Monte Carlo methods I

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Abstract

Contents of the lecture:

- The uniform distribution and frequencies.
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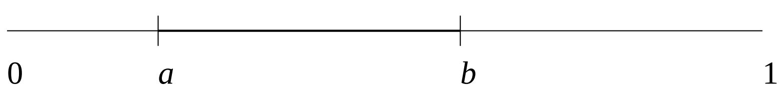
The uniform distribution and frequency

Let ξ denotes the random variable that has uniform distribution on the interval $[0,1]$. Then for any $0 \le a < b \le 1$ we have

$$\mathsf{P}\{a \le \xi \le b\} = b - a.$$



Let $\xi_1, \xi_2, \dots, \xi_N$ be the *sequence* of independent random variables that have uniform distribution on the interval $[0,1]$. Let n of these N variables belong to the interval $[a,b]$. The number

$$\nu_N = \frac{n}{N}$$

is called the *frequency* of the event $a \le \xi \le b$.

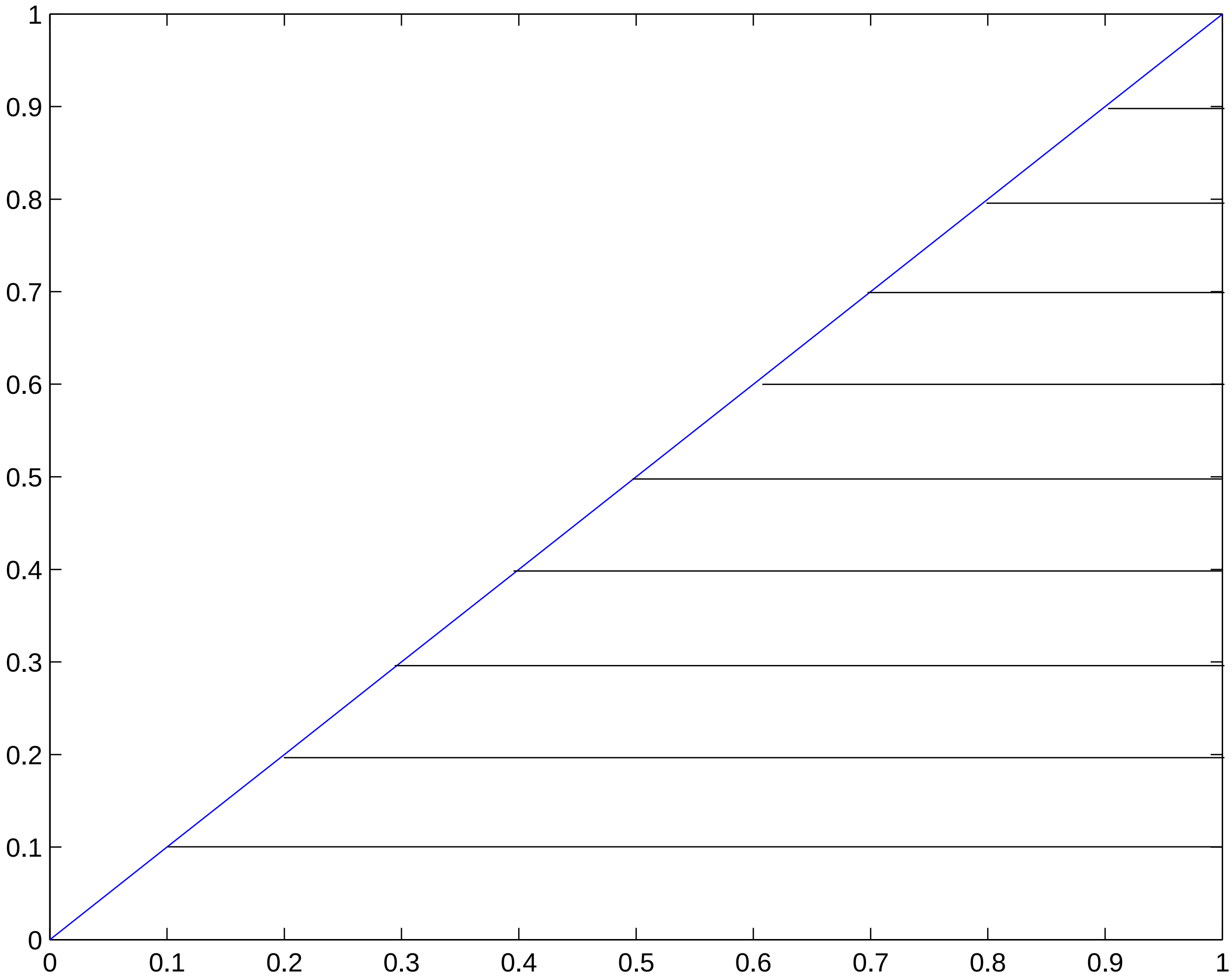
We have:

$$\lim_{N\to\infty} \nu_N = \mathsf{P}\{a \le \xi \le b\} = b - a,$$

i.e., *the frequency of the event $a \le \xi \le b$ converges to the probability of this event*.

Two-dimensional case

For example, we want to calculate the area of the triangle $\{(x, y)\colon\ 0 \le x \le 1, 0 \le y \le x\}$.



Let ξ_k and η_k, $1 \le k \le N$ be independent random variables that have uniform distribution on the interval $[0, 1]$. They define N random points (ξ_k, η_k) in the square $0 \le x, y \le 1$. Let n of these points belong to the set A. Define the frequency

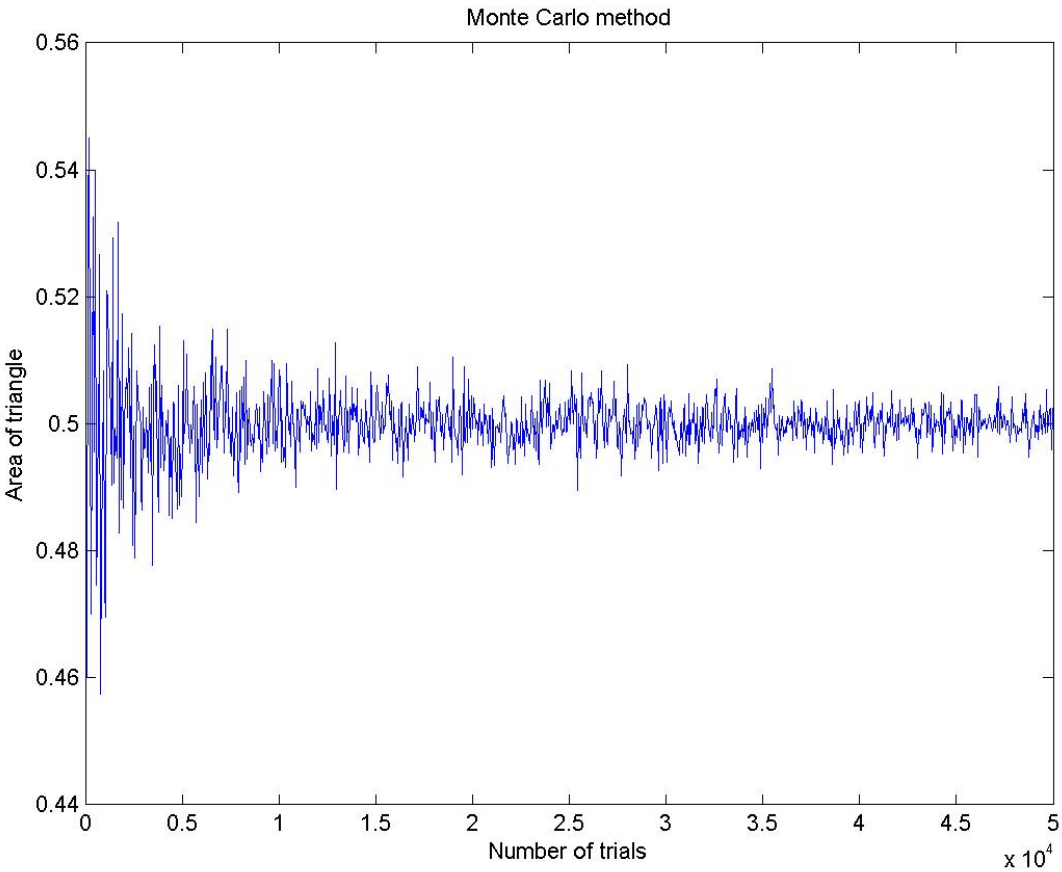
$$\nu_N = \frac{n}{N}$$

Then the sequence ν_N converges to the value of the area of the set A as $N \to \infty$.

MATLAB code

```
% File: triangle.m
% Estimating area of triangle
% Author: Anatoliy Malyarenko
% E-mail: anatoliy.malyarenko@mdh.se
N=50:50:50000;
res=zeros(size(N));
for k=1:length(N)
    temp=rand(50*k,2);
    res(k)=sum(temp(:,1)>temp(:,2))/(50*k);
end;
plot(N,res);
xlabel('Number of trials');

ylabel('Area of triangle');
title('Monte Carlo method');
```



Example: pricing options

Example 1. A security is presently selling for a price of \$30, the nominal interest rate is 8%, the security's volatility is 20%. The strike price is \$34. Find the cost of a call option expiring in three months.

Solution. Let ξ_1, ξ_2, ... be the sequence of independent and equally distributed random variables. Then

$$\lim_{N\to\infty} \frac{\xi_1 + \xi_2 + \cdots + \xi_N}{N} = \mathsf{E}\xi_1.$$

This is the *law of large numbers*.

Let W be a normal random variable with mean $(r - \sigma^2/2)T$ and variance $\sigma^2 T$. Then

$$C = e^{-rT}\mathsf{E}[\max\{S_0 e^W - X, 0\}].$$

Now we can write MATLAB program

MATLAB program

```
% File: bs.m
% Estimating Black-Scholes price
% Author: Anatoliy Malyarenko
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T=0.25; r=0.08; sigma=0.2;
X=34; S0=30;
price=blsprice(S0,X,r,T,sigma);
N=50:50:50000;
result=zeros(size(N));
for k=1:length(N)
    temp=1:50*k;
    temp=randn(size(temp));
    temp=temp*sigma*sqrt(T);
    temp=temp+(r-sigma^2/2)*T;
    temp=exp(temp)*S0;
    temp=max(temp-X,zeros(size(temp)));
    result(k)=sum(temp)/(50*k);
    result(k)=result(k)*exp(-r*T);
end;
plot(N,result);
```

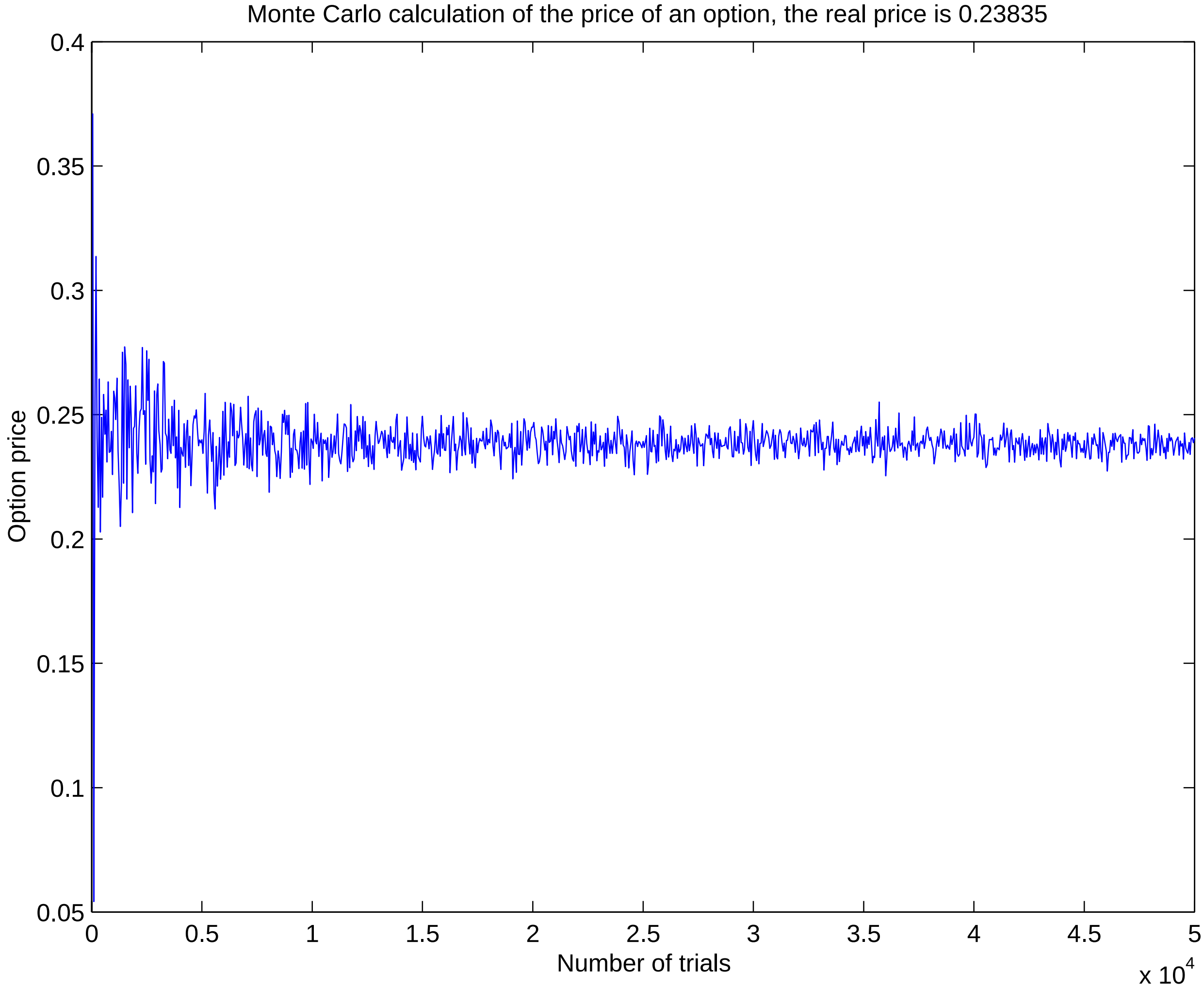

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Problems

1. Consider a five-month European put option when the initial price of the stock is \$50, the strike price is changed from \$48 to \$52 with step \$0.1, the risk-free interest is 10% per annum, and the volatility is 10% per annum. Estimate the option price using Monte Carlo method. Plot the results of estimation for $N = 50000$ trials.

Hint. Let W be a normal random variable with mean $(r-\sigma^2/2)T$ and variance $\sigma^2 T$. Then

$$P = e^{-rT}\mathsf{E}[\max\{X - S_0 e^W, 0\}].$$

2. (For pass with distinction). The *four-dimensional ball* is defined as

$$B = \{\, \mathbf{x} = (x_1, x_2, x_3, x_4) \colon x_1^2 + x_2^2 + x_3^2 + x_4^2 \le 1 \,\}.$$

Estimate the volume of the set B. Plot the results of estimation for $N = 10000 : 50 : 50000$ trials.