Linear systems

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Abstract

Contents of the lecture:

- ☞ Gaussian elimination
- ☞ QR decomposition.
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The simple example

Example 1. Consider the linear system

$$\begin{aligned} 2x + \quad y &= 3 \\ 2y &= 2 \end{aligned} \tag{1}$$

It can be written as $Ax = b$, where

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \qquad b = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

The system (1) is *very* easy to solve, because the matrix A is *triangular*. Indeed, from the second equation we have $y = 1$. Substituting this value to the first equation, we obtain $2x = 2$, or $x = 1$.

Gaussian elimination

Is it possible to transform linear system to the system with triangular matrix?

The answer is: in most cases *yes*. The corresponding process is called *Gaussian elimination*.

Example 2. Consider the linear system

$$\begin{aligned} 2x+ \quad y &= 0 \\ 3x- \quad y &= 5 \end{aligned} \tag{2}$$

Multiply the first equation by $3/2$ and subtract it from the second equation. We obtain

$$\begin{aligned} 2x+ \qquad y &= 0 \\ - \quad 5/2 \cdot y &= 5 \end{aligned}$$

Exceptional case 1 — no solutions

Consider the linear system

$$\begin{aligned} x+ \quad y &= 2 \\ 2x+ \quad 2y &= 5 \end{aligned}$$

Multiply the first equation by 2 and subtract it from the second equation. We obtain:

$$\begin{aligned} x+ \quad y &= 2 \\ 0y &= 1 \end{aligned}$$

Exceptional case 2 — many solutions

Consider the linear system

$$\begin{aligned} x+ \quad y &= 2 \\ 2x+ \quad 2y &= 4 \end{aligned}$$

Multiply the first equation by 2 and subtract it from the second equation. We obtain:

$$\begin{aligned} x+ \quad y &= 2 \\ 0y &= 0 \end{aligned}$$

Orthogonal matrices

Definition 1. The matrix A is called *orthogonal* if $AA^T = I$.

Example 3.

$$A = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}$$

Indeed,

$$\begin{aligned} AA^T &= \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix} \\ &= \begin{pmatrix} (1/2)(1/2) + (\sqrt{3}/2)(\sqrt{3}/2) & -(1/2)(\sqrt{3}/2) + (\sqrt{3}/2)(1/2) \\ -(\sqrt{3}/2)(1/2) + (1/2)(\sqrt{3}/2) & (\sqrt{3}/2)(\sqrt{3}/2) + (1/2)(1/2) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

QR decomposition

Theorem 1. *Any matrix A can be represented as*

$$A = QR,$$

where the matrix Q is orthogonal, and the matrix R is upper triangular.

Why QR decomposition is useful? For any linear system $Ax = b$ we can write

$$QRx = b.$$

Denote Rx by y. Then $Qy = b$, and

$$y = Q^{-1}b = Q^T b.$$

The system $Rx = y$ is easy to solve.

QR decomposition in MATLAB

```
>> A=[2 1;3 -1];
>> [Q,R]=qr(A)
Q =
   -0.5547   -0.8321
   -0.8321    0.5547

R =
   -3.6056    0.2774
         0   -1.3868
```

Now we solve the system (2) by QR decomposition.

```
>> b=[0;5];
>> y=Q'*b;
>> x=R\y

x =
    1.0000
   -2.0000
```

Problems

1. Today is September 30, 2002. Three bonds are available on the market (Table 1). Find weights for a bond portfolio with duration 10 years and convexities from 150 to 160 with step 0.1, using QR decomposition. Show your results graphically.

	Bond 1	Bond 2	Bond 3
Maturity	15.06.2015	1.03.2025	1.03.2020
Coupon rate	0.07	0.08	0.065
Yield	0.059	0.075	0.050

Table 1: Available bonds, problem 1

2. (For pass with distinction). Today is September 30, 2002. Three bonds are available on the market (Table 2). Find weights for a bond portfolio with convexities from 145 to 155 with step 0.1, using QR decomposition. Show your results graphically. **Your MATLAB script should not contain loops**.

	Bond 1	Bond 2	Bond 3
Maturity	1.03.2025	15.01.2013	1.08.2017
Coupon rate	0.07	0.08	0.075
Face value	250	100	200
Clean price	264.00	108.36	232.07

Table 2: Available bonds, problem 2