Numerical integration

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Abstract

Contents of the lecture:

- ☞ Black–Scholes model.
- ☞ The trapezoidal rule.
- ☞ Simpson's rule.
- ☞ Error handling in MATLAB.
- ☞ Error analysis.
- ☞ The MATLAB function `quad`.
- ☞ Problems.

Notation

X	Strike price
T	Maturity
S_0	Stock price
r	Risk-free interest rate
σ	Volatility
C	Call option price
P	Put option price

Black–Scholes formula

$$C = S_0 N(d_1) - Xe^{-rT} N(d_2),$$
$$P = Xe^{-rT} N(-d_2) - S_0 N(-d_1),$$

where

$$d_1 = \frac{\log(S_0/X) + (r + \sigma^2/2)T}{\sigma\sqrt{T}},$$
$$d_2 = d_1 - \sigma\sqrt{T},$$
$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy.$$

The question is: *how to calculate* $N(x)$?

We can write:

$$N(x) = \begin{cases} \frac{1}{2} + \frac{1}{\sqrt{2\pi}} \int_0^x e^{-y^2/2} dy, & x \geq 0, \\ \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \int_x^0 e^{-y^2/2} dy, & x < 0. \end{cases} \quad (1)$$

We need to calculate integrals *numerically*.

The trapezoidal rule

One way to approximate a definite integral is to use n trapezoids, as shown in Figure 1.

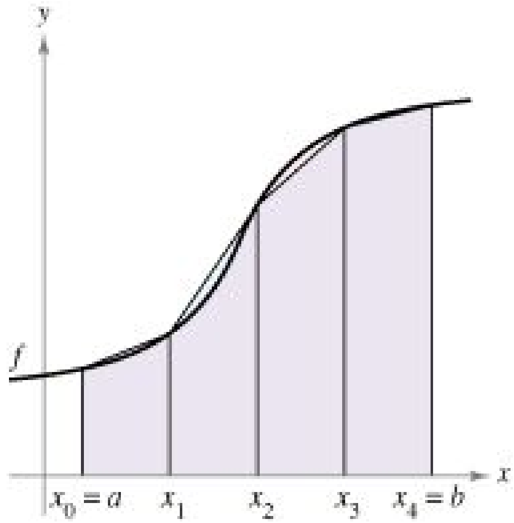

Figure 1: The area of the region can be approximated using four trapezoids

In the development of the method, assume that f is positive and continuous on the interval $[a, b]$. So, the definite integral

$$\int_a^b f(x)\,dx$$

represents the area of the region bounded by the graph of f and the x-axis, from $x = a$ to $x = b$. First, partition the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b - a)/n$, such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

Then form a trapezoid for each subinterval, as shown in Figure 2.

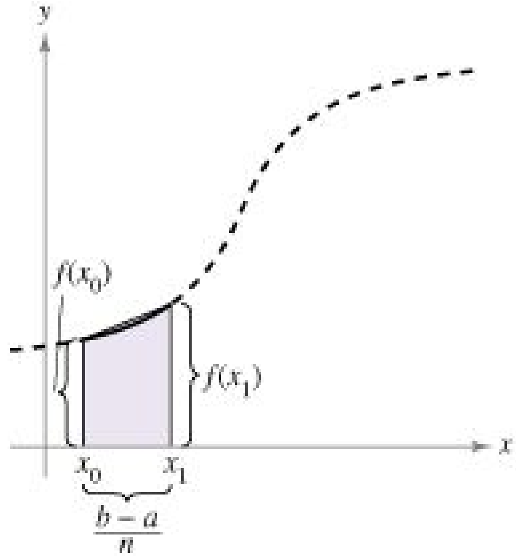

Figure 2: The area of the first trapezoid is $\left[\frac{f(x_0 + f(x_1)}{2}\right]\left(\frac{b-a}{n}\right)$

The area of the ith trapezoid is $\left[\frac{f(x_{i-1} + f(x_i)}{2}\right]\left(\frac{b-a}{n}\right)$. This implies that the sum of the areas of the n trapezoids is

$$\begin{aligned}
&\left(\frac{b-a}{n}\right)\left[\frac{f(x_0 + f(x_1)}{2} + \cdots + \frac{f(x_{n-1} + f(x_n)}{2}\right] \\
&= \left(\frac{b-a}{2n}\right)[f(x_0) + f(x_1) + f(x_1) + f(x_2) + \ldots \\
&\quad + f(x_{n-1}) + f(x_n)] \\
&= \left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)].
\end{aligned}$$

We obtained the *Trapezoidal Rule*:

$$\int_a^b f(x)\,dx \approx \left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)].$$

Observe that the coefficients in the Trapezoidal Rule have the following pattern

$$1\ 2\ 2\ 2\ \ldots\ 2\ 2\ 1$$

MATLAB example 1

This is the MATLAB function that returns the approximate value of the integral $\int_0^{\pi} \sin x\, dx$, calculated with the Trapezoidal Rule with N subintervals.

```
function y=trapezoidal(n) %Function definition
% TRAPEZOIDAL The Trapezoidal Rule
% TRAPEZOIDAL(N) returns the approximate value
% of the integral from 0 to pi of sin(x),
% calculated with the Trapezoidal Rule with
% N subintervals
% Author: Anatoliy Malyarenko
% Mail:   anatoliy.malyarenko@mdh.se
x=linspace(0,pi,n+1); %x0, x1, ... , xn
x=sin(x);
y=2*sum(x)-x(1)-x(n+1); %sin(x0)+2 sin(x1)+ ...
y=y*pi/(2*n); %multiply by Delta x
```

Function call: $n = 4$ and $n = 8$

We type in the Command Window

```
>> y=trapezoidal(4)
```

MATLAB returns

```
y =
    1.8961
```

Indeed, when $n = 4$, $\Delta x = \pi/4$, and you obtain (Figure 3)

$$\begin{aligned}\int_0^{\pi} \sin x\, dx &\approx \frac{\pi}{8}\left(\sin 0 + 2\sin\frac{\pi}{4} + 2\sin\frac{\pi}{2} + 2\sin\frac{3\pi}{4} + \sin\pi\right)\\ &= \frac{\pi}{8}(0 + \sqrt{2} + 2 + \sqrt{2} + 0)\\ &= \frac{\pi(1+\sqrt{2})}{4}\\ &\approx 1.8961.\end{aligned}$$

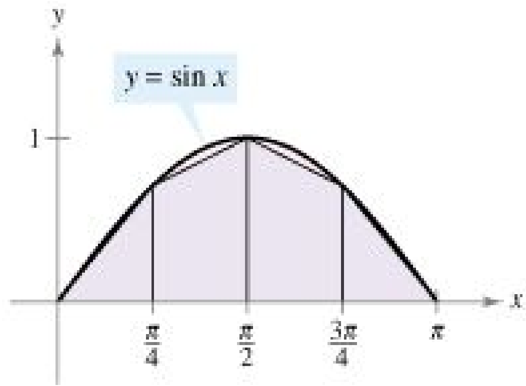

Figure 3: Four subintervals

We type in the Command Window

```
>> y=trapezoidal(8)
```

MATLAB returns

```
y =
    1.9742
```

This calculation is illustrated in Figure 4.

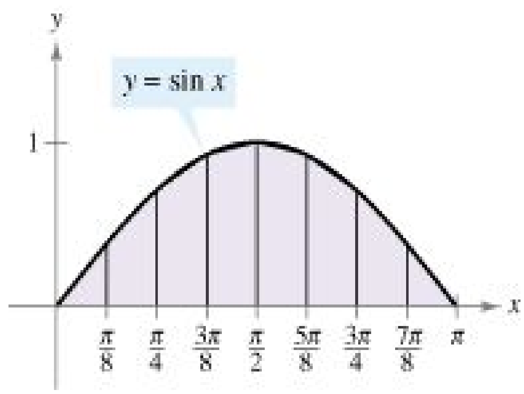

Figure 4: Eight subintervals

Simpson's Rule

One way to view the trapezoidal approximation of a definite integral is to say that on each subinterval you approximate f by a *first*-degree polynomial. In Simpson's Rule, named after the English mathematician Thomas Simpson (1710–1761), you take this procedure one step further and approximate f by *second*-degree polynomials.

Before presenting Simpson's Rule, we list a theorem for evaluating integrals of polynomials of degree 2 (or less).

Theorem 1 (Integral of $\mathbf{p(x) = Ax^2 + Bx + C}$). *If* $p(x) = Ax^2 + Bx + C$, *then*

$$\int_a^b p(x)\,dx = \left(\frac{b-a}{6}\right)\left[p(a) + 4p\left(\frac{a+b}{2}\right) + p(b)\right].$$

Proof of Theorem 1

Proof.

$$\begin{aligned}\int_a^b p(x)\,dx &= \int_a^b (Ax^2 + Bx + C)\,dx \\ &= \left[\frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx\right]_a^b \\ &= \frac{A(b^3 - a^3)}{3} + \frac{B(b^2 - a^2)}{2} + C(b-a) \\ &= \left(\frac{b-a}{6}\right)[2A(a^2 + ab + b^2) + 3B(b+a) + 6C].\end{aligned}$$

By expansion and collecting of terms, the expression inside the brackets becomes

$$\underbrace{(Aa^2 + Ba + C)}_{p(a)} + \underbrace{4\left[A\left(\frac{b+a}{2}\right)^2 + B\left(\frac{b+a}{2}\right) + C\right]}_{4p\left(\frac{a+b}{2}\right)}$$

$$+ \underbrace{(Ab^2 + Bb + C)}_{p(b)}.$$

□

Developing Simpson's Rule

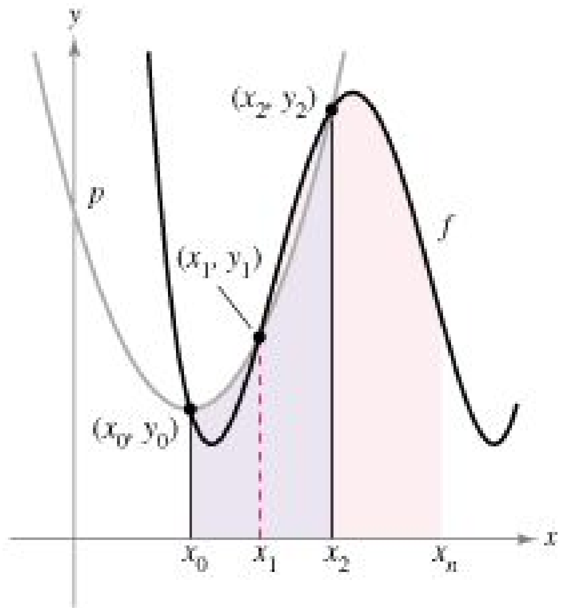

Figure 5: $\int_{x_0}^{x_2} p(x)\,dx \approx \int_{x_0}^{x_2} f(x)\,dx$

To develop Simpson's Rule for approximating a definite integral, you again partition the interval $[a, b]$ into n subintervals, each of width $\Delta x = (b-a)/n$. This time, however, n is required to be even, and the subintervals are grouped in pairs such that

$$a = \underbrace{x_0 < x_1 < x_2}_{[x_0,x_2]} < \underbrace{x_3 < x_4}_{[x_2,x_4]} < \cdots < \underbrace{x_{n-2} < x_{n-1} < x_n}_{[x_{n-2},x_n]} = b.$$

On each (double) subinterval you can approximate f by a polynomial p of degree less than or equal to 2. For example, on the subinterval $[x_0, x_2]$, choose the polynomial of least degree passing through the points (x_0, y_0), (x_1, y_1), and (x_2, y_2), as shown in Figure 5. Now, using p

as approximation of f on this subinterval, you have, by Theorem 1,

$$\begin{aligned}\int_{x_0}^{x_2} f(x)\,dx &\approx \int_{x_0}^{x_2} p(x)\,dx \\ &= \frac{x_2 - x_0}{6}\left[p(x_0) + 4p\left(\frac{x_0 + x_2}{2}\right) + p(x_2)\right] \\ &= \frac{2[(b-a)/n]}{6}[p(x_0) + 4p(x_1) + p(x_2)] \\ &= \frac{b-a}{3n}[f(x_0) + 4f(x_1) + f(x_2)].\end{aligned}$$

Repeating this procedure on the entire interval $[a, b]$ produces the Simpson's Rule:

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots \\ + 4f(x_{n-1}) + f(x_n)].$$

Observe that the coefficients in the Simpson's Rule have the following pattern

$$1\ 4\ 2\ 4\ 2\ 4\ \dots\ 4\ 2\ 4\ 1$$

MATLAB example 2

This is the MATLAB function that returns the approximate value of the integral $\int_0^\pi \sin x\,dx$, calculated with the Simpson's Rule with N subintervals.

```
function y=simpsons(n) %Function definition
% SIMPSONS The Simpson's Rule
% SINPSONS(N) returns the approximate value
% of the integral from 0 to pi of sin(x),
% calculated with the Simpsons Rule with
% N subintervals
% Author: Anatoliy Malyarenko
% Mail:   anatoliy.malyarenko@mdh.se
if (mod(n,2) ~= 0)
    error('N must be even'); %Error handler
```

```
end
x=linspace(0,pi,n+1); %x0, x1, ... , xn
x=sin(x);
y=4*sum(x)-3*x(1)-3*x(n+1); %sin(x0)+4 sin(x1)+ ...
for k=3:2:n-1
    y=y-2*x(k); %sin(x0)+4 sin(x1)+2 sin(x2)+...
end
y=y*pi/(3*n);
```

Error handling in MATLAB

In Simpson's Rule, n is required to be even. We check this in the following fragment of the function:

```
if (mod(n,2) ~= 0)
    error('N must be even'); %Error handler
end
```

The MATLAB function `mod` returns the modulus after division of n by 2. If n is odd or non-integer, it returns non-zero number, and MATLAB function `error` displays the text 'N must be even' and causes an error exit from function to the keyboard.

Type in the Command Window

```
>> y=simpsons(9)
```

MATLAB returns

```
??? Error using ==> simpsons
N must be even
```

Function call: $n = 4$ and $n = 8$

We type in the Command Window

```
>> y=simpsons(4)
```

MATLAB returns

```
y =
    2.0046
```

If we type in the Command Window

```
>> y=simpsons(8)
```

MATLAB returns

```
y =
    2.0003
```

Error analysis

If you must use an approximation technique, it is important to know how accurate you can expect the approximation to be. The following theorem, which we list without proof, gives the formulas for estimating the errors involved in the use of Simpson's Rule and the Trapezoidal Rule.

Theorem 2 (Error in the Trapezoidal Rule and Simpson's Rule)**.** *If f has a continuous second derivative on $[a,b]$, then the absolute error E in approximating $\int_a^b f(x)\,dx$ by the Trapezoidal Rule is*

$$E \leq \frac{(b-a)^3}{12n^2} \max |f''(x)|, \quad x \in [a,b].$$

Moreover, if f has a continuous fourth derivative on $[a,b]$, then the absolute error E in approximating $\int_a^b f(x)\,dx$ by the Simpson's Rule is

$$E \leq \frac{(b-a)^5}{180n^4} \max |f^{(4)}(x)|, \quad x \in [a,b].$$

Example: the approximate error in the Trapezoidal Rule

Example 1. Determine a value of n such that the Trapezoidal Rule will approximate the value of $\int_0^1 \sqrt{1+x^2}\,dx$ with an absolute error that is less than 0.01.

Solution. Begin by letting $f(x) = \sqrt{1+x^2}$ and finding the second derivative of f.

$$f'(x) = x(1+x^2)^{-1/2} \quad \text{and} \quad f''(x) = (1+x^2)^{-3/2}.$$

The maximal value of $|f''(x)|$ on the interval $[0,1]$ is $f''(0) = 1$. So, by Theorem 2, you can write

$$E \leq \frac{(b-a)^3}{12n^2}|f''(0)| = \frac{1}{12n^2}(1) = \frac{1}{12n^2}.$$

To obtain an absolute error E that is less than 0.01, you must choose n such that $1/(12n^2) \leq 1/100$.

$$100 \leq 12n^2 \quad \Rightarrow \quad n \geq \sqrt{100/12} \approx 2.89.$$

Therefore, you can choose $n = 3$ (because n must be grater than or equal 2.89) and apply the Trapezoidal Rule, as shown in Figure 6, to obtain

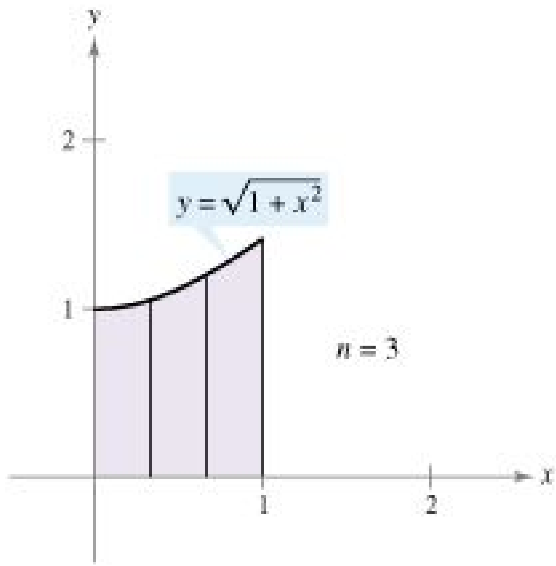

Figure 6: $1.144 \leq \int_0^1 \sqrt{1 + x^2}\, dx \leq 1.164$

$$\int_0^1 \sqrt{1 + x^2}\, dx \approx 1.154.$$

So, with an error no larger than 0.01, you know that

$$1.144 \leq \int_0^1 \sqrt{1 + x^2}\, dx \leq 1.164.$$

□

The MATLAB function quad

The MATLAB function `quad` numerically evaluate integrals. The MATLAB command `Q = quad(FUN,A,B)` tries to approximate the integral of function `FUN` from `A` to `B` to within an error of 10^{-6}.

For example, type in the Command Window

```
>> q=quad('sin',0,pi)
```

MATLAB returns

```
q =
    2.0000
```

For this particular integral, you could have found an antiderivative and determined that the exact value of the integral is 2.

Problems

1. Write two MATLAB functions. The first function must evaluate the approximate value of the integral $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$, using Simpson's rule. The second function must evaluate the same value, using the MATLAB function `quad`. Assuming that the second function gives the exact answer, draw the graphs of the absolute and relative errors of the Simpson's approximation for `n=10` and `x=0:0.01:1`.
2. (For pass with distinction). Prove that you can find a polynomial $p(x) = Ax^2 + Bx + C$ that passes through any three points (x_1, y_1), (x_2, y_2), and (x_3, y_3), where the x_i are distinct.