparative statics

The option price formulas are price functions of five parameters: S , τ , X , r and σ . To understand better the pricing behaviors of European vanilla options, we analyze the comparative statics, that is, the rate of change of the option price with respect to these parameters. We commonly use different greek letters to denote different types of comparative statics, so these rates of change are also called greeks of option price.

Delta - derivative with respect to asset price

The *delta* Δ of the value of a derivative security is defined to be $\frac{\partial V}{\partial S}$ where V is the value of the derivative security and S is the asset price. Delta plays a crucial role in the hedging of portfolios. Recall that in the Black-Scholes riskless hedging procedure, a covered call position is maintained by creating a riskless portfolio where the writer of a call sells one unit of call and holds Δ units of asset. From the call price formula (3.3.10a), the delta of the value of a European call option is found to be

$$\begin{aligned}\Delta_c &= \frac{\partial c}{\partial S} = N(d_1) + S \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \frac{\partial d_1}{\partial S} - X e^{-r\tau} \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \frac{\partial d_2}{\partial S} \\ &= N(d_1) + \frac{1}{\sigma\sqrt{2\pi\tau}} [e^{-\frac{d_1^2}{2}} - e^{-(r\tau + \ln \frac{S}{X})} e^{-\frac{d_2^2}{2}}] \\ &= N(d_1) > 0.\end{aligned}\quad (3.3.22)$$

Knowing that a European call can be replicated by Δ units of asset and riskless asset in the form of money market account, the factor $N(d_1)$ in front of S in the call price formula thus gives the hedge ratio Δ . From the put-call parity relation, the delta of the value of a European vanilla put option is

$$\Delta_p = \frac{\partial p}{\partial S} = \Delta_c - 1 = N(d_1) - 1 = -N(-d_1) < 0. \quad (3.3.23)$$

The delta of the value of a call is always positive since an increase in the asset price will increase the probability of a positive terminal payoff resulting in a higher call value. The reverse argument is used to explain why the delta of the value of a put is always negative. The negativity of $\frac{\partial p}{\partial S}$ means that a long position in a put option should be hedged by a continuously varying long position in the underlying asset.

Both call and put deltas are functions of S and τ . Note that Δ_c is an increasing function of S since $\frac{\partial}{\partial S} N(d_1)$ is always positive. Also, the value of Δ_c is bounded between 0 and 1. The curve of Δ_c against S changes concavity at $S_c = X \exp\left(-\left(r + \frac{3\sigma^2}{2}\right)\tau\right)$ so that the curve is concave upward for $0 \leq S < S_c$ and concave downward for $S_c < S < \infty$. To estimate the limiting values of $\frac{\partial \tau}{\partial S}$ at $\tau \rightarrow \infty$ and $\tau \rightarrow 0^+$, it is necessary to observe the following properties of the normal distributive function $N(x)$:

$$\lim_{x \rightarrow \infty} N(x) = 1, \quad \lim_{x \rightarrow -\infty} N(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow 0} N(x) = \frac{1}{2}. \quad (3.3.24a)$$

Note that $d_1 \rightarrow \infty$ when $\tau \rightarrow \infty$ for all values of S . Also, when $\tau \rightarrow 0^+$, we have (i) $d_1 \rightarrow \infty$ if $S > X$, (ii) $d_1 \rightarrow 0$ if $S = X$ and (iii) $d_1 \rightarrow -\infty$ if $S < X$. Hence, we can deduce that

$$\lim_{\tau \rightarrow \infty} \frac{\partial c}{\partial S} = 1 \quad \text{for all values of } S, \quad (3.3.24b)$$

while

$$\lim_{\tau \rightarrow 0^+} \frac{\partial c}{\partial S} = \begin{cases} 1 & \text{if } S > X \\ \frac{1}{2} & \text{if } S = X \\ 0 & \text{if } S < X \end{cases}. \quad (3.3.24c)$$

The variation of the delta of the call value with respect to asset price S and time to expiry τ are shown in Figs. 3.3 and 3.4, respectively.

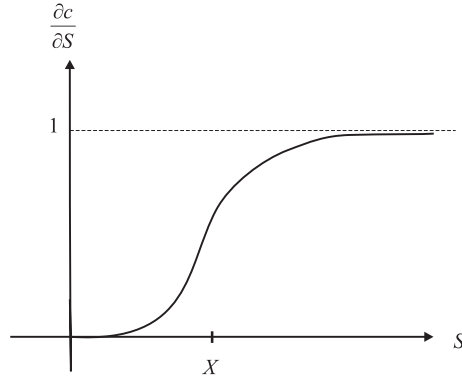


Fig. 3.3 Variation of the delta of the European call value with respect to the asset price S . The curve changes concavity at $S = Xe^{-(r+\frac{3\sigma^2}{2})\tau}$.

Elasticity with respect to asset price

We define the *elasticity* of the call price with respect to the asset price as $\left(\frac{\partial c}{\partial S}\right) \left(\frac{S}{c}\right)$. The elasticity parameter gives the measure of the percentage change in call price for a unit percentage change in the asset price. For a European call on a non-dividend paying asset, the elasticity e_c is found to be

$$e_c = \left(\frac{\partial c}{\partial S}\right) \left(\frac{S}{c}\right) = \frac{SN(d_1)}{SN(d_1) - Xe^{-r\tau}N(d_2)} > 1. \quad (3.3.25)$$

With $e_c > 1$, this implies that a call option is riskier in percentage change than the underlying asset. It can be shown that the elasticity of a call has a very high value when the asset price is small (out-of-the-money) and it decreases monotonically with an increase of the asset price. At sufficiently high value of S , the elasticity tends asymptotically to one since $c \sim S$ as $S \rightarrow \infty$.

The elasticity of the European put price is defined similarly by

$$e_p = \left(\frac{\partial p}{\partial S} \right) \left(\frac{S}{p} \right). \quad (3.3.26)$$

It can be shown that put's elasticity is always negative but its absolute value can be less than or greater than one (see Problem 3.12). Therefore, a European put option may or may not be riskier than the underlying asset.

For both put and call options, their elasticity increases in absolute value when the corresponding options become more out-of-the-money and closer to expiry.

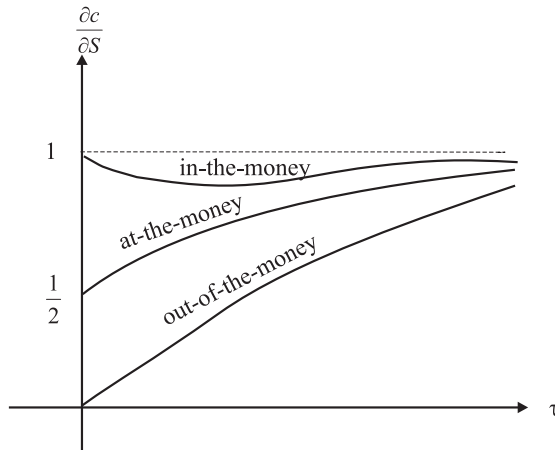


Fig. 3.4 Variation of the delta of the European call value with respect to time to expiry τ . The delta value always tends to one from below when the time to expiry tends to infinity. The delta value tends to different asymptotic limits as time comes close to expiry, depending on the moneyness of the option.

Derivative with respect to strike price

In Sec. 1.2, we argue that the European call (put) price is a decreasing (increasing) function of the strike price. These properties can be verified for the Black-Scholes call and price functions by computing the corresponding derivatives as follows:

$$\begin{aligned} \frac{\partial c}{\partial X} &= S \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \frac{\partial d_1}{\partial X} - X e^{-r\tau} \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \frac{\partial d_2}{\partial X} - e^{-r\tau} N(d_2) \\ &= -e^{-r\tau} N(d_2) < 0, \end{aligned} \quad (3.3.27)$$

and

$$\begin{aligned}\frac{\partial p}{\partial X} &= \frac{\partial c}{\partial X} + e^{-r\tau} \quad (\text{from the put-call parity relation}) \\ &= e^{-r\tau}[1 - N(d_2)] = e^{-r\tau}N(-d_2) > 0.\end{aligned}\quad (3.3.28)$$

Theta - derivative with respect to time

The *theta* Θ of the value of a derivative security V is defined as $\frac{\partial V}{\partial t}$, where t is the calendar time. The theta of the European vanilla call and put prices are found, respectively, to be

$$\begin{aligned}\Theta_c &= \frac{\partial c}{\partial t} = -\frac{\partial c}{\partial \tau} \\ &= -\left[S \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}} \frac{\partial d_1}{\partial \tau} + rXe^{-r\tau}N(d_2) - Xe^{-r\tau} \frac{1}{\sqrt{2\pi}} e^{-\frac{d_2^2}{2}} \frac{\partial d_2}{\partial \tau} \right] \\ &= -\frac{1}{\sqrt{2\pi}} \frac{Se^{-\frac{d_1^2}{2}}\sigma}{2\sqrt{\tau}} - rXe^{-r\tau}N(d_2) < 0,\end{aligned}\quad (3.3.29)$$

and

$$\begin{aligned}\Theta_p &= \frac{\partial p}{\partial t} = -\frac{\partial p}{\partial \tau} = -\frac{\partial c}{\partial \tau} + rXe^{-r\tau} \quad (\text{from the put-call parity relation}) \\ &= -\frac{1}{\sqrt{2\pi}} \frac{Se^{-\frac{d_1^2}{2}}\sigma}{2\sqrt{\tau}} + rXe^{-r\tau}N(-d_2).\end{aligned}\quad (3.3.30)$$

In Sec. 1.2, we deduce that longer-lived American options are worth more than their shorter-lived counterparts. Since an American call option on a non-dividend paying asset will not be exercised early, the above property also holds for European call options on a non-dividend paying asset. The negativity of $\frac{\partial c}{\partial t}$ confirms the above observation. The theta has its greatest absolute value when the call option is at-the-money since the option may become in-the-money or out-of-the-money at an instant later. Also, the theta has a small absolute value when the option is sufficiently out-of-the-money since it will be quite unlikely for the option to become in-the-money at a later time. Further, the theta tends asymptotically to $-rXe^{-r\tau}$ at sufficiently high value of S . The variation of the theta of the European call value with respect to asset price is sketched in Fig. 3.5.

The sign of the theta of the value of a European put option may be positive or negative depending on the relative magnitude of the two terms with opposing signs in Eq. (3.3.30). When the put is deep in-the-money, S assumes a small value so that $N(-d_2)$ tends to one, then the second term is dominant over the first term. In this case, the theta is positive. The positivity of theta is consistent with the observation that the European put value can be below the intrinsic value $X - S$ when S is sufficiently small, which will then grow to $X - S$ at expiry. On the other hand, when the option is at-the-money

or out-of-the-money, there will be a higher chance of a positive payoff for the put option as the time to expiration is lengthened. The European put value then becomes a decreasing function of time and so $\frac{\partial p}{\partial t} < 0$. For American put options, the corresponding theta is always negative. This is because a longer-lived American put is always worth more than its shorter-lived counterpart.

Actually, the details of the sign behaviors of the theta of a European put option is very complicated. The hints to its full analysis are given in Problem 3.14.

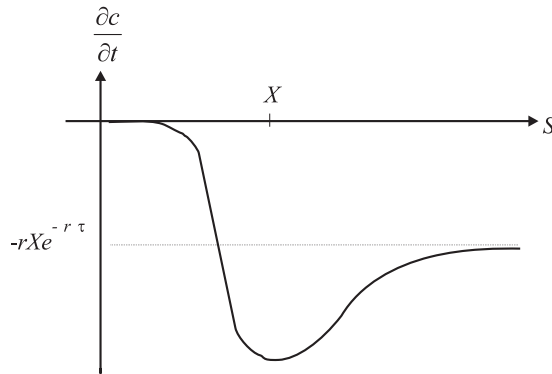


Fig. 3.5 Variation of the theta of the value of a European call option with respect to asset price S . The theta value tends asymptotically to $-rXe^{-r\tau}$ from below when the asset price is sufficiently high.

Gamma - second order derivative with respect to asset price

The *gamma* Γ of the value of a derivative security V is defined as the rate of change of the delta with respect to the asset price S , that is, $\Gamma = \frac{\partial^2 V}{\partial S^2}$. The gammas of the European call and put options are the same since their deltas differ by a constant. The gamma of the value of a European put/call option is given by

$$\Gamma_p = \Gamma_c = \frac{e^{-\frac{d^2}{2}}}{S\sigma\sqrt{2\pi\tau}} > 0. \quad (3.3.31)$$

The curve of Γ_c against S resembles a slightly skewed bell-shaped curve centered at $S = X$ above the S -axis. Since the gamma is always positive for any European call or put option, this explains why option price curves plotted against asset price are always concave upward. From calculus, we know that gamma assumes a small value when the curvature of the option value curve is small. A small value of gamma implies that delta changes slowly with asset

price and so portfolio adjustment required to keep a portfolio delta neutral can be made less frequently.

Vega - derivative with respect to volatility

In the Black-Scholes model, we assume the volatility of the underlying asset to be constant. In reality, the volatility changes over time. Sometimes, we may be interested to see how the option value responds to changes in volatility value. The *vega* \wedge of the value of a derivative security is defined to be the rate of change of the value of the derivative security with respect to the volatility of the underlying asset. For the European vanilla call and put options, their vegas are found to be

$$\wedge_c = \frac{\partial c}{\partial \sigma} = SN'(d_1) \frac{\partial d_1}{\partial \sigma} - Xe^{-r\tau} N'(d_2) \frac{\partial d_2}{\partial \sigma} = \frac{S\sqrt{\tau}e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} > 0, \quad (3.3.32)$$

and

$$\wedge_p = \frac{\partial p}{\partial \sigma} = \frac{\partial c}{\partial \sigma} + \frac{\partial}{\partial \sigma}(Xe^{-r\tau} - S) = \wedge_c. \quad (3.3.33)$$

The above results indicate that both European call and put prices increase with increasing volatility. Since the increase in volatility will lead to a wider spread of the terminal asset price, there is a higher chance that the option may end up either deeper in-the-money or deeper out-of-the-money. However, there is no increase in penalty for the option to be deeper out-of-the-money but the payoff increases when the option expires deeper in-the-money. Due to this non-symmetry in the payoff pattern, the vega of any option is always positive.

Rho - derivative with respect to interest rate

A higher interest rate lowers the present value of the cost of exercising the European call option at expiration (the effect is similar to the lowering of the strike price) and so this increases the call price. Reverse effect holds for the put price. The rho ρ of the value of a derivative security is defined to be the rate of change of the value of the derivative security with respect to the interest rate. The rhos of the European call and put values are given, respectively, by

$$\begin{aligned} \rho_c &= \frac{\partial c}{\partial r} = SN'(d_1) \frac{\partial d_1}{\partial r} + \tau Xe^{-r\tau} N(d_2) - Xe^{-r\tau} N'(d_2) \frac{\partial d_2}{\partial r} \\ &= \tau Xe^{-r\tau} N(d_2) > 0 \quad \text{for } r > 0 \text{ and } X > 0, \end{aligned} \quad (3.3.34)$$

and

$$\begin{aligned} \rho_p &= \frac{\partial p}{\partial r} = \frac{\partial c}{\partial r} - \tau Xe^{-r\tau} \quad (\text{from the put-call parity relation}) \\ &= -\tau Xe^{-r\tau} N(-d_2) < 0 \quad \text{for } r > 0 \text{ and } X > 0. \end{aligned} \quad (3.3.35)$$

The signs of ρ_c and ρ_p confirm the above claims on the impact of changing interest rate on the call and put prices.

3.4 Extension of option pricing models

In this section, we would like to extend the original Black-Scholes formulation by relaxing some of the assumptions in the model. It is a common practice that assets pay *dividends* either as discrete payments or continuous yields. We examine the modification of the governing differential equation and price formulas required for the inclusion of continuous dividend yield and discrete dividends. The analytic techniques of solving option pricing models with time dependent parameters are also presented. We also derive the price formulas of *futures options* where the underlying asset is a futures contract. Lastly, we consider the valuation of the *chooser option*, which has the feature that the holder can choose whether the option is a call or a put after a specified period of time has lapsed from the starting date of the option contract.

3.4.1 Options on dividend-paying assets

The dividends received by holding an underlying asset may be stochastic or deterministic. The modeling of stochastic dividends is more complicated since we have to assume the dividend to be another random variable independent of the asset price. Here, we only consider dividends which are deterministic. This is not an unreasonable assumption in many situations. Concerning the effect of dividends on the asset price, we have shown using the principle of no arbitrage that the asset price falls immediately right after an ex-dividend date by the same amount as the dividend payment (see Sec. 1.2.1).

Continuous dividend yield models

First, we consider the effect of continuous dividend yield on the value of a European call option. Let q denote the constant continuous dividend yield, that is, the holder receives dividend of amount equal to $qS dt$ within the interval dt , where S is the asset price. The asset price dynamics is assumed to follow the Geometric Brownian Motion

$$\frac{dS}{S} = \rho dt + \sigma dZ, \quad (3.4.1)$$

where ρ and σ^2 are respectively the expected rate of return and variance rate of the asset price. To derive the governing differential equation, we form a riskless hedging portfolio by short selling one unit of the European call and long holding Δ units of the underlying asset. The differential of the portfolio value Π is given by

$$\begin{aligned}
d\Pi &= -dc + \Delta dS + q\Delta S dt \\
&= \left(-\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + q\Delta S \right) dt + \left(\Delta - \frac{\partial c}{\partial S} \right) dS. \quad (3.4.2)
\end{aligned}$$

The last term $q\Delta S dt$ is the wealth added to the portfolio due to the dividend payment received. By choosing $\Delta = \frac{\partial c}{\partial S}$, we obtain a riskless hedge for the portfolio. By the usual no-arbitrage argument, the hedged portfolio should earn the riskless interest rate. We then have

$$d\Pi = \left(-\frac{\partial c}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + qS \frac{\partial c}{\partial S} \right) dt = r \left(-c + S \frac{\partial c}{\partial S} \right) dt, \quad (3.4.3)$$

which leads to the following modified version of the Black-Scholes equation

$$\frac{\partial c}{\partial \tau} = \frac{\sigma^2}{2} S^2 \frac{\partial^2 c}{\partial S^2} + (r - q)S \frac{\partial c}{\partial S} - rc, \quad \tau = T - t, \quad 0 < S < \infty, \quad \tau > 0. \quad (3.4.4)$$

The terminal payoff of the European call option on a continuous dividend paying asset is identical to that of the non-dividend paying counterpart.

Risk neutral drift rate

From the modified Black-Scholes equation (3.4.4), we can deduce that the risk neutral drift rate of the price process of an asset paying dividend yield q is $r - q$. One can also show this result via the martingale pricing approach. Suppose all the dividend yields received are used to purchase additional units of asset, then the wealth process of holding one unit of asset initially is given by

$$\widehat{S}_t = e^{qt} S_t, \quad (3.4.5a)$$

where e^{qt} represents the growth factor in the number of units. Suppose S_t follows the price dynamics as defined in Eq. (3.4.1), then the wealth process \widehat{S}_t follows

$$\frac{d\widehat{S}_t}{\widehat{S}_t} = (\rho + q) dt + \sigma dZ. \quad (3.4.5b)$$

We would like to find the equivalent risk neutral measure Q under which the discounted wealth process \widehat{S}_t^* is Q -martingale. In a similar manner, we choose $\gamma(t)$ in the Radon-Nikodym derivative to be

$$\gamma(t) = \frac{\rho + q - r}{\sigma} \quad (3.4.6a)$$

so that \widetilde{Z} is Brownian process under Q and

$$d\widetilde{Z} = dZ + \frac{\rho + q - r}{\sigma} dt. \quad (3.4.6b)$$

Now, \widehat{S}_t^* becomes Q -martingale since

$$\frac{d\widehat{S}_t^*}{\widehat{S}_t^*} = \sigma d\widetilde{Z}. \quad (3.4.6c)$$

The asset price S_t under the equivalent risk neutral measure Q becomes

$$\frac{dS_t}{S_t} = (r - q) dt + \sigma d\widetilde{Z}. \quad (3.4.6d)$$

Hence, the risk neutral drift rate of S_t is $r - q$.

Analogy with foreign currency options

The continuous yield model is also applicable to *options on foreign currencies* where the continuous dividend yield can be considered as the yield due to the interest earned by the foreign currency at the foreign interest rate r_f . In the pricing model for a foreign currency call option, we can simply set $q = r_f$ in Eq. (3.4.4). This is consistent with the observation that the risk neutral drift rate of the exchange rate process under the domestic equivalent martingale measure is $r - r_f$ [see Eq. (3.2.25)].

Call and put price formulas

The price of a European call option on a continuous dividend paying asset can be obtained by a simple modification of the Black-Scholes call price formula (3.3.10a,b) as follows: changing S to $Se^{-q\tau}$ in the price formula. This rule of transformation is justified since the drift rate of the dividend yield paying asset under the risk neutral measure is $r - q$. Now, the European call price formula with continuous dividend yield q is found to be

$$c = Se^{-q\tau} N(\widehat{d}_1) - Xe^{-r\tau} N(\widehat{d}_2), \quad (3.4.7a)$$

where

$$\widehat{d}_1 = \frac{\ln \frac{S}{X} + (r - q + \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}, \quad \widehat{d}_2 = \widehat{d}_1 - \sigma\sqrt{\tau}. \quad (3.4.7b)$$

Similarly, the European put formula with continuous dividend yield q can be deduced from the Black-Scholes put price formula to be

$$p = Xe^{-r\tau} N(-\widehat{d}_2) - Se^{-q\tau} N(-\widehat{d}_1). \quad (3.4.8)$$

Note that the new put and call prices satisfy the put-call parity relation [see Eq. (1.2.18)]

$$p = c - Se^{-q\tau} + Xe^{-r\tau}. \quad (3.4.9a)$$

Furthermore, the following *put-call symmetry relation* can also be deduced from the above call and put price formulas

$$c(S, \tau; X, r, q) = p(X, \tau; S, q, r), \quad (3.4.9b)$$

that is, the put price formula can be obtained from the corresponding call price formula by interchanging S with X and r with q in the formula. To provide an intuitive argument behind the put-call symmetry relation, we recall that a call option entitles its holder the right to exchange the riskless asset for the risky asset, and vice versa for a put option. The dividend yield earned from the risky asset is q while that from the riskless asset is r . If we interchange the roles of the riskless asset and risky asset in a call option, the call becomes a put option, thus giving the justification for the put-call symmetry relation.

The call and put price formulas of foreign currency options mimic the above price formulas, except that the dividend yield q is replaced by the foreign interest rate r_f (Garman and Kohlhagen, 1983). Here, the risky price process is replaced by the exchange rate process F , where F represents the domestic currency price of a unit of foreign currency.

Time dependent parameters

So far, we have assumed constant value for the dividend yield, interest rate and volatility. Suppose these parameters now become deterministic functions of time, the Black-Scholes equation has to be modified as follows

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2(\tau)}{2} S^2 \frac{\partial^2 V}{\partial S^2} + [r(\tau) - q(\tau)] S \frac{\partial V}{\partial S} - r(\tau)V, \quad 0 < S < \infty, \quad \tau > 0, \quad (3.4.10a)$$

where V is the price of the derivative security. When we apply the following transformations: $y = \ln S$ and $w = e^{\int_0^\tau r(u) du} V$, Eq. (3.4.10a) becomes

$$\frac{\partial w}{\partial \tau} = \frac{\sigma^2(\tau)}{2} \frac{\partial^2 w}{\partial y^2} + \left[r(\tau) - q(\tau) - \frac{\sigma^2(\tau)}{2} \right] \frac{\partial w}{\partial y}. \quad (3.4.10b)$$

Consider the following form of the fundamental solution

$$f(y, \tau) = \frac{1}{\sqrt{2\pi s(\tau)}} \exp\left(-\frac{[y + e(\tau)]^2}{2s(\tau)}\right), \quad (3.4.11)$$

it can be shown that $f(y, \tau)$ satisfies the parabolic equation

$$\frac{\partial f}{\partial \tau} = \frac{1}{2} s'(\tau) \frac{\partial^2 f}{\partial y^2} + e'(\tau) \frac{\partial f}{\partial y}. \quad (3.4.12)$$

Suppose we let

$$s(\tau) = \int_0^\tau \sigma^2(u) du \quad (3.4.13a)$$

$$e(\tau) = \int_0^\tau [r(u) - q(u)] du - \frac{s(\tau)}{2}, \quad (3.4.13b)$$

by comparing Eqs. (3.4.10b, 3.4.12), one can deduce that the fundamental solution of Eq. (3.4.10b) is given by

$$\phi(y, \tau) = \frac{1}{\sqrt{2\pi \int_0^\tau \sigma^2(u) du}} \exp\left(-\frac{\{y + \int_0^\tau [r(u) - q(u) - \frac{\sigma^2(u)}{2}] du\}^2}{2 \int_0^\tau \sigma^2(u) du}\right). \quad (3.4.14)$$

Given the initial condition $w(y, 0)$, the solution to Eq. (3.4.10b) can be expressed as

$$w(y, \tau) = \int_{-\infty}^{\infty} w(\xi, 0) \phi(y - \xi, \tau) d\xi. \quad (3.4.15)$$

Note that the time dependency of the coefficients $r(\tau)$, $q(\tau)$ and $\sigma^2(\tau)$ will not affect the spatial integration with respect to ξ . The result of integration will be similar in analytic form to that obtained for the constant coefficient models, except that we have to make the following respective substitutions in the option price formulas

$$\begin{aligned} r &\text{ is replaced by } \frac{1}{\tau} \int_0^\tau r(u) du \\ q &\text{ is replaced by } \frac{1}{\tau} \int_0^\tau q(u) du \\ \sigma^2 &\text{ is replaced by } \frac{1}{\tau} \int_0^\tau \sigma^2(u) du. \end{aligned}$$

For example, the European call price formula is modified as follows:

$$c = S e^{-\int_0^\tau q(u) du} N(\tilde{d}_1) - X e^{-\int_0^\tau r(u) du} N(\tilde{d}_2) \quad (3.4.16a)$$

where

$$\tilde{d}_1 = \frac{\ln \frac{S}{X} + \int_0^\tau [r(u) - q(u) + \frac{\sigma^2(u)}{2}] du}{\sqrt{\int_0^\tau \sigma^2(u) du}}, \quad \tilde{d}_2 = \tilde{d}_1 - \sqrt{\int_0^\tau \sigma^2(u) du}, \quad (3.4.16b)$$

for the option model with time dependent parameters. The European put price formula can be deduced in a similar manner. In conclusion, the Black-Scholes call and put formulas are also applicable to models with time dependent parameters except that the interest rate r , the dividend yield q and the variance rate σ^2 in the Black-Scholes formulas are replaced by the corresponding average value of the instantaneous interest rate, dividend yield and variance rate for the remaining life of the option.

Discrete dividends

Suppose the underlying asset pays N discrete dividends at known payment times t_1, t_2, \dots, t_N of amount D_1, D_2, \dots, D_N , respectively. A common assumption for the valuation of options with known discrete dividends is to take the asset price to consist of two components: a riskless component that will be used to pay the known dividends during the life of the option and a

risky component which follows the lognormal process. The riskless component at a given time is taken to be the present value of all future dividends discounted from the ex-dividend dates to the present at the riskless interest rate. One can then apply the Black-Scholes formulas by setting the asset price equal the risky component and letting the volatility parameter be the volatility of the stochastic process followed by the risky component (which is not quite the same as that followed by the whole asset price). The value of the risky component \tilde{S}_t is taken to be

$$\begin{aligned} \tilde{S}_t &= S_t - D_1 e^{-r\tau_1} - D_2 e^{-r\tau_2} - \dots - D_N e^{-r\tau_N} & \text{for } t < t_1 \\ \tilde{S}_t &= S_t - D_2 e^{-r\tau_2} - \dots - D_N e^{-r\tau_N} & \text{for } t_1 < t < t_2 \\ &\vdots & \\ \tilde{S}_t &= S_t & \text{for } t > t_N \end{aligned} \quad (3.4.17)$$

where S_t is the current asset price, $\tau_i = t_i - t$, $i = 1, 2, \dots, N$. It is customary to take the volatility of the risky component to be approximately given by the volatility of the whole asset price multiplied by the factor $\frac{S_t}{S_t - D}$, where D is the present value of lumped future discrete dividends.

Note that the asset price may not fall by the same amount as the whole dividend due to tax and other considerations. In the above discussion, the “dividend” should be interpreted as the decline in the asset price on the ex-dividend date caused by the dividend, rather than the actual amount of dividend payment.

3.4.2 Futures options

The underlying asset in a futures option is a futures contract. When a futures call option is exercised, the holder acquires from the writer of the option a long position in the underlying futures contract plus a cash inflow equal to the excess of the spot futures price over the strike price. Since the newly opened futures contract has zero value, the value of the futures option upon exercise is the above cash inflow. For example, suppose the strike price of an October futures call option on 10,000 ounces of gold is \$340 per ounce. On the expiration date of the option (say, August 15), the spot gold futures price is \$350 per ounce. The holder of the call option then receives \$100,000 [= 10,000 × (\$350 − \$340)], plus a long position in a futures contract to buy 10,000 ounces of gold on the October delivery date. The position of the futures contract can be immediately closed out at no cost, if the option holder chooses. The maturity dates of the futures option and the underlying futures may or may not coincide. Note that the maturity date of the futures should not be earlier than that of the option.

The trading of futures options is more popular than the trading of options on the underlying asset since futures contracts are more liquid and easier to

trade than the underlying asset. Futures and futures options are often traded in the same exchange. Most futures options are settled in cash without the delivery of the underlying futures. For most commodities and bonds, the futures price is readily available from trading in the futures exchange whereas the spot price of the commodity or bond may have to be obtained through a dealer.

We would like derive the governing differential equation for the value of a futures option based on the Black-Scholes formulation. The interest rate is assumed to be *constant* so that the futures price becomes equal the forward price. Assume that the price dynamics of the underlying asset follows the Geometric Brownian motion. Since the futures price is given by a deterministic time function times the asset price, the volatility of the futures price should be the same as that of the underlying asset. Hence, we can write the dynamics of the futures price F as

$$\frac{dF}{F} = \rho_F dt + \sigma dZ, \quad (3.4.18)$$

where ρ_F is the expected rate of return of the futures and σ is the constant volatility of the asset price. Let $V(F, t)$ denote the value of the futures option. Now, we consider a portfolio which contains α units of the futures in the long position and one unit of the futures option in the short position. The value of the portfolio Π is given by

$$\Pi = -V, \quad (3.4.19)$$

since there is no cost incurred to enter into a futures contract. In time dt , the change in the value of the portfolio is

$$d\Pi = -dV + \alpha dF, \quad (3.4.20)$$

where dV can be computed by Ito's lemma as

$$dV = \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} F^2 \frac{\partial^2 V}{\partial F^2} + \rho_F F \frac{\partial V}{\partial F} \right) dt + \sigma F \frac{\partial V}{\partial F} dZ \quad (3.4.21)$$

and dF is given by Eq. (3.4.18). With the judicious choice of $\alpha = \frac{\partial V}{\partial F}$, the portfolio becomes riskless. The riskless portfolio should earn the riskless interest rate and this leads to

$$d\Pi = - \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} F^2 \frac{\partial^2 V}{\partial F^2} \right) dt = r\Pi dt = -rV dt. \quad (3.4.22)$$

Hence, the governing equation for the value of the futures option is given by

$$\frac{\partial V}{\partial \tau} = \frac{\sigma^2}{2} F^2 \frac{\partial^2 V}{\partial F^2} - rV, \quad \tau = T - t. \quad (3.4.23)$$

If we compare Eq. (3.4.23) with the corresponding governing equation for the value of a European option on an asset that pays continuous dividend yield at the rate q , then Eq. (3.4.23) can be obtained by setting $q = r$ in Eq. (3.4.4). Recall that the expected rate of growth of the continuous dividend paying asset under the risk neutral measure is $r - q$. Since it costs nothing to enter into the futures contract, so the expected gains to the holder of a futures contract should be zero under the risk neutral measure. The setting of $q = r$ is consistent with zero rate of return of a futures.

Using a similar argument as for the continuous yield model, we can obtain the prices of European futures call and put options by simply substituting $q = r$ in the price formulas of the corresponding call and put options on a continuous dividend paying asset. It then follows that the prices of European futures call option and put option are, respectively, given by

$$c = e^{-r\tau} [FN(\tilde{d}_1) - XN(\tilde{d}_2)] \quad (3.4.24a)$$

and

$$p = e^{-r\tau} [XN(-\tilde{d}_2) - FN(-\tilde{d}_1)], \quad (3.4.24b)$$

where

$$\tilde{d}_1 = \frac{\ln \frac{F}{X} + \frac{\sigma^2}{2} \tau}{\sigma \sqrt{\tau}}, \quad \tilde{d}_2 = \tilde{d}_1 - \sigma \sqrt{\tau}, \quad (3.4.24c)$$

F and X are the futures price and the strike price of the options, respectively. The corresponding put-call parity relation is

$$p + Fe^{-r\tau} = c + Xe^{-r\tau}. \quad (3.4.25)$$

Since the futures price of any asset is the same as its spot price at maturity of the futures, a European futures option must be worth the same as the corresponding European option on the underlying asset if the option and the futures contract are set to have the same date of maturity. Recall that τ is the time to expiry of the futures option. Since the futures and option have the same maturity, the futures price is equal to $F = Se^{r\tau}$. If we substitute $F = Se^{r\tau}$ into Eqs. (3.4.24a,b), then the resulting price formulas become the usual Black-Scholes price formulas for European vanilla options.

Black model

The above option price formulas (3.4.24a,b) of futures options are first obtained by Black (1976). Black extends this framework of analysis to price bond options and interest rate caps (see Sec. 9.1). Since the bond forward price and forward interest rate are more readily observable in the market, Black attempts to express the option price formula in terms of the forward price of the underlying state variable.

Consider a European call option whose underlying is a stochastic variable ξ_t , which is assumed to follow the lognormal process. Let F denote the forward price for a forward contract with ξ_t as underlying and maturity date

T (which coincides with option's maturity date). In a more general setting, we can relax the assumption of deterministic interest rate and allow for its stochastic fluctuations. Let $B(t, T)$ denote the price at time t of a zero coupon unit par bond maturing at T . Suppose we use $B(t, T)$ as the numeraire, the call option price with strike X is given by

$$c(\xi, \tau) = B(t, T)E_{Q^T}[\max(\xi_T - X, 0)], \quad \tau = T - t, \quad (3.4.26)$$

where E_{Q^T} is the expectation under the T -forward measure conditional on the filtration \mathcal{F}_t and $\xi_t = \xi$ (see Problem 3.5). Under the same T -forward measure, $\xi_t/B(t, T)$ is Q^T -martingale so that

$$\xi_t = B(t, T)E_{Q^T}[\xi_T]. \quad (3.4.27a)$$

By applying no-arbitrage argument (see Problem 3.1), we deduce that the forward price at current time t is given by

$$F = \xi_t/B(t, T). \quad (3.4.27b)$$

By combining Eqs. (3.4.27a,b), the forward price F is given by the expectation of ξ_T under Q^T , that is,

$$F = E_{Q^T}[\xi_T]. \quad (3.4.28a)$$

Since $\xi_t/B(t, T)$ is Q^T -martingale, we can write the process followed by F under the T -forward measure as

$$\frac{dF}{F} = \sigma_F dZ_T, \quad (3.4.28b)$$

where Z_T is a Brownian process under Q^T . That is, the forward price F is Q^T -martingale. Here, σ_F corresponds to the volatility of F , or called the forward volatility of ξ_t . Alternatively, σ_F can be interpreted as the volatility of ξ_t under the T -forward measure. By performing the usual expectation calculations in Eq. (3.4.26) and observing the relations in Eq. (3.4.28a,b), we obtain

$$c(\xi, \tau) = B(t, T)[FN(\bar{d}_1) - XN(\bar{d}_2)] \quad (3.4.29a)$$

where

$$\bar{d}_1 = \frac{\ln \frac{F}{X} + \frac{\sigma_F^2}{2}\tau}{\sigma_F\sqrt{\tau}} \quad \text{and} \quad \bar{d}_2 = \bar{d}_1 - \sigma_F\sqrt{\tau}. \quad (3.4.29b)$$

Under constant interest rate, $B(t, T)$ equals $e^{-r\tau}$, $\tau = T - t$, then the forward volatility and usual volatility are the same. In this case, price formula (3.4.29a) reduces to that in Eq. (3.4.24a). When the interest rate is stochastic, σ_F has dependence on the volatility of ξ_t and the volatility of bond price. Since the volatility of bond price is time dependent, so σ_F is also time dependent. The call price formula is then modified by applying the following modification: $\sigma_F^2\tau$ is replaced by $\int_0^\tau \sigma_F^2(u) du$ accordingly.

3.4.3 Chooser options

A standard *chooser option* (or called *as-you-like-it option*) entitles the holder to choose, at a predetermined time T_c in the future, whether the option is a standard European call or put with a common strike price X for the remaining time to expiration $T - T_c$. The payoff of the chooser option on the date of choice T_c is

$$V(S_{T_c}, T_c) = \max(c(S_{T_c}, T - T_c; X), p(S_{T_c}, T - T_c; X)), \quad (3.4.30)$$

where $T - T_c$ is the time to expiry in both call and put price formulas above, and S_{T_c} is the asset price at time T_c . Suppose the underlying asset pays a continuous dividend yield at the rate q . By the put-call parity relation, the above payoff function can be expressed as

$$\begin{aligned} V(S_{T_c}, T_c) &= \max(c, c + Xe^{-r(T-T_c)} - S_{T_c}e^{-q(T-T_c)}) \\ &= c + e^{-q(T-T_c)} \max(0, Xe^{-(r-q)(T-T_c)} - S_{T_c}). \end{aligned} \quad (3.4.31)$$

Hence, the chooser option is equivalent to the combination of one call with exercise price X and time to expiration T and $e^{-q(T-T_c)}$ units of put with strike price $Xe^{-(r-q)(T-T_c)}$ and time to expiration T_c . For notational convenience, we take the current time to be zero. Applying the Black-Scholes pricing approach, the value of the standard chooser option at current time is found to be (Rubinstein, 1992)

$$\begin{aligned} V(S, 0) &= Se^{-qT} N(x) - Xe^{-rT} N(x - \sigma\sqrt{T}) + e^{-q(T-T_c)} \\ &\quad [Xe^{-(r-q)(T-T_c)} e^{-rT_c} N(-y + \sigma\sqrt{T_c}) - Se^{-qT_c} N(-y)] \\ &= Se^{-qT} N(x) - Xe^{-rT} N(x - \sigma\sqrt{T}) \\ &\quad + Xe^{-rT} N(-y + \sigma\sqrt{T_c}) - Se^{-qT} N(-y), \end{aligned} \quad (3.4.32a)$$

where S is the current asset price and

$$x = \frac{\ln \frac{S}{X} + (r - q + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}, \quad y = \frac{\ln \frac{S}{X} + (r - q)T + \frac{\sigma^2}{2}T_c}{\sigma\sqrt{T_c}}. \quad (3.4.32b)$$

3.4.4 Compound options

A compound option is simply *an option on an option*. There are four main types of compound options, namely, a call on a call, a call on a put, a put on a call and a put on a put. A compound option has two strike prices and two expiration dates. As an illustration, we consider a call on a call where both calls are European-style. On the first expiration date T_1 , the holder of the compound option has the right to buy the underlying call option by paying the first strike price X_1 . The underlying call option again gives the right to the holder to buy the underlying asset by paying the second strike price X_2

on a later expiration date T_2 . Let $c(S, t)$ denote the value of the compound call-on-a-call option, where S is the asset price at current time t . The value of the underlying call option at the first expiration time T_1 is denoted by $\tilde{c}(S_1, T_1)$, where S_{T_1} is the asset price at time T_1 . Note that the compound option will be exercised at T_1 only when $\tilde{c}(S_{T_1}, T_1) > X_1$.

Assume the usual Black-Scholes pricing approach, we would like to derive the analytic price formula for a European call-on-a-call option. First, the value of the underlying call option at time T_1 is given by the Black-Scholes call formula

$$\tilde{c}(S_{T_1}, T_1) = S_{T_1}N(d_1) - X_2e^{-r(T_2-T_1)}N(d_2), \quad (3.4.33a)$$

where

$$d_1 = \frac{\ln \frac{S_{T_1}}{X_2} + (r + \frac{\sigma^2}{2})(T_2 - T_1)}{\sigma\sqrt{T_2 - T_1}}, \quad d_2 = d_1 - \sigma\sqrt{T_2 - T_1}. \quad (3.4.33b)$$

Let \tilde{S}_{T_1} denote the critical value for S_{T_1} , above which the compound option will be exercised at T_1 . The value of \tilde{S}_{T_1} is obtained by solving the following non-linear algebraic equation

$$\tilde{c}(\tilde{S}_{T_1}, T_1) = X_1. \quad (3.4.34)$$

The payoff function of the compound call-on-a call option at time T_1 is

$$c(S_{T_1}, T_1) = \max(\tilde{c}(S_{T_1}, T_1) - X_1, 0). \quad (3.4.35)$$

The value of the compound option for $t < T_1$ is given by the following risk neutral valuation calculation, where

$$\begin{aligned} c(S, t) &= e^{-r(T_1-t)}E_Q[\max(\tilde{c}(S_{T_1}, T_1) - X_1, 0)] \\ &= e^{-r(T_1-t)} \int_0^\infty \max(\tilde{c}(S_{T_1}, T_1) - X_1, 0) \psi(S_{T_1}; S) dS_{T_1} \\ &= e^{-r(T_1-t)} \int_{\tilde{S}_{T_1}}^\infty [S_{T_1}N(d_1) - X_2e^{-r(T_2-T_1)}N(d_2) - X_1] \\ &\quad \psi(S_{T_1}; S) dS_{T_1}. \end{aligned} \quad (3.4.36)$$

Here, Q denotes the risk neutral measure and the transition density function $\psi(S_{T_1}; S)$ is given by Eq. (3.3.14). The last term in Eq. (3.4.36) is easily recognized as

$$\text{3rd term} = -X_1e^{-r(T_1-t)}E_Q[\mathbf{1}_{\{S_{T_1} \geq \tilde{S}_{T_1}\}}] = -X_1e^{-r(T_1-t)}N(a_2), \quad (3.4.37a)$$

where

$$a_2 = \frac{\ln \frac{S}{\tilde{S}_{T_1}} + (r - \frac{\sigma^2}{2})(T_1 - t)}{\sigma\sqrt{T_1 - t}}. \quad (3.4.37b)$$

Here, $X_1 e^{-r(T_1-t)} N(a_2)$ represents the present value of the expected payment at time T_1 conditional on the first call being exercised.

If we define the random variables Y_1 and Y_2 to be the logarithm of the price ratios $\frac{S_{T_1}}{S}$ and $\frac{S_{T_2}}{S}$, respectively, then Y_1 and Y_2 are Brownian increments over the overlapping intervals $[t, T_1]$ and $[t, T_2]$. The second term in Eq. (3.4.36) can be expressed as

$$\begin{aligned} \text{2nd term} &= -X_2 e^{-r(T_2-t)} E_Q[\mathbf{1}_{\{S_{T_1} \geq \tilde{S}_{r_1}\}} \mathbf{1}_{\{S_{T_2} \geq X_2\}}] \\ &= -X_2 e^{-r(T_2-t)} Q \left[Y_1 \geq \ln \frac{\tilde{S}_{T_1}}{S}, Y_2 \geq \ln \frac{X_2}{S} \right]. \end{aligned} \quad (3.4.38)$$

To evaluate the above probability, it is necessary to find the joint density function of Y_1 and Y_2 . The correlation coefficient between Y_1 and Y_2 is found to be [see Eq. (2.3.18c)]

$$\rho = \sqrt{\frac{T_1 - t}{T_2 - t}}. \quad (3.4.39)$$

The Brownian increments, Y_1 and Y_2 , are bivariate normally distributed. Their respective mean are $\left(r - \frac{\sigma^2}{2}\right)(T_1 - t)$ and $\left(r - \frac{\sigma^2}{2}\right)(T_2 - t)$, and respective variance are $\sigma^2(T_1 - t)$ and $\sigma^2(T_2 - t)$, while the correlation coefficient ρ is given by Eq. (3.4.39). Suppose we define the standard normal random variables Y'_1 and Y'_2 by

$$Y'_1 = \frac{Y_1 - \left(r - \frac{\sigma^2}{2}\right)(T_1 - t)}{\sigma\sqrt{T_1 - t}} \quad \text{and} \quad Y'_2 = \frac{Y_2 - \left(r - \frac{\sigma^2}{2}\right)(T_2 - t)}{\sigma\sqrt{T_2 - t}},$$

and let

$$b_2 = \frac{\ln \frac{S}{X_2} + \left(r - \frac{\sigma^2}{2}\right)(T_2 - t)}{\sigma\sqrt{T_2 - t}}, \quad (3.4.40)$$

then we can express the second term in the following form

$$\begin{aligned} \text{2nd term} &= -X_2 e^{-r(T_2-t)} \int_{-a_2}^{\infty} \int_{-b_2}^{\infty} \frac{1}{2\pi} \frac{1}{\sqrt{1-\rho^2}} \\ &\quad \exp\left(-\frac{y_1'^2 - 2\rho y_1' y_2' + y_2'^2}{2(1-\rho^2)}\right) dy_2' dy_1' \\ &= -X_2 e^{-r(T_2-t)} N_2(a_2, b_2; \rho), \end{aligned} \quad (3.4.41)$$

where $N_2(a_2, b_2; \rho)$ is the standard bivariate normal distribution function with correlation coefficient ρ . Note that $N_2(a_2, b_2; \rho)$ can be interpreted as the probability that $S_{T_1} > \tilde{S}_{T_1}$ at time T_1 and $S_{T_2} > X_2$ at time T_2 , given that

the asset price at time t equals S . Hence, $X_2 e^{-r(T_2-t)} N_2(a_2, b_2; \rho)$ represents the present value of expected payment made at time T_2 conditional on both calls being exercised.

Consider the first term in Eq. (3.4.36)

$$\text{1st term} = e^{-r(T_1-t)} \int_{\tilde{S}_{T_1}}^{\infty} S_{T_1} N(d_1) \psi(S_{T_1}; S) dS_{T_1}, \quad (3.4.42)$$

by following the analytic procedures outlined in Problem 3.27, we can show that

$$\text{1st term} = SN_2(a_1, b_1; \rho), \quad (3.4.43a)$$

where

$$a_1 = a_2 + \sigma \sqrt{T_1 - t} \quad \text{and} \quad b_1 = b_2 + \sigma \sqrt{T_2 - t}. \quad (3.4.43b)$$

Combining the above results, the price of a European call-on-a-call compound option is found to be

$$c(S, t) = SN_2(a_1, b_1; \rho) - X_2 e^{-r(T_2-t)} N_2(a_2, b_2; \rho) - X_1 e^{-r(T_1-t)} N(a_2), \quad (3.4.44)$$

where the parameter \tilde{S}_{T_1} contained in a_2 is obtained by solving Eq. (3.4.34).

In a similar manner, for a European put-on-a-put, we can deduce that its price is given by

$$\begin{aligned} p(S, t) &= e^{-r(T_1-t)} \int_0^{\tilde{S}_{T_1}} \{X_1 - [X_2 e^{-r(T_2-T_1)} N(-d_2) - S_{T_1} N(-d_1)]\} \\ &\quad \psi(S_{T_1}; S) dS_{T_1} \\ &= X_1 e^{-r(T_1-t)} N(-a_2) - X_2 e^{-r(T_2-t)} N_2(-a_2, -b_2; \rho) \\ &\quad + SN_2(-a_1, -b_1; \rho). \end{aligned} \quad (3.4.45)$$

Here, \tilde{S}_{T_1} is the critical value for S_{T_1} below which the first put is exercised at T_1 .

The compound option models are first used by Geske (1977) to find the value of an option on a firm's stock, where the firm is assumed to be defaultable. The firm consists of claims to future cash flows by the bondholders and the stock holders. When the firm defaults, the common stock entitles its holder to have the right but not the obligation to sell the entire firm to the bondholders for a strike price equal to the par value of the bond. Therefore, an option on a share of the common stock can be considered as a compound option since the stock received upon exercising the option can be visualized as an option on the firm value.

3.5 Beyond the Black-Scholes pricing framework

In the Black-Scholes-Merton option pricing framework, it is assumed that the portfolio composition changes continuously according to a dynamic hedging strategy at zero transaction costs. In the presence of transaction costs associated with buying and selling of the asset, the continuous portfolio adjustment required by the Black-Scholes-Merton model will lead to an infinite number of transactions and so infinite total transaction costs. As a hedger, one has to strike the balance between transaction costs required for rebalancing the portfolio and the implied costs of hedging errors. The presence of transaction costs implies that absence of arbitrage no longer leads to a single option price but rather a range of feasible prices. An option can be overpriced or underpriced up to the extent where the profit obtained by an arbitrageur is offset by the transaction costs. The proportional transaction costs model will be presented in Sec. 3.5.1.

Another assumption in the Black-Scholes-Merton framework is the continuity of the asset price path. Numerous empirical studies have revealed that asset price may jump discontinuously, say, due to the arrival of sudden news. In Sec. 3.5.2, we consider the jump-diffusion model proposed by Merton (1976). Under the assumption that the jump components are uncorrelated with the market (jump risks can be diversified away), we can derive the governing equation for the derivative price. Also, when the random jump arrivals follow a Poisson process and the logarithm of the jump ratio is normally distributed, it is possible to obtain closed form option price formulas under the jump-diffusion model.

The Black-Scholes-Merton model gives the option price as a function of volatility and quantifies the randomness in the asset price dynamics through a constant volatility parameter. Instead of computing the option price given the volatility value using the Black-Scholes price formula, we solve for the volatility from the observed market option price. The volatility value implied by an observed option price is called the *implied volatility*. If the option pricing model were perfect, the implied volatility would be the same for all option market prices. However, empirical studies have revealed that the implied volatilities depend on the strike price and the maturity of the options. Such phenomena are called the volatility smiles. Suppose the constant volatility assumption is relaxed, we can model volatility either as a deterministic volatility function (local volatility) or as a mean-reverting stochastic process. By following the local volatility approach, we derive the Dupire equation that governs option prices with maturity and strike price as independent variables. Issues of implied volatilities and local volatilities will be discussed in Sec. 3.5.3.

The local volatility model is in general too restrictive to describe the behaviors of volatility variations. In the literature, there have been extensive research efforts to develop different types of volatility models. In the stochastic volatility models, volatility itself is modeled as a mean reverting Ito process

(Hull White, 1987). The mean reverting characteristics of volatility agree with our intuition that the level of volatility should revert to its mean level of its long-run distribution. The pricing of options under the stochastic volatility assumption is quite challenging. When the asset price process and the volatility process are uncorrelated, it can be shown that the price of a European option is the Black-Scholes price integrated over the probability distribution of the average variance rate for the remaining life of the option. However, when the two processes are correlated, the analytic solution can be obtained via the Fourier transform method (Heston, 1993). The book by Fouque *et al.* (2000) provides a comprehensive review of different aspects of stochastic volatility. Another class of volatility models which have gained popularity in recent years are the family of GARCH (generalized autoregressive conditional heteroskedasticity) models (Duan, 1995). In the GARCH models, the variance rate at current time step is a weighted average of a constant long-run average variance rate, the variance rate at the previous time steps and the most recent information about the variance rate. Continued research efforts are directed to explore better volatility models to explain the volatility smile and extract useful market information from the smile itself.

3.5.1 Transaction costs models

How to construct hedging strategy that best replicates the payoff of a derivative security in the presence of transaction costs? Recall that one can create a portfolio containing Δ units of the underlying asset and a certain amount of riskless asset in the form of money market account which replicates the payoff of the option. By the portfolio replication argument, the value of an option is equal to the initial cost of setting up the replicating portfolio which mimics the payoff of the option. The replicating procedure may require changing risky asset into money market account or vice versa continuously in a self-financing manner. Leland (1985) proposes a modification to the Black-Scholes model where the portfolio is adjusted at regular time intervals so that the total transaction costs of the replicating strategy is bounded. His model further assumes proportional transaction costs where the costs in buying and selling the asset are proportional to the monetary value of the transaction. Let k denote the round trip transaction cost per unit dollar of transaction. Suppose α units of assets are bought ($\alpha > 0$) or sold ($\alpha < 0$) at the price S , then the transaction cost is given by $\frac{k}{2} |\alpha| S$ in either buying or selling.

In the following proportional transaction costs option pricing model (Leland, 1985; Whalley and Wilmott, 1993), the asset price dynamics is assumed to follow the lognormal distribution where the volatility is taken to be constant. Also, we assume that the underlying asset pays no dividends during the life of the option. We consider a hedged portfolio of the writer of the option, where he is shorting one unit of option and long holding Δ units of the underlying asset. The value of this hedged portfolio at time t is given by

$$\Pi(t) = -V(S, t) + \Delta S, \quad (3.5.1)$$

where $V(S, t)$ is the value of the option and S is the asset price at time t . Let δt denote the fixed and finite time interval between successive revisions of the portfolio. After one time interval δt , the change in the value of the portfolio is

$$\delta \Pi = -\delta V + \Delta \delta S - \frac{k}{2} |\delta \Delta| S, \quad (3.5.2)$$

where δS is the change in asset price and $\delta \Delta$ is the change in the number of units of asset held in the portfolio. By Ito's lemma, the change in option value in time δt is given by

$$\delta V = \frac{\partial V}{\partial S} \delta S + \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) \delta t. \quad (3.5.3)$$

In order to cancel stochastic terms in Eqs. (3.5.2, 3.5.3), one chooses $\Delta = \frac{\partial V}{\partial S}$ so as to hedge against the risk due to stock price fluctuation. The change in the number of units of asset in time δt is given by

$$\delta \Delta = \frac{\partial V}{\partial S}(S + \delta S, t + \delta t) - \frac{\partial V}{\partial S}(S, t). \quad (3.5.4)$$

By Ito's lemma, the leading order of $|\delta \Delta|$ is found to be

$$|\delta \Delta| \approx \sigma S \left| \frac{\partial^2 V}{\partial S^2} \right| |\delta Z|, \quad (3.5.5)$$

where $\delta Z = \tilde{x} \sqrt{\delta t}$, \tilde{x} is the standard normal variable. It can be shown that the expectation of the reflected Brownian process $|\delta Z|$ is given by

$$E(|\delta Z|) = \sqrt{\frac{2}{\pi}} \sqrt{\delta t} \quad (3.5.6)$$

(see Problem 2.25). This hedged portfolio should expect to earn a return from this hedged portfolio same as that of a riskless deposit. This gives

$$E[\delta \Pi] = r \left(-V + \frac{\partial V}{\partial S} S \right) \delta t. \quad (3.5.7a)$$

By putting all the above results together, Eq. (3.5.7a) can be rewritten as

$$\left(-\frac{\partial V}{\partial t} - \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} - \frac{k}{2} \sigma S^2 \sqrt{\frac{2}{\pi \delta t}} \left| \frac{\partial^2 V}{\partial S^2} \right| \right) \delta t = r \left(-V + \frac{\partial V}{\partial S} S \right) \delta t. \quad (3.5.7b)$$

If we define the Leland number to be $Le = \sqrt{\frac{2}{\pi}} \left(\frac{k}{\sigma \sqrt{\delta t}} \right)$, we obtain

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\sigma^2}{2} Le S^2 \left| \frac{\partial^2 V}{\partial S^2} \right| + rS \frac{\partial V}{\partial S} - rV = 0, \quad (3.5.8)$$

which is the governing equation of the *proportional transaction costs model*. Note that the additional term $\frac{\sigma^2}{2} Le S^2 \left| \frac{\partial^2 V}{\partial S^2} \right|$ is non-linear, except when $\Gamma = \frac{\partial^2 V}{\partial S^2}$ does not change sign for all S . Hence, the above equation is in general non-linear. Since Γ represents the degree of mishedging of the portfolio, it is not surprising to observe that Γ is involved in the transaction costs term. One may rewrite Eq. (3.5.8) in the form which resembles the Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{\tilde{\sigma}^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0, \quad (3.5.9)$$

where the modified volatility is given by

$$\tilde{\sigma}^2 = \sigma^2 [1 + Le \operatorname{sign}(\Gamma)]. \quad (3.5.10)$$

Equation (3.5.9) becomes mathematically ill-posed when $\tilde{\sigma}^2$ becomes negative. This occurs when $\Gamma < 0$ and $Le > 1$. However, it is known that Γ is always positive for the simple European call and put options in the absence of transaction costs. If we postulate the same sign behavior for Γ in the presence of transaction costs, then $\tilde{\sigma}^2 = \sigma^2(1 + Le) > \sigma^2$. Now, Eq. (3.5.9) becomes linear under such assumptions so that the Black-Scholes formulas become applicable except that the modified volatility $\tilde{\sigma}$ should be used as the volatility parameter. We can deduce $V(S, t)$ to be an increasing function of Le since we expect a higher option value for high value of modified volatility. Financially speaking, the more frequent the rebalancing (smaller δt) the higher the transaction costs and so the writer of an option should charge higher for the price of the option. Let $V(S, t; \tilde{\sigma})$ and $V(S, t; \sigma)$ denote the option values obtained from the Black-Scholes formula with volatility values $\tilde{\sigma}$ and σ , respectively. The total transaction costs associated with the replicating strategy is then given by

$$\mathcal{T} = V(S, t; \tilde{\sigma}) - V(S, t; \sigma). \quad (3.5.11)$$

When Le is small, \mathcal{T} can be approximated by

$$\mathcal{T} \approx \frac{\partial V}{\partial \sigma} (\tilde{\sigma} - \sigma), \quad (3.5.12)$$

where $\tilde{\sigma} - \sigma \approx \frac{k}{\sqrt{2\pi\delta t}}$. Note that $\frac{\partial V}{\partial \sigma}$ is the same for both call and put options and the vega value is given by Eqs (3.3.32–33). Hence, for $Le \ll 1$, the total transaction costs for either a call or a put is approximately given by

$$\mathcal{T} \approx \frac{kSe^{-\frac{d_1^2}{2}}}{2\pi} \sqrt{\frac{T-t}{\delta t}}, \quad (3.5.13)$$

where $T - t$ is the time to expiry and d_1 is defined in Eq. (3.3.10b).

Rehedging at regular time intervals is one of the many possible hedging strategies. The natural question is: How would we characterize the optimality condition of a given hedging strategy? The usual approach is to define an appropriate utility function, which is used as the reference for which optimization is being taken. For the discussion of utility-based hedging strategies in the presence of transaction costs, one may refer to the papers by Hodges and Neuberger (1989), Davis *et al.* (1993). In their models, they attempt to find optimal portfolio policies which maximize expected utility over an infinite horizon. Neuberger (1994) shows that it is possible to use arbitrage strategies to set tight and preference-free bounds on option prices in the presence of transaction costs when the underlying asset follows a pure jump process. Other aspects of option pricing models with transaction costs are discussed in the works by Bensaid *et al.* (1992) and Grannan and Swindle (1996).

3.5.2 Jump-diffusion models

In the Black-Scholes option pricing model, we assume that trading takes place continuously in time and the asset price dynamics has a continuous sample path. There have been numerous empirical studies on asset price dynamics that show occasional jumps in asset price. Such jumps may reflect the arrival of new important information on the firm or its industry or economy as a whole.

Merton (1976) initiates the modeling of the asset price dynamics by a combination of normal fluctuation and abnormal jumps. The normal fluctuation is modeled by the Geometric Brownian process and the associated sample paths are continuous. The jumps are modeled by Poisson distributed events where their arrivals are assumed to be independent and identically distributed with intensity λ . That is, the probability that a jump event occurs over the time interval $(t, t + dt)$ is equal to λdt . We may define the Poisson process dq by

$$dq = \begin{cases} 0 & \text{with probability } 1 - \lambda dt \\ 1 & \text{with probability } \lambda dt \end{cases}. \quad (3.5.14)$$

Here, λ is interpreted as the mean number of arrivals per unit time.

Let J denote the jump ratio of the asset price upon the arrival of a jump event, that is, S jumps immediately to JS when $dq = 1$. The jump ratio itself is a random quantity with density function f_J . For example, suppose we assume $\ln J$ to be a Gaussian distribution with mean μ_J and variance σ_J^2 , then

$$E[J - 1] = \exp\left(\mu_J + \frac{\sigma_J^2}{2}\right) - 1. \quad (3.5.15)$$

Assume that the asset price dynamics is a combination of the Geometric Brownian motion and the Poisson jump process, the asset price process is then governed by

$$\frac{dS}{S} = \rho dt + \sigma dZ + (J - 1) dq, \quad (3.5.16)$$

where ρ and σ are the drift rate and volatility of the Geometric Brownian motion, respectively. The change in stock price upon the arrival of a jump event is $(J - 1)S$.

Imagine that a writer of an option follows the Black-Scholes hedging strategy, where he is long holding Δ units of the underlying asset and shorting one unit of the option. Let $V(S, t)$ denote the price function of the option. The portfolio value Π and its differential $d\Pi$ are given by

$$\Pi = \Delta S - V(S, t) \quad (3.5.17a)$$

and

$$\begin{aligned} d\Pi = & - \left(\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} \right) dt + \left(\Delta - \frac{\partial V}{\partial S} \right) (\rho S dt + \sigma S dZ) \\ & + \{ \Delta(J - 1)S - [V(JS, t) - V(S, t)] \} dq. \end{aligned} \quad (3.5.17b)$$

There are two sources of risk, and they come from the diffusion component dZ and the jump component dq . To hedge the diffusion risk, we may choose $\Delta = \frac{\partial V}{\partial S}$, like the usual Black-Scholes hedge ratio. How about the jump risk? Merton (1976) argues that if the jump component is firm specific and uncorrelated with the market (non-systematic risk), then the jump risk should not be priced into the option. In this case, the beta (from the Capital Asset Pricing model) of the portfolio is zero. Since the expected return on all zero-beta securities is equal to the riskless interest rate, we then have

$$E_J[d\Pi] = r\Pi dt, \quad (3.5.18)$$

where the expectation E_J is taken over the jump ratio J . Combining with Eq. (3.5.17b), we obtain the following governing equation of the option price function $V(S, t)$ under the jump-diffusion asset price process

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + (r - \lambda E_J[J - 1]) S \frac{\partial V}{\partial S} - rV \\ + \lambda E_J[V(JS, t) - V(S, t)] = 0. \end{aligned} \quad (3.5.19)$$

To solve for $V(S, t)$, one has to specify the distribution for J . Let $k = E_J[J - 1]$ and define the random variable X_n which has the same distribution as the product of n independent and identically distributed random variables, each identically distributed to J (with $X_0 = 1$). We write $V_{BS}(S, t)$ as the Black-Scholes price function of the same option contract. Merton (1976) shows that the representation of the solution to Eq. (3.5.19) in terms of expectations is given by

$$V(S, \tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda\tau} (\lambda\tau)^n}{n!} \{ E_{X_n} [V_{BS}(SX_n e^{-\lambda k\tau}, \tau)] \}, \quad \tau = T - t, \quad (3.5.20)$$

where E_{X_n} is the expectation over the distribution of X_n . The hints of the proof of the above representation are given in Problem 3.33.

In general, it is not easy to obtain closed form price formula for options under the jump-diffusion models, except for a few exceptions. When the jump ratio J follows the lognormal distribution, it is possible to obtain a closed form price formula for a European call option (see Problem 3.34). Also, Das and Foresi (1996) obtain closed form price formulas for bond and option prices when the interest rate follows the jump-diffusion models.

3.5.3 Implied and local volatilities

The option prices obtained from the Black-Scholes pricing framework are functions of five parameters: asset price S , strike price X , interest rate r , time to expiry τ and volatility σ . Except for the volatility parameter, the other four parameters are observable quantities. The difficulties of setting volatility value in the valuation formulas lie in the fact that the input value should be the forecast value over the remaining life of the option rather than an estimated volatility value (*historical volatility*) from the past market data of the asset price. The volatility value implied by an observed market option price (*implied volatility*) indicates a consensual view about the volatility level determined by the market. In particular, several implied volatility values obtained simultaneously from different options on the same underlying asset provide an extensive market view about the volatility of the stochastic movement of that asset. Such information may be useful for a trader to set the volatility value for the underlying asset of an option that he is interested in. In financial markets, it becomes a common practice for traders to quote an option's market price in terms of implied volatility, σ_{imp} . In essence, σ_{imp} becomes a convenient means of quoting option prices.

Since σ cannot be solved explicitly in terms of S , X , r , τ and option price V from the pricing formulas, the determination of the implied volatility must be accomplished by an iterative algorithm as commonly performed for the root-finding procedure for a non-linear equation. Since the option price is known to be an increasing function of the volatility value [see Eqs. (3.3.32–33)], the iterative algorithm becomes much simplified. We discuss the *Newton-Raphson method* for the calculation of the implied volatility.

Applied to the implied volatility calculations, the Newton-Raphson iterative scheme is given by

$$\sigma_{n+1} = \sigma_n - \frac{V(\sigma_n) - V_{market}}{V'(\sigma_n)}, \quad (3.5.21)$$

where σ_n denotes the n th iterative for σ_{imp} . Provided that the first iterate σ_1 is properly chosen, the limit of the sequence $\{\sigma_n\}$ converges to the unique solution σ_{imp} . The Newton-Raphson method enjoys its popularity due to its quadratic rate of convergence property, that is, $\sigma_{n+1} - \sigma_{imp} = K(\sigma_n - \sigma_{imp})^2$

for some K independent of n . Equation (3.5.21) may be rewritten in the following form

$$\frac{\sigma_{n+1} - \sigma_{imp}}{\sigma_n - \sigma_{imp}} = 1 - \frac{V(\sigma_n) - V(\sigma_{imp})}{\sigma_n - \sigma_{imp}} \frac{1}{V'(\sigma_n)} = 1 - \frac{V'(\sigma_n^*)}{V'(\sigma_n)}, \quad (3.5.22)$$

where σ_n^* lies between σ_n and σ_{imp} , by virtue of the Mean Value Theorem in calculus. Manaster and Koehler (1982) propose to choose the first iterate σ_1 such that $V'(\sigma)$ is maximized by $\sigma = \sigma_1$. Recall from Eq. (3.3.32) that

$$V'(\sigma) = \frac{S\sqrt{\tau} e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}} > 0 \quad \text{for all } \sigma, \quad (3.5.23a)$$

and so

$$V''(\sigma) = \frac{S\sqrt{\tau} d_1 d_2 e^{-\frac{d_1^2}{2}}}{\sqrt{2\pi}\sigma} = \frac{V'(\sigma) d_1 d_2}{\sigma}, \quad (3.5.23b)$$

where d_1 and d_2 are defined in Eq. (3.3.10b). Therefore, the critical points of the function $V'(\sigma)$ are given by $d_1 = 0$ and $d_2 = 0$, which lead respectively to

$$\sigma^2 = -2 \frac{\ln \frac{S}{X} + r\tau}{\tau} \quad \text{and} \quad \sigma^2 = 2 \frac{\ln \frac{S}{X} + r\tau}{\tau}. \quad (3.5.24)$$

The above two values of σ^2 both give $V'''(\sigma) < 0$. Hence, we can choose the first iterate σ_1 to be

$$\sigma_1 = \sqrt{\left| \frac{2}{\tau} \left(\ln \frac{S}{X} + r\tau \right) \right|}, \quad (3.5.25)$$

which satisfies the requirement of maximizing $V'(\sigma)$. Setting $n = 1$ in Eq. (3.5.22), we then have

$$0 < \frac{\sigma_2 - \sigma_{imp}}{\sigma_1 - \sigma_{imp}} < 1. \quad (3.5.26a)$$

In general, suppose we can establish (see Problem 3.36)

$$0 < \frac{\sigma_{n+1} - \sigma_{imp}}{\sigma_n - \sigma_{imp}} < 1, \quad n \geq 1, \quad (3.5.26b)$$

then the sequence $\{\sigma_n\}$ is monotonic and bounded and so $\{\sigma_n\}$ converges to the unique solution σ_{imp} . In conclusion, if we start with the first iterate σ_1 given by Eq. (3.5.25), then the sequence $\{\sigma_n\}$ generated by Eq. (3.5.21) will converge to σ_{imp} monotonically with a quadratic rate of convergence.

Volatility smiles

The Black-Scholes model assumes a lognormal probability distribution of the asset price at future time. Since volatility is the only unobservable parameter in the Black-Scholes model, the model gives the option price as a function of

volatility. It is more interesting to examine the dependence of volatility on the strike price.

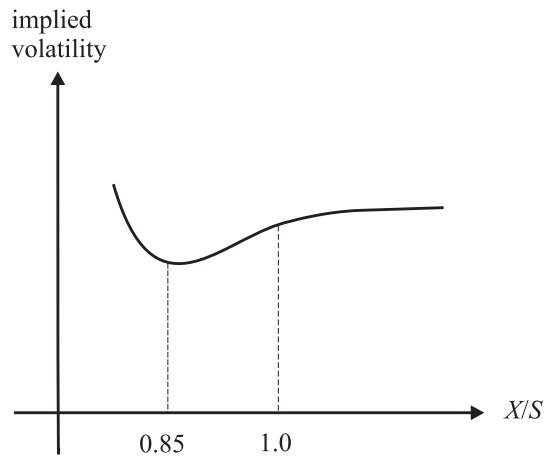


Fig. 3.6a A typical pattern of pre-crash smile. The implied volatility curve is convex with a dip.

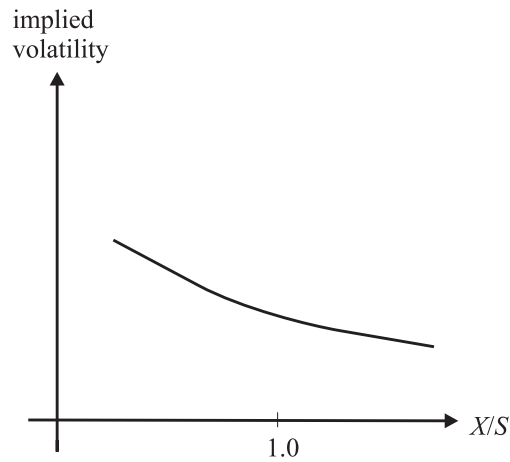


Fig. 3.6b A typical pattern of post-crash smile. The implied volatility drops against X/S , indicating that out-of-the-money puts ($X/S < 1$) are traded at higher implied volatility than out-of-the-money calls ($X/S > 1$).

If we plot the implied volatility of exchange-traded options, like index options, against their strike price for fixed maturity, the curve is typically

convex in shape, rather than a straight horizontal line as suggested by the simple Black-Scholes model. This phenomenon is commonly called the *volatility smile* by market practitioners. The general behaviors of these smiles differ depending on whether the market data were taken before or after the October, 1987 market crash. Figs. 3.6a,b show the shapes of typical pre-crash smile and post-crash smile of exchange-traded European index options. The implied volatility values are obtained by averaging options of different maturities.

In real market situation, it is a common occurrence that when the asset price is high, volatility tends to decrease, making it less probable for higher asset price to be realized. On the other hand, when the asset price is low, volatility tends to increase, that is, it is more probable that the asset price plummets further down. Suppose we plot the true probability distribution of the asset price and compare with the lognormal distribution, one observes that the left-hand tail of the true distribution is thicker than that of the lognormal one, while the reverse situation occurs at the right-hand tail (see Fig. 3.7).

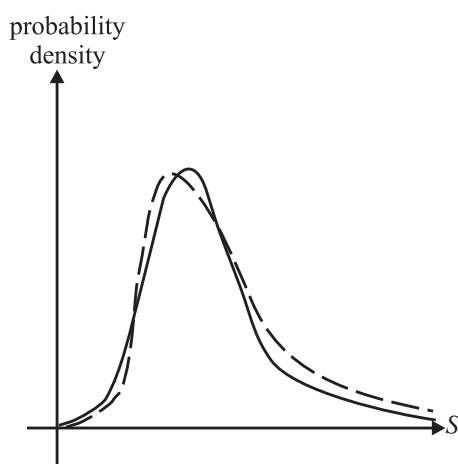


Fig. 3.7 Comparison of the true probability density of asset price (solid curve) implied from market data and the lognormal distribution (dotted curve). The true probability density is thicker at the left tail and thinner at the right tail.

As reflected from the implied probabilities calculated from the market data of option prices, this market behavior of higher probability of large decline in stock index is better known to market practitioners after the October, 1987 market crash. In other words, the market price of out-of-the-money calls (puts) became cheaper (more expensive) than the Black-Scholes price after

the 1987 crash because of the thickening (thinning) of the left- (right-) hand tail of the true probability distribution. In common market situation, out-of-the-money stock index puts are traded at higher implied volatilities than out-of-the-money stock index calls.

Local volatility

Instead of introducing stochastic volatility which requires assumptions about investor's risk preferences, one may choose to stay within the Black-Scholes one-factor diffusion framework but allowing the volatility function to be time or state dependent or both. If the volatility function is assumed to be time dependent, it can be shown easily that $\sigma(T)$ can be deduced from the known information of implied volatility $\sigma_{imp}(t, T)$ that is available for all $T > t$ (see Problem 3.37). Now, suppose European option prices at all strikes and maturities are available so that $\sigma_{imp}(t, T; X)$ can be computed, can we find a state-time dependent volatility function $\sigma(S, t)$ that gives the theoretical Black-Scholes option prices which are consistent with the market option prices. In the literature, $\sigma(S, t)$ is called the *local volatility function*.

Given that market European option prices are all available, Breeden and Litzenberger (1979) show that the risk neutral probability distribution of the asset price can be recovered. Let $\psi(S_T, T; S_t, t)$ denote the transition density function of the asset price. Conditional on $S_t = S$ at time t , the price at time t of a European call with maturity date T and strike price X is given by

$$c(S, t; X, T) = e^{-r(T-t)} \int_X^\infty (S_T - X) \psi(S_T, T; S_t, t) dS_T. \quad (3.5.27)$$

If we differentiate c with respect to X , we obtain

$$\frac{\partial c}{\partial X} = -e^{-r(T-t)} \int_X^\infty \psi(S_T, T; S_t, t) dS_T; \quad (3.5.28)$$

and differentiate once more, we have

$$\psi(X, T; S_t, t) = e^{r(T-t)} \frac{\partial^2 c}{\partial X^2}. \quad (3.5.29)$$

The above equation indicates that the transition density function can be inferred completely from the market prices of options with the same maturity and different strikes, without knowing the volatility function.

The Black-Scholes equation that governs the European call price can be considered as a backward equation since it involves the backward state and time variables. Can we find the forward version of the option pricing equation that involves the forward state variables? Such equation does exist, and it is commonly known as the Dupire equation (Dupire, 1994).

Assuming that the asset price dynamics under the risk neutral measure is governed by

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma(S_t, t) dZ \quad (3.5.30)$$

and write the call price function in the form of $c = c(X, T)$, the Dupire equation takes the form

$$\frac{\partial c}{\partial T} = -qc - (r - q)X \frac{\partial c}{\partial X} + \frac{\sigma^2(X, T)}{2} X^2 \frac{\partial^2 c}{\partial X^2}. \quad (3.5.31)$$

The Black-Scholes equation and Dupire equation somewhat resemble a pair of backward and forward Fokker-Planck equations.

To derive the Dupire equation, we start with the differentiation with respect to T of Eq. (3.5.29) to obtain

$$\frac{\partial \psi}{\partial T} = e^{r(T-t)} \left(r \frac{\partial^2 c}{\partial X^2} + \frac{\partial^2}{\partial X^2} \frac{\partial c}{\partial T} \right). \quad (2.5.32)$$

Recall that $\psi(X, T; S, t)$ satisfies the forward Fokker-Planck equation, where

$$\frac{\partial \psi}{\partial T} = \frac{\partial^2}{\partial X^2} \left[\frac{\sigma^2(X, T)}{2} X^2 \psi \right] - \frac{\partial}{\partial X} [(r - q)X \psi]. \quad (3.5.33)$$

Combining Eqs. (3.5.29,32,33) and eliminating the common factor $e^{r(T-t)}$, we have

$$\begin{aligned} & r \frac{\partial^2 c}{\partial X^2} + \frac{\partial^2}{\partial X^2} \frac{\partial c}{\partial T} \\ &= \frac{\partial^2}{\partial X^2} \left[\frac{\sigma^2(X, T)}{2} X^2 \frac{\partial^2 c}{\partial X^2} \right] - \frac{\partial}{\partial X} \left[(r - q)X \frac{\partial^2 c}{\partial X^2} \right]. \end{aligned} \quad (3.5.34)$$

Integrating the above equation with respect to X twice, we obtain

$$\begin{aligned} & \frac{\partial c}{\partial T} + rc + (r - q) \left(X \frac{\partial c}{\partial X} - c \right) \\ &= \frac{\sigma^2(X, T)}{2} X^2 \frac{\partial^2 c}{\partial X^2} + \alpha(T)X + \beta(T), \end{aligned} \quad (3.5.35)$$

where $\alpha(T)$ and $\beta(T)$ are arbitrary functions of T . Since all functions involving c in the above equation vanish as X tends to infinity, hence $\alpha(T)$ and $\beta(T)$ must be zero. Grouping the remaining terms in the equation, we obtain the Dupire equation. Alternatively, we may express the local volatility $\sigma(X, T)$ explicitly in terms of the call price function and its derivatives, where

$$\sigma^2(X, T) = \frac{2 \left[\frac{\partial c}{\partial T} + qc + (r - q)X \frac{\partial c}{\partial X} \right]}{X^2 \frac{\partial^2 c}{\partial X^2}}. \quad (3.5.36)$$

Suppose a sufficiently large number of market option prices are available at many maturities and strikes, we can estimate the local volatility from the above equation by approximating the derivatives of c with respect to X

and T using the market data. However, in real market conditions, market prices of options are available only at limited number of maturities and strikes. Given a finite number of market option prices, how to construct a discrete binomial tree that simulates the asset price movement based on the one-factor local volatility assumption? Unlike the constant volatility binomial tree, the implied binomial tree will be distorted in shape. The upward and downward moves and their associated probabilities are determined by an induction procedure such that the implied tree gives the numerical estimated option prices that agree with the observed option prices. In other words, the tree structure is *implied* by the market data. Unfortunately, the number of nodes in the binomial tree is in general far more than the number of available option prices. This would cause numerical implementation of the implied binomial tree highly unstable. For a discussion of the implied tree techniques, one may read the pioneering papers by Derman and Kani (1994) and Rubinstein (1994).

3.6 Problems

- 3.1** Consider a forward contract on an underlying commodity, find the portfolio consisting of the underlying commodity and bond (bond's maturity coincides with forward's maturity) that replicates the forward contract. Show that the hedge ratio Δ is always equal to one. Give the financial argument to justify why the hedge ratio is one. Let $B(t, T)$ denote the price at current time t of the unit-par zero-coupon bond maturing at time T and S denote the price of commodity at time t . Show that the forward price $F(S, \tau)$ is given by

$$F(S, \tau) = S/B(t, T), \quad \tau = T - t.$$

- 3.2** Consider a portfolio containing Δ units of asset and M dollars of riskless asset in the form of money market account. The portfolio is dynamically adjusted so as to replicate an option. Let S and $V(S, t)$ denote the value of the underlying asset and the option, respectively. Let r denote the riskless interest rate and Π denote the value of the *self-financing* replicating portfolio. When the self-financing trading strategy is adopted, explain why

$$\Pi = \Delta S + M \quad \text{and} \quad d\Pi = \Delta dS + rM dt,$$

where r is the riskless interest rate. Here, the differential term $S d\Delta$ does not enter into $d\Pi$. Assume that the asset price dynamics follows the Geometric Brownian process:

$$\frac{dS}{S} = \rho dt + \sigma dZ.$$

Using the condition that the option value and the value of the replicating portfolio should match at all times, show that the number of units of asset held must be given by

$$\Delta = \frac{\partial V}{\partial S}.$$

How to proceed further in order to obtain the Black-Scholes equation for V ?

3.3 The following statement is quoted from Black's paper (1989):

"... the expected return on a warrant (call option) should depend on the risk of the warrant in the same way that a common stock's expected return depends on its risk ..."

Explain the meaning of the above statement in relation to the concept of risk neutrality.

3.4 Suppose the cost of carry of a commodity is b . Show that the governing differential equation for the price of the option on the commodity based on the Black-Scholes formulation is given by

$$\frac{\partial V}{\partial t} + \frac{\sigma^2}{2} S^2 \frac{\partial^2 V}{\partial S^2} + bS \frac{\partial V}{\partial S} - rV = 0,$$

where $V(S, t)$ is the price of the option, σ and r are the constant volatility and riskless interest rate respectively. Find the put-call parity relation for the price functions of the European put and call options on the commodity.

3.5 Let $B = B(t; T)$ be the price at current time t of a unit-par zero-coupon bond with maturity date T and $M = M(t)$ be the value of a money market account. Both B and M are riskless assets which grow at the riskless rate r , that is,

$$dB = rB dt \quad \text{and} \quad dM = rM dt,$$

where r can be stochastic. Show that the arbitrage price $V(t)$ of a derivative is equal to

(i) expected discounted payoff

$$V(t) = E_Q \left[V(T) \exp \left(- \int_t^T r(u) du \right) \right]$$

if the money market account is used as the numeraire; or

(ii) discounted expected payoff

$$V(t) = B(t; T) E_{Q^T} [V(T)]$$

if the bond price is used as the numeraire.

We usually call Q the risk neutral measure and Q^T the T -forward measure.

3.6 Suppose the price of an asset follows the diffusion process

$$dS = \mu(S, t) dt + \sigma(S, t) dZ.$$

Show that the corresponding governing equation for the price of a derivative security V contingent on the above asset takes the form

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S, t)\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,$$

where r is the riskless interest rate. Again, the derivative price V does not depend on the instantaneous mean $\mu(S, t)$ of the diffusion process.

3.7 Let the dynamics of the random variable $f(S)$ be governed by

$$dS = \mu(S, t) dt + \sigma(S, t) dZ.$$

For a twice differentiable function $f(S)$, the differential of $f(S)$ is given by

$$df = \left[\mu(S, t)\frac{\partial f}{\partial S} + \frac{\sigma^2(S, t)}{2}\frac{\partial^2 f}{\partial S^2} \right] dt + \sigma(S, t)\frac{\partial f}{\partial S} dZ.$$

We let $\psi(S, t; S_0, t_0)$ denote the transition density function of S at the future time t , conditional on value S_0 at an earlier time t_0 . By considering the time-derivative of the expected value of $f(S)$ and equating $\frac{d}{dt}E[f(S)]$ and $E\left[\frac{df(S)}{dt}\right]$, where

$$\frac{d}{dt}E[f(S)] = \int_{-\infty}^{\infty} f(\xi)\frac{\partial\psi}{\partial t}(\xi, t; S_0, t_0) d\xi \quad (\text{i})$$

$$E\left[\frac{df(S)}{dt}\right] = \int_{-\infty}^{\infty} \left[\mu(\xi, t)\frac{\partial f}{\partial \xi} + \frac{\sigma^2(\xi, t)}{2}\frac{\partial^2 f}{\partial \xi^2} \right] \psi(\xi, t; S_0, t_0) d\xi, \quad (\text{ii})$$

show that $\psi(S, t; S_0, t_0)$ is governed by the following forward Fokker-Planck equation:

$$\frac{\partial\psi}{\partial t} + \frac{\partial}{\partial S}[\mu(S, t)\psi] - \frac{\partial^2}{\partial S^2}\left[\frac{\sigma^2(S, t)}{2}\psi\right] = 0.$$

Hint: Perform parts integration of the integral in Eq. (ii).

3.8 To derive the backward Fokker-Planck equation, we consider

$$\psi(S, t; S_0, t_0) = \int_{-\infty}^{\infty} \psi(S, t; \xi, u)\psi(\xi, u; S_0, t_0) d\xi$$

where u is some intermediate time satisfying $t_0 < u < t$. Taking the differentiation with respect to u on both sides, we obtain

$$0 = \int_{-\infty}^{\infty} \frac{\partial \psi}{\partial u}(S, t; \xi, u) \psi(\xi, u; S_0, t_0) d\xi \\ + \int_{-\infty}^{\infty} \psi(S, t; \xi, u) \frac{\partial \psi}{\partial u}(\xi, u; S_0, t_0) d\xi.$$

From the forward Fokker-Planck equation derived in Problem 3.7, we obtain

$$\int_{-\infty}^{\infty} \frac{\partial \psi}{\partial u}(S, t; \xi, u) \psi(\xi, u; S_0, t_0) d\xi \\ = - \int_{-\infty}^{\infty} \left\{ - \frac{\partial}{\partial \xi} [\mu(\xi, u) \psi(\xi, u; S_0, t_0)] \right. \\ \left. + \frac{\partial^2}{\partial \xi^2} \left[\frac{\sigma^2(\xi, u)}{2} \psi(\xi, u; S_0, t_0) \right] \right\} \psi(S, t; \xi, u) d\xi.$$

By performing parts integration of the last integral and taking the limit $u \rightarrow t_0$, show that $\psi(S, t; S_0, t_0)$ satisfies

$$\frac{\partial \psi}{\partial t_0} + \mu(S_0, t_0) \frac{\partial \psi}{\partial S_0} + \frac{\sigma^2(S_0, t_0)}{2} \frac{\partial^2 \psi}{\partial S_0^2} = 0.$$

Hint: $\psi(\xi, u; S_0, t_0) \rightarrow \delta(\xi - S_0)$ as $u \rightarrow t_0$.

- 3.9** Let Q^* denote the equivalent martingale measure where the asset price S_t is used as the numeraire. Suppose S_t follows the lognormal distribution with drift rate r and volatility σ under Q^* , where r is the riskless interest rate. By using Eq. (3.2.11), show that

$$\frac{dQ^*}{dQ} = \frac{S_T}{S_0} e^{-rT} = e^{-\frac{\sigma^2}{2}T + \sigma Z_T},$$

where Q is the martingale measure with the money market account as the numeraire and Z_T is a Brownian motion under Q . Using the Girsanov Theorem, show that

$$Z_T^* = Z_T - \sigma T$$

is a Brownian motion under Q^* . Explain why

$$E_Q[\mathbf{1}_{\{S_T \geq X\}}] = N\left(\frac{\ln \frac{S_0}{X} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\right),$$

then deduce that [see Eq. (3.3.16b)]

$$E_{Q^*}[S_T \mathbf{1}_{\{S_T \geq X\}}] = e^{rT} S_0 N\left(\frac{\ln \frac{S_0}{X} + \left(r + \frac{\sigma^2}{2}\right) T}{\sigma \sqrt{T}}\right).$$

- 3.10** From the Black-Scholes price function $c(S, \tau)$ for a European vanilla call, show that the limiting values of the call price at vanishing volatility and infinite volatility are the lower and upper bounds of the European call price respectively, namely,

$$\lim_{\sigma \rightarrow 0^+} c(S, \tau) = \max(S - Xe^{-r\tau}, 0),$$

and

$$\lim_{\sigma \rightarrow \infty} c(S, \tau) = S.$$

Give the appropriate financial interpretation of the above results.

- 3.11** Show that when a European option is currently out-of-the-money, then higher volatility of the asset price or longer time to expiry makes it more likely for the option to expire in-the-money. What would be the impact on the value of delta? Do we have the same effect or opposite effect when the option is currently in-the-money? Also, give the financial interpretation of the asymptotic behaviors of the delta curves in Fig. 3.4 at the limits $\tau \rightarrow 0^+$ and $\tau \rightarrow \infty$.
- 3.12** Show that when the European call price is a convex function of the asset price, the elasticity of the call price is always greater than or equal to one. Give the financial argument to explain why the elasticity of the price of a European option increases in absolute value when the option becomes more out-of-the-money and closer to expiry. Can you think of a situation where the European put's elasticity has absolute value less than one, that is, the European put option is less riskier than the underlying asset?
- 3.13** Suppose the greeks of the value of a derivative security are defined by

$$\Theta = \frac{\partial f}{\partial t}, \quad \Delta = \frac{\partial f}{\partial S}, \quad \Gamma = \frac{\partial^2 f}{\partial S^2}.$$

- Find the relation between Θ and Γ for a delta-neutral portfolio where $\Delta = 0$.
 - Show that the theta may become positive for an in-the-money European call option on a continuous dividend paying asset when the dividend yield is sufficiently high.
 - Explain by financial argument why the theta value tends asymptotically to $-rXe^{-r\tau}$ from below when the asset value is sufficiently high.
- 3.14** Let $P_\alpha(\tau)$ denote the European put price normalized by the asset price, that is,

$$P_\alpha(\tau) = p(S, \tau)/S = \alpha e^{-r\tau} N(-d_-) - N(-d_+),$$

where

$$\begin{aligned} \gamma_- &= \frac{r - \frac{\sigma^2}{2}}{\sigma}, & \gamma_+ &= \frac{r + \frac{\sigma^2}{2}}{\sigma}, & \alpha &= \frac{X}{S}, & \beta &= \frac{\ln \frac{1}{\alpha}}{\sigma}, \\ d_- &= \gamma_- \sqrt{\tau} + \frac{\beta}{\sqrt{\tau}}, & d_+ &= \gamma_+ \sqrt{\tau} + \frac{\beta}{\sqrt{\tau}}. \end{aligned}$$

The derivative of $P_\alpha(\tau)$ is found to be

$$P'_\alpha(\tau) = \alpha e^{-r\tau} \left[-rN(-d_-) + n(-d_-) \frac{\sigma}{2\sqrt{\tau}} \right].$$

Define $f(\tau)$ by the relation: $P'_\alpha(\tau) = \alpha e^{-r\tau} f(\tau)$, and the quadratic polynomial $p_2(\tau)$ by

$$p_2(\tau) = \gamma_- \gamma_+ \tau^2 - [\beta(\gamma_- + \gamma_+) + 1]\tau + \beta^2.$$

Let τ_1 and τ_2 denote the two real roots of $p_2(\tau)$, where $\tau_1 < \tau_2$, and let $\tau_0 = \frac{\beta^2 \sigma}{2r\beta + \sigma}$. The sign behaviors of $P'_\alpha(\tau)$ exhibit the following properties (Dai and Kwok, 2005).

1. When $r \leq 0$, $P'_\alpha(\tau) > 0$ for all $\tau \geq 0$.
2. When $r > 0$ and $\beta \geq 0$ (equivalent to $S \geq X$), there exists unique $\tau^* > 0$ at which $P'_\alpha(\tau)$ changes sign, and that $P'_\alpha(\tau) > 0$ for $\tau < \tau^*$ and $P'_\alpha(\tau) < 0$ for $\tau > \tau^*$.
3. When $r > 0$ and $\beta < 0$ (equivalent to $S < X$), there are two possibilities:
 - (a) There may exist a time interval (τ_1^*, τ_2^*) such that $P'_\alpha(\tau) > 0$ when $\tau \in (\tau_1^*, \tau_2^*)$ and $P'_\alpha(\tau) \leq 0$ if otherwise. This occurs only when either one of the following conditions is satisfied.
 - (i) $\gamma_- < 0$ and $f(\tau_2) > 0$;
 - (ii) $\gamma_- > 0$, $\beta(\gamma_- + \gamma_+) + 1 > 0$, $\Delta = \beta^2 \sigma^2 + 1 + \frac{4\beta r}{\sigma} > 0$ and $f(\tau_1) > 0$;
 - (iii) $\gamma_- = 0$, $\beta(\gamma_- + \gamma_+) + 1 > 0$ and $f(\tau_0) > 0$.
 - (b) When none of the above conditions (i)–(iii) hold, then $P'_\alpha(\tau) \leq 0$ for all $\tau \geq 0$.

3.15 Show that the value of a European call option satisfies

$$c(S, \tau; X) = S \frac{\partial c}{\partial S}(S, \tau; X) + X \frac{\partial c}{\partial X}(S, \tau; X)$$

Hint: The call price function is a linear homogeneous function of S and X , that is,

$$c(\lambda S, \tau; \lambda X) = \lambda c(S, \tau; X).$$

- 3.16** Consider a European capped call option whose terminal payoff function is given by

$$c_M(S, 0; X, M) = \min(\max(S - X, 0), M),$$

where X is the strike price and M is the cap. Show that the value of the European capped call is given by

$$c_M(S, \tau; X, M) = c(S, \tau; X) - c(S, \tau; X + M),$$

where $c(S, \tau; X + M)$ is the value of a European vanilla call with strike price $X + M$.

- 3.17** Consider the value of a European call option written by an issuer whose only asset is α (< 1) units of the underlying asset. At expiration, the terminal payoff of this call is then given by

$$\begin{aligned} S_T - X & \text{ if } \alpha S_T \geq S_T - X \geq 0 \\ \alpha S_T & \text{ if } S_T - X > \alpha S_T \end{aligned}$$

and zero otherwise. Show that the value of this European call option is given by

$$c_L(S, \tau; X, \alpha) = c(S, \tau; X) - (1 - \alpha)c\left(S, \tau; \frac{X}{1 - \alpha}\right), \quad \alpha < 1,$$

where $c\left(S, \tau; \frac{X}{1 - \alpha}\right)$ is the value of a European vanilla call with strike price $\frac{X}{1 - \alpha}$.

- 3.18** Consider the price functions of European call and put options on an underlying asset which pays a dividend yield at the rate q , show that their deltas and thetas are given by

$$\begin{aligned} \frac{\partial c}{\partial S} &= e^{-q\tau} N(\hat{d}_1) \\ \frac{\partial p}{\partial S} &= e^{-q\tau} [N(\hat{d}_1) - 1] \\ \frac{\partial c}{\partial t} &= -\frac{Se^{-q\tau}\sigma N'(\hat{d}_1)}{2\sqrt{\tau}} + qSe^{-q\tau}N(\hat{d}_1) - rXe^{-r\tau}N(\hat{d}_2) \\ \frac{\partial p}{\partial t} &= -\frac{Se^{-q\tau}\sigma N'(\hat{d}_1)}{2\sqrt{\tau}} - qSe^{-q\tau}N(-\hat{d}_1) + rXe^{-r\tau}N(-\hat{d}_2) \end{aligned}$$

where \hat{d}_1 and \hat{d}_2 are given by Eq. (3.4.7b). Deduce the expressions for the gammas, vegas and rhos for the above call and put option prices.

- 3.19** Deduce the corresponding put-call parity relation when the parameters in the European option models are time dependent, namely, volatility

of the asset price is $\sigma(t)$, dividend yield is $q(t)$ and riskless interest rate is $r(t)$.

- 3.20** Explain why the option price should be continuous across a dividend date though the asset price experiences a jump. Using no-arbitrage principle, deduce the following jump condition:

$$V(S(t_d^+), t_d^+) = V(S(t_d^-), t_d^-)$$

where V denotes option price, t_d^- and t_d^+ denote the time just before and after the dividend date.

- 3.21** Suppose the dividends and interest incomes are taxed at the rate R but capital gains taxes are zero. Find the price formulas for the European put and call on an asset which pays a continuous dividend yield at the constant rate q , assuming that the riskless interest rate r is also constant.

Hint: Explain why the riskless interest rate r and dividend yield q should be replaced by $r(1 - R)$ and $q(1 - R)$, respectively, in the Black-Scholes formulas.

- 3.22** Consider a futures on an underlying asset which pays N discrete dividends between t and T and let D_i denote the amount of the i th dividend paid on the ex-dividend date t_i . Show that the futures price is given by

$$F(S, t) = Se^{r(T-t)} - \sum_{i=1}^N D_i e^{r(T-t_i)},$$

where S is the current asset price and r is the riskless interest rate. Consider a European call option on the above futures. Show that the governing differential equation for the price of the call, $c_F(F, t)$, is given by (Brenner *et al.*, 1985)

$$\frac{\partial c_F}{\partial t} + \frac{\sigma^2}{2} \left[F + \sum_{i=1}^N D_i e^{r(T-t_i)} \right]^2 \frac{\partial^2 c_F}{\partial F^2} - r c_F = 0.$$

- 3.23** A *forward start* option is an option which comes into existence at some future time T_1 and expires at T_2 ($T_2 > T_1$). The strike price is set equal the asset price at T_1 such that the option is at-the-money at the future option's initiation time T_1 . Consider a forward start call option whose underlying asset has value S at current time t and constant dividend yield q , show that the value of the forward start call is given by

$$e^{-qT_1} c(S, T_2 - T_1; S)$$

where $c(S, T_1 - T_1; S)$ is the value of an at-the-money call (strike price same as asset price) with time to expiry $T_2 - T_1$.

Hint: The value of an at-the-money call option is proportional to the asset price.

- 3.24** Show that the payoff function of a chooser option on the date of choice T_c can be alternatively decomposed into the following form:

$$\begin{aligned} V(S_{T_c}, T_c) &= \max(p + S_{T_c} e^{-q(T-T_c)} - X e^{-r(T-T_c)}, p) \\ &= p + e^{-q(T-T_c)} \max(S_{T_c} - X e^{-(r-q)(T-T_c)}, 0) \end{aligned}$$

[see Eq. (3.4.31)]. Find the alternative representation of the price formula of the chooser option based on the above decomposition. Show that your new formula agrees with that given by Eqs. (3.4.32a,b).

- 3.25** Suppose the holder of the chooser option can make the choice of either a call or a put at any time between now and a later cutoff date T_c . Is it optimal for the holder to make the choice at some time before T_c ?

Hint: Apparently, the price function of the chooser option depends on T_c [see Eqs. (3.4.32a,b)]. Check whether the price function is an increasing or decreasing function of T_c . Consider the extreme case where T_c coincides with the expiration date of the option.

- 3.26** Consider a chooser option which entitles the holder to choose, on the choice date T_c periods from now, whether the option is a European call with exercise price X_1 and time to expiration $T_1 - T_c$ or a European put with exercise price X_2 and time to expiration $T_2 - T_c$. Show that the price of the chooser option at the current time (taken to be time zero) is given by (Rubinstein, 1992)

$$\begin{aligned} &S e^{-qT_1} N_2(x, y_1; \rho_1) - X_1 e^{-rT_1} N_2(x - \sigma\sqrt{T_c}, y_1 - \sigma\sqrt{T_1}; \rho_1) \\ &- S e^{-qT_2} N_2(-x, -y_2; \rho_2) + X_2 e^{-rT_2} N_2(-x + \sigma\sqrt{T_c}, -y_2 + \sigma\sqrt{T_2}; \rho_2), \end{aligned}$$

where q is the continuous dividend yield of the underlying asset. The parameters are defined by

$$\begin{aligned} x &= \frac{\ln \frac{S}{X} + (r - q + \frac{\sigma^2}{2})T_c}{\sigma\sqrt{T_c}}, & \rho_1 &= \sqrt{\frac{T_c}{T_1}}, & \rho_2 &= \sqrt{\frac{T_c}{T_2}}, \\ y_1 &= \frac{\ln \frac{S}{X_1} + (r - q + \frac{\sigma^2}{2})T_1}{\sigma\sqrt{T_1}}, & y_2 &= \frac{\ln \frac{S}{X_2} + (r - q + \frac{\sigma^2}{2})T_2}{\sigma\sqrt{T_2}}. \end{aligned}$$

Here, X solves the following non-linear algebraic equation

$$\begin{aligned} &X e^{-q(T_1-T_c)} N(z_1) - X_1 e^{-r(T_1-T_c)} N(z_1 - \sigma\sqrt{T_1 - T_c}) \\ &+ X e^{-q(T_2-T_c)} N(-z_2) - X_2 e^{-r(T_2-T_c)} N(-z_2 + \sigma\sqrt{T_2 - T_c}) = 0, \end{aligned}$$

where

$$z_1 = \frac{\ln \frac{X}{X_1} + (r - q + \frac{\sigma^2}{2})(T_1 - T_c)}{\sigma \sqrt{T_1 - T_c}},$$

$$z_2 = \frac{\ln \frac{X}{X_2} + (r - q + \frac{\sigma^2}{2})(T_2 - T_c)}{\sigma \sqrt{T_2 - T_c}}.$$

Hint: The two overlapping standard Brownian increments $Z(T_c)$ and $Z(T_1)$ have joint normal distribution with zero means, unit variances and correlation coefficient $\sqrt{\frac{T_c}{T_1}}$, $T_c < T_1$.

3.27 Show that the first term in Eq. (3.4.36) can be expressed as

$$\begin{aligned} & e^{-r(T_1-t)} \int_{\tilde{S}_{T_1}}^{\infty} S_{T_1} N_1(d_1) \psi(S_{T_1}; S) dS_{T_1} \\ &= e^{-r(T_1-t)} \int_{\ln \tilde{S}_{T_1}}^{\infty} \int_{\ln X_2}^{\infty} \frac{1}{2\pi} \frac{1}{\sigma \sqrt{T_1-t}} \frac{1}{\sigma \sqrt{T_2-T_1}} S e^{r(T_1-t)} \\ & \quad \exp\left(-\frac{\left\{y - \left[\ln S + \left(r + \frac{\sigma^2}{2}\right)(T_1-t)\right]\right\}^2}{2\sigma^2(T_1-t)}\right) \\ & \quad \exp\left(-\frac{\left\{x - \left[y + \left(r + \frac{\sigma^2}{2}\right)(T_2-T_1)\right]\right\}^2}{2\sigma^2(T_2-T_1)}\right) dx dy, \end{aligned}$$

where $x = \ln S_{T_2}$ and $y = \ln S_{T_1}$. By comparing with the second term in Eq. (3.4.36), show that the above integral reduces to the result given in Eq. (3.4.43a,b).

Hint: The second term in Eq. (3.4.36) can be expressed as

$$\begin{aligned} & -X_2 e^{-r(T_2-t)} \int_{\ln \tilde{S}_{T_1}}^{\infty} \int_{\ln X_2}^{\infty} \frac{1}{2\pi} \frac{1}{\sigma \sqrt{T_1-t}} \frac{1}{\sigma \sqrt{T_2-T_1}} \\ & \quad \exp\left(-\frac{\left\{y - \left[\ln S + \left(r - \frac{\sigma^2}{2}\right)(T_1-t)\right]\right\}^2}{2\sigma^2(T_1-t)}\right) \\ & \quad \exp\left(-\frac{\left\{x - \left[y + \left(r - \frac{\sigma^2}{2}\right)(T_2-T_1)\right]\right\}^2}{2\sigma^2(T_2-T_1)}\right) dx dy. \end{aligned}$$

3.28 Explain why the sum of prices of the call-on-a-call and call-on-a-put is equal to the price of the call with expiration T_2 . Show that the price of a European call-on-a-put is given by

$$c(S, t) = X_2 e^{-r(T_2-t)} N_2(a_2, -b_2; -\rho) - S N_2(a_1, -b_1; -\rho) - e^{-r(T_1-t)} X_1 N(a_2),$$

where a_1, b_1, a_2 and b_2 are defined in Sec. 3.4.4.

Hint: Use the relation

$$N_2(a, b; \rho) + N_2(a, -b; -\rho) = N(a).$$

3.29 Find the valuation formulas for the following European compound options:

- (a) put-on-a-call option when the underlying asset pays a continuous dividend yield q ;
- (b) call-on-a-put option when the underlying asset is a futures;
- (c) put-on-a-put option when the underlying asset has a constant cost of carry b .

3.30 Consider a contingent claim whose value at maturity T is given by

$$\min(S_{T_0}, S_T),$$

where T_0 is some intermediate time before maturity, $T_0 < T$, and S_T and S_{T_0} are the asset price at T and T_0 , respectively. Show that the value of the contingent claim at time t is given by

$$V_t = S_t [N(d_1) - e^{-r(T-T_0)} N(d_2)],$$

where S_t is the asset price at time t and

$$d_1 = \frac{r(T - T_0) + \frac{\sigma^2}{2}(T - T_0)}{\sigma\sqrt{T - T_0}}, \quad d_2 = d_1 - \sigma\sqrt{T - T_0}.$$

3.31 Show that the total transaction costs in Leland's model (Leland, 1985) increases (decreases) with the strike price X when $X < X^*$ ($X > X^*$), where

$$X^* = S e^{(r + \frac{\sigma^2}{2})(T-t)}.$$

Hint: Use the result $\frac{\partial}{\partial X} \left(\frac{\partial V}{\partial \sigma} \right) = \frac{S}{\sqrt{2\pi(T-t)} \sigma} d_1 \exp\left(-\frac{d_1^2}{2}\right)$.

3.32 Suppose the transaction costs are proportional to the number of units of asset traded rather than the dollar value of the asset traded as in the original Leland's model. Find the corresponding governing equation for the price of a derivative based on this new transaction costs assumption.

3.33 By writing $P_n(\tau) = e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!}$ and $\tilde{S}_n = S X_n e^{-\lambda k\tau}$, and considering

$$V(S, \tau) = \sum_{n=0}^{\infty} P_n(\tau) E_{X_n} [V_{BS}(\tilde{S}_n, \tau)]$$

[see Eq. (3.5.20)], show that

$$\begin{aligned} \frac{\partial V}{\partial \tau} = & -\lambda V - \lambda k S \frac{\partial V}{\partial S} + \sum_{n=0}^{\infty} P_n(\tau) E_{X_n} \left[\frac{\partial V_{BS}}{\partial \tau}(\tilde{S}_n, \tau) \right] \\ & + \lambda \sum_{m=0}^{\infty} P_m(\tau) E_{X_{m+1}} [V_{BS}(\tilde{S}_{m+1}, \tau)]. \end{aligned}$$

Furthermore, by observing that

$$E_J[V(JS, \tau)] = \sum_{n=0}^{\infty} P_n(\tau) E_{X_{n+1}} [V_{BS}(\tilde{S}_{n+1}, \tau)],$$

show that $V(S, \tau)$ satisfies the governing equation (3.5.19). Also, show that $V(S, \tau)$ and $V_{BS}(S, \tau)$ satisfy the same terminal payoff condition.

- 3.34** Suppose $\ln J$ is normally distributed with standard deviation σ_J , show that the price of a European vanilla option under the jump-diffusion model can be expressed as (Merton, 1976)

$$V(S, \tau) = \sum_{n=0}^{\infty} \frac{e^{-\lambda' \tau} (\lambda' \tau)^n}{n!} V_{BS}(S, \tau; \sigma_n, r_n),$$

where

$$\lambda' = \lambda(1+k), \sigma_n^2 = \sigma^2 + \frac{n\sigma_J^2}{\tau} \quad \text{and} \quad r_n = r - \lambda k + \frac{n \ln(1+k)}{\tau}.$$

- 3.35** Consider the expression for $d\Pi$ given in Eq. (3.5.17b), show that the variance of $d\Pi$ is given by

$$\begin{aligned} \text{var}(d\Pi) = & \left(\Delta - \frac{\partial V}{\partial S} \right)^2 \sigma^2 S^2 dt \\ & + \lambda E_J [\{ \Delta(J-1)S - [V(JS, t) - V(S, t)] \}^2] dt. \end{aligned}$$

Suppose we try to hedge the diffusion and jump risks as much as possible by minimizing $\text{var}(d\Pi)$, show that this can be achieved by choosing Δ such that

$$\Delta = \frac{\lambda E_J [(J-1)\{V(JS, t) - V(S, t)\}] + \sigma^2 S \frac{\partial V}{\partial S}}{\lambda S E_J [(J-1)^2] + \sigma^2 S}.$$

With this choice of Δ , find the corresponding governing equation for the option price function under the jump-diffusion asset price dynamics.

- 3.36** Suppose $V(\sigma)$ is the option price function with volatility σ as the independent variable. Show that

$$V''(\sigma) = \frac{V'(\sigma)\tau}{4\sigma^3} (\sigma_1^4 - \sigma^4) \quad \text{for all } \sigma,$$

where σ_1 is given by Eq. (3.5.25). Hence, deduce that $V'' > 0$ if $\sigma_1 > \sigma_{imp}$ and $V'' < 0$ if $\sigma_1 < \sigma_{imp}$, where σ_{imp} is the implied volatility. Explain why $V(\sigma)$ is strictly convex if $\sigma_1 > \sigma_{imp}$ and strictly concave if $\sigma_1 < \sigma_{imp}$, and deduce that

$$0 < \frac{\sigma_{n+1} - \sigma_{imp}}{\sigma_n - \sigma_{imp}} < 1$$

for both cases (Manaster and Koehler, 1982).

- 3.37** Assume that the time dependent volatility function $\sigma(T)$ is deterministic. Suppose we write $\sigma_{imp}(t, T)$ as the implied volatility obtained from the time- t price of a European option with time- T maturity, for all $T > t$. Show that

$$\sigma(T) = \sqrt{\sigma_{imp}^2(t, T) + 2(T-t)\sigma_{imp}(t, T) \frac{\partial}{\partial T} \sigma_{imp}(t, T)}.$$

In real situation, we may have the implied volatility available at discrete times $T_i, i = 1, 2, \dots, N$. Assuming the volatility $\sigma(T)$ to be piecewise constant over each time interval $[T_{i-1}, T_i], i = 1, 2, \dots, N$, show that

$$\sigma(u) = \sqrt{\frac{(T_i - t)\sigma_{imp}^2(t, T_i) - (T_{i-1} - t)\sigma_{imp}^2(t, T_{i-1})}{T_i - T_{i-1}}}$$

for $T_{i-1} < u < T_i$.

Hint: The implied volatility σ_{imp} and local volatility σ are related by

$$\sigma_{imp}^2(t, T)(T-t) = \int_t^T \sigma^2(u) du.$$

- 3.38** We would like to compute $d(S_T - X)^+$, where S_t follows the Geometric Brownian process

$$\frac{dS_t}{S_t} = (r - q) dt + \sigma(S_t, t) dZ.$$

The function $(S_T - X)^+$ has a discontinuity at $S_T = X$. Rossi (2002) proposes to approximate $(S_T - X)^+$ by the following function $f(S_T)$ whose first derivative is continuous, where

$$f(S_T) = \begin{cases} 0 & \text{if } S_T < X \\ \frac{(S_T - X)^2}{2\epsilon} & \text{if } X \leq S_T \leq X + \epsilon \\ S_T - X - \frac{\epsilon}{2} & \text{if } S_T > X + \epsilon \end{cases}$$

Here, ϵ is a small positive quantity. By applying Ito's lemma, show that

$$f(S_T) = f(S_0) + \int_0^T f''(S_t) dS_t + \frac{1}{2} \int_0^T \sigma^2(S_t, t) S_t^2 f''(S_t) dt.$$

By taking the limit $\epsilon \rightarrow 0$, explain why

$$\begin{aligned} \int_0^T f'(S_t) dS_t &\rightarrow \int_0^T \mathbf{1}_{\{S_t \geq X\}} dS_t \\ \int_0^T \sigma^2(S_t, t) S_t^2 f''(S_t) dt &\rightarrow \int_0^T \sigma^2(S_t, t) S_t^2 \delta(S - X) dt. \end{aligned}$$

Lastly, show that

$$d(S_T - X)^+ = \mathbf{1}_{\{S_T \geq X\}} dS_T + \frac{\sigma^2(S_T, T)}{2} S_T^2 \delta(S_T - X) dT.$$