Approximation of functions and American options

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Abstract

Contents of the lecture:

- Approximation of functions.
- Gorner's scheme.
- American options.
- Binomial pricing.
- The MATLAB function `binprice`.
- Problems.

Another way to calculate $\mathbf{N(x)}$

Recall that

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2}\, dy.$$

It is easy to see that $N(-x) = 1 - N(x)$. To calculate $N(x)$ for $x \geq 0$, we use the next approximation:

$$N(x) \approx 1 - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sum_{j=1}^{5} a_j d^j,$$

where $d = \dfrac{1}{1 + 0.2316419x}$, and

$$a_1 = 0.31938153, \qquad a_2 = -0.356563782,$$

$$a_3 = 1.781477937, \qquad a_4 = -1.821255978,$$

$$a_5 = 1.330274429.$$

MATLAB function

```
function y=normal(x) %Function definition
% NORMAL The distribution function of the
% standard normal random variable
% NORMAL(X) returns the value of
% P{xi<x}, where xi is the standard normal
% random variable
% Author: Anatoliy Malyarenko
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if (x>=0)
    flag = 0;
    z = x;
else
    flag = 1;
    z = -x;
end
d = 1/(1+0.3316419*z);
a = [0
     0.31938153
     -0.356563782
     1.781477937
     -1.821255978
     1.330274429];
% Gorner's scheme
p = a(6);
for k = 5:-1:1
    p = p*d+a(k);
end
y = 1 - exp(-z^2/2)*p/sqrt(2*pi);
if (flag)
    y = 1 - y;
end
```

Gorner's scheme for calculating the value of $p = a_n d^n + a_{n-1} d^{n-1} + \cdots + a_1 d_1 + a_0$ works as follows:

Initial value	$p = a_n$
After the first passage	$p = a_n d + a_{n-1}$
After the second passage	$p = a_n d^2 + a_{n-1} d + a_{n-2}$
...	...
After the *n*th passage	$p = a_n d^n + \cdots + a_1 d_1 + a_0$

American options

Unlike their European counterparts, American options can be exercised at any date prior to expiration.

In this case, Black–Scholes formula *does not work*.

It can be shown that a vanilla American call option on a non-dividend-paying stock will never be exercised early, and its value is the same as the European call option.

Our problem is: to find approximate value of the American *put* option.

Binomial pricing

① Divide the life of the option into n equal periods of length T/n.

② Suppose that the price of a security can change only at the times $t_k = kT/n$, $k = 1, 2, \dots, n$

③ Suppose that the option can be exercised only at one of the times t_k.

④ Suppose that the security price S_{k+1} at $k+1$ time periods later is either uS_k or dS_k.

⑤ The multipliers u and d can be computed as

$$u = e^{\sigma\sqrt{T/n}}, \qquad d = e^{-\sigma\sqrt{T/n}}.$$

⑥ If l of the first n price movements were increases and $n - l$ were decreases, then the possible values of the price of the put option at time $t_n = T$ are equal to

$$S_n(l) = \max\{X - u^l d^{n-l} S_0, 0\}.$$

⑦ The return V_1 if we exercise the option in moment t_k at node l is equal to

$$V_1 = X - u^l d^{k-l} S_0.$$

⑧ The return V_2 if we do not exercise the option in moment t_k at node l is equal to

$$V_2 = e^{-rT/n}[pS_{k+1}(l+1) + (1-p)S_{k+1}(l)]$$

where $p = \dfrac{1 + rt/n - d}{u - d}$.

⑨ The value $S_k(l)$ of the price of the put option at time t_k at node l is equal to

$$S_k(l) = \max\{V_1, V_2\}.$$

Going backward in time (decreasing k), we calculate $S_0(0)$. This is the approximate value of the price of the American put option.

MATLAB realisation

In MATLAB, this model is realised in a function `binprice`.

```
[pr,opt]=binprice(S0,X,r,T,dt,sig,flag);
```

where `sig` is the volatility, `flag` is equal to 0 for put option and 1 for call option. `pr` is the matrix of possible values of the stock, and `opt` is the matrix of the prices of the option.

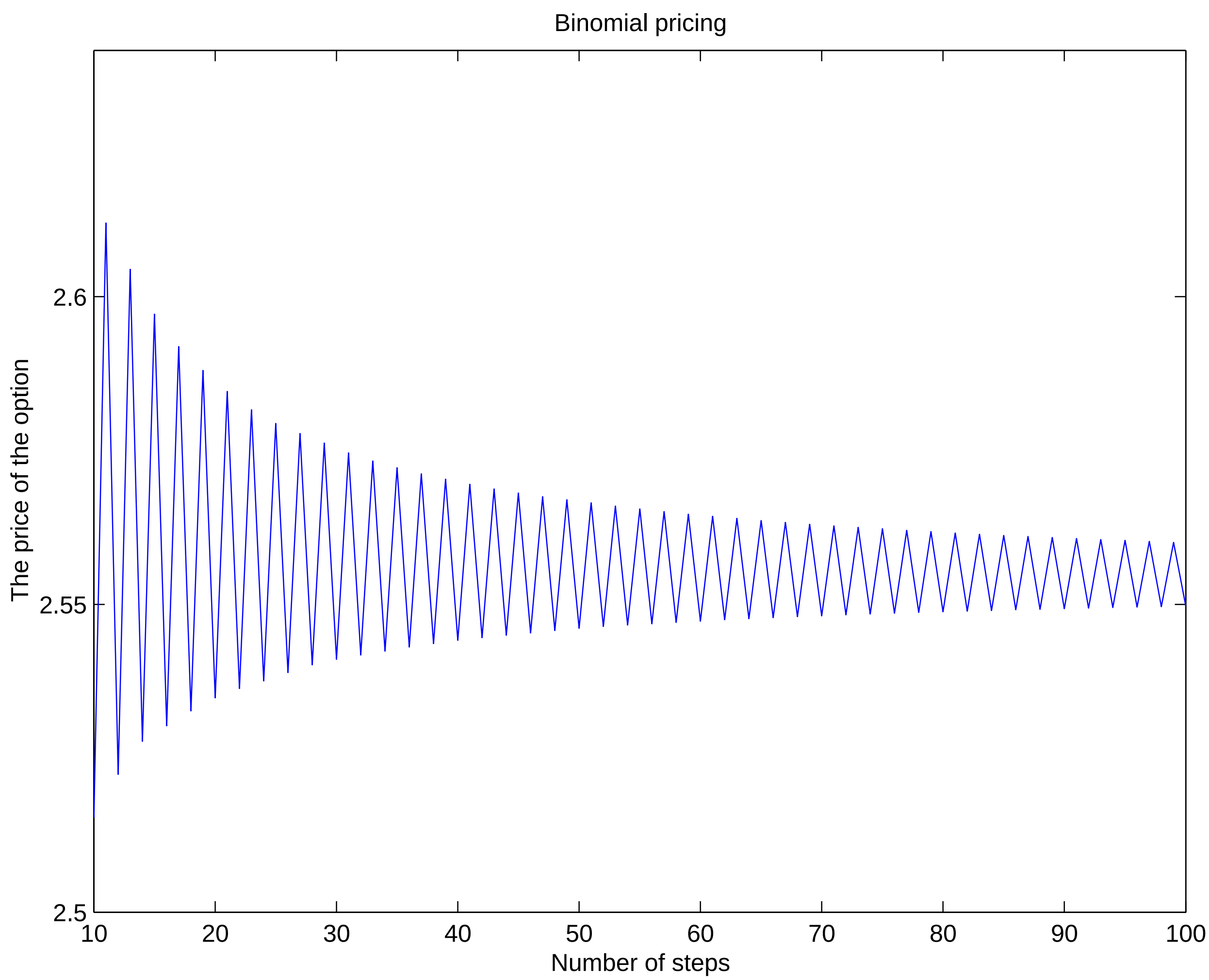
For example, consider a five-month American put option when the initial price of the stock is \$40, the strike price is \$40, the risk-free interest is 8% per annum, and the volatility is 30% per annum. Divide the life of the option into N equal parts, where N changes from 10 to 100 with step 1. Build the graph that shows the dependence between N and the price of the option.

MATLAB solution

```
% File: american.m
% Pricing American put option
% Author: Anatoliy Malyarenko
% E-mail: amo@mdh.se
S0 = 40; X = 40; r = 0.08; T = 5/12;
sigma = 0.3; N = 10:100;
price=zeros(size(N));
for k = 1:length(N)
```

```
    dt = T/N(k);
    [stockpr,optpr] = binprice(S0,X,r,...
                      T,dt,sigma,0);
    price(k)=optpr(1,1);
end;
plot(N,price);
xlabel('Number of steps');

ylabel('The price of the option');
title('Binomial pricing');
```



Problems

1. Consider a five-month ($t = 0.4167$) American put option when the initial price of the stock is $\$50$, the strike price is changed from $\$48$ to $\$52$ with step $\$0.1$, the risk-free interest is 10% per annum, and the volatility is 10% per annum. Divide the life of the option into 100 equal periods. Build a graph that shows the dependence between the strike price and the price of the option.
2. (For pass with distinction). Write your own function for pricing vanilla American put option on a non-dividend paying stock. Solve Problem 1 using your function instead of standard MATLAB function `blsprice`.