CHAPTER 2
Concepts of Financial Economics and Asset Price Dynamics

In the last chapter, we observe how the application of the no arbitrage argument *enforces* the forward price of a forward contract. The forward price is not given by the expectation of the asset price at maturity of the forward, that is, it is independent on the asset price dynamics over the life of the contract. The no arbitrage argument turns out to be the basis of the pricing models for various types of derivatives considered in this text. We also observe that a call can be replicated by a put and a forward [see Eq. (1.3.2)]. Indeed, it will be shown in Sec. 3.1 that an option can be replicated *dynamically* by a portfolio containing the underlying asset and the riskless bond. Assuming frictionless market and no premature termination of the option contract, suppose the option's payoff matches with that of the replicating portfolio at maturity, one can show by no arbitrage argument that the value of the option is equal to the value of the replicating portfolio at all times throughout the life of the option. If every derivative can be replicated by a portfolio of the fundamental assets in the market, then the market is said to be complete. In other words, we price a derivative based on the prices of other marketed assets that replicate the derivative.

From the theory of financial economics, we show that the condition of no arbitrage is equivalent to the existence of an equivalent martingale measure. Under the equivalent martingale measure, all discounted price processes of the risky assets are martingales. Further, if the market is complete (all contingent claims can be replicated), then the equivalent martingale measure is unique. The above statements are the essence of the *Fundamental Theorem of Asset Pricing*. It can be shown that the replication based price of any contingent claim can be obtained by calculating the discounted expected value of its terminal payoff under the equivalent martingale probability measure (Harrison and Kreps, 1979). This approach has come to be known as the risk neutral pricing. The term risk neutrality is used since all assets in the market offer the same return as the riskfree security under this probability, so an investor who is neutral to risk and faces with this probability would be indifferent among various assets. The concepts of *replicable contingent claims*, *absence of arbitrage* and *risk neutrality* form the cornerstones of modern option pricing theory.

In the first two sections, we limit our discussion of securities model to the discrete framework. We start with the single period securities models in Sec. 2.1. The notions of the law of one price, non-dominant trading strategy, linear pricing measure and absence of arbitrage are discussed. Every attempt has been made to have the financial economics concepts self contained. The use of the Separating Hyperplane Theorem leads to the identification of the risk neutral measure for the valuation of contingent claims under the assumption of no arbitrage. In Sec. 2.2, the discussion is extended to multi-period securities models. The readers will be shown how to construct the information structures of securities models. Various definitions in probability theory will be presented, like filtrations, measurable random variables, conditional expectations and martingales. In multi-period situation, the risk neutral probability measure is defined in terms of martingales. The highlight of the first two sections is the derivation of the *Fundamental Theorem of Asset Pricing*. More detailed exposition on the related concepts of financial economics can be found in the books by Pliska (1997) and LeRoy and Werner (2001).

In general, the price of a derivative depends primarily on the stochastic process of the price of the underlying asset. Most asset price processes are modeled by the Ito processes. For equity prices, they are fairly described by the Geometric Brownian processes, a popular class of Ito processes. In Sec. 2.3, we provide a brief exposition on the Brownian process. We start with the discrete random walk model and treat the Brownian process as the continuous limit of the random walk process. The forward Fokker-Planck equation that governs the transition density function for Brownian processes is also developed. In the last section, we introduce some basic tools in stochastic calculus, in particular, the notion of stochastic integrals and stochastic differentials. We explain the non-differentiability of Brownian paths. We provide an intuitive proof of the Ito lemma, which is an essential tool in performing calculus operations on functions of stochastic state variables. We also discuss the Feynman-Kac representation, Radon-Nikodym derivative and the Girsanov Theorem. The Girsanov Theorem provides an effective tool to transform Ito processes with general drifts into martingales. All these preliminaries in stochastic calculus are essential to develop the option pricing theory and to derive option price formulas in later chapters.

2.1 Single period securities models

The no arbitrage approach is one of the cornerstones in the development of pricing theory of financial derivatives. In simple language, arbitrage refers to the possibility of making an investment gain with no chance of loss (the rigorous definition of arbitrage will be given later). In the theoretical development of pricing models, it is commonly assumed that there are no arbitrage opportunities in well functioning and competitive financial markets.

In this section, we discuss the concepts of no arbitrage principle via single period securities models, where investment decisions on a finite set of M securities are made at initial time $t = 0$ and the payoff is attained at terminal time $t = 1$. Though single period models appear to be not quite realistic representation of the complex world of investment activities, however, a lot of fundamental concepts in financial economics can be revealed from the analysis of single period securities models. Also, single period investment models approximate quite well the buy-and-hold investment strategies.

2.1.1 Law of one price, dominant trading strategies and linear pricing measures

In single period securities models, the initial prices of M risky securities, denoted by $S_1(0), \cdots, S_M(0)$, are positive scalars that are known at $t = 0$. However, their values at $t = 1$ are random variables. These random variables are defined with respect to a sample space $\Omega = \{\omega_1, \omega_2, \cdots, \omega_K\}$ of K possible outcomes (or states of world). At $t = 0$, the investors know the list of all possible outcomes, but which outcome does occur is revealed only at the end of the investment period $t = 1$. Further, a probability measure P satisfying $P(\omega) > 0$, for all $\omega \in \Omega$, is defined on Ω.

We use S to denote the price process $\{S(t) : t = 0, 1\}$, where $S(t)$ is the row vector $S(t) = (S_1(t)\ S_2(t) \cdots S_M(t))$. The possible values of the asset price process at $t = 1$ are listed in the following $K \times M$ matrix

$$S(1; \Omega) = \begin{pmatrix} S_1(1;\omega_1) & S_2(1;\omega_1) & \cdots & S_M(1;\omega_1) \\ S_1(1;\omega_2) & S_2(1;\omega_2) & \cdots & S_M(1;\omega_2) \\ \cdots & \cdots & \cdots & \cdots \\ S_1(1;\omega_K) & S_2(1;\omega_K) & \cdots & S_M(1;\omega_K) \end{pmatrix}. \tag{2.1.1}$$

Since the assets are limited liability securities, the entries in $S(1; \Omega)$ are non-negative scalars.

We also assume the existence of a strictly positive riskless security or bank account, whose value is denoted by S_0. Without loss of generality, we take $S_0(0) = 1$ and the value at time 1 to be $S_0(1) = 1 + r$, where $r \geq 0$ is the deterministic interest rate over one period. The reciprocal of $S_0(1)$ is called the discount factor over the period. We define the discounted price process by

$$S^*(t) = S(t)/S_0(t), \quad t = 0, 1, \tag{2.1.2a}$$

that is, we use the riskless security as the *numeraire* or *accounting unit*. Accordingly, the payoff matrix of the discounted price processes of the M risky assets and the riskless security can be expressed in the form

$$\widehat{S}^*(1; \Omega) = \begin{pmatrix} 1 & S_1^*(1;\omega_1) & \cdots & S_M^*(1;\omega_1) \\ 1 & S_1^*(1;\omega_2) & \cdots & S_M^*(1;\omega_2) \\ \cdots & \cdots & \cdots & \cdots \\ 1 & S_1^*(1;\omega_K) & \cdots & S_M^*(1;\omega_K) \end{pmatrix}. \tag{2.1.2b}$$

The first column in $\widehat{S}^*(1;\Omega)$ (all entries are equal to one) represents the discounted payoff of the riskless secuirty at all states of world. Also, we define the vector of discounted price processes associated with the riskless security and the M risky securities by

$$\widehat{S}^*(t) = (1 \quad S_1^*(t) \cdots S_M^*(t)), \qquad t = 0, 1. \tag{2.1.2c}$$

An investor adopts a *trading strategy* by selecting a portfolio of the assets at time 0. The number of units of asset m held in the portfolio from $t = 0$ to $t = 1$ is denoted by $h_m, m = 0, 1, \cdots, M$. The scalars h_m can be positive (long holding), negative (short selling) or zero (no holding).

Let $V = \{V_t : t = 0, 1\}$ denote the value process that represents the total value of the portfolio over time. It is seen that

$$V_t = h_0 S_0(t) + \sum_{m=1}^{M} h_m S_m(t), \quad t = 0, 1. \tag{2.1.3}$$

The gain due to the investment on the m^{th} risky security is given by $h_m[S_m(1) - S_m(0)] = h_m \Delta S_m, m = 1, \cdots, M$. Let G be the random variable that denotes the total gain generated by investing in the portfolio. We then have

$$G = h_0 r + \sum_{m=1}^{M} h_m \Delta S_m. \tag{2.1.4}$$

If there is no withdrawal or addition of funds within the investment horizon, then

$$V_1 = V_0 + G. \tag{2.1.5}$$

Suppose we use the bank account as the numeraire, and define the discounted value process by $V_t^* = V_t / S_0(t)$ and discounted gain by $G^* = V_1^* - V_0^*$, we then have

$$V_t^* = h_0 + \sum_{m=1}^{M} h_m S_m^*(t), \quad t = 0, 1; \tag{2.1.6a}$$

$$G^* = V_1^* - V_0^* = \sum_{m=1}^{M} h_m \Delta S_m^*. \tag{2.1.6b}$$

Dominant trading strategies

Let $\mathcal{H}$ denote the trading strategy that involves the choice of the number of units of assets held in the portfolio. A trading strategy is said to be *dominant* if there exists another trading strategy $\widehat{\mathcal{H}}$ such that

$$V_0 = \widehat{V}_0 \quad \text{and} \quad V_1(\omega) > \widehat{V}_1(\omega) \quad \text{for all } \omega \in \Omega. \tag{2.1.7}$$

Here, $\widehat{V}_0$ and $\widehat{V}_1$ denote the portfolio value of $\widehat{\mathcal{H}}$ at $t = 0$ and $t = 1$, respectively. Financially speaking, both strategies $\mathcal{H}$ and $\widehat{\mathcal{H}}$ start with the same initial investment amount but the dominant strategy $\mathcal{H}$ leads to a higher gain with certainty.

Suppose $\mathcal{H}$ dominates $\widehat{\mathcal{H}}$, we define a new trading strategy $\widetilde{\mathcal{H}} = \mathcal{H} - \widehat{\mathcal{H}}$. Let $\widetilde{V}_0$ and $\widetilde{V}_1$ denote the portfolio value of $\widetilde{\mathcal{H}}$ at $t = 0$ and $t = 1$, respectively. From Eq. (2.1.7), we then have $\widetilde{V}_0 = 0$ and $\widetilde{V}_1(\omega) > 0$ for all $\omega \in \Omega$. This trading strategy is dominant since it dominates the strategy which starts with zero value and does no investment at all. A securities model that allows the existence of a dominant trading strategy is not realistic since an investor starting with no money should not be guaranteed of ending up with positive returns by adopting a particular trading strategy. Equivalently, one can show that a dominant trading strategy is one that can transform a strictly negative wealth at $t = 0$ into a non-negative wealth at $t = 1$ (see Problem 2.1). Later, we show how the non-existence of dominant strategies is closely related to the existence of a linear pricing measure.

The two important concepts in the analysis of securities pricing are *linearity* and *positivity*. In simple words, linearity means if two portfolios A and B have payoff vectors as represented by $\mathbf{p}_A$ and $\mathbf{p}_B$, and their portfolio values at the current time are V_A and V_B, respectively, then the current value of the portfolio with payoff vector $\alpha\mathbf{p}_A + \beta\mathbf{p}_B$ will be given by $\alpha V_A + \beta V_B$, where α and β are scalar constants. Linearity of pricing is related to the *law of one price* [see Eq. (2.1.8)]. Positivity of pricing refers to the positivity of state prices, and this relates to the concepts of linearly pricing measure [see Eq. (2.1.9)]. In most of the subsequent expositions, we use the riskless security as the numeraire and consider discounted value processes of the risky securities. With this choice of numeraire, the linear pricing measure can be interpreted as a probability measure [see Eqs. (2.1.10a,b)].

Asset span, law of one price and state prices

Consider the following numerical example, where the number of possible states is taken to be 3. First, we consider two risky securities whose discounted payoff vectors are $\mathbf{S}_1^*(1) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{S}_2^*(1) = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$. The payoff vectors are used to form the payoff matrix $S^*(1) = \begin{pmatrix} 1 & 3 \\ 2 & 1 \\ 3 & 2 \end{pmatrix}$. Let the current discounted prices be represented by the row vector $\mathbf{S}^*(0) = (1 \quad 2)$. We write $\mathbf{h}$ as the column vector whose entries are the weights of the securities in the portfolio. The current portfolio value and the discounted portfolio payoff are given by $\mathbf{S}^*(0)\mathbf{h}$ and $S^*(1)\mathbf{h}$, respectively. As $S_0^*(0) = 1$, the current portfolio value and discounted portfolio value are the same.

The set of all portfolio payoffs via different holding of securities is called the *asset span* $\mathcal{S}$. The asset span is seen to be the column space of the payoff matrix $S^*(1)$. In this example, the asset span consists of all vectors of the form $h_1 \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + h_2 \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}$, where h_1 and h_2 are scalars.

To these two securities in the portfolio, we may add a third security or even more securities. The newly added securities may or may not fall within the asset span. If the added security lies inside $\mathcal{S}$, then its payoff can be expressed as a linear combination of $\mathbf{S}_1^*(1)$ and $\mathbf{S}_2^*(1)$. In this case, it is said to be a *redundant security*. Since there are only 3 possible states, the dimension of the asset span cannot be more than 3, that is, the maximal number of non-redundant securities is 3. Suppose we add the third security whose discounted payoff is $\mathbf{S}_3^*(1) = \begin{pmatrix} 1 \\ 3 \\ 4 \end{pmatrix}$, it can be easily checked that it is a non-redundant security. The new asset span [the subspace in $\mathbb{R}^3$ spanned by $\mathbf{S}_1^*(1), \mathbf{S}_2^*(1)$ and $\mathbf{S}_3^*(1)$] will be the whole $\mathbb{R}^3$. Any further security added must be redundant since its discounted payoff vector must lie inside the new asset span. A securities model is said to be *complete* if every payoff vector lies inside the asset span. This occurs if and only if the dimension of the asset span equals the number of possible states.

The law of one price states that all portfolios with the same payoff have the same price. Consider two portfolios with different portfolio weights $\mathbf{h}$ and $\mathbf{h}'$. Suppose these two portfolios have the same discounted payoff, that is, $S^*(1)\mathbf{h} = S^*(1)\mathbf{h}'$, then the law of one price infers that $\mathbf{S}^*(0)\mathbf{h} = \mathbf{S}^*(0)\mathbf{h}'$. It is quite straightforward to show that a necessary and sufficient condition for the law of one price to hold is that a portfolio with zero payoff must have zero price. Also, if the law of one price fails, then it is possible to have two trading strategies $\mathbf{h}$ and $\mathbf{h}'$ such that $S^*(1)\mathbf{h} = S^*(1)\mathbf{h}'$ but $\mathbf{S}^*(0)\mathbf{h} > \mathbf{S}^*(0)\mathbf{h}'$. Let $G^*(\omega)$ and $G^{*\prime}(\omega)$ denote the respective discounted gain corresponding to the trading strategies $\mathbf{h}$ and $\mathbf{h}'$. We then have $G^{*\prime}(\omega) > G^*(\omega)$ for all $\omega \in \Omega$, so there exists a dominant trading strategy. Hence, the non-existence of dominant trading strategy implies the law of one price. However, the converse statement does not hold (see Problem 2.4).

Given a discounted portfolio payoff $\mathbf{x}$ that lies inside the asset span, the payoff can be generated by some linear combination of the securities in the securities model. We have $\mathbf{x} = S^*(1)\mathbf{h}$ for some $\mathbf{h} \in \mathbb{R}^M$. The current value of the portfolio is $\mathbf{S}^*(0)\mathbf{h}$, where $\mathbf{S}^*(0)$ is the price vector. We may consider $\mathbf{S}^*(0)\mathbf{h}$ as a pricing functional $F(\mathbf{x})$ on the payoff $\mathbf{x}$. If the law of one price holds, then the pricing functional is single-valued. Furthermore, it is a linear functional, that is,

$$F(\alpha_1\mathbf{x}_1 + \alpha_2\mathbf{x}_2) = \alpha_1 F(\mathbf{x}_1) + \alpha_2 F(\mathbf{x}_2) \tag{2.1.8}$$

for any scalars α_1 and α_2 and payoffs $\mathbf{x}_1$ and $\mathbf{x}_2$ (see Problem 2.5).

Let $\mathbf{e}_k$ denote the k^{th} coordinate vector in the vector space $\mathbb{R}^K$, where $\mathbf{e}_k$ assumes the value 1 in the k^{th} entry and zero in all other entries. The vector $\mathbf{e}_k$ can be considered as the discounted payoff vector of a security, and it is called the Arrow security of state k. Suppose the securities model is complete and the law of one price holds, then the pricing functional F assigns unique value to each Arrow security. We write $s_k = F(\mathbf{e}_k)$, which is called the state price of state k (see Problem 2.6).

Linear pricing measure

We consider securities models with the inclusion of the riskfree security. A non-negative row vector $\mathbf{q} = (q(\omega_1) \cdots q(\omega_K))$ is said to be a linear pricing measure if for every trading strategy we have

$$V_0^* = \sum_{k=1}^{K} q(\omega_k) V_1^*(\omega_k). \tag{2.1.9}$$

The linear pricing measure exhibits the following properties. First, suppose we take the holding amount of each risky security to be zero, thereby $h_1 = h_2 = \cdots = h_M = 0$, then

$$V_0^* = h_0 = \sum_{k=1}^{K} q(\omega_k) h_0 \tag{2.1.10a}$$

so that

$$\sum_{k=1}^{K} q(\omega_k) = 1. \tag{2.1.10b}$$

Next, by taking the portfolio weights to be zero except for the m^{th} security, we have

$$S_m^*(0) = \sum_{k=1}^{K} q(\omega_k) S_m^*(1;\omega_k), \quad m = 1, \cdots, M. \tag{2.1.11}$$

Since we have taken $q(\omega_k) \geq 0, k = 1, \cdots, K$, and their sum is one, we may interpret $q(\omega_k)$ as a probability measure on the sample space Ω. Note that $q(\omega_k)$ is not related to the actual probability of occurrence of the state k, though the current discounted security price is given by the expectation of the discounted security payoff one period later under the linear pricing measure [see Eq. (2.1.11)]. In matrix, form, Eq. (2.1.11) can be expressed as

$$\widehat{\mathbf{S}}^*(0) = \mathbf{q}\widehat{S}^*(1;\Omega), \quad \mathbf{q} \geq \mathbf{0}. \tag{2.1.12}$$

As a numerical example, we consider a securities model with 2 risky securities and the riskfree security, and there are 3 possible states. The current discounted price vector $\widehat{\mathbf{S}}^*(0)$ is $(1 \quad 4 \quad 2)$ and the discounted payoff matrix at $t = 1$ is $\widehat{S}^*(1) = \begin{pmatrix} 1 & 4 & 3 \\ 1 & 3 & 2 \\ 1 & 2 & 4 \end{pmatrix}$. Here, the law of one price holds since the only solution to $\widehat{S}^*(1)\mathbf{h} = \mathbf{0}$ is $\mathbf{h} = \mathbf{0}$. This is because the columns of $\widehat{S}^*(1)$ are independent so that the dimension of the nullspace of $\widehat{S}^*(1)$ is zero. We would like to see whether linear pricing measure exists for the given securities model. By virtue of Eqs. (2.1.10b) and (2.1.11), the linear pricing probabilities $q(\omega_1), q(\omega_2)$ and $q(\omega_3)$, if exist, should satisfy the following equations:

$$\begin{aligned} 1 &= q(\omega_1) + q(\omega_2) + q(\omega_3) \\ 4 &= 4q(\omega_1) + 3q(\omega_2) + 2q(\omega_3) \\ 2 &= 3q(\omega_1) + 2q(\omega_2) + 4q(\omega_3). \end{aligned} \tag{2.1.13a}$$

Solving the above equations, we obtain $q(\omega_1) = q(\omega_2) = 2/3$ and $q(\omega_3) = -1/3$. Since not all the pricing probabilities are non-negative, the linear pricing measure does not exist for this securities model.

Do dominant trading strategies exist for the above securities model? That is, can we find trading strategy $(h_1 \ h_2)$ such that $V_0^* = 4h_1 + 2h_2 = 0$ but $V_1^*(\omega_k) > 0, k = 1, 2, 3$? This is equivalent to ask whether there exist h_1 and h_2 such that $4h_1 + 2h_2 = 0$ and

$$\begin{aligned} 4h_1 + 3h_2 &> 0 \\ 3h_1 + 2h_2 &> 0 \\ 2h_1 + 4h_2 &> 0. \end{aligned} \tag{2.1.13b}$$

In Fig. 2.1, we show the region containing the set of points in the (h_1, h_2)-plane that satisfy inequalities (2.1.13b). The region is found to be lying on the top right sides above the two bold lines: (i) $3h_1 + 2h_2 = 0, h_1 < 0$ and (ii) $2h_1 + 4h_2 = 0, h_1 > 0$. It is seen that all the points on the dotted half line: $4h_1 + 2h_2 = 0, h_1 < 0$ represent dominant trading strategies that start with zero wealth but end with positive wealth with certainty.

Suppose the initial discounted price vector is changed from $(4 \quad 2)$ to $(3 \quad 3)$, the new set of linear pricing probabilities will be determined by

$$\begin{aligned} 1 &= q(\omega_1) + q(\omega_2) + q(\omega_3) \\ 3 &= 4q(\omega_1) + 3q(\omega_2) + 2q(\omega_3) \\ 3 &= 3q(\omega_1) + 2q(\omega_2) + 4q(\omega_3), \end{aligned} \tag{2.1.14}$$

which is seen to have the solution: $q(\omega_1) = q(\omega_2) = q(\omega_3) = 1/3$. Now, all the pricing probabilities have non-negative values, the row vector $\mathbf{q} = (1/3 \quad 1/3 \quad 1/3)$ represents a linear pricing measure. Referring to Fig. 2.1, we observe that the line $3h_1 + 3h_2 = 0$ always lies outside the region above the two bold lines. Hence, with respect to this new securities model, we cannot find $(h_1 \quad h_2)$ such that $3h_1 + 3h_2 = 0$ together with h_1 and h_2 satisfying inequalities (2.1.13b). Since a linear pricing measure exists, by virtue of Eq.

(2.1.12), we expect that the initial price vector $(3 \quad 3)$ can be expressed as some linear combination of the 3 vectors: $(4 \quad 3)$, $(3 \quad 2)$ and $(2 \quad 4)$ with non-negative weights. Actually, we have $(3 \quad 3) = \frac{1}{3}(4 \quad 3) + \frac{1}{3}(3 \quad 2) + \frac{1}{3}(2 \quad 4)$, where the weights are the linear pricing probabilities.

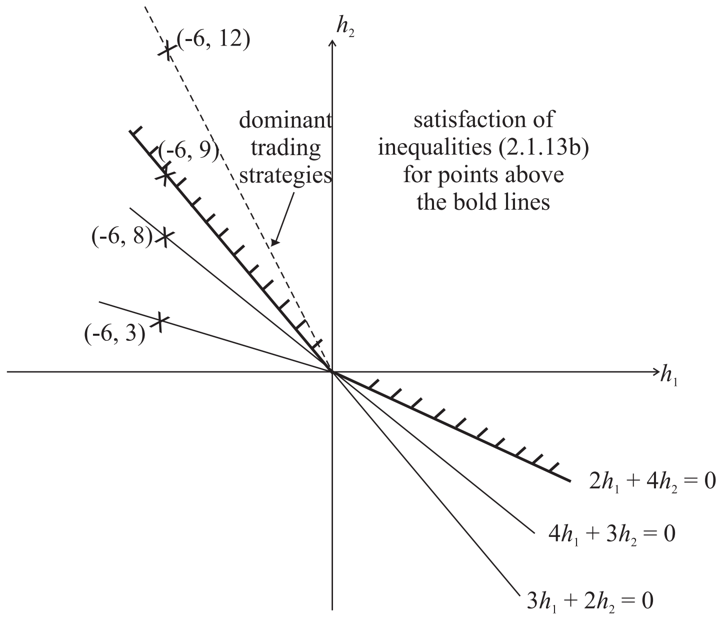

Fig. 2.1 The region above the two bold lines represents trading strategies that satisfy inequalities (2.1.13b). The trading strategies that lie on the dotted line: $4h_1 + 2h_2 = 0, h_1 < 0$ are dominant trading strategies.

Apparently, one may conjecture that the existence of linear pricing measure is related to the non-existence of dominant trading strategies. Indeed, we have the following theorem.

Theorem 2.1
There exists a linear pricing measure if and only if there are no dominant trading strategies.

The above linear pricing measure theorem can be seen to be a direct consequence of the Farkas Lemma.

Farkas Lemma
There does not exist $\mathbf{h} \in \mathbb{R}^M$ such that

$$\widehat{S}^*(1; \Omega)\mathbf{h} > \mathbf{0} \quad \text{and} \quad \widehat{\mathbf{S}}^*(0)\mathbf{h} = 0$$

if and only if there exists $\mathbf{q} \in \mathbb{R}^K$ such that

$$\widehat{\mathbf{S}}^*(0) = \mathbf{q}\widehat{S}^*(1;\Omega) \quad \text{and} \quad \mathbf{q} \geq \mathbf{0}.$$

2.1.2 Arbitrage opportunities and risk neutral probability measures

Suppose $\mathbf{S}^*(0)$ in the above securities model is modified to $(3 \quad 2)$ and consider the trading strategy: $h_1 = -2$ and $h_2 = 3$. We observe that $V_0^* = 0$ and the possible discounted payoffs at $t = 1$ are: $V_1^*(\omega_1) = 1, V_1^*(\omega_2) = 0$ and $V_1^*(\omega_3) = 8$. This represents a trading strategy that starts with zero wealth, guarantees no loss, and ends up with a strictly positive wealth in some states (not necessarily in all states). The occurrence of such investment opportunity is called an arbitrage opportunity. Formally, we define an *arbitrage opportunity* to be some trading strategy that has the following properties: (i) $V_0^* = 0$, (ii) $V_1^*(\omega) \geq 0$ and $EV_1^*(\omega) > 0$, where E is the expectation under the actual probability measure P. Readers should be watchful for the difference between dominant strategy and arbitrage opportunity, where the existence of a dominant strategy requires a portfolio with initial zero wealth to end up with a *strictly* positive wealth in all states. Therefore, the existence of a dominant trading strategy implies the existence of an arbitrage opportunity, but the converse is not necessarily true. In other words, the absence of arbitrage implies the non-existence of dominant trading strategy and in turn implying that the law of one price holds.

Existence of arbitrage opportunities is unreasonable from the economic standpoint. The natural question: What would be the necessary and sufficient condition for the non-existence of arbitrage opportunities? The answer is related to the existence of a pricing measure, called the risk neutral probability measure. In financial markets with no arbitrage opportunities, we will show that every investor should use such risk neutral probability measure (though not necessarily unique) to find the fair value of a portfolio, irrespective to the risk preference of the investor.

Risk neutral probability measure

The example just mentioned above represents the presence of an arbitrage opportunity but non-existence of dominant trading strategy [since $V_1^*(\omega) = 0$ for some ω]. The linear pricing measure vector is found to be $(0 \quad 1 \quad 0)$, where some of the linear pricing probabilities are zero. In order to exclude arbitrage opportunities, we need a bit stronger condition on the linear pricing probabilities, namely, the probabilities must be strictly positive.

A probability measure Q on Ω is a risk neutral probability measure if it satisfies (i) $Q(\omega) > 0$ for all $\omega \in \Omega$, and (ii) $E_Q[\Delta S_m^*] = 0, m = 1, \cdots, M$, where E_Q denotes the expectation under Q. Note that $E_Q[\Delta S_m^*] = 0$ is equivalent to $S_m^*(0) = \sum_{k=1}^{K} Q(\omega_k) S_m^*(1;\omega_k)$, which takes similar form as Eq.(2.1.11).

Indeed, a linear pricing measure becomes a risk neutral probability measure if the probability masses are all positive.

The existence of a risk neutral measure is directly related to the exclusion of arbitrage opportunities, the details of which are stated in the following theorem.

Theorem 2.2
No arbitrage opportunities exist if and only if there exists a risk neutral probability measure Q.

The proof of Theorem 2.2 requires the Separating Hyperplane Theorem. A brief intuition of the theorem is given here. First, we present the definitions of hyperplane and convex sets in a vector space. Let $\mathbf{f}$ be a vector in $\mathbb{R}^n$. The hyperplane $H = [\mathbf{f}, \alpha]$ in $\mathbb{R}^n$ is defined to be the collection of those vectors $\mathbf{x}$ in $\mathbb{R}^n$ whose projection onto $\mathbf{f}$ has magnitude α. For example, the collection of vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ satisfying $x_1 + 2x_2 + 3x_3 = 2$ is a hyperplane in $\mathbb{R}^3$, where $\mathbf{f} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\alpha = 2$. A set C in $\mathbb{R}^n$ is said to be convex if for any pair of vectors $\mathbf{x}$ and $\mathbf{y}$ in C, all convex combinations of $\mathbf{x}$ and $\mathbf{y}$ represented by the form $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}, 0 \leq \lambda \leq 1$, also lie in C. For example, the set $C = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \right\}$ is a convex set in $\mathbb{R}^3$. The hyperplane $[\mathbf{f}, \alpha]$ separates the sets A and B in $\mathbb{R}^n$ if there exists α such that $\mathbf{f} \cdot \mathbf{x} \geq \alpha$ for all $\mathbf{x} \in A$ and $\mathbf{f} \cdot \mathbf{y} < \alpha$ for all $\mathbf{y} \in B$. For example, the hyperplane $\left[\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, 0 \right]$ separates the two disjoint convex sets $A = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \right\}$ and $B = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 < 0, x_2 < 0, x_3 < 0 \right\}$ in $\mathbb{R}^3$.

The *Separating Hyperplane Theorem* states that if A and B are two nonempty disjoint convex sets in a vector space V, then they can be separated by a hyperplane. A pictorial interpretation of the Separating Hyperplane Theorem for the vector space $\mathbb{R}^2$ is shown in Fig. 2.2.

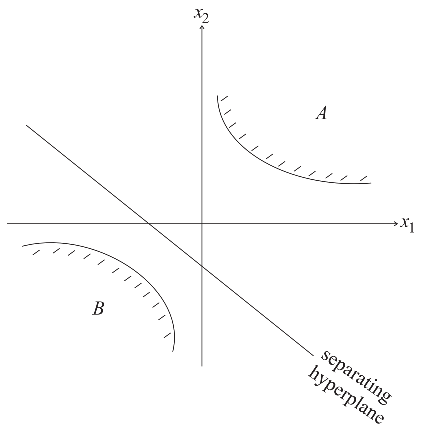

Fig. 2.2 The hyperplane (represented by a line in $\mathbb{R}^2$) separates the two convex sets A and B in $\mathbb{R}^2$.

Proof of Theorem 2.2
"$\Leftarrow$ part". Assume a risk neutral probability measure Q exists, that is, $\widehat{\mathbf{S}}^*(0) = \boldsymbol{\pi}\widehat{S}^*(1;\Omega)$, where $\boldsymbol{\pi} = (Q(\omega_1)\cdots Q(\omega_K))$. Consider a trading strategy $\mathbf{h} = (h_1 \quad \cdots \quad h_M)^T \in \mathbb{R}^M$ such that $S^*(1;\Omega)\mathbf{h} \geq 0$ in all $\omega \in \Omega$ and with strict inequality in some states. Now consider $\widehat{\mathbf{S}}^*(0)\mathbf{h} = \boldsymbol{\pi}\widehat{S}^*(1;\Omega)\mathbf{h}$, it is seen that $\widehat{\mathbf{S}}^*(0)\mathbf{h} > 0$ since all entries in $\boldsymbol{\pi}$ are strictly positive and entries in $\widehat{S}^*(1;\Omega)\mathbf{h}$ are either zero or strictly positive. Hence, no arbitrage opportunities exist.

"$\Rightarrow$ part". First, we define the subset U in $\mathbb{R}^{K+1}$ which consists of vectors of the form $\begin{pmatrix} -\widehat{\mathbf{S}}^*(0)\mathbf{h} \\ \widehat{\mathbf{S}}^*(1;\omega_1)\mathbf{h} \\ \vdots \\ \widehat{\mathbf{S}}^*(1;\omega_K)\mathbf{h} \end{pmatrix}$, where $\widehat{\mathbf{S}}^*(1;\omega_k)$ is the k^{th} row in $\widehat{S}^*(1;\Omega)$ and $\mathbf{h} \in \mathbb{R}^M$ represents a trading strategy. This subset is seen to be a convex subspace. Consider another subset $\mathbb{R}_+^{K+1}$ defined by

$$\mathbb{R}_+^{K+1} = \{\mathbf{x} = (x_0\ x_1 \cdots x_K)^T \in \mathbb{R}^{K+1} : x_i \geq 0 \quad \text{for all} \quad 0 \leq i \leq K\},$$

which is a convex set in $\mathbb{R}^{K+1}$. We claim that the non-existence of arbitrage opportunities implies that U and $\mathbb{R}_+^{K+1}$ can only have the zero vector in common.

Assume the contrary, suppose there exists a non-zero vector $\mathbf{x} \in U \cap \mathbb{R}_+^{K+1}$. Since there is a trading strategy vector $\mathbf{h}$ assoicated with every vector in U, it suffices to show that the trading strategy $\mathbf{h}$ associated with $\mathbf{x}$ always represents an arbitrage opportunity. We consider the following two cases: $-\widehat{\mathbf{S}}^*(0)\mathbf{h} = 0$ or $-\widehat{\mathbf{S}}^*(0)\mathbf{h} > 0$. When $\widehat{\mathbf{S}}^*(0)\mathbf{h} = 0$, since $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{x} \in R_+^{K+1}$, then the entries $\widehat{\mathbf{S}}(1;\omega_k)\mathbf{h}, k = 1, 2, \cdots K$, must be all greater than or equal to zero, with at least one strict inequality. In this case, $\mathbf{h}$ is seen to represent an arbitrage opportunity. When $\widehat{\mathbf{S}}^*(0)\mathbf{h} < 0$, all the entries $\widehat{\mathbf{S}}(1;\omega_k)\mathbf{h}, k = 1, 2, \cdots, K$ must be all non-negative. Correspondingly, $\mathbf{h}$ represents a dominant trading strategy (see Problem 2.1) and in turns $\mathbf{h}$ is an arbitrage opportunity.

Since $U \cap R_+^{K+1} = \{\mathbf{0}\}$, by the Separating Hyperplane Theorem, there exists a hyperplane that separates $\mathbb{R}_+^{K+1}\backslash\{\mathbf{0}\}$ and U. Let $\mathbf{f} \in \mathbb{R}^{K+1}$ be the normal to this hyperplane, then we have $\mathbf{f} \cdot \mathbf{x} > \mathbf{f} \cdot \mathbf{y}$, where $\mathbf{x} \in \mathbb{R}_+^{K+1}\backslash\{\mathbf{0}\}$ and $\mathbf{y} \in U$. [*Remark*: We may have $\mathbf{f} \cdot \mathbf{x} < \mathbf{f} \cdot \mathbf{y}$, depending on the orientation of the normal. However, the final conclusion remains unchanged.] Since U is a linear subspace so that a negative multiple of $\mathbf{y} \in U$ also belongs to U, the condition $\mathbf{f} \cdot \mathbf{x} > \mathbf{f} \cdot \mathbf{y}$ holds only if $\mathbf{f} \cdot \mathbf{y} = 0$ for all $\mathbf{y} \in U$. We then have $\mathbf{f} \cdot \mathbf{x} > 0$ for all $\mathbf{x}$ in $\mathbb{R}_+^{K+1}\backslash\{\mathbf{0}\}$. This requires all entries in $\mathbf{f}$ to be strictly positive. Also, from $\mathbf{f} \cdot \mathbf{y} = 0$, we have

$$- f_0\widehat{\mathbf{S}}^*(0)\mathbf{h} + \sum_{k=1}^{K} f_k\widehat{\mathbf{S}}^*(1;\omega_k)\mathbf{h} = 0 \qquad (2.1.15a)$$

for all $\mathbf{h} \in \mathbb{R}^M$, where $f_j, j = 0, 1, \cdots, K$ are the entries of $\mathbf{f}$. We then deduce that

$$\widehat{\mathbf{S}}^*(0) = \sum_{k=1}^{K} Q(\omega_k)\widehat{\mathbf{S}}^*(1;\omega_k), \text{ where } Q(\omega_k) = f_k/f_0. \qquad (2.1.15b)$$

Lastly, we consider the first component in the vectors on both sides of the above equation. They both correspond to the current price and discounted payoff of the riskless security, and all are equal to one. We then obtain

$$1 = \sum_{k=1}^{K} Q(\omega_k). \qquad (2.1.15c)$$

Here, we obtain the risk neutral probabilities $Q(\omega_k), k = 1, \cdots, K$, whose sum is equal to one and they are all strictly positive since $f_j > 0, j = 0, 1, \cdots, K$.

Remark

Corresponding to each risky asset, Eq. (2.1.15b) dictates that

$$S_m^*(0) = \sum_{k=1}^{K} Q(\omega_k)S_m^*(1;\omega_k), \quad m = 1, 2, \cdots, M. \qquad (2.1.16)$$

Hence, the current price of any risky security is given by the expectation of the discounted payoff under the risk neutral measure Q.

Calculation of risk neutral measures

Consider the earlier securities model with the riskfree security and only one risky security, where $\widehat{S}(1;\Omega) = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix}$ and $\widehat{\mathbf{S}}(0) = (1 \quad 3)$. The risk neutral probability measure $\boldsymbol{\pi} = (Q(\omega_1) \quad Q(\omega_2) \quad Q(\omega_3))$, if exists, will be determined by the following system of equations

$$(Q(\omega_1) \quad Q(\omega_2) \quad Q(\omega_3)) \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix} = (1 \quad 3). \tag{2.1.17}$$

Since there are more unknowns than the number of equations, the solution is not unique. The solution is found to be $\boldsymbol{\pi} = (\lambda \quad 1-2\lambda \quad \lambda)$, where λ is a free parameter. In order that all risk neutral probabilities are all strictly positive, we must have $0 < \lambda < 1/2$. We would expect that uniqueness of the risk neutral measure is directly related to the completeness of the securities model. Suppose we add the second risky security with discounted payoff $\mathbf{S}_2^*(1) = \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ and current discounted value $S_2^*(0) = 3$. With this new addition the securities model becomes complete (the asset span of the two risky securities and the riskfree security is the whole $\mathbb{R}^3$ space). With the new equation $3Q(\omega_1) + 2Q(\omega_2) + 4Q(\omega_3) = 3$ added to the system (2.1.17), this new securities model is seen to have the unique risk neutral measure $(1/3 \quad 1/3 \quad 1/3)$.

Let W be a subspace in $\mathbb{R}^K$ which consists of discounted gains corresponding to some trading strategy $\mathbf{h}$. In the above securities model the discounted gains of the first and second risky securities are $\begin{pmatrix} 4 \\ 3 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix} - \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, respectively. Therefore, the corresponding discounted gain subspace is given by

$$W = \left\{ h_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + h_2 \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \text{ where } h_1 \text{ and } h_2 \text{ are scalars} \right\}. \tag{2.1.18}$$

For any risk neutral probability measure Q, we have

$$\begin{aligned} E_Q G^* &= \sum_{k=1}^{K} Q(\omega_k) \left[\sum_{m=1}^{M} h_m \Delta S_m^*(\omega_k) \right] \\ &= \sum_{m=1}^{M} h_m E_Q[\Delta S_m^*] = 0, \end{aligned} \tag{2.1.19}$$

where $\Delta S^*_m(\omega_k)$ is the discounted gain on the m^{th} risky security when the state ω_k occurs. Therefore, the risk neutral probability vector $\boldsymbol{\pi}$ must lie in the orthogonal complement $W^\perp$. Since the sum of risk neutral probabilities must be one and all probability values must be positive, the risk neutral probability vector $\boldsymbol{\pi}$ must lie in the following subset

$$P^+ = \{\mathbf{y} \in \mathbb{R}^K : y_1 + y_2 + \cdots + y_K = 1 \quad \text{and} \quad y_k > 0, k = 1, \cdots K\}. \tag{2.1.20}$$

Let R denote the set of all risk neutral measures. From the above arguments, we see that $R = P^+ \cap W^\perp$.

In the above numerical example, $W^\perp$ is the line through the origin in $\mathbb{R}^3$ which is perpendicular to $(1 \quad 0 \quad -1)$ and $(0 \quad -1 \quad 1)$. The line should assume the form $\lambda(1 \quad 1 \quad 1)$ for some scalar λ. Together with the constraints that sum of components equals one and each component is positive, we obtain the risk neutral probability vector $\boldsymbol{\pi} = (1/3 \quad 1/3 \quad 1/3)$. The risk neutral measure of this securities model is unique since the securities model is complete.

2.1.3 Valuation of contingent claims

A contingent claim can be considered as a random variable Y that represents a terminal payoff whose value depends on the occurrence of a particular state ω_k, where $\omega_k \in \Omega$. Suppose the holder of the contingent claim is promised to receive the preset payoff, how much should the writer of such contingent claim charge at $t = 0$ so that the price is *fair* to both parties.

Consider the securities model with the riskfree security whose values at $t = 0$ and $t = 1$ are $S_0(0) = 1$ and $S_0(1) = 1.1$, respectively, and a risky security with $S_1(0) = 3$ and $\mathbf{S}_1(1) = \begin{pmatrix} 4.4 \\ 3.3 \\ 2.2 \end{pmatrix}$. The set of $t = 1$ payoffs that can be generated by certain trading strategy is given by $h_0 \begin{pmatrix} 1.1 \\ 1.1 \\ 1.1 \end{pmatrix} + h_1 \begin{pmatrix} 4.4 \\ 3.3 \\ 2.2 \end{pmatrix}$ for some scalars h_0 and h_1. For example, the contingent claim $\begin{pmatrix} 5.5 \\ 4.4 \\ 3.3 \end{pmatrix}$ can be generated by the trading strategy: $h_0 = 1$ and $h_1 = 1$, while the other contingent claim $\begin{pmatrix} 5.5 \\ 4.0 \\ 3.3 \end{pmatrix}$ cannot be generated by any trading strategy associated with the given securities model. A contingent claim Y is said to be *attainable* if there exists some trading strategy $\mathbf{h}$, called the *replicating portfolio*, such that $V_1 = Y$ for all possible states occurring at $t = 1$.

What should be the price at $t = 0$ of the attainable contingent claim $\begin{pmatrix} 5.5 \\ 4.4 \\ 3.3 \end{pmatrix}$? One may propose that the price at $t = 0$ of the replicating portfolio is given by $V_0 = h_0 S_0(0) + h_1 S_1(0) = 1 \times 1 + 1 \times 3 = 4$. As discussed in previous sub-section, suppose there are no arbitrage opportunities (equivalent to the existence of a risk neutral probability measure), then the law of one price holds and so V_0 is unique. The price at $t = 0$ of the contingent claim Y is simply V_0, the price that is implied by the arbitrage pricing theory. If otherwise, suppose the price p of the contingent claim at $t = 0$ is greater than V_0, an arbitrageur can lock in a riskfree profit of amount $p - V_0$ by shorting selling the contingent claim and buying the replicating portfolio. The arbitrage strategy is reversed if $p < V_0$. In this securities model, we have shown earlier that risk neutral probability measures do exist (though not unique). By the above argument, the initial price of the contingent claim $\begin{pmatrix} 5.5 \\ 4.4 \\ 3.3 \end{pmatrix}$ is unique and it is found to be $V_0 = 4$.

Consider a given attainable contingent claim Y which is generated by certain trading strategy. The associated discounted gain G^* of the trading strategy is given by $G^* = \sum_{m=1}^{M} h_m \Delta S_m^*$. Now, suppose a risk neutral probability measure Q associated with the securities model exists, we have

$$V_0 = E_Q V_0^* = E_Q[V_1^* - G^*]. \tag{2.1.21a}$$

Since $E_Q[G^*] = 0$ and $V_1^* = Y/S_0(1)$, we obtain

$$V_0 = E_Q[Y/S_0(1)]. \tag{2.1.21b}$$

Recall that the existence of the risk neutral probability measure implies the law of one price. Does $E_Q[Y/S_0(1)]$ assume the same value for every risk neutral probability measure Q? This must be true by virtue of the law of one price since we cannot have two different values for V_0 corresponding to the same contingent claim Y. This gives the *risk neutral valuation principle*: The price at $t = 0$ of an attainable claim Y is given by the expectation under any risk neutral measure Q of the discounted value of the contingent claim. Actually, one can show that a rather strong result: If $E_Q[Y/S_0(1)]$ takes the same value for every Q, then the contingent claim Y is attainable [for proof, see Pliska's text (1997)].

Readers are reminded that if the law of one price does not hold for a given securities model, we cannot define a unique price for an attainable contingent claim (see Problem 2.12).

State prices

Suppose we take Y to be the following contingent claim: $Y^* = Y/S_0(1)$ equals one if $\omega = \omega_k$ for some $\omega_k \in \Omega$ and zero otherwise. This is just the Arrow security $\mathbf{e}_k$ corresponding to the state ω_k. We then have

$$E_Q[Y/S_0(1)] = \boldsymbol{\pi}\mathbf{e}_k = Q(\omega_k). \tag{2.1.22}$$

The price of the Arrow security with discounted payoff $\mathbf{e}_k$ is called the state price for state $\omega_k \in \Omega$. The above result shows that the state price for ω_k is equal to the risk neutral probability for the same state.

Any contingent claim Y can be written as a linear combination of these basic Arrow securities. Suppose $Y^* = Y/S_0 = \sum_{k=1}^{K} \alpha_k \mathbf{e}_k$, then the price at $t = 0$ of the contingent claim is equal to $\sum_{k=1}^{K} \alpha_k Q(\omega_k)$. For example, suppose

$$Y^* = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix} \quad \text{and} \quad \widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & 4 \\ 1 & 3 \\ 1 & 2 \end{pmatrix}, \tag{2.1.23a}$$

we have seen that the risk neutral probability is given by

$$\boldsymbol{\pi} = (\lambda \quad 1 - 2\lambda \quad \lambda), \text{ where } 0 < \lambda < 1/2. \tag{2.1.23b}$$

The price at $t = 0$ of the contingent claim is given by

$$V_0 = 5\lambda + 4(1 - 2\lambda) + 3\lambda = 4, \tag{2.1.23c}$$

which is independent of λ. This verifies the earlier claim that $E_Q[Y/S_0(1)]$ assumes the same value for any risk neutral measure Q.

Complete markets

Recall that a securities model is complete if every contingent claim Y lies in the asset span, that is, Y can be generated by some trading strategy. Consider the augmented terminal payoff matrix

$$\widehat{S}(1;\Omega) = \begin{pmatrix} S_0(1;\omega_1) & S_1(1;\omega_1) & \cdots & S_M(1;\omega_1) \\ \vdots & \vdots & & \vdots \\ S_0(1;\omega_K) & S_1(1;\omega_K) & \cdots & S_M(1;\omega_K) \end{pmatrix}, \tag{2.1.24}$$

we deduce from linear algebra theory that Y always lies in the asset span if and only if the column space of $\widehat{S}(1;\Omega)$ is equal to $\mathbb{R}^K$. Since the dimension of the column space of $\widehat{S}(1;\Omega)$ cannot be greater than $M + 1$, therefore a necessary condition for market completeness is that $M + 1 \geq K$. Under market completeness, if the set of risk neutral probability measures is non-empty, then it must be a singleton (see Problem 2.11). Furthermore, when $\widehat{S}(1;\Omega)$ has independent columns and the asset span is the whole $\mathbb{R}^K$, then

$M + 1 = K$. In this case, the trading strategy that generates Y must be unique since there are no redundant securities. On the other hand, when the asset span is the whole $\mathbb{R}^K$ but some securities are redundant, the trading strategy that generates Y would not be unique. However, the price at $t = 0$ of the contingent claim is unique under arbitrage pricing, independent of the chosen trading strategy. This is a consequence of the law of one price, which holds since risk neutral measure exists.

When the dimension of the column space $\widehat{S}(1;\Omega)$ is less than K, then not all contingent claims can be attainable. In this case, a non-attainable contingent claim cannot be priced using arbitrage pricing theory. However, we may specify an interval $(V_-(Y), V_+(Y))$ where a reasonable price at $t = 0$ of the contingent claim should lie. The lower and upper bounds are given by

$$V_+(Y) = \inf\{E_Q[\widetilde{Y}/S_0(1)] : \widetilde{Y} \geq Y \text{ and } \widetilde{Y} \text{ is attainable}\} \quad (2.1.24a)$$

$$V_-(Y) = \sup\{E_Q[\widetilde{Y}/S_0(1)] : \widetilde{Y} \leq Y \text{ and } \widetilde{Y} \text{ is attainable}\}. \quad (2.1.24b)$$

Here, $V_+(Y)$ is the minimum value among all prices of attainable contingent claims that dominate the non-attainable claim Y, while $V_-(Y)$ is the maximum value among all prices of attainable contingent claims that are dominated by Y. Suppose $V(Y) > V_+(Y)$, then an arbitrageur can lock in riskless profit by selling the contingent claim to receive $V(Y)$ and use $V_+(Y)$ to construct the replicating portfolio that generates the attainable $\widetilde{Y}$ as defined in Eq. (2.1.24a). The upfront positive gain is $V(Y) - V_+(Y)$. At $t = 1$, the payoff from the replicating portfolio always dominates that of Y so that no loss at expiry is also ensured.

2.1.4 Principles of binomial option pricing model

We would like to illustrate the risk neutral valuation of contingent claims using the renowned binomial option pricing model. In the binomial model, the asset price movement is simulated by a discrete binomial random walk model (see Sec. 2.3.1 for a more detailed discussion on random walk models). Here, we limit our discussion to the one-period binomial model, and defer the analysis of the multi-period binomial model later (see Sec. 2.2.4). We will show that the option price obtained from the binomial model depends only on the riskless interest rate but independent on the actual expected rate of return of the asset price.

Formulation of the replicating portfolio

We follow the derivation of the discrete binomial model presented by Cox, Ross and Rubinstein (1979). They showed that by buying the asset and borrowing cash (in the form of riskless investment) in appropriate proportions, one can replicate the position of a call. Let S denote the current asset price. Under the binomial random walk model, the asset price after one period Δt will be either uS or dS with probability q and $1 - q$, respectively (see Fig.

2.3). We assume $u > 1 > d$ so that uS and dS represent the up-move and down-move of the asset price, respectively. The jump parameters u and d will be related to the asset price dynamics, the detailed discussion of which will be relegated to Sec. 7.1.1. Let R denote the growth factor of riskless investment over one period so that \$1 invested in a riskless money market account will grow to \$$R$ after one period. In order to avoid riskless arbitrage opportunities, we must have $u > R > d$ (see Problem 2.14).

Suppose we form a portfolio which consists of α units of asset and cash amount B in the form of riskless investment (money market account). After one period of time $\triangle t$, the value of the portfolio becomes (see Fig. 2.3)

$$\begin{cases} \alpha uS + RM & \text{with probability } q \\ \alpha dS + RM & \text{with probability } 1-q. \end{cases}$$

The portfolio is used to replicate the long position of a call option on a non-dividend paying asset. As there are two possible states of the world: asset price goes up or down, the call is thus a contingent claim. Suppose the current time is only one period $\triangle t$ prior to expiration. Let c denote the current call price, and c_u and c_d denote the call price after one period (which is the expiration time in the present context) corresponding to the up-move and down-move of the asset price, respectively. Let X denote the strike price of the call. The payoff of the call at expiry is given by

$$\begin{cases} c_u = \max(uS - X, 0) & \text{with probability } q \\ c_d = \max(dS - X, 0) & \text{with probability } 1-q. \end{cases}$$

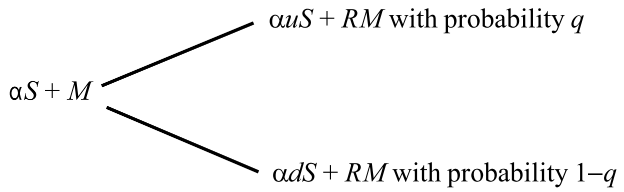

Fig. 2.3 Evolution of the asset price S and money market account M after one time period under the binomial model. The risky asset value may either go up to uS or go down to dS, while the riskless investment amount M grows to RM.

The above portfolio containing the risky asset and money market account is said to replicate the long position of the call if and only if the values of the portfolio and the call option match for each possible outcome, that is,

$$\alpha uS + RM = c_u \quad \text{and} \quad \alpha dS + RM = c_d. \tag{2.1.25}$$

The unknowns are α and M in the above linear system of equations. It occurs that the number of unknowns (related to the number of units of asset and cash amount) and the number of equations (two possible states of the world under the binomial model) are equal. Solving the equations, we obtain

$$\alpha = \frac{c_u - c_d}{(u-d)S} \geq 0, \qquad M = \frac{uc_d - dc_u}{(u-d)R} \leq 0. \tag{2.1.26}$$

Since M is always non-positive, the replicating portfolio involves buying the asset and borrowing cash in the proportions given by Eq. (2.1.26). The number of units of asset held is seen to be the ratio of the difference of call values $c_u - c_d$ to the difference of asset values $uS - dS$.

Under the present one-period binomial model of asset price dynamics, we observe that the call option can be replicated by a portfolio of basic securities: risky asset and riskfree money market account.

Binomial option pricing formula

By the principle of no arbitrage, the current value of the call must be the same as that of the replicating portfolio. What happens if it were not? Suppose the current value of the call is less than the portfolio value, then we could make a riskless profit by buying the cheaper call and selling the more expensive portfolio. The net gain from the above two transactions is secured since the portfolio value and call value will cancel each other off one period later. The argument can be reversed if the call is worth more than the portfolio. Therefore, the current value of the call is given by the current value of the portfolio, that is,

$$\begin{aligned} c = \alpha S + M &= \frac{\frac{R-d}{u-d} c_u + \frac{u-R}{u-d} c_d}{R} \\ &= \frac{pc_u + (1-p)c_d}{R} \qquad \text{where} \quad p = \frac{R-d}{u-d}. \end{aligned} \tag{2.1.27}$$

Note that the probability q, which is the subjective probability about upward or downward movement of the asset price, does not appear in the call value formula (2.1.27). The parameter p can be shown to be $0 < p < 1$ since $u > R > d$ and so p can be interpreted as a probability. Further, from the relation

$$puS + (1-p)dS = \frac{R-d}{u-d}\, uS + \frac{u-R}{u-d}\, dS = RS, \tag{2.1.28}$$

one can interpret the result as follows: the expected rate of returns on the asset with p as the probability of upside move is just equal to the riskless interest rate. Let $S^{\Delta t}$ be the random variable that denotes the asset price one period later. We may express Eq. (2.1.28) as

$$S = \frac{1}{R} E^*(S^{\Delta t} | S), \tag{2.1.29}$$

where E^* is expectation under this probability measure. According to the definition given in Sec. 2.1.2, we may view p as the *risk neutral probability*.

Similarly, the call price formula (2.1.27) can be interpreted as the expectation of the payoff of the call option at expiry under the risk neutral probability measure discounted at the riskless interest rate [see Eq. (2.1.21b) for comparison]. The binomial call value formula (2.1.27) can be expressed as

$$c = \frac{1}{R} E^* \left(c^{\Delta t} | S \right), \tag{2.1.30}$$

where c denotes the call value at the current time, and $c^{\Delta t}$ denotes the random variable representing the call value one period later.

Besides applying the principle of replication of claims, the binomial option pricing formula can also be derived using the riskless hedging principle or via the concept of state prices (see Problems 2.15 and 2.16).

2.2 Filtrations, martingales and multi-period models

In this section, we extend our discussion of securities models to multi-period, where there are $T+1$ trading dates: $t = 0, 1, \cdots, T, T > 1$. Similar to an one-period model, we have a finite sample space Ω of K elements, $\Omega = \{\omega_1, \omega_2, \cdots, \omega_K\}$, which represents the possible states of the world. There is a probability measure P defined on the sample space with $P(\omega) > 0$ for all $\omega \in \Omega$. The securities model consists of M risky securities whose price processes are non-negative stochastic processes, as denoted by $S_m = \{S_m(t); t = 0, 1, \cdots, T\}, m = 1, \cdots, M$. In addition, there is a riskfree security whose price process $S_0(t)$ is deterministic, with $S_0(t)$ strictly positive and possibly non-decreasing. We may consider $S_0(t)$ as a money market account, and the quantity $r_t = \dfrac{S_0(t) - S_0(t-1)}{S_0(t-1)}, t = 1, \cdots, T$, is visualized as the interest rate over the time interval $(t-1, t)$.

In this section, we would like to show that the concepts of arbitrage opportunity and risk neutral valuation can be carried over from single-period models to multi-period models. However, we need to specify how the investors learn about the true state of the world on intermediate trading dates in a multi-period model. Accordingly, we have to construct some information structure that models how information is revealed to investors in terms of the subsets of the sample space Ω. We show how information structure can be described by a filtration and understand how security price processes can be adapted to a given filtration. After then, we introduce martingales, which are defined with reference to conditional expectations. In the multi-period setting, the risk neutral probability measures are defined in terms of martingales. The highlight of this section is the derivation of the Fundamental Theorem of Asset Pricing. The last part of this section will be devoted to the multi-period binomial models.

2.2.1 Information structures and filtrations

Consider the sample space $\Omega = \{\omega_1, \omega_2, \cdots, \omega_{10}\}$ with 10 elements. We can construct various partitions of the set Ω. A *partition* of Ω is a collection $\mathcal{P} = \{B_1, B_2, \cdots B_n\}$ such that $B_j, j = 1, \cdots, n$, are subsets of Ω and $B_i \cap B_j = \phi, i \neq j$, and $\bigcup_{j=1}^{n} B_j = \Omega$. Each of the sets $B_1, \cdots, B_n$ is called an *atom* of the partition. For example, we may form the partitions as

$$
\begin{aligned}
\mathcal{P}_0 &= \{\Omega\} \\
\mathcal{P}_1 &= \{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}\}\} \\
\mathcal{P}_2 &= \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7, \omega_8, \omega_9\}, \{\omega_{10}\}\} \\
\mathcal{P}_3 &= \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}, \{\omega_9\}, \{\omega_{10}\}\}.
\end{aligned}
$$

In the above, we have defined a finite sequence of partitions of Ω, which have the property that they are nested with successive refinements of one another. Each set belonging to $\mathcal{P}_k$ splits into smaller sets which are elements of $\mathcal{P}_{k+1}$.

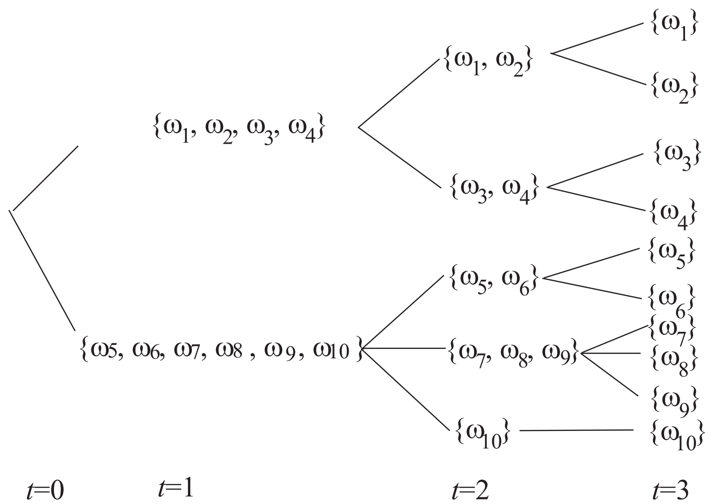

Fig. 2.4 Information tree of a three-period securities model with 10 possible states. The partitions form a sequence of successively finer partitions.

Consider a three-period securities model that consists of the above sequence of successively finer partitions: $\{\mathcal{P}_k : k = 0, 1, 2, 3\}$. The pair $(\Omega, \mathcal{P}_k)$ is called a *filtered space*, which consists of a sample space Ω and a sequence of partitions of Ω. The filtered space is used to model the unfolding of information through time. At time $t = 0$, the investors know only the set of all possible outcomes, so $\mathcal{P}_0 = \{\Omega\}$. At time $t = 1$, the investors get a bit more information: the actual state ω is in either $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ or $\{\omega_5, \omega_5, \omega_7, \omega_8, \omega_9, \omega_{10}\}$. In the next trading date $t = 2$, more information is revealed, say, ω is in the set $\{\omega_7, \omega_8, \omega_9\}$. On the last date $t = 3$, we have $\mathcal{P}_3 = \{\{\omega_i\}, i = 1, \cdots, 10\}$. Each set of $\mathcal{P}_3$ consists of a single element of Ω, so the investors have full information of which particular state has occurred. The information submodel of this three-period securities model can be represented by the information tree shown in Fig. 2.4.

Algebra

Let Ω be a finite set and $\mathcal{F}$ be a collection of subsets of Ω. The collection $\mathcal{F}$ is an *algebra* on Ω if
(i) $\Omega \in \mathcal{F}$
(ii) $B \in \mathcal{F} \Rightarrow B^c \in \mathcal{F}$
(iii) B_1 and $B_2 \in \mathcal{F} \Rightarrow B_1 \cup B_2 \in \mathcal{F}$.

Given an algebra $\mathcal{F}$ on Ω, one can always find a unique collection of disjoint subsets B_n such that each $B_n \in \mathcal{F}$ and the union of the subsets equals Ω. The algebra $\mathcal{F}$ generated by a partition $\mathcal{P} = \{B_1, \cdots, B_n\}$ is a set of subsets of Ω. Actually, when Ω is a finite sample space, there is a one-to-one correspondence between partitions of Ω and algebras on Ω. The information structure defined by a sequence of partitions can be visualized as a sequence of algebras. We define a *filtration* $\mathbb{F} = \{\mathcal{F}_k; k = 0, 1, \cdots, T\}$ to be a nested sequence of algebras satisfying $\mathcal{F}_k \subseteq \mathcal{F}_{k+1}$.

As an example, given the algebra $\mathcal{F} = \{\phi, \{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}, \{\omega_1, \omega_2, \omega_3\}, \{\omega_2, \omega_3, \omega_4\}, \{\omega_1, \omega_4\}, \{\omega_1, \omega_2, \omega_3, \omega_4\}\}$, the corresponding partition $\mathcal{P}$ is found to be $\{\{\omega_1\}, \{\omega_2, \omega_3\}, \{\omega_4\}\}$. The atoms of $\mathcal{P}$ are $B_1 = \{\omega_1\}, B_2 = \{\omega_2, \omega_3\}$ and $B_3 = \{\omega_4\}$. A non-empty event whose occurrence to be revealed through revelation of $\mathcal{P}$ would be an union of atoms in $\mathcal{P}$. Take the event $A = \{\omega_1, \omega_2, \omega_3\}$, which is the union of B_1 and B_2. Given that $B_2 = \{\omega_2, \omega_3\}$ of $\mathcal{P}$ has occurred, we can decide whether A or its complement A^c has occurred. However, for another event $\widetilde{A} = \{\omega_1, \omega_2\}$, even though we know that B_2 has occurred, we cannot determine whether $\widetilde{A}$ or $\widetilde{A}^c$ has occurred.

Next, we define a probability measure P defined on an algebra $\mathcal{F}$. The probability measure P is a function

$$P : \mathcal{F} \rightarrow [0, 1]$$

such that
1. $P(\Omega) = 1$.
2. If $B_1, B_2, \cdots$ are pairwise disjoint sets belonging to $\mathcal{F}$, then

$$P(B_1 \cup B_2 \cup \cdots) = P(B_1) + P(B_2) + \cdots.$$

Equipped with a probability measure, the elements of $\mathcal{F}$ are called measurable events. Given the sample space Ω, an algebra $\mathcal{F}$ and a probability measure P defined on Ω, the triplet $(\Omega, \mathcal{F}, P)$ together with the filtration $\mathbb{F}$ is called a *filtered probability space*.

Equivalent measures
Given two probability measures P and P' defined on the same measurable space $(\Omega, \mathcal{F})$, suppose that

$$P(\omega) > 0 \iff P'(\omega) > 0, \quad \text{for all } \omega \in \Omega,$$

then P and P' are said to be equivalent measures. In other words, though the two equivalent measures may not agree on the assignment of probability values to individual events, but they always agree as to which events are possible or impossible.

Measurability of random variables
Consider an algebra $\mathcal{F}$ generated by a partition $\mathcal{P} = \{B_1, \cdots, B_n\}$, a random variable X is said to be measurable with respect to $\mathcal{F}$ (denoted by $X \in \mathcal{F}$) if $X(\omega)$ is constant for all $\omega \in B_i, B_i$ is any element in $\mathcal{P}$. For example, consider the algebra $\mathcal{F}_1$ generated by $\mathcal{P}_1 = \{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8, \omega_9, \omega_{10}\}\}$. If $X(\omega_1) = 3$ and $X(\omega_4) = 5$, then X is not measurable with respect to $\mathcal{F}_1$.

Consider an example where $\mathcal{P} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5\}\}$ and X is measurable with respect to the algebra $\mathcal{F}$ generated by $\mathcal{P}$. Let $X(\omega_1) = X(\omega_2) = 3, X(\omega_3) = X(\omega_4) = 5$ and $X(\omega_5) = 7$. Suppose the random experiment associated with the random variable X is performed, giving $X = 5$. This tells the information that the event $\{\omega_3, \omega_4\}$ has occurred. One can argue that the information of outcome from the random experiment is revealed through the random variable X. We may say that $\mathcal{F}$ is being generated by X.

A stochastic process $S_m = \{S_m(t); t = 0, 1, \cdots, T\}$ is said to be *adapted to the filtration* $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\}$ if the random variables $S_m(t)$ is $\mathcal{F}_t$-measurable for each $t = 0, 1, \cdots, T$. For the bank account process $S_0(t)$, the interest rate is normally known at the beginning of the period so that $S_0(t)$ is $\mathcal{F}_{t-1}$-measurable, $t = 1, \cdots, T$. In this case, we say that the process $S_0(t)$ is *predictable*.

2.2.2 Conditional expectations and martingales

Consider the filtered probability space defined by the triplet $(\Omega, \mathcal{F}, P)$ together with the filtration $\mathbb{F}$. Recall that a random variable is a mapping $\omega \to X(\omega)$ that assigns a real number $X(\omega)$ to each $\omega \in \Omega$. A random variable is said to be simple if X can be decomposed into the form

$$X(\omega) = \sum_{j=1}^{n} a_j I_{B_j}(\omega) \tag{2.2.1}$$

where $\{B_1, \cdots B_n\}$ is a finite partition of Ω with each $B_j \in \mathcal{F}$ and the indicator of B_j is defined by

$$I_{B_j}(\omega) = \begin{cases} 1 & \text{if } \omega \in B_j \\ 0 & \text{if otherwise} \end{cases}. \tag{2.2.2}$$

The expectation of X with respect to the probability measure P is defined as

$$E[X] = \sum_{j=1}^{n} a_j E[I_{B_j}(\omega)] = \sum_{j=1}^{n} a_j P[B_j], \tag{2.2.3}$$

where $P(B_j)$ is the probability that a state ω contained in B_j occurs. The conditional expectation of X given that event B has occurred is defined to be

$$\begin{aligned} E[X|B] &= \sum_x xP[X = x|B] \\ &= \sum_x xP[X = x, B]/P[B] \\ &= \frac{1}{P[B]} \sum_{\omega \in B} X(\omega)P[\omega]. \end{aligned} \tag{2.2.4}$$

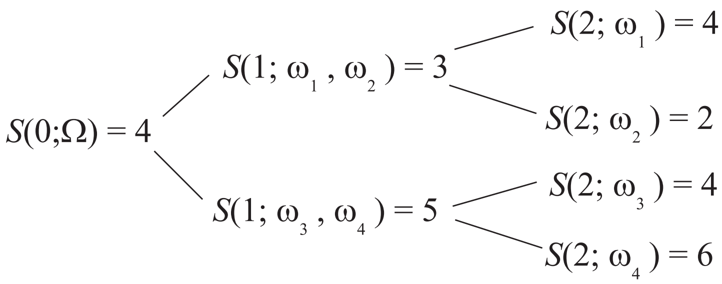

Fig. 2.5 The asset price process of a two-period securities model.

As a numerical example, consider the sample space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ and the algebra is generated by the partition $\mathcal{P} = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$. The probabilities of occurrence of the states are given by $P[\omega_1] = 0.2, P[\omega_2] = 0.3, P[\omega_3] = 0.35$ and $P[\omega_4] = 0.15$. Consider the two-period price process S whose values are given by

$$\begin{aligned} &S(1;\omega_1) = 3, \quad S(1;\omega_2) = 3, \quad S(1;\omega_3) = 5, \quad S(1;\omega_4) = 5, \\ &S(2;\omega_1) = 4, \quad S(2;\omega_2) = 2, \quad S(2;\omega_3) = 4, \quad S(2;\omega_4) = 6, \end{aligned}$$

where the tree representation is shown in Fig. 2.5. The conditional expectations $E[S(2)|\ S(1) = 3]$ and $E[S(2)|S(1) = 5]$ are computed using Eq. (2.2.4) as follows:

$$\begin{aligned} E[S(2)|S(1) = 3] &= \frac{S(2;\omega_1)P[\omega_1] + S(2;\omega_2)P[\omega_2]}{P[\omega_1] + P[\omega_2]} \\ &= (4 \times 0.2 + 2 \times 0.3)/0.5 = 2.8; \end{aligned} \tag{2.2.5a}$$

$$\begin{aligned} E[S(2)|S(1) = 5] &= \frac{S(2;\omega_3)P[\omega_3] + S(2;\omega_4)P[\omega_4]}{P[\omega_3] + P[\omega_4]} \\ &= (4 \times 0.35 + 6 \times 0.15)/0.5 = 4.6. \end{aligned} \tag{2.2.5b}$$

Interpretation of $E[X|\mathcal{F}]$

It is quite often that we would like to consider all conditional expectations of the form $E[X|B]$ where the event B runs through the algebra $\mathcal{F}$. We define the quantity $E[X|\mathcal{F}]$ by

$$E[X|\mathcal{F}]I_B = E[X|B] \quad \text{for all } B \in \mathcal{F}. \tag{2.2.6}$$

We see that $E[X|\mathcal{F}]$ is actually a random variable that is measurable with respect to the algebra $\mathcal{F}$. In the above numerical example, suppose we write $\mathcal{F}_1 = \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}\}$, then

$$E[S(2)|\mathcal{F}_1] = \begin{cases} 2.8 & \text{if } \omega_1 \text{ or } \omega_2 \text{ occurs} \\ 4.6 & \text{if } \omega_3 \text{ or } \omega_4 \text{ occurs} \end{cases}. \tag{2.2.7}$$

Since $E[X|\mathcal{F}]$ is a random variable, we may compute its expectation. We find that

$$\begin{aligned} E[E[X|\mathcal{F}]] &= \sum_{B\in\mathcal{F}} E[X|B]P[B] = \sum_{B\in\mathcal{F}}\sum_{\omega\in B} X(\omega)(P[\omega]/P[B])P[B] \\ &= \sum_{B\in\mathcal{F}}\sum_{\omega\in B} X(\omega)P[\omega] = E[X]. \end{aligned} \tag{2.2.8a}$$

The above result can be generalized as follows. If $\mathcal{F}_1 \subset \mathcal{F}_2$, then

$$E[E[X|\mathcal{F}_2]|\mathcal{F}_1] = E[X|\mathcal{F}_1]. \tag{2.2.8b}$$

If we condition first on the information up to $\mathcal{F}_2$ and later on the information $\mathcal{F}_1$ at an earlier time, then it is the same as conditioning originally on $\mathcal{F}_1$. This is called the *tower property* of conditional expectations.

Suppose that the random variable X is $\mathcal{F}$-measurable, we would like to show $E[XY|\mathcal{F}] = XE[Y|\mathcal{F}]$ for any random variable Y. Using Eq. (2.2.1), we may write $X = \sum_{B_j\in\mathcal{P}} a_j I_{B_j}$, where $\mathcal{P}$ is the partition corresponding to the algebra $\mathcal{F}$. Further, by Eq. (2.2.6), we obtain

$$\begin{aligned} E[XY|\mathcal{F}] &= \sum_{B_j\in\mathcal{P}} E[XY|B_j]I_{B_j} = \sum_{B_j\in\mathcal{P}} E[a_jY|B_j]I_{B_j} \\ &= \sum_{B_j\in\mathcal{P}} a_jE[Y|B_j]I_{B_j} = XE[Y|\mathcal{F}]. \end{aligned} \tag{2.2.9}$$

When we take the conditional expectation with respect to the filtration $\mathcal{F}$, we can treat X as constant if X is known with regard to the information provided by $\mathcal{F}$. The proofs of other properties on conditional expectations are relegated as exercises (see Problem 2.18).

Martingales

The term martingale has its origin in gambling. It refers to the gambling tactics of doubling the stake when losing in order to recoup oneself. In the studies of stochastic processes, martingales are defined in relation to an adapted stochastic process.

Consider a filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\}$. An adapted stochastic process $S = \{S(t); t = 0, 1 \cdots, T\}$ is said to be martingale if it observes

$$E[S(t+s)|\mathcal{F}_t] = S(t) \quad \text{for all } t \geq 0 \text{ and } s \geq 0. \tag{2.2.10}$$

We define an adapted stochastic process S to be a supermartingale if

$$E[S(t+s)|\mathcal{F}_t] \leq S(t) \quad \text{for all } t \geq 0 \text{ and } s \geq 0; \tag{2.2.11a}$$

and a submartingale if

$$E[S(t+s)|\mathcal{F}_t] \geq S(t) \quad \text{for all } t \geq 0 \text{ and } s \geq 0. \tag{2.2.11b}$$

It is straightforward to deduce the following properties:
1. All martingales are supermartingales, but not vice versa. The same observation is applied to submartingales.
2. An adapted stochastic process S is a submartingale if and only if $-S$ is a supermartingale; S is a martingale if and only if it is both a supermartingale and a submartingale.

Martingales are related to models of fair gambling. For example, let X_n represent the amount of money a player possesses at stage n of the game. The martingale property means that the expected amount of the player would have at stage $n+1$ given that $X_n = \alpha_n$, is equal to α_n, regardless of his past history of fortune.

It must be emphasized that a martingale is defined with respect to a filtration (information set) and with respect to some probability measure. In later sections we will show that martingales are found to be very useful tools in option pricing theory. Indeed, the risk neutral valuation approach in option pricing theory is closely related to the theory of martingales. In Sec. 2.2.3, we show that the necessary and sufficient condition for the exclusion of arbitrage opportunities in a securities model is the existence of a risk neutral pricing measure constructed from the martingale property of the asset price processes.

Martingale transforms

Suppose S is a martingale and H is a predictable process with respect to the filtration $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\}$, we define the process

$$G_t = \sum_{u=1}^{t} H_u \Delta S_u, \tag{2.2.12}$$

where $\Delta S_u = S_u - S_{u-1}$. One then deduces that $\Delta G_u = G_u - G_{u-1} = H_u \Delta S_u$. If S and H represent the asset price process and trading strategy, respectively, then G can be visualized as the gain process. Note that trading strategy is a predictable process, that is H_t is $\mathcal{F}_{t-1}$-measurable. This is because the number of units held for each security is determined at the beginning of the trading period by taking into account all the information available up to that time.

We call G to be the martingale transform of S by H, as G itself is also a martingale. To show the claim, it suffices to show that $E[G_{t+s}|\mathcal{F}_t] = G_t, t \geq 0, s \geq 0$. We consider

$$\begin{aligned} E[G_{t+s}|\mathcal{F}_t] &= E[G_{t+s} - G_t + G_t|\mathcal{F}_t] \\ &= E[H_{t+1}\Delta S_{t+1} + \cdots + H_{t+s}\Delta S_{t+s}|\mathcal{F}_t] + E[G_t|\mathcal{F}_t] \\ &= E[H_{t+1}\Delta S_{t+1}|\mathcal{F}_t] + \cdots + E[H_{t+s}\Delta S_{t+s}|\mathcal{F}_t] + G_t. \quad (2.2.13) \end{aligned}$$

Consider the typical term $E[H_{t+u}\Delta S_{t+u}|\mathcal{F}_t]$, we can express it as $E[E[H_{t+u} \Delta S_{t+u}|\mathcal{F}_{t+u-1}]|\mathcal{F}_t]$, by the result in Eq. (2.2.8b). Further, since H_{t+u} is $\mathcal{F}_{t+u-1}$-measurable and S is a martingale, by virtue of Eqs. (2.2.9,10), we have

$$E[H_{t+u}\Delta S_{t+u}|\mathcal{F}_{t+u-1}] = H_{t+u}E[\Delta S_{t+u}|\mathcal{F}_{t+u-1}] = 0. \quad (2.2.14)$$

Collecting all the calculations, we obtain the desired result.

2.2.3 Multi-period securities models

Once we are equipped with the knowledge of filtrations, adapted stochastic processes and martingales, we now move to the discussion of the fundamentals of financial economics of multi-period securities models. In particular, we will consider the relation between exclusion of arbitrage opportunities and existence of the martingale measure (risk neutral probability measure).

We start with the prescription of a discrete n-period securities model with M risky securities. Like the discrete single-period model, there is a sample space $\Omega = \{\omega_1, \omega_2, \cdots, \omega_K\}$ of K possible states of the world. Let S denote the asset price process $\{S(t); t = 0, 1, \cdots, n\}$, where $S(t)$ is the row vector $S(t) = (S_1(t)\ S_2(t) \cdots S_M(t))$ and whose components are security prices. Also, there is a bank account process $S_0(t)$, whose value is given by

$$S_0(t) = (1 + r_0)(1 + r_1) \cdots (1 + r_{t-1}), \quad (2.2.15)$$

where r_u is the interest rate applied over one time period starting at time $u, u = 0, 1, \cdots, t-1$. A trading strategy is the rule taken by an investor that specifies the investor's position in each security at each time and in each state of the world based on the available information as prescribed by a filtration. Hence, one can visualize a trading strategy as an adapted stochastic process. We prescribe a trading strategy by a vector stochastic process

$H(t) = (h_1(t)\ h_2(t) \cdots h_M(t))^T, t = 1, 2, \cdots, n$ (represented as a column vector), where $h_m(t)$ is the number of units held in the portfolio for the m^{th} security from time $t-1$ to time t. Also, the amount of bank account held at time $t-1$ is given by $h_0(t)S_0(t)$. Note that $h_m(t)$ should be $\mathcal{F}_{t-1}$-measurable, $m = 0, 1, \cdots, M$.

The value of the portfolio is a stochastic process given by

$$V(t) = h_0(t)S_0(t) + \sum_{m=1}^{M} h_m(t)S_m(t), \quad t = 1, 2, \cdots, n, \qquad (2.2.16)$$

which gives the portfolio value at the moment right after the asset prices are observed but before changes in portfolio weights are made.

We write $\Delta S_m(t) = S_m(t) - S_m(t-1)$ as the change in value of one unit of the m^{th} security between times $t-1$ and t. The cumulative gain associated with investing in the m^{th} security from time zero to time t is given by

$$\sum_{u=1}^{t} h_m(u)\Delta S_m(u).$$

We define the gain process $G(t)$ to be the total cumulative gain in holding the portfolio consisting of the M risky securities and the bank account up to time t. The value of $G(t)$ is found to be

$$G(t) = \sum_{u=1}^{t} h_0(u)\Delta S_0(u) + \sum_{m=1}^{M}\sum_{u=1}^{t} h_m(u)\Delta S_m(u), \quad t = 1, 2, \cdots, n. \ (2.2.17)$$

If we define the discounted price process $S_m^*(t)$ by

$$S_m^*(t) = S_m(t)/S_0(t), \quad t = 0, 1, \cdots, n, m = 1, 2, \cdots, M, \quad (2.2.18a)$$

and write $\Delta S_m^*(t) = S_m^*(t) - S_m^*(t-1)$, then the discounted value process $V^*(t)$ and discounted gain process $G^*(t)$ are given by

$$V^*(t) = h_0(t) + \sum_{m=1}^{M} h_m(t)S_m^*(t), \quad t = 1, 2, \cdots n, \qquad (2.2.18b)$$

$$G^*(t) = \sum_{m=1}^{M}\sum_{u=1}^{t} h_m(u)\Delta S_m^*(u), \quad t = 1, 2, \cdots, n. \qquad (2.2.18c)$$

Once the asset prices, $S_m(t), m = 1, 2, \cdots, M$, are revealed to the investor, he changes the trading strategy from $H(t)$ to $H(t+1)$ as a response to the arrival of the new information. In general, there may be addition or withdrawal of fund from the portfolio. However, if there were no such addition and withdrawal, then the portfolio value remains the same; and correspondingly, we observe

$$V(t) = h_0(t+1)S_0(t) + \sum_{m=1}^{M} h_m(t+1)S_m(t). \qquad (2.2.19)$$

The purchase of additional units of one particular security is financed by the sales of other securities. In this case, the trading strategy is said to be *self-financing*.

If there were no addition or withdrawal of funds at all trading times, then the cumulative change of portfolio value $V(t) - V(0)$ should be equal to the gain $G(t)$ associated with price changes of the securities on all trading dates. Hence, we expect that a trading strategy H is self-financing if and only if

$$V(t) = V(0) + G(t). \qquad (2.2.20)$$

To show the claim, we rewrite Eqs. (2.2.16a) and (2.2.19) as

$$V(u) = h_0(u)S_0(u) + \sum_{m=1}^{M} h_m(u)S_m(u) \qquad (2.2.21a)$$

$$V(u-1) = h_0(u)S_0(u-1) + \sum_{m=1}^{M} h_m(u)S_m(u-1), \qquad (2.2.21b)$$

respectively, then subtract the two equations to obtain

$$V(u) - V(u-1) = h_0(u)\Delta S_0(u) + \sum_{m=1}^{M} h_m(u)\Delta S_m(u). \quad (2.2.21c)$$

Summing the above equation from $u = 1$ to $u = t$ and applying the result in Eq. (2.2.17), we obtain the result in Eq. (2.2.20). In a similar manner, we can use Eqs. (2.2.18b,c) to show that H is self-financing if and only if

$$V^*(t) = V^*(0) + G^*(t). \qquad (2.2.22)$$

No arbitrage principle and martingale measure

The definition of arbitrage opportunity for single period securities model (see Sec. 2.1.2) is extended to multi-period models. A trading strategy H represents an arbitrage opportunity if and only if the value process $V(t)$ and H satisfy the following properties:

(i) $V(0) = 0$,

(ii) $V(T) \geq 0$ and $EV(T) > 0$, and

(iii) H is self-financing.

Equivalently, we may state that the self-financing trading strategy H is an arbitrage opportunity if and only if (i) $G^*(T) \geq 0$ and (ii) $EG^*(T) > 0$. Like that in single period models, we expect that arbitrage opportunity does not exist if and only if there exists a risk neutral probability measure. In multi-period models, risk neutral probabilities are defined in terms of martingales.

Martingale measure
The measure Q is called a martingale measure (or called risk neutral probability measure) if it has the following properties:
1. $Q(\omega) > 0$ for all $\omega \in \Omega$.
2. Every discounted price process S_m^* in the securities model is a martingale under $Q, m = 1, 2, \cdots, M$, that is,

$$E_Q[S_m^*(t+s)|\mathcal{F}_t] = S_m^*(t) \quad \text{for all } t \geq 0 \text{ and } s \geq 0. \tag{2.2.23}$$

We call the discounted price process $S_m^*(t)$ to be a Q-martingale.

As a numerical example, we determine the martingale measure Q associated with the two-period securities model shown in Fig. 2.5. Let $r \geq 0$ be the constant riskless interest rate over one period, and write $Q(\omega_j)$ as the martingale measure associated with the state $\omega_j, j = 1, 2, 3, 4$. By invoking Eq. (2.2.23), we obtain the following equations for $Q(\omega_1), \cdots, Q(\omega_4)$:
(i) $t = 0$ and $s = 1$

$$4 = \frac{3}{1+r}[Q(\omega_1) + Q(\omega_2)] + \frac{5}{1+r}[Q(\omega_3) + Q(\omega_4)] \tag{2.2.24a}$$

(ii) $t = 0$ and $s = 2$

$$\begin{aligned} 4 = {} & \frac{4}{(1+r)^2}Q(\omega_1) + \frac{2}{(1+r)^2}Q(\omega_2) \\ & + \frac{4}{(1+r)^2}Q(\omega_3) + \frac{6}{(1+r)^2}Q(\omega_4) \end{aligned} \tag{2.2.24b}$$

(iii) $t = 1$ and $s = 1$

$$3 = \frac{4}{1+r}\frac{Q(\omega_1)}{Q(\omega_1) + Q(\omega_2)} + \frac{2}{1+r}\frac{Q(\omega_2)}{Q(\omega_1) + Q(\omega_2)} \tag{2.2.24c}$$

$$5 = \frac{4}{1+r}\frac{Q(\omega_3)}{Q(\omega_3) + Q(\omega_4)} + \frac{6}{1+r}\frac{Q(\omega_4)}{Q(\omega_3) + Q(\omega_4)}. \tag{2.2.24d}$$

It may be quite tedious to solve the above equations simultaneously. The calculation procedure can be simplified by observing that $Q(\omega_j)$ is given by the product of the conditional probabilities along the path from the node at $t = 0$ to the node ω_j at $t = 2$. First, we start with the conditional probability p associated with the upper branch $\{\omega_1, \omega_2\}$. The corresponding conditional probability p is given by

$$4 = \frac{3}{1+r}p + \frac{5}{1+r}(1-p) \tag{2.2.25a}$$

so that $p = \dfrac{1-4r}{2}$. Similarly, the conditional probability p' associated with the branch $\{\omega_1\}$ from the node $\{\omega_1, \omega_2\}$ is given by

$$3 = \frac{4}{1+r}p' + \frac{2}{1+r}(1-p') \tag{2.2.25b}$$

giving $p' = \dfrac{1-3r}{2}$. In a similar manner, the conditional probability p'' associated with $\{\omega_3\}$ from $\{\omega_3, \omega_4\}$ is found to be $\dfrac{1-5r}{2}$. The martingale probabilities are then found to be

$$\begin{aligned}
Q(\omega_1) &= \; pp' = \frac{1-4r}{2}\frac{1-3r}{2}, \\
Q(\omega_2) &= \; p(1-p') = \frac{1-4r}{2}\frac{1+3r}{2}, \\
Q(\omega_3) &= \; (1-p)p'' = \frac{1+4r}{2}\frac{1-5r}{2} \\
Q(\omega_4) &= \; (1-p)(1-p'') = \frac{1+4r}{2}\frac{1+5r}{2}.
\end{aligned} \tag{2.2.26}$$

These martingale probabilities can be shown to satisfy Eqs. (2.2.24a-d). In order that the martingale probabilities remain positive, we have to impose the restriction: $r < 0.2$.

Martingale property of value processes
Suppose H is a self-financing trading strategy and Q is a martingale measure with respect to a filtration $\mathcal{F}$, then the value process $V(t)$ is a Q-martingale. To show the claim, we use the relation

$$V^*(t) = V^*(0) + G^*(t) \tag{2.2.27a}$$

since H is self-financing [see Eq. (2.2.22)]; and deduce that

$$\begin{aligned}
V^*(t+1) - V^*(t) &= \; G^*(t+1) - G^*(t) \\
&= \; [S^*(t+1) - S^*(t)]H(t+1).
\end{aligned} \tag{2.2.27b}$$

As H is a predictable process, $V^*(t)$ is the martingale transform of the Q-martingale $S^*(t)$. Hence, $V^*(t)$ itself is also a Q-martingale.

The above result can be applied to show that the existence of martingale measure Q implies the non-existence of arbitrage opportunities. To prove the claim, suppose H is any self-financing trading strategy with $V^*(T) \geq 0$ and $EV^*(T) > 0$. As $Q(\omega) > 0$, we then have $E_Q V^*(T) > 0$. Furthermore, since $V^*(T)$ is a Q-martingale, we have $V^*(0) = E_Q V^*(T) > 0$. Therefore, H cannot be an arbitrage opportunity.

The converse of the above claim remains to be valid, that is, the non-existence of arbitrage opportunities implies the existence of a martingale measure. The intuition behind the proof can be outlined as follows. If there are no arbitrage opportunities in the multi-period model, then there will be no arbitrage opportunities in any underlying single period. Since each single period does not admit arbitrage opportunities, one can construct the one-period risk neutral conditional probabilities. The martingale probability measure $Q(\omega)$ is then obtained by multiplying all the risk neutral conditional probabilities along the path from the node at $t = 0$ to the terminal node (T, ω). The rigorous proofs of these arguments are quite technical, the details of which can be found in the text by Bingham and Kiesel (1998).

We summarize the result into the following theorem.

Theorem 2.3
A multi-period securities model is arbitrage free if and only if there exists a probability measure Q such that the discounted asset price processes are Q-martingales.

Most of the results on valuation of contingent claims in the single period models can be extended to the multi-period models. First, the martingale measure is unique if and only if the multi-period securities model is complete. Here, completeness implies that all contingent claims ($\mathcal{F}_T$-measurable random variables) can be replicated by a self-financing trading strategy. In an arbitrage free complete market, the arbitrage price of a contingent claim is then given by the discounted expected value under the martingale measure of the unique portfolio that replicates the claim. Let Y denote a contingent claim at maturity T and $V(t)$ denote the arbitrage price of the contingent claim at time $t, t < T$. We then have

$$V(t) = \frac{S_0(t)}{S_0(T)} E_Q[Y|\mathcal{F}_t], \tag{2.2.28}$$

where $S_0(t)$ is the riskless asset and the ratio $S_0(t)/S_0(T)$ is the discount factor over the period from t to T. Rigorous justification to these results can be found in Bingham and Kiesel's text (1998).

2.2.4 Multi-period binomial models

We extend the one-period binomial model to its multi-period version. We start with the two-period binomial model. The corresponding dynamics of the binomial process for the asset price and the call price are shown in Fig. 2.6. It is assumed that the jump ratios of the asset price, u and d, stay the same value for all binomial steps.

Let c_{uu} denote the call value at two periods beyond the current time with two consecutive upward moves of the asset price and similar notational interpretation for c_{ud} and c_{dd}. Based on a similar argument as depicted in Eq. (2.1.27), the call values c_u and c_d are related to c_{uu}, c_{ud} and c_{dd} as follows:

$$c_u = \frac{pc_{uu} + (1-p)c_{ud}}{R} \text{ and } c_d = \frac{pc_{ud} + (1-p)c_{dd}}{R}. \tag{2.2.29a}$$

Subsequently, by substituting the above results into Eq. (2.1.27), the call value at the current time which is two periods from expiry is found to be

$$c = \frac{p^2 c_{uu} + 2p(1-p)c_{ud} + (1-p)^2 c_{dd}}{R^2}, \qquad (2.2.29b)$$

where the corresponding terminal payoff values are given by

$$c_{uu} = \max(u^2S - X, 0), c_{ud} = \max(udS - X, 0), c_{dd} = \max(d^2S - X, 0). \qquad (2.2.30)$$

Note that the coefficients $p^2, 2p(1-p)$ and $(1-p)^2$ represent the respective risk neutral probability of having two up jumps, one up jump and one down jump, and two down jumps in two moves of the binomial process.

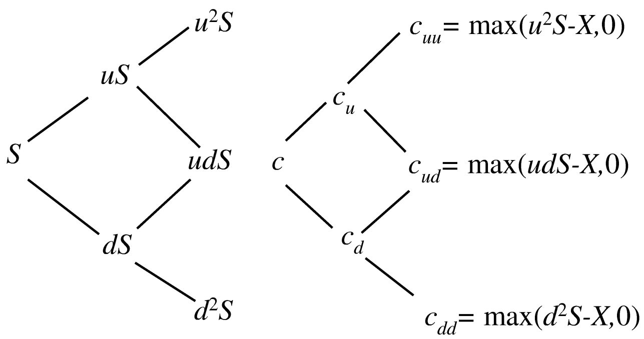

Fig. 2.6 Dynamics of the asset price and call price in a two-period binomial model.

The extension of the binomial model to the n-period case should be quite straightforward. With n binomial steps, the risk neutral probability of having j up jumps and $n-j$ down jumps is given by $C_j^n p^j (1-p)^{n-j}$, where $C_j^n = \dfrac{n!}{j!(n-j)!}$ is the binomial coefficient. The corresponding terminal payoff when j up jumps and $n-j$ down jumps occur is seen to be $\max(u^j d^{n-j} S - X, 0)$. The call value obtained from the n-period binomial model is given by

$$c = \frac{\displaystyle\sum_{j=0}^{n} C_j^n p^j (1-p)^{n-j} \max(u^j d^{n-j} S - X, 0)}{R^n}. \qquad (2.2.31)$$

We define k to be the smallest non-negative integer such that $u^k d^{n-k} S \geq X$, that is, $k \geq \dfrac{\ln \frac{X}{Sd^n}}{\ln \frac{u}{d}}$. It is seen that

$$\max(u^j d^{n-j} S - X, 0) = \begin{cases} 0 & \text{when} \quad j < k \\ u^j d^{n-j} S - X & \text{when} \quad j \geq k. \end{cases} \qquad (2.2.32)$$

The integer k gives the minimum number of upward moves required for the asset price in the multiplicative binomial process in order that the call expires in-the-money. The call formula can then be simplified as

$$c = S\sum_{j=k}^{n} C_j^n p^j (1-p)^{n-j} \frac{u^j d^{n-j}}{R^n} - XR^{-n}\sum_{j=k}^{n} C_j^n p^j (1-p)^{n-j}. \quad (2.2.33)$$

The last term in above equation can be interpreted as the expectation value of the payment made by the holder at expiration discounted by the factor R^{-n}, and $\sum_{j=k}^{n} C_j^n p^j (1-p)^{n-j}$ is seen to be the probability (under the risk neutral measure) that the call will expire in-the-money. The above probability is related to the *complementary binomial distribution function* defined by

$$\Phi(n,k,p) = \sum_{j=k}^{n} C_j^n p^j (1-p)^{n-j}. \quad (2.2.34)$$

Note that $\Phi(n,k,p)$ gives the probability for at least k successes in n trials of a binomial experiment, where p is the probability of success in each trial. Further, if we write $p' = \frac{up}{R}$ so that $1-p' = \frac{d(1-p)}{R}$, then the call price formula for the n-period binomial model can be expressed as

$$c = S\Phi(n,k,p') - XR^{-n}\Phi(n,k,p). \quad (2.2.35)$$

The first term gives the discounted expectation of the asset price at expiration given that the call expires in-the-money and the second term gives the present value of the expected cost incurred by exercising the call.

Using the argument of discounted expectation of the payoff of a contingent claim under the risk neutral measure, the call price for the n-period binomial model can be expressed in the following canonical form

$$c = \frac{1}{R^n} E^*(c_T) = \frac{1}{R^n} E^*\left[\max(S_T - X, 0)\right], \quad T = t + n\Delta t, \quad (2.2.36)$$

where c_T is the payoff, $\max(S_T - X, 0)$, of the call at expiration time T and $\frac{1}{R^n}$ is the discount factor over n periods. Here, the expectation operator E^* is taken under the risk neutral measure rather than the true probability measure associated with the actual (physical) asset price process.

Numerical implementation

The n-period binomial model can be represented schematically by a n-step tree structure (see Fig. 2.7 for a 3-step tree). The binomial tree will be symmetrical about S if $ud = 1$, skewed upward if $ud > 1$ and skewed downward if $ud < 1$. At the time level that is m time steps marching forward from the current time in the binomial tree, there are $m+1$ nodes. The asset price at the node obtained by j upward moves and $m - j$ downward moves equals $Su^j d^{m-j}, j = 0, 1, \cdots m$. The possible option values at expiration are known since the payoff function at expiry is given in the option contract. Rather than using the multiplicative binomial formula (2.2.31), the following stepwise backward induction procedure is more effective in numerical implementation. First, we compute option values at the nodes that are one time step from expiration using the binomial formula (2.1.27). Once option values at one time step from expiration are known, we proceed two time steps from expiration and repeat the same numerical procedure. This backward induction procedure is similar in spirit as the procedure used in the derivation of Eqs. (2.2.28-29). After performing n backward steps in the tree, we come to the starting node (tip of the tree) at which the option value is desired.

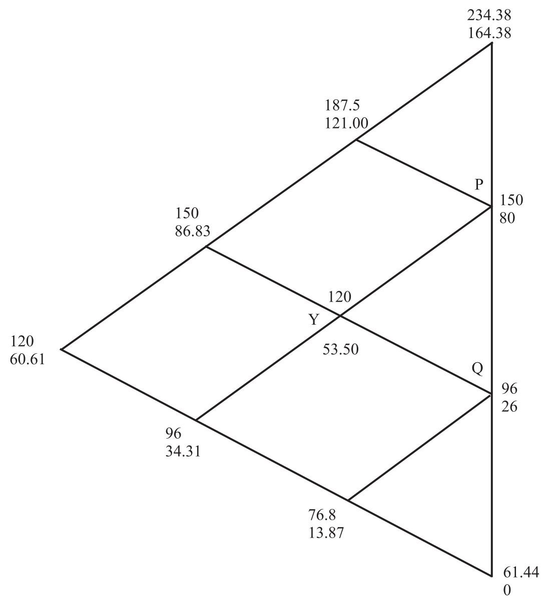

Fig. 2.7 Illustration of the binomial calculations with 3 time steps for a European call with strike price $X = 70$. The top figures are asset prices and the bottom figures are option values.

As a numerical example, suppose we have chosen the following values for the binomial parameters: $u = 1.25, d = 0.8$, and the discount factor for one period $= 1/R = 0.95$. According to Eq. (2.1.27), we have

$$p = \frac{R - d}{u - d} = \left(\frac{1}{0.95} - 0.8\right) \Big/ (1.25 - 0.8) = 0.5614. \tag{2.2.37}$$

The strike price of the call is taken to be 70 and the asset price S at the current time is 120. The binomial tree with three time steps is illustrated in

Fig. 2.7. The upper and lower figures at the nodes denote the asset prices and option values, respectively. For example, the option values at nodes P and Q are, respectively, $\max(150-70,0)=80$ and $\max(96-70,0)=26$. The option value at node Y is computed by

$$\begin{aligned} c_Y &= \frac{1}{R}[pc_P + (1-p)c_Q] \\ &= 0.95(0.5614 \times 80 + 0.4386 \times 26) \\ &= 53.50 \text{ (2 decimal places).} \end{aligned} \tag{2.2.38}$$

Working backward from the expiration time to the current time, the current option value at $S = 120$ is found to be 60.61 (see Fig. 2.7).

2.3 Asset price dynamics and stochastic processes

In this section, we discuss the stochastic models for the simulation of the asset price movement. The asset price movement is said to follow a *stochastic process* if its value changes over time in an uncertain manner. The study of stochastic processes is concerned with the investigation of the structure of families of random variables X_t, where t is a parameter (t is usually interpreted as the time parameter) running over some index set $\mathcal{T}$. If the index set $\mathcal{T}$ is discrete, then the stochastic process $\{X_t, t \in \mathcal{T}\}$ is called a discrete stochastic process, and if the index set $\mathcal{T}$ is continuous, then $\{X_t, t \in \mathcal{T}\}$ is called a continuous stochastic process. In other words, a discrete-time stochastic process for the asset price is one where the asset price can change at some discrete fixed times while the asset price which follows a continuous-time stochastic process can change its value at any time. Further, the value taken by the random variable X_t can be either discrete or continuous, and the corresponding stochastic process is called discrete-valued or continuous-valued, respectively. In reality, stock prices can change only at discrete values and during periods when the stock exchange is open. However, for simplicity, we assume continuous-valued, continuous-time stochastic processes for asset price movement models for most of the later exposition so that analytic tools in stochastic calculus can be employed.

A *Markovian process* is a stochastic process that, given the value of X_s, the value of $X_t, t > s$, depends only on X_s but not on the values taken by $X_u, u < s$. If the asset prices follow a Markovian process, then only the present asset prices are relevant for predicting their future values. This Markovian property of asset prices is consistent with the *weak form of market efficiency*, which assumes that the present value of an asset price already impounds all information in past prices and the particular path taken by the asset price to reach the present value is irrelevant. If the past history was indeed relevant, that is, a particular pattern might have a higher chance of price increases, then investors would bid up the asset price once such a pattern occurs and the profitable advantage would be eliminated.

We start with the discussion of the discrete random walk model and deduce its continuum limit. We obtain the Fokker-Planck equation that governs the probability density function of the continuous random walk motion. We then present the formal definition of a Brownian motion and discuss some of the properties of Brownian processes.

2.3.1 Random walk models

We describe the unrestricted, one-dimensional discrete random walk and consider the continuum limit of the discrete random walk problem to yield the continuous random walk model. Suppose a particle starts at the origin of the x-axis and it jumps either to the left or the right of the same length δ. We define x_i to be a random variable which takes the value δ or $-\delta$ when the particle at the ith step moves to the right or the left, respectively. Assume that the jump probabilities are stationary, that is, these probabilities are the same at all times. We then write the probabilities as

$$P[x_i = \delta] = p, \quad P[x_i = -\delta] = q, \tag{2.3.1}$$

where $p + q = 1, p$ and q are independent of i. The individual jumps are assumed to be independent of each other so that the random variables x_i are independent. This discrete random walk problem is then a discrete Markovian process (see Fig. 2.8).

Define the discrete sum process

$$X_n = x_1 + x_2 + \cdots + x_n, \tag{2.3.2}$$

which gives the position of the particle at the nth step. Since the expected value of x_i is

$$E[x_i] = \delta p - \delta q = (p - q)\delta, \quad i = 1, 2, \cdots, n, \tag{2.3.3a}$$

therefore

$$E[X_n] = E\left[\sum_{i=1}^{n} x_i\right] = \sum_{i=1}^{n} E[x_i] = (p - q)\delta n. \tag{2.3.3b}$$

As x_i's are independent, we have

$$\text{var}(X_n) = n \ \text{var}(x_i). \tag{2.3.3c}$$

The variance of x_i is

$$\text{var}(x_i) = [\delta^2 p + (-\delta)^2 q] - (E[x_i])^2 = \delta^2 - (p - q)^2\delta^2 = 4pq\delta^2 \tag{2.3.4a}$$

so that

$$\text{var}(X_n) = 4pq\delta^2 n. \tag{2.3.4b}$$

We call $X_{n+k}-X_n$ an increment of the discrete random walk model. Since X_n is a sum process of independent and identically distributed (iid) random variables, it observes the properties of stationary and independent increments.

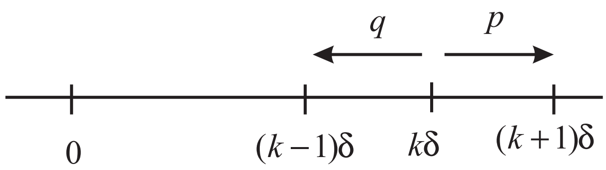

Fig. 2.8 A pictorial representation of the discrete random walk model. Suppose the particle is at position $x = k\delta$ after $i-1$ steps, $|k| \leq i-1$. In the ith step, it moves to the right with probability p or the left with probability q.

Continuum limit

Next, we take the continuum limit of infinitesimally small step size of the above discrete model to yield the continuous random walk model. Suppose there are r steps per unit time, then according to Eqs. (2.3.3b, 2.3.4b), the mean displacement of the particle per unit time μ is $(p-q)\delta r$ and the variance of the observed displacement around the mean position per unit time σ^2 is $4pq\delta^2 r$. Let $\lambda = 1/r$, which is the time interval between two successive steps, and let $u(x,t)$ denote the probability that the particle takes the position x at time t. Now, we write $X_n = x$ and $n\lambda = t$ so that

$$u(x,t) = P[X_n = x] \quad \text{at } t = n\lambda. \tag{2.3.5}$$

The probability function $u(x,t)$ satisfies the recurrence relation

$$u(x,t+\lambda) = pu(x-\delta,t) + qu(x+\delta,t). \tag{2.3.6}$$

In the continuum limit, we take $\delta \to 0$ and $r \to \infty$ so that $\lambda \to 0$. Now, consider the Taylor expansion of relation (2.3.6):

$$\begin{aligned} u(x,t)+\lambda\frac{\partial u}{\partial t}(x,t) + O(\lambda^2) = \; & p\left[u(x,t) - \delta\frac{\partial u}{\partial x}(x,t) + \frac{\delta^2}{2}\frac{\partial^2 u}{\partial x^2}(x,t) + O(\delta^3)\right] \\ & + q\left[u(x,t) + \delta\frac{\partial u}{\partial x}(x,t) + \frac{\delta^2}{2}\frac{\partial^2 u}{\partial x^2}(x,t) + O(\delta^3)\right], \end{aligned} \tag{2.3.7a}$$

and upon simplification, we obtain

$$\frac{\partial u}{\partial t} = \left[(q-p)\frac{\delta}{\lambda}\right]\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\delta^2}{\lambda}\right)\frac{\partial^2 u}{\partial x^2} + O(\lambda) + O\left((q-p)\frac{\delta^3}{\lambda}\right). \tag{2.3.7b}$$

We take the limits $\delta, \lambda \to 0$ such that the mean displacement and variance per unit time are given by

$$(p-q)\frac{\delta}{\lambda} = \mu \text{ and } 4pq\frac{\delta^2}{\lambda} = \sigma^2, \tag{2.3.8}$$

where μ and σ^2 are finite quantities. One can deduce from detailed asymptotic analysis that the probability values p and q should not be infinitesimal quantities, that is, $p = O(1), q = O(1)$ and $p + q = 1$. Consequently, we can deduce from $4pq\frac{\delta^2}{\lambda} = \sigma^2$ that $\frac{\delta^2}{\lambda} = O(1)$ and $\frac{\delta}{\lambda} = O\left(\frac{1}{\delta}\right)$. Further, from $(p-q)\frac{\delta}{\lambda} = \mu$ and $p+q = 1$, the asymptotic expansion of p and q up to $O(\delta)$ must take the following forms:

$$p \approx \frac{1}{2}(1 + k\delta) \text{ and } q \approx \frac{1}{2}(1 - k\delta) \tag{2.3.9}$$

for some k to be determined. We then have $4pq \approx 1$ and so

$$\lim_{\delta,\lambda\to 0} \frac{\delta^2}{\lambda} = \sigma^2. \tag{2.3.10}$$

Lastly, from $(p-q)\frac{\delta}{\lambda} = \mu$ and condition (2.3.10), one deduces that $p-q \approx \frac{\mu}{\sigma^2}\delta$ and so $k = \frac{\mu}{\sigma^2}$. The asymptotic expansion of p and q are then found to be

$$p \approx \frac{1}{2}\left(1 + \frac{\mu}{\sigma^2}\delta\right) \text{ and } q \approx \frac{1}{2}\left(1 - \frac{\mu}{\sigma^2}\delta\right). \tag{2.3.11}$$

Note that $p \to \frac{1}{2}$ and $q \to \frac{1}{2}$ when taking the asymptotic limit $\delta \to 0$; otherwise, the drift rate would become infinite. Since $\frac{\delta^2}{\lambda} = O(1)$, the last term in Eq. (2.3.7b) becomes $O\left((q-p)\frac{\delta^3}{\lambda}\right) = O(\lambda)$. Consequently, by taking the limits $\delta, \lambda \to 0$ in Eq. (2.3.7b), we obtain the following partial differential equation

$$\frac{\partial u}{\partial t} = -\mu\frac{\partial u}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 u}{\partial x^2} \tag{2.3.12}$$

for the probability density function $u(x,t)$ of the continuous random walk motion with drift.

The above differential equation is called the *forward Fokker-Planck equation*. The drift rate is μ and the diffusion rate is σ^2. In time t, the mean displacement of the particle is μt and the variance of the observed displacement around the mean position is $\sigma^2 t$.

From the Central Limit Theorem in probability theory, one can show that the continnum limit of the probability density of the discrete random variable X_n defined in Eq. (2.3.2) tends to that of a normal random variable with the same mean and variance. The probability density function of the normal random variable X with mean μt and variance $\sigma^2 t$ is given by

$$f_X(x,t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(x-\mu t)^2}{2\sigma^2 t}\right) = n\left(\frac{x-\mu t}{\sigma\sqrt{t}}\right), \tag{2.3.13}$$

where the function

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{2.3.14a}$$

is called the standard normal density function. The cumulative normal distribution function $N(x)$ is defined to be

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt. \tag{2.3.14b}$$

From partial differential equation theory, $f_X(x,t)$ satisfies the following initial value problem

$$\frac{\partial u}{\partial t} + \mu \frac{\partial u}{\partial x} - \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial x^2} = 0, \ -\infty < x < \infty, \ t > 0, \tag{2.3.15}$$

with initial condition: $u(x, 0^+) = \delta(x)$, where $u(x, 0^+)$ signifies $\lim_{t\to 0^+} u(x,t)$. Here, $\delta(x)$ represents the Dirac function.

The above result has the following probabilistic interpretation. Conditional on the event that the particle starts at the position $x = 0$ initially, $f_X(x,t)\Delta x$ gives the probability that the particle would stay within $[x, x+\Delta x]$ at some future time t. This is why $f_X(x,t)$ is usually called the *transition density function*. The initial condition $u(x,0^+) = \delta(x)$ indicates that the particle stays at $x = 0$ almost surely. Also, the continuous random walk model inherits the properties of stationary and independent increments from the discrete random walk model.

2.3.2 Brownian motions

The *Brownian motion* refers to the ceaseless, irregular random motion of small particles immersed in a liquid or gas, as observed by R. Brown in 1827. The phenomena can be explained by the perpetual collisions of the particles with the molecules of the surrounding medium. Brownian motion is sometimes known as the *Wiener stochastic process*.

The formal definition of a Brownian motion with drift is presented below.

Definition

The *Brownian motion with drift* is a stochastic process $\{X(t); t \geq 0\}$ with the following properties:

(i) Every increment $X(t+s) - X(s)$ is normally distributed with mean μt and variance $\sigma^2 t; \mu$ and σ are fixed parameters.

(ii) For every $t_1 < t_2 < \cdots < t_n$, the increments $X(t_2) - X(t_1), \cdots, X(t_n) - X(t_{n-1})$ are independent random variables with distributions given in (i).

(iii) $X(0) = 0$ and the sample paths of $X(t)$ are continuous.

Note that $X(t+s)-X(s)$ is independent of the past history of the random path, that is, the knowledge of $X(\tau)$ for $\tau < s$ has no effect on the probability distribution for $X(t+s) - X(s)$. This is precisely the Markovian character of the Brownian motion.

Standard Brownian motion

For the particular case $\mu = 0$ and $\sigma^2 = 1$, the Brownian motion is called the *standard Brownian motion* (or *standard Wiener process*). The corresponding probability distribution for the standard Wiener process $\{Z(t); t \geq 0\}$ is given by [see Eq. (2.3.13)]

$$\begin{aligned} P[Z(t) \leq z | Z(t_0) = z_0] &= P[Z(t) - Z(t_0) \leq z - z_0] \\ &= \frac{1}{\sqrt{2\pi(t-t_0)}} \int_{-\infty}^{z-z_0} \exp\left(-\frac{x^2}{2(t-t_0)}\right) dx \\ &= N\left(\frac{z-z_0}{\sqrt{t-t_0}}\right). \end{aligned} \tag{2.3.16}$$

Some other properties

(a) $E[Z(t)^2] = \text{var}(Z(t)) + E[Z(t)]^2 = t.$ (2.3.17*a*)

(b) $E[Z(t)Z(s)] = \min(t, s).$ (2.3.17*b*)

To show the result in (b), we assume $t > s$ (without loss of generality) and consider

$$\begin{aligned} E[Z(t)Z(s)] &= E[\{Z(t) - Z(s)\}Z(s) + Z(s)^2] \\ &= E[\{Z(t) - Z(s)\}Z(s)] + E[Z(s)^2]. \end{aligned} \tag{2.3.18a}$$

Since $Z(t) - Z(s)$ and $Z(s)$ are independent and both $Z(t) - Z(s)$ and $Z(s)$ have zero mean, so

$$E[Z(t)Z(s)] = E[Z(s)^2] = s = \min(t, s). \tag{2.3.18b}$$

Overlapping Brownian increments

When $t > s$, the correlation coefficient ρ between the two overlapping Brownian increments $Z(t)$ and $z(s)$ is given by

$$\rho = \frac{E[Z(t)Z(s)]}{\sqrt{\text{var}(Z(t))}\sqrt{\text{var}(Z(s))}} = \frac{s}{\sqrt{st}} = \sqrt{\frac{s}{t}}. \tag{2.3.19}$$

Since both $Z(t)$ and $Z(s)$ are normally distributed with zero mean and variance t and s, respectively, the probability distribution of the overlapping Brownian increments is given by the bivariate normal distribution function.

If we define $X_1 = Z(t)/\sqrt{t}$ and $X_2 = Z(s)/\sqrt{s}$, then X_1 and X_2 become standard normal random variables. We then have

$$P[Z(t) \leq z_t, Z(s) \leq z_s] = P[X_1 \leq z_t/\sqrt{t}, X_2 \leq z_s/\sqrt{s}] \\ = N_2(z_t/\sqrt{t}, z_s/\sqrt{s}; \sqrt{s/t}) \qquad (2.3.20a)$$

where the bivariate normal distribution function is given by

$$N_2(x_1, x_2; \rho) = \int_{-\infty}^{x_2}\int_{-\infty}^{x_1} \frac{1}{2\pi\sqrt{1-\rho^2}} \\ \exp\left(-\frac{\xi_1^2 - 2\rho\xi_1\xi_2 + \xi_2^2}{2(1-\rho^2)}\right) d\xi_1 d\xi_2. \qquad (2.3.20b)$$

Geometric Brownian motion

Let $X(t)$ denote the Brownian motion with drift parameter $\mu \geq 0$ and variance parameter σ^2. The stochastic process defined by

$$Y(t) = e^{X(t)}, \quad t \geq 0, \qquad (2.3.21)$$

is called the *Geometric Brownian motion*. Obviously, the value taken by $Y(t)$ is non-negative. Since $X(t) = \ln Y(t)$ is a Brownian motion, by properties (i) and (ii), we deduce that $\ln Y(t) - \ln Y(0)$ is normally distributed with mean μt and variance $\sigma^2 t$. For common usage, $\frac{Y(t)}{Y(0)}$ is said to be lognormally distributed. From the density function of $X(t)$ given in Eq. (2.3.13), the density function of $\frac{Y(t)}{Y(0)}$ is deduced to be

$$f_Y(y,t) = \frac{1}{y\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(\ln y - \mu t)^2}{2\sigma^2 t}\right). \qquad (2.3.22)$$

The mean of $Y(t)$ conditional on $Y(0) = y_0$ is found to be

$$\begin{aligned} & E[Y(t)|Y(0) = y_0] \\ = \; & y_0 \int_0^\infty y f_Y(y,t)\, dy \\ = \; & y_0 \int_{-\infty}^\infty \frac{e^x}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{(x-\mu t)^2}{2\sigma^2 t}\right) dx, \quad x = \ln y, \\ = \; & y_0 \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{[x - (\mu t + \sigma^2 t)]^2 - 2\mu t\sigma^2 t - \sigma^4 t^2}{2\sigma^2 t}\right) dx \\ = \; & y_0 \exp\left(\mu t + \frac{\sigma^2 t}{2}\right). \end{aligned} \qquad (2.3.23a)$$

Similarly, the variance of $Y(t)$ conditional on $Y(0) = y_0$ is found to be

$$\begin{aligned}
&\operatorname{var}(Y(t)|Y(0)=y_0)\\
&= y_0^2 \int_0^\infty y^2 f_Y(y,t)\ dy - \left[y_0 \exp\left(\mu t + \frac{\sigma^2 t}{2}\right)\right]^2\\
&= y_0^2 \left\{\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left(-\frac{[x-(\mu t+2\sigma^2 t)]^2 - 4\mu t\sigma^2 t - 4\sigma^4 t^2}{2\sigma^2 t}\right) dx\right.\\
&\qquad \left. - \left[\exp\left(\mu t + \frac{\sigma^2 t}{2}\right)\right]^2\right\}\\
&= y_0^2 \exp(2\mu t + \sigma^2 t)[\exp(\sigma^2 t) - 1]. \qquad (2.3.23b)
\end{aligned}$$

Further, for every $t_1 < t_2 < \cdots < t_n$, the successive ratios $Y(t_2)/Y(t_1), \cdots, Y(t_n)/Y(t_{n-1})$ are independent random variables, that is, the percentage changes over non-overlapping time intervals are independent.

2.4 Stochastic calculus: Ito's Lemma and Girsanov's Theorem

The price of a derivative is a function of the underlying asset price, and the asset price process is modeled by a stochastic state variable. In order to construct pricing models for derivatives, it is necessary to develop calculus tools that allow us to perform mathematical operations, like composition, differentiation, integration, etc. on functions of stochastic random variables. In this section, we define stochastic integrals and stochastic differentials of functions that involve the Brownian random variables. In particular, we develop the Ito differentiation rule that computes the differentials of functions of random variables. The Feynman-Kac representation formula is derived, which gives a stochastic representation to the solution of a parabolic partial differential equation. We also discuss the notion of Radon-Nikodym derivatives and the Girsanov Theorem that effect the change of equivalent probability measures.

2.4.1 Stochastic integrals

Brownian motions are the continuous limit of discrete random walk models. Intuitively, one may visualize Brownian paths to be continuous (though the rigorous mathematical proof of the continuity property is not trivial). However, Brownian paths are seen to be non-smooth; and in fact, they are not differentiable. The non-differentiability property can be shown easily by proving the finiteness of the quadratic variation of a Brownian motion. This stems from the well known result in elementary calculus that differentiability implies vanishing of the quadratic variation of the function.

Quadratic variation of Brownian motions

Suppose we form a partition π of the time interval $[0, T]$ by the discrete points

$$0 = t_0 < t_1 < \cdots < t_n = T,$$

and let $\delta t_{max} = \max_k (t_k - t_{k-1})$. We write $\Delta t_k = t_k - t_{k-1}$, and define the corresponding quadratic variation of the standard Brownian motion $Z(t)$ by

$$Q_\pi = \sum_{k=1}^{n} [Z(t_k) - Z(t_{k-1})]^2. \tag{2.4.1}$$

Next, we show that the quadratic variation of $Z(t)$ over $[0, T]$ is given by

$$Q_{[0,T]} = \lim_{\delta t_{max} \to 0} Q_\pi = T. \tag{2.4.2}$$

To prove the above claim, it suffices to show that

$$\lim_{\delta t_{max} \to 0} E[Q_\pi] = T \quad \text{and} \quad \lim_{\delta t_{max} \to 0} \text{var}(Q_\pi - T) = 0. \tag{2.4.3}$$

First, we consider

$$\begin{aligned} &E[Q_\pi] \\ &= \sum_{k=1}^{n} E[\{Z(t_k) - Z(t_{k-1})\}^2] \\ &= \sum_{k=1}^{n} \text{var}(Z(t_k) - Z(t_{k-1})) \quad \text{since } Z(t_k) - Z(t_{k-1}) \text{ has zero mean} \\ &= \text{var}(Z(t_n) - Z(t_0)) \quad \text{since } Z(t_k) - Z(t_{k-1}), k = 1, \cdots, n \text{ are independent} \\ &= t_n - t_0 = T \end{aligned} \tag{2.4.4}$$

so that the first result in Eq. (2.4.3) is established. Next, we consider

$$\begin{aligned} \text{var}(Q_\pi - T) = E\Bigg[&\sum_{k=1}^{n} \sum_{\ell=1}^{n} \left\{ [Z(t_k) - Z(t_{k-1})]^2 - \Delta t_k \right\} \\ &\left\{ [Z(t_\ell) - Z(t_{\ell-1})]^2 - \Delta t_\ell \right\} \Bigg]. \end{aligned} \tag{2.4.5a}$$

Since the increments $[Z(t_k) - Z(t_{k-1})] - \Delta t_k, k = 1, \cdots, n$ are independent, only those terms corresponding to $k = \ell$ in the above series survive, so we have

$$\begin{aligned}\mathrm{var}(Q_\pi - T) &= E\left[\sum_{k=1}^{n}\left\{[Z(t_k) - Z(t_{k-1})]^2 - \Delta t_k\right\}^2\right] \\ &= \sum_{k=1}^{n} E\left[\{Z(t_k) - Z(t_{k-1})\}^4\right] \\ &\qquad - 2\Delta t_k \sum_{k=1}^{n} E\left[\{Z(t_k) - Z(t_{k-1})\}^2\right] + \Delta t_k^2. \quad (2.4.5b)\end{aligned}$$

Since $Z(t_k) - Z(t_{k-1})$ is normally distributed with zero mean and variance Δt_k, its fourth order moment is known to be (see Problem 2.24)

$$E[\{Z(t_k) - Z(t_{k-1})\}^4] = 3\Delta t_k^2, \qquad (2.4.5c)$$

so

$$\mathrm{var}(Q_\pi - T) = \sum_{k=1}^{n}[3\Delta t_k^2 - 2\Delta t_k^2 + \Delta t_k^2] = 2\sum_{k=1}^{n} \Delta t_k^2. \qquad (2.4.5d)$$

In taking the limit $\delta t_{max} \to 0$, we observe that $\mathrm{var}(Q_\pi - T) \to 0$, thus we obtain the second result in Eq. (2.4.3). By virtue of $\lim_{n\to\infty} \mathrm{var}(Q_\pi - T) = 0$, we say that T is the *mean square limit* of Q_π.

Remark

1. In general, the quadratic variation of the Brownian motion with variance rate σ^2 over the time interval $[t_1, t_2]$ is given by

$$Q_{[t_1,t_2]} = \sigma^2(t_2 - t_1). \qquad (2.4.6)$$

2. If we write $dZ(t) = Z(t) - Z(t - dt)$, where $dt \to 0$, then we can deduce from the above calculations that

$$E[dZ(t)^2] = dt \quad \text{and} \quad \mathrm{var}(dZ(t)^2) = 2\ dt^2. \qquad (2.4.7)$$

Since dt^2 is a higher order infinitesimally small quantity, we may claim that the random quantity $dZ(t)^2$ converges in the mean square sense to the deterministic quantity dt.

Definition of stochastic integration

Let $f(t)$ be an arbitrary function of t and $Z(t)$ be the standard Brownian motion. First, we consider the definition of the stochastic integral $\int_0^T f(t)\ dZ(t)$ as a limit of the following partial sums (defined in the usual Riemann-Stieltjes sense):

$$\int_0^T f(t)\ dZ(t) = \lim_{n\to\infty} \sum_{k=1}^{n} f(\xi_k)[Z(t_k) - Z(t_{k-1})] \qquad (2.4.8)$$

where the discrete points $0 < t_0 < t_1 < \cdots < t_n = T$ form a partition of the interval $[0, T]$ and ξ_k is some immediate point between t_{k-1} and t_k. The limit is taken in the mean square sense. Unfortunately, the limit depends on how the immediate points are chosen. For example, suppose we take $f(t) = Z(t)$ and choose $\xi_k = \alpha t_k + (1-\alpha)t_{k-1}, 0 < \alpha < 1$, for all k. We consider

$$\begin{aligned} &E\left[\sum_{k=1}^{n} Z(\xi_k)[Z(t_k) - Z(t_{k-1})\right] \\ &= \sum_{k=1}^{n} E\left[Z(\xi_k)Z(t_k) - Z(\xi_k)Z(t_{k-1})\right] \\ &= \sum_{k=1}^{n}[\min(\xi_k, t_k) - \min(\xi_k, t_{k-1})] \qquad \text{[see Eq. (2.3.17)]} \\ &= \sum_{k=1}^{n}(\xi_k - t_{k-1}) = \alpha\sum_{k=1}^{n}(t_k - t_{k-1}) = \alpha T, \end{aligned} \tag{2.4.9}$$

so that the expected value of the stochastic integral depends on the choice of immediate points.

A function is said to be *non-anticipative* with respect to the Brownian motion $Z(t)$ if the value of the function at time t is determined by the path history of $Z(t)$ up to time t. In finance, the investor's action is non-anticipative in nature since he makes the investment decision before the asset prices move. It seems natural to define the stochastic integration by taking $\xi_k = t_{k-1}$ (left-hand point in each sub-interval) so that integration is taken to be non-anticipatory. The Ito definition of stochastic integral is given by

$$\int_0^T f(t)\ dZ(t) = \lim_{n\to\infty}\sum_{k=1}^{n} f(t_{k-1})[Z(t_k) - Z(t_{k-1})], \tag{2.4.10}$$

where the limit is taken in the mean square sense and $f(t)$ is non-anticipative with respect to $Z(t)$.

As an example, consider the evaluation of the Ito stochastic integral $\int_0^T Z(t)\ dZ(t)$. A naive evaluation according to the usual integration rule gives

$$\int_0^T Z(t)\ dZ(t) = \frac{1}{2}\int_0^T \frac{d}{dt}[Z(t)]^2\ dt = \frac{Z(T)^2 - Z(0)^2}{2}, \tag{2.4.11a}$$

which unfortunately gives a wrong result (see explanation below). According to the definition in Eq. (2.4.10), we have

$$\int_0^T Z(t)\ dZ(t) = \lim_{n\to\infty}\sum_{k=1}^{n} Z(t_{k-1})[Z(t_k) - Z(t_{k-1})]$$

$$
\begin{aligned}
&= \lim_{n\to\infty} \frac{1}{2} \sum_{k=1}^{n} (\{Z(t_{k-1}) + [Z(t_k) - Z(t_{k-1})]\}^2 \\
&\qquad\qquad - Z(t_{k-1})^2 - [Z(t_k) - Z(t_{k-1})]^2) \\
&= \frac{1}{2} \lim_{n\to\infty} [Z(t_n)^2 - Z(t_0)^2] \\
&\quad - \frac{1}{2} \lim_{n\to\infty} \sum_{k=1}^{n} [Z(t_k) - Z(t_{k-1})]^2 \\
&= \frac{Z(T)^2 - Z(0)^2}{2} - \frac{T}{2} \qquad \text{[by Eq. (2.4.3)]}. \qquad (2.4.11b)
\end{aligned}
$$

Rearranging the terms, we may rewrite the above result as

$$2\int_0^T Z(t)\ dZ(t) + \int_0^T\ dt = \int_0^T \frac{d}{dt}[Z(t)]^2\ dt, \tag{2.4.12a}$$

or in differential form,

$$2Z(t)\ dZ(t) + dt = d[Z(t)]^2. \tag{2.4.12b}$$

Unlike the usual differential rule, we have the extra term dt. This comes from the finiteness of the quadratic variation of the Brownian motion, since $|Z(t_k) - Z(t_{k-1})|$ is of order $\sqrt{\Delta t_k}$ and $\lim_{n\to\infty} \sum_{k=1}^{n} [Z(t_k) - Z(t_{k-1})]^2$ remains finite on taking the limit. Apparently, it is necessary to develop new differential rules that deal with the computation of differentials of stochastic functions.

2.4.2 Ito's Lemma and stochastic differentials

Once we have defined stochastic integrals, we can give a formal definition of a class of continuous stochastic processes, called Ito processes. Let $\mathcal{F}_t$ be the natural filtration generated by the standard Brownian motion $Z(t)$ through the observation of the trajectory of $Z(t)$. Let $\mu(t)$ and $\sigma(t)$ be adapted to $\mathcal{F}_t$ with $\int_0^T |\mu(t)|\ dt < \infty$ and $\int_0^T \sigma^2(t)\ dt < \infty$ (almost surely) for all T, then the process $X(t)$ defined by

$$X(t) = X(0) + \int_0^t \mu(s)\ ds + \int_0^t \sigma(s)\ dZ(s), \tag{2.4.13}$$

is called an *Ito process*. The differential form of the above equation is given as

$$dX(t) = \mu(t)\ dt + \sigma(t)\ dZ(t). \tag{2.4.14}$$

Ito's Lemma

Suppose $f(x,t)$ is a deterministic twice continuously differentiable function and the stochastic process Y is defined by $Y = f(X,t)$, where $X(t)$ is governed by Eq. (2.4.14). How to compute the differential $dY(t)$? We have seen the justification why $dZ(t)^2$ converges in the mean square sense to dt [see Eq. (2.4.7)]. Hence, the second order term dX^2 also contributes to the differential dY. The Ito formula of computing the differential of the stochastic function $f(X,t)$ is given by

$$dY = \left[\frac{\partial f}{\partial t}(X,t) + \mu(t)\frac{\partial f}{\partial x}(X,t) + \frac{\sigma^2(t)}{2}\frac{\partial^2 f}{\partial x^2}\right] dt + \sigma(t)\frac{\partial f}{\partial x}(X,t)\; dZ. \qquad (2.4.15)$$

The regorous proof of the Ito formula is quite technical, so only a heuristic proof is provided here. We expand ΔY by the Taylor series up to the second order terms as follows:

$$\Delta Y = \frac{\partial f}{\partial t}\Delta t + \frac{\partial f}{\partial x}\Delta X + \frac{1}{2}\left(\frac{\partial^2 f}{\partial t^2}\Delta t^2 + 2\frac{\partial^2 f}{\partial x \partial t}\Delta X \Delta t + \frac{\partial^2 f}{\partial x^2}\Delta X^2\right) + O(\Delta X^3, \Delta t^3). \qquad (2.4.16a)$$

In the limit $\Delta X \to 0$ and $\Delta t \to 0$, we apply the multiplication rules where $dZ^2 = dt, dZdt = 0$ and $dt^2 = 0$ so that

$$dY = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX + \frac{\sigma^2(t)}{2}\frac{\partial^2 f}{\partial x^2}dt. \qquad (2.4.16b)$$

Writing out in full in terms of dZ and dt, we obtain the Ito formula (2.4.15).

As a simple verification, when we apply the Ito formula to $f = Z^2$, we obtain the result in Eq. (2.4.12b) immediately. As an another example, we consider the stochastic function

$$S(t) = S_0 e^{\left(r - \frac{\sigma^2}{2}\right)t + \sigma Z(t)}. \qquad (2.4.17)$$

Suppose we write

$$X(t) = \left(r - \frac{\sigma^2}{2}\right)t + \sigma Z(t) \qquad (2.4.18a)$$

so that

$$S(t) = S_0 e^{X(t)}. \qquad (2.4.18b)$$

Now, the respective partial derivatives of S are

$$\frac{\partial S}{\partial t} = 0, \quad \frac{\partial S}{\partial X} = S \quad \text{and} \quad \frac{\partial^2 S}{\partial X^2} = S. \qquad (2.4.19)$$

By the Ito lemma, we obtain

$$dS = \left(r - \frac{\sigma^2}{2} + \frac{\sigma^2}{2}\right) S\ dt + \sigma S\ dZ \tag{2.4.20a}$$

or

$$\frac{dS}{S} = r\ dt + \sigma\ dZ. \tag{2.4.20b}$$

Conversely, we observe that $S(t)$ defined in Eq. (2.4.17) is the solution to the stochastic differential equation (2.4.20b). Since $E[X(t)] = \left(r - \frac{\sigma^2}{2}\right)t$ and $\mathrm{var}(X(t)) = \sigma^2 t$, the mean and variance of $\ln \frac{S(t)}{S_0}$ are found to be $\left(r - \frac{\sigma^2}{2}\right)t$ and $\sigma^2 t$, respectively.

Multi-dimensional version of Ito's lemma

Suppose $f(x_1, \cdots, x_n, t)$ is a multi-dimensional twice continuously differentiable function and the stochastic process Y_n is defined by

$$Y_n = f(X_1, \cdots, X_n, t), \tag{2.4.21a}$$

where the process $X_j(t)$ follows the Ito process

$$dX_j(t) = \mu_j(t)\ dt + \sigma_j(t)\ dZ_j(t), \quad j = 1, 2, \cdots, n. \tag{2.4.21b}$$

The Brownian motions $Z_j(t)$ and $Z_k(t)$ are assumed to be correlated with correlation coefficient ρ_{jk} so that $dZ_j\ dZ_k = \rho_{jk}\ dt$. In a similar manner, we expand ΔY_n up to the second order term in ΔX_j:

$$\begin{aligned}\Delta Y_n = {} & \frac{\partial f}{\partial t}(X_1, \cdots, X_n, t)\ \Delta t + \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(X_1, \cdots, X_n, t)\ \Delta X_j \\ & + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k}(X_1, \cdots, X_n, t)\ \Delta X_j\ \Delta X_k \\ & + O(\Delta t \Delta X_j) + O(\Delta t^2).\end{aligned} \tag{2.4.22}$$

In the limits $\Delta X_j \to 0, j = 1, 2, \cdots, n$, and $\Delta t \to 0$, we neglect the higher order terms in $O(\Delta t \Delta X_j)$ and $O(\Delta t^2)$ and observe $dX_j\ dX_k = \sigma_j(t)\sigma_k(t)\rho_{jk}\ dt$. We then obtain the following multi-dimensional version of the Ito lemma:

$$\begin{aligned}dY_n = {} & \left[\frac{\partial f}{\partial t}(X_1, \cdots, X_n, t) + \sum_{j=1}^{n} \mu_j(t) \frac{\partial f}{\partial x_j}(X_1, \cdots, X_n t)\right. \\ & \left. + \frac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \sigma_j(t)\sigma_k(t)\rho_{jk} \frac{\partial^2 f}{\partial x_j \partial x_k}(X_1, \cdots, X_n, t)\right] dt \\ & + \sum_{j=1}^{n} \sigma_j(t) \frac{\partial f}{\partial x_j}(X_1, \cdots, X_n, t)\ dZ_j.\end{aligned} \tag{2.4.23}$$

Feynman-Kac representation formula
Suppose the Ito process $X(t)$ is governed by the stochastic differential equation

$$dX(s) = \mu(X(s), s)\ ds + \sigma(X(s), s)\ dZ(s), \quad t \leq s \leq T, \quad (2.4.24)$$

with initial condition: $X(t) = x$. Consider a smooth function $F(X(t), t)$, by virtue of the Ito lemma, the differential of which is given by

$$dF = \left[\frac{\partial F}{\partial t} + \mu(X, t)\frac{\partial F}{\partial X} + \frac{\sigma^2(X, t)}{2}\frac{\partial^2 F}{\partial X^2}\right] dt + \sigma\frac{\partial F}{\partial X}\ dZ. \quad (2.4.25)$$

We define the infinitesimal generator $\mathcal{A}$ associated with the Ito process $X(t)$ by

$$\mathcal{A} = \mu(X, t)\frac{\partial}{\partial X} + \frac{\sigma^2(X, t)}{2}\frac{\partial^2}{\partial X^2}. \quad (2.4.26)$$

Suppose F satisfies the parabolic partial differential equation

$$\frac{\partial F}{\partial t} + \mathcal{A}F = 0 \quad (2.4.27)$$

with terminal condition: $F(X(T), T) = h(X(T))$, then dF becomes

$$dF = \sigma\frac{\partial F}{\partial X}\ dZ. \quad (2.4.28)$$

Supposing that $\sigma\dfrac{\partial F}{\partial X}$ is non-anticipative with the Brownian motion $Z(t)$, we can express the above stochastic differential form into the following integral form

$$F(X(s), s) = F(X(t), t) + \int_t^s \sigma(X(u), u)\frac{\partial F}{\partial X}(X(u), u)\ dZ(u). \quad (2.4.29)$$

The stochastic integral can be viewed as a sum of inhomogeneous consecutive Gaussian increments with mean zero, hence it has zero conditional expectation. By taking the conditional expectation and setting $s = T$ and $F(X(T), T) = h(X(T))$, we then obtain the following Feynman-Kac representation formula

$$F(x, t) = E^{x,t}[h(X(T))], \quad t < T, \quad (2.4.30)$$

where $F(x, t)$ satisfies the partial differential equation (2.4.27). The process $X(t)$ is initialized at the fixed point x at time t and it follows the Ito process defined in Eq. (2.4.24).

2.4.3 Change of measure: Girsanov's Theorem

Consider an Ito process defined either in differential form

$$dX(t) = \mu(t)\ dt + \sigma(t)\ dZ(t) \tag{2.4.31a}$$

or in integral form

$$X(t) = X(0) + \int_0^t \mu(s)\ ds + \int_0^t \sigma(s)\ dZ(s) \tag{2.4.31b}$$

with non-zero drift term $\mu(t)$. We write $M(t) = \displaystyle\int_0^t \sigma(s)\ dZ(s)$ and note that

$$M(T) = M(t) + \int_t^T \sigma(s)\ dZ(s), \qquad T > t. \tag{2.4.32}$$

Suppose we take the conditional expectation of $M(T)$ given the history up to the time t (denoted by the operator E_t), we obtain

$$E_t[M(T)] = M(t) \tag{2.4.33}$$

since the stochastic integral in Eq. (2.4.32) has zero conditional expectation. Hence, $M(t)$ is a martingale. However, $X(t)$ is not a martingale if $\mu(t)$ is non-zero.

In Chap. 3, we will consider the valuation of contingent claims under the risk neutral measure. With respect to the risk neutral measure, the discounted price of the underlying asset becomes a martingale. The valuation procedure often requires the transformation of an underlying price process with drift into a martingale, but under a different measure. Such transformation can be performed effectively by the use of Girsanov's Theorem. Before stating the theorem, we define the Radon-Nikodym derivative which relates the transformation between two equivalent probability measures.

Radon-Nikodym derivatives
Let Z and $\widetilde{Z}$ denote two Gaussian random variables with unit variance and respective means 0 and μ. We define $dP(\xi)$ and $d\widetilde{P}(\xi)$ as

$$dP(\xi) = \ P\left[\xi - \frac{d\xi}{2} < Z < \xi + \frac{d\xi}{2}\right] = f_Z(\xi)\ d\xi \tag{2.4.34a}$$

$$d\widetilde{P}(\xi) = \ P\left[\xi - \frac{d\xi}{2} < \widetilde{Z} < \xi + \frac{d\xi}{2}\right] = f_{\widetilde{Z}}(\xi)\ d\xi \tag{2.4.34b}$$

where

$$f_Z(\xi) = \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} \quad \text{and} \quad f_{\widetilde{Z}}(\xi) = \frac{1}{\sqrt{2\pi}} e^{-(\xi-\mu)^2/2} \tag{2.4.35}$$

are the density function of Z and $\widetilde{Z}$, respectively. We set $f(\xi)$ equal the ratio $f_{\widetilde{Z}}(\xi)/f_Z(\xi)$ so that

$$d\widetilde{P}(\xi) = f(\xi)\ dP(\xi) = e^{-\frac{\mu^2}{2}+\mu\xi}\ dP(\xi). \tag{2.4.36}$$

Note that the multiplication of $dP(\xi)$ by $f(\xi)$ transforms one measure dP to another measure $d\widetilde{P}$. The two measures correspond to the two separate Gaussian distributions with the same variance but different means. The two measures P and $\widetilde{P}$ are said to be equivalent since $dP(\xi) > 0$ if and only if $d\widetilde{P}(\xi) > 0$. In probability theory, $f(\xi) = \dfrac{d\widetilde{P}(\xi)}{dP(\xi)}$ is called the Radon-Nikodym derivative of the two equivalent probability measures.

Girsanov Theorem

We state without proof a version of the Girsanov Theorem, which is a useful tool to effect a change of measure on a stochastic process. The application of the Girsanov Theorem in identifying an equivalent martingale measure for pricing contingent claims will be demonstrated in Sec. 3.2.

Theorem 2.4

Consider a stochastic process $\gamma(t)$ which satisfies the Novikov condition:

$$E[e^{\int_0^t \frac{1}{2}\gamma(s)^2\ ds}] < \infty, \tag{2.4.37}$$

and consider the Radon-Nikodym derivative:

$$\frac{d\widetilde{P}}{dP} = \rho(t) \tag{2.4.38a}$$

where

$$\rho(t) = \exp\left(\int_0^t -\gamma(s)\ dZ(s) - \frac{1}{2}\int_0^t \gamma(s)^2\ ds\right). \tag{2.4.38b}$$

Here, $Z(t)$ is a Brownian motion under the measure P (called P-Brownian motion). Under the measure $\widetilde{P}$, the stochastic process

$$\widetilde{Z}(t) = Z(t) + \int_0^t \gamma(s)\ ds \tag{2.4.39}$$

is $\widetilde{P}$-Brownian motion.

The proof of the Girsanov Theorem can be found in the text by Karatzas and Shreve (1991). As an illustration, suppose we take $\gamma(t)$ to be a constant so that the Radon-Nikodym derivative is

$$\frac{d\widetilde{P}}{dP} = \exp\left(-\gamma Z(T) - \frac{\gamma^2 T}{2}\right). \tag{2.4.40}$$

Recall that if X is the normal random variable with mean μ and variance σ^2, then the expectation of $\exp(\alpha X)$ is given by [see Problem 2.24]

$$E[\exp(\alpha X)] = \exp\left(\alpha\mu + \frac{\alpha^2\sigma^2}{2}\right), \quad \text{for any } \alpha. \tag{2.4.41}$$

Let $Z(T)$ be P-Brownian motion and consider $\widetilde{Z}(T) = Z(T) + \gamma T$, we would like to show that $\widetilde{Z}(T)$ is $\widetilde{P}$-Brownian motion. By virtue of Eq. (2.4.41), it suffices to show that $E_{\widetilde{P}}[\exp(\gamma\widetilde{Z}(T))] = \exp\left(\frac{\gamma^2 T}{2}\right)$, thus verifying that $\widetilde{Z}(T)$ has zero mean and variance T under $\widetilde{P}$. We consider

$$\begin{aligned} E_{\widetilde{P}}[\exp(\gamma\widetilde{Z}(T))] &= E_{\widetilde{P}}[\exp(\gamma Z(T) + \gamma^2 T] \\ &= E_P\left[\frac{d\widetilde{P}}{dP}\exp(\gamma Z(T) + \gamma^2 T)\right] \\ &= E_P\left[\exp\left(\gamma Z(T) - \frac{\gamma^2 T}{2}\right)\exp(\gamma Z(T) + \gamma^2 T)\right] \\ &= \exp\left(\frac{\gamma^2 T}{2}\right), \end{aligned} \tag{2.4.42}$$

hence $\widetilde{Z}(T)$ is $\widetilde{P}$-Brownian motion. Note that the factor $\frac{d\widetilde{P}}{dP}$ is included when we effect the change of probability measure from $\widetilde{P}$ to P in the expectation calculations.

2.5 Problems

2.1 Show that a dominant trading strategy exists if and only if there exists a trading strategy satisfying $V_0 < 0$ and $V_1(\omega) \geq 0$ for all $\omega \in \Omega$.
Hint: Consider the dominant trading strategy $\mathcal{H} = (h_0\ h_1 \cdots h_M)^T$ satisfying $V_0 = 0$ and $V_1(\omega) > 0$ for all $\omega \in \Omega$. Take $G^*_{\min} = \min_{\omega} G^*(\omega) > 0$ and define a new trading strategy with $\widehat{h}_m = h_m, m = 1, \cdots, M$ and $\widehat{h}_0 = -G^*_{\min} - \sum_{m=1}^{M} h_m S^*_m(0)$.

2.2 Consider a portfolio with one risky security and the riskfree security. Suppose the price of the risky asset at time 0 is 4 and the possible values of the $t = 1$ price are $1.1, 2.2$ and 3.3 (3 possible states of the world at the end of a single trading period). Let the riskfree interest rate r be 0.1 and take the price of the riskfree security at $t = 0$ to be unity.

(a) Show that the trading strategy: $h_0 = 4$ and $h_1 = -1$ is a dominant trading strategy that starts with zero wealth and ends with positive wealth with certainty.
(b) Find the discounted gain G^* over the single trading period.
(c) Find a trading strategy that starts with negative wealth and ends with non-negative wealth with certainty.

2.3 Show that if the law of one price does not hold, then every payoff in the asset span can be bought at any price.

2.4 Construct a securities model such that it satisfies the law of one price but admits a dominant trading strategy.

2.5 Define the pricing functional $F(\mathbf{x})$ on the asset span $\mathcal{S}$ by $F(\mathbf{x}) = \{y : y = \mathbf{S}^*(0)\mathbf{h}$ for some $\mathbf{h}$ such that $\mathbf{x} = S^*(1)\mathbf{h}$, where $\mathbf{x} \in \mathcal{S}\}$. Show that if the law of one price holds, then F is a *linear* functional.

2.6 Given the discounted terminal payoff matrix

$$S^*(1) = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 3 \\ 3 & 2 & 4 \end{pmatrix},$$

and the current discounted price vector $\mathbf{S}^*(0) = (1 \quad 2 \quad 4)$, find the state price of the Arrow security with discounted payoff $\mathbf{e}_k$, $k = 1, 2, 3$.

2.7 Construct a securities model with 2 risky securities and riskfree security and 3 possible states of world such that the law of one price holds but there are dominant trading strategies.

2.8 Show that if there exists a dominant trading strategy, then there exists an arbitrage opportunity. Outline the thought that underlies the construction of a securities model such that there exists an arbitrary opportunity but dominant trading strategy does not exist.

2.9 Show that $\mathbf{h}$ is an arbitrage if and only if the discounted gain G^* satisfies (i) $G^* \geq 0$ and (ii) $EG^* > 0$.

2.10 Suppose a betting game has 3 possible outcomes. If a gambler bets on outcome i, then he receives a net gain of d_i dollars for one dollar betted, $i = 1, 2, 3$. The payoff matrix thus takes the form (discounting is not required in a betting game)

$$S(1; \Omega) = \begin{pmatrix} d_1 + 1 & 0 & 0 \\ 0 & d_2 + 1 & 0 \\ 0 & 0 & d_3 + 1 \end{pmatrix}.$$

Find the condition on d_i such that a risk neutral probability measure exists for the above betting game (visualized as an investment model).

2.11 Suppose the set of risk neutral measures for a given securities model is non-empty. Based on the property that a contingent claim X is attainable if and only if $E_Q[X/S_0(1)]$ is invariant under any risk neutral measure Q, show that a securities model is complete if and only if there exists a unique risk neutral measure.

2.12 Consider the following securities model with discounted payoffs of the securities at $t = 1$ given by the augmented payoff matrix

$$\widehat{S}^*(1;\Omega) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 1 & 5 & 6 & 7 \end{pmatrix},$$

where the first column gives the discounted payoff of the riskfree security. Let the augmented initial price vector $\widehat{\mathbf{S}}^*(0)$ be $(1 \quad 3 \quad 5 \quad 9)$. Show that the law of one price does not hold for this securities model. Show that the contingent claim with discounted payoff $\begin{pmatrix} 6 \\ 8 \\ 12 \end{pmatrix}$ is attainable and find the set of all possible trading securities that generate the payoff. Can we find the price at $t = 0$ of this contingent claim?

2.13 Let P be the true probability measure, where $P(\omega)$ denotes the actual probability that the state ω occurs. Define the *state price density* by the random variable $L(\omega) = Q(\omega)/P(\omega)$, where Q is a risk neutral measure. Use R_m to denote the return of the risky security m, where $R_m = [S_m(1) - S_m(0)]/S_m(0), m = 1, \cdots, M$. Show that $E_Q[R_m] = r, m = 1, \cdots, M$, where r is the interest over one period. Let $E_P[R_m]$ denote the expectation of R_m under the actual probability measure P, show that

$$E_P[R_m] - r = -\text{cov}(R_m, L),$$

where cov denotes the covariance operator.

2.14 Suppose $u > d > R$ in the discrete binomial model. Show that an investor can lock in a riskless profit by borrowing cash as much as possible to purchase the asset, and selling the asset after one period and returning the loan. When $R > u > d$, what should be the corresponding strategy in order to take arbitrage?

2.15 We can also derive the binomial formula using the *riskless hedging principle* (see Sec. 3.1.1). Suppose we have a call which is one period from expiry and would like to create a perfectly hedged portfolio with a long position of one unit of the underlying asset and a short position of m units of call. Let c_u and c_d denote the payoff of the call at expiry corresponding to the upward and downward movement of the asset price, respectively. Show that the number of calls to be sold short in the portfolio should be

$$m = \frac{S(u-d)}{c_u - c_d}$$

in order that the portfolio is perfectly hedged. The hedged portfolio should earn the risk-free interest rate. Let R denote the growth factor of the value of a perfectly hedged portfolio in one period. Show that the binomial option pricing formula for the call as deduced from the riskless hedging principle is given by

$$c = \frac{pc_u + (1-p)c_d}{R} \quad \text{where} \quad p = \frac{R-d}{u-d}.$$

2.16 Let Π_u and Π_d denote the state prices corresponding to the states of asset value going up and going down, respectively. The state prices can also be interpreted as state contingent discount rates. If no arbitrage opportunities are available, then all securities (including the bond, the asset and the call option) must have returns with the same state contingent discount rates Π_u and Π_d. Hence, the respective relations for the bond price, asset price and call option value with Π_u and Π_d are given by

$$\begin{aligned} 1 &= \Pi_u R + \Pi_d R \\ S &= \Pi_u uS + \Pi_d dS \\ c &= \Pi_u c_u + \Pi_d c_d. \end{aligned}$$

By solving for Π_u and Π_d from the first two equations and substituting the solutions into the third equation, show that the binomial call price formula over one period is given by

$$c = \frac{pc_u + (1-p)c_d}{R} \quad \text{where} \quad p = \frac{R-d}{u-d}.$$

2.17 Consider the sample space $\Omega = \{-3, -2, -1, 1, 2, 3\}$ and the algebra $\mathcal{F} = \{\phi, \{-3, -2\}, \{-1, 1\}, \{2, 3\}, \{-3, -2, -1, 1\}, \{-3, -2, 2, 3\}, \{-1, 1, 2, 3\}, \Omega\}$. For each of the following random variables, determine whether it is $\mathcal{F}$-measurable:
(i) $X(\omega) = \omega^2$, (ii) $X(\omega) = \max(\omega, 2)$.
Find a random variable that is $\mathcal{F}$-measurable.

2.18 Let X be a random variable defined on $(\Omega, \mathcal{F}, P)$ and $\mathcal{F}_1 \subset \mathcal{F}_2$ are sub-algebras of $\mathcal{F}$. Prove the following properties on conditional expectations:
(a) $E[XI_B] = E[I_B E[X|\mathcal{F}]]$ for all $B \in \mathcal{F}$,
(b) $E[E[X|\mathcal{F}_1]|\mathcal{F}_2] = E[X|\mathcal{F}_1]$.

2.19 Let $X = \{X_t; t = 0, 1, \cdots, T\}$ be a stochastic process adapted to the filtration $\mathbb{F} = \{\mathcal{F}_t; t = 0, 1, \cdots, T\}$. Show that X is a martingale if and only if $X_t = E[X_T|\mathcal{F}_t] = 0, t = 0, 1, \cdots, T-1$. Does the property: $E[X_{t+1} - X_t|\mathcal{F}_t] = 0, t = 0, 1, \cdots, T-1$ imply that X is a martingale?

2.20 Consider the binomial experiment with probability of success $p, 0 < p < 1$. We let N_k denote the number of successes after k independent trials. Define the discrete process Y_k by $N_k - kp$, the excess number of successes above the mean kp. Show that Y_k is a martingale.

2.21 Consider the two-period securities model shown in Fig. 2.5. Suppose the riskless interest rate r violates the restriction $r < 0.2$, say, $r = 0.3$. Construct an arbitrage opportunity associated with the securities model.

2.22 Deduce the price formula for a European put option with terminal payoff $\max(X - S, 0)$ for the n-period binomial model.
Hint: See Eqs. (2.2.33-35).

2.23 Consider the continuous random walk model discussed in Sec. 2.3.1 [see Eq. (2.3.13)]. Suppose the particle starts initially at $x = a_0$, find the probability that the particle stays above $x = a$ at time t. Express your answer in terms of

$$n(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \quad \text{and} \quad N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2}\, dt.$$

2.24 Let X be a normally distributed random variable with mean μ and variance σ^2. Show that the higher central moments of the normal random variable are given by

$$\mu_n(X;\mu) = E[(X-\mu)^n] = \begin{cases} 0, & n \text{ odd} \\ (n-1)(n-3)\cdots 3\cdot 1\sigma^n, & n \text{ even.} \end{cases}$$

For the lognormal random variable $Z = e^X$, show that the corresponding higher moments are

$$\mu_n(Z;0) = E(Z^n) = \exp\left(n\mu + \frac{n^2\sigma^2}{2}\right), \qquad n = 1, 2, \cdots.$$

Hint: See Eqs. (2.3.21-22).

2.25 Suppose $\{X(t), t \geq 0\}$ is the standard Brownian motion, its corresponding reflected Brownian motion is defined by

$$Y(t) = |X(t)|, \quad t \geq 0.$$

Show that $Y(t)$ is also Markovian and its mean and variance are respectively

$$E[Y] = \sqrt{\frac{2t}{\pi}}$$

and

$$\text{var}(Y) = \left(1 - \frac{2}{\pi}\right) t.$$

2.26 Suppose $Z(t)$ is a standard Brownian process, show that the following processes defined by

$$X_1(t) = kZ(t/k^2), \quad k > 0$$
$$X_2(t) = \begin{cases} tZ\left(\frac{1}{t}\right) & \text{for } t > 0 \\ 0 & \text{for } t = 0 \end{cases}$$

and

$$X_3(t) = Z(t+h) - Z(h), \quad h > 0$$

are also Brownian processes.

Hint: To show that $X_i(t)$ is a Brownian process, $i = 1, 2, 3$, it is necessary to show that

$$X_i(t+s) - X_i(s)$$

is normally distributed with zero mean, and

$$E[[X_i(t+s) - X_i(s)]^2] = t.$$

Also, the increments over disjoint time intervals are independent, and $X_i(t)$ is continuous at $t = 0$. For $X_2(t)$, we need to establish

$$P\left[\lim_{t\to\infty} \frac{X(t)}{t} = 0 \middle| x(0) = 0\right] = 1.$$

2.27 Consider the Brownian motion with drift defined by

$$X(t) = \mu t + \sigma Z(t), \quad X(0) = 0, Z(t) \text{ is the standard Brownian motion,}$$

find $E[X(t)|X(t_0)]$, $\text{var}(X(t)|X(t_0))$ and $\text{cov}(X(t_1), X(t_2))$.

2.28 Assume that the price of an asset follows the Geometric Brownian motion with expected return of 10% per annum and a volatility of 20% per annum. Suppose the asset price at present is \$100, find the expected value and variance of the asset price half a year from now and its 90% confidence limits.

2.29 Let $Z(t)$ denote the standard Brownian motion. Show that

(a) $dZ(t)^{2+n} = 0$, for any positive integer n,

(b) $$\int_{t_0}^{t_1} Z(t)^n \, dZ(t) = \frac{1}{n+1}[Z(t_1)^{n+1} - Z(t_0)^{n+1}] - \frac{n}{2}\int_{t_0}^{t_1} Z(t)^{n-1} \, dt,$$

for any positive integer n,

(c) $E[Z^4(t)] = 3t^2$,

(d) $E[e^{\alpha Z(t)}] = e^{\alpha^2 t/2}$.

2.30 Let the stochastic process $X(t), t \geq 0$, be defined by

$$X(t) = \int_0^t e^{\alpha(t-u)} \, dZ(u),$$

where $Z(t)$ is the standard Wiener process. Show that

$$\text{cov}(X(s), X(t)) = \frac{e^{\alpha(s+t)} - e^{\alpha|s-t|}}{2\alpha}, \quad s \geq 0, t \geq 0.$$

2.31 Let $Z(t), t \geq 0$, be the standard Wiener process, $f(t)$ and $g(t)$ are differentiable functions over $[a, b]$. Show that

$$E\left[\int_a^b f'(t)[Z(t) - Z(a)] \, dt \int_a^b g'(t)[Z(t) - Z(a)] \, dt\right]$$
$$= \int_a^b [f(b) - f(t)][g(b) - g(t)] \, dt.$$

Hint: Interchange the order of expectation and integration, and observe

$$E[[Z(t) - Z(a)][Z(s) - Z(a)]] = \min(t, s) - a.$$

2.32 Show that

$$\sigma \int_t^T [Z(u) - Z(t)] \, du$$

has zero mean and variance $\sigma^2(T-t)^3/3$.
Hint: Consider

$$\begin{aligned}
&\text{var}\left(\int_t^T [Z(u) - Z(t)] \, du\right) \\
&= E\left[\int_t^T \int_t^T [Z(u) - Z(t)][Z(v) - Z(t)] \, dudv\right] \\
&= \int_t^T \int_t^T E[\{Z(u) - Z(t)\}\{Z(v) - Z(t)\}] \, dudv \\
&= \int_t^T \int_t^T [\min(u, v) - t] \, dudv.
\end{aligned}$$

2.33 Show that

$$N_2(a, b; \rho) + N_2(a, -b; -\rho) = N(a),$$

where N and N_2 denote the standard univariate and bivariate normal distribution functions, respectively. Also, show that

$$N_2(a, b; \rho) = \int_{-\infty}^{a} n(x) N\left(\frac{b - \rho x}{\sqrt{1-\rho^2}}\right) dx,$$

where $n(x)$ is the standard normal density function.

2.34 Suppose the stochastic variables S_1 and S_2 follow the Geometric Brownian processes where

$$\frac{dS_i}{S_i} = \mu_i \, dt + \sigma_i \, dZ_i, \qquad i = 1, 2.$$

Let ρ_{12} denote the correlation coefficient between the Wiener processes dZ_1 and dZ_2. Let $f = S_1 S_2$, show that f also follows the Geometric Brownian process of the form

$$\frac{df}{f} = \mu \, dt + \sigma \, dZ$$

where $\mu = \mu_1 + \mu_2 + \rho_{12}\sigma_1\sigma_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2 + 2\rho_{12}\sigma_1\sigma_2$. Similarly, let $g = \dfrac{S_1}{S_2}$, show that

$$\frac{dg}{g} = \widetilde{\mu} \, dt + \widetilde{\sigma} \, dz$$

where $\widetilde{\mu} = \mu_1 - \mu_2 - \rho_{12}\sigma_1\sigma_2 + \sigma_2^2$ and $\widetilde{\sigma}^2 = \sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2$.

2.35 Suppose the function $F(x, t)$ satisfies

$$\frac{\partial F}{\partial t} + \mu(x, t)\frac{\partial F}{\partial x} + \frac{\sigma^2(x, t)}{2}\frac{\partial^2 F}{\partial x^2} - rF = 0$$

with terminal condition: $F(X(T), T) = h(X(T))$. Show that

$$F(x, t) = e^{-r(T-t)} E_t[h(X(T))|X(t) = x], \quad t < T,$$

where $X(t)$ follows the Ito process

$$dX(t) = \mu(X(t), t) \; dt + \sigma(X(t), t) \; dZ(t).$$

2.36 Define the discrete random variable X by

$$X(\omega) = \begin{cases} 2 & \text{if } \omega = \omega_1 \\ 3 & \text{if } \omega = \omega_2 \\ 4 & \text{if } \omega = \omega_3 \end{cases},$$

where the sample space $\Omega = \{\omega_1, \omega_2, \omega_3\}, P[\omega_1] = P[\omega_2] = P[\omega_3] = 1/3$. Find a new probability measure $\widetilde{P}$ such that the mean becomes $E_{\widetilde{P}}[X] = 3.5$ while the variance remains unchanged. Is $\widetilde{P}$ unique?

2.37 Given that $S(t)$ is a Geometric Brownian motion which follows

$$\frac{dS}{S} = \mu\ dt + \sigma\ dZ$$

where Z is P-Brownian motion. Find another measure $\widetilde{P}$ by specifying the Radon-Nikodym derivative $\dfrac{d\widetilde{P}}{dP}$ such that $S(t)$ is governed by

$$\frac{dS}{S} = \mu'\ dt + \sigma\ d\widetilde{Z}$$

under the measure $\widetilde{P}$, where $\widetilde{Z}$ is $\widetilde{P}$-Brownian motion and μ' is the new drift rate.