Valuation of fixed-income securities

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Abstract

Contents of the lecture:

- ☞ Bonds.
- ☞ Linear systems.
- ☞ Examples and problems.

Fixed-income securities

Fixed-income securities are debt instruments which do not imply any ownership of a firm on the part of the buyer.

Example: the *simple fixed-coupon bond*. It is characterised by the next parameters:

- ☞ The *settlement date*.
- ☞ The *maturity date*.
- ☞ The *period*.
- ☞ The *basis*.
- ☞ The *end-of-month rule*.
- ☞ The *issue date*.
- ☞ The *first coupon date* and *last coupon date*.

Terminology

The *settlement date* of a bond is the date when money first changes hands; i.e., when a buyer pays for a bond. It need not coincide with the *issue date*, which is the date a bond is first offered for sale.

The *first coupon date* and *last coupon date* are the dates when the first and last coupons are paid, respectively. Although bonds typically pay periodic annual or semiannual coupons, the length of the first and last coupon periods may differ from the standard coupon period.

The *maturity date* of a bond is the date when the issuer returns the final face value, also known as the *redemption value* or *par value*, to the buyer.

The *period* of a bond refers to the frequency with which the issuer of a bond makes coupon payments to the holder.

Period Value	**Payment Schedule**
0	No coupons. (Zero coupon bond.)
1	Annual
2	Semiannual
3	Tri-annual
4	Quarterly
6	Bi-monthly
12	Monthly

Table 1: Period of a bond

The *basis* of a bond refers to the basis or day-count convention for a bond. Basis is normally expressed as a fraction in which the numerator determines the number of days between two dates, and the denominator determines the number of days in the year. For example, the numerator of *actual/actual* means that when determining the number of days between two dates, count the actual number of days; the denominator means that you use the actual number of days in the given year in any calculations (either 365 or 366 days depending on whether or not the given year is a leap year).

The *end-of-month rule* affects a bond's coupon payment structure. When the rule is in effect, a security that pays a coupon on the last actual day of a month will always pay coupons on the last day of the month. This means, for example, that a semiannual bond that pays a coupon on February 28 in non-leap years will pay coupons on August 31 in all years and on February 29 in leap years.

The simple fixed-coupon bond: example

For instance: given a face value of $\$100$, a maturity of five years and semiannual payment, a coupon rate of 8% implies that we will receive a payment of $\$4$ every six months and a final payment of $\$104$.

We see that when we buy a bond, we actually buy a cash flow. The term *foxed income* stems from the fact that in simple bonds the cash flow is fixed and known from the beginning (at least if we assume that the issuer will not default). Since there is a well-developed secondary market for bonds, there is no need to buy a bond right when it is issued, nor to keep it to maturity.

The basic problems

There are two basic problems one has to deal with:

☞ Valuing bonds, i.e., determining a fair price for them.

☞ Managing a portfolio of bonds, shaped according to one's particular needs.

Valuing bonds

Let $t_0, t_1, \ldots, t_n$ and $C_0, C_1, \ldots, C_n$ be a cash flow of the bond. The present value of this cash flow can be calculated as (Lecture 5)

$$P = \sum_{k=0}^{n} \frac{C_{t_k}}{(1+r)^{t_k - t_0}}. \tag{1}$$

What rate r should we use in pricing? If the bond is default-free, as in the case of government bonds, this should be the *prevailing risk-free interest rate, no more, no less*. To see why, we may use a common principle in finance, i.e., the *no-arbitrage* principle.

The no-arbitrage principle

Consider a zero-coupon bond, with a face value F, a maturity of one year, and a price P on the issue date. According to (1) we have

$$P = \frac{F}{1+r}.$$

Assume that the bond is underpriced, i.e., it sells for a price P_1 such that

$$P_1 < P = \frac{F}{1+r}.$$

and that we may take out a loan at the risk-free interest rate r. Then we can borrow an amount L and use it to purchase L/P_1 bonds. Then, at maturity, we must pay $L(1+r)$ to our money lender, and we get an amount FL/P_1 when the face value is refunded for each bond. The cash flow at maturity will be

$$L\frac{F}{P_1} - L(1+r) = L\left(\frac{F}{P_1} - 1 - r\right) > 0.$$

Hence, we pay nothing in the beginning and receive a positive amount in the future. This is what is called *arbitrage*.

Assume that the bond is overpriced:

$$P_1 > P = \frac{F}{1+r}.$$

In this case we should borrow the bond itself rather than the cash needed to buy it. Let us assume that we borrow bonds for a total value L, we sell them at price P_1 and we invest the money we obtain. At maturity, we get $L(1+r)$ from our investment, and we have to pay the face value F to the owner for each bond that we have borrowed. Hence the cash flow at maturity is again positive:

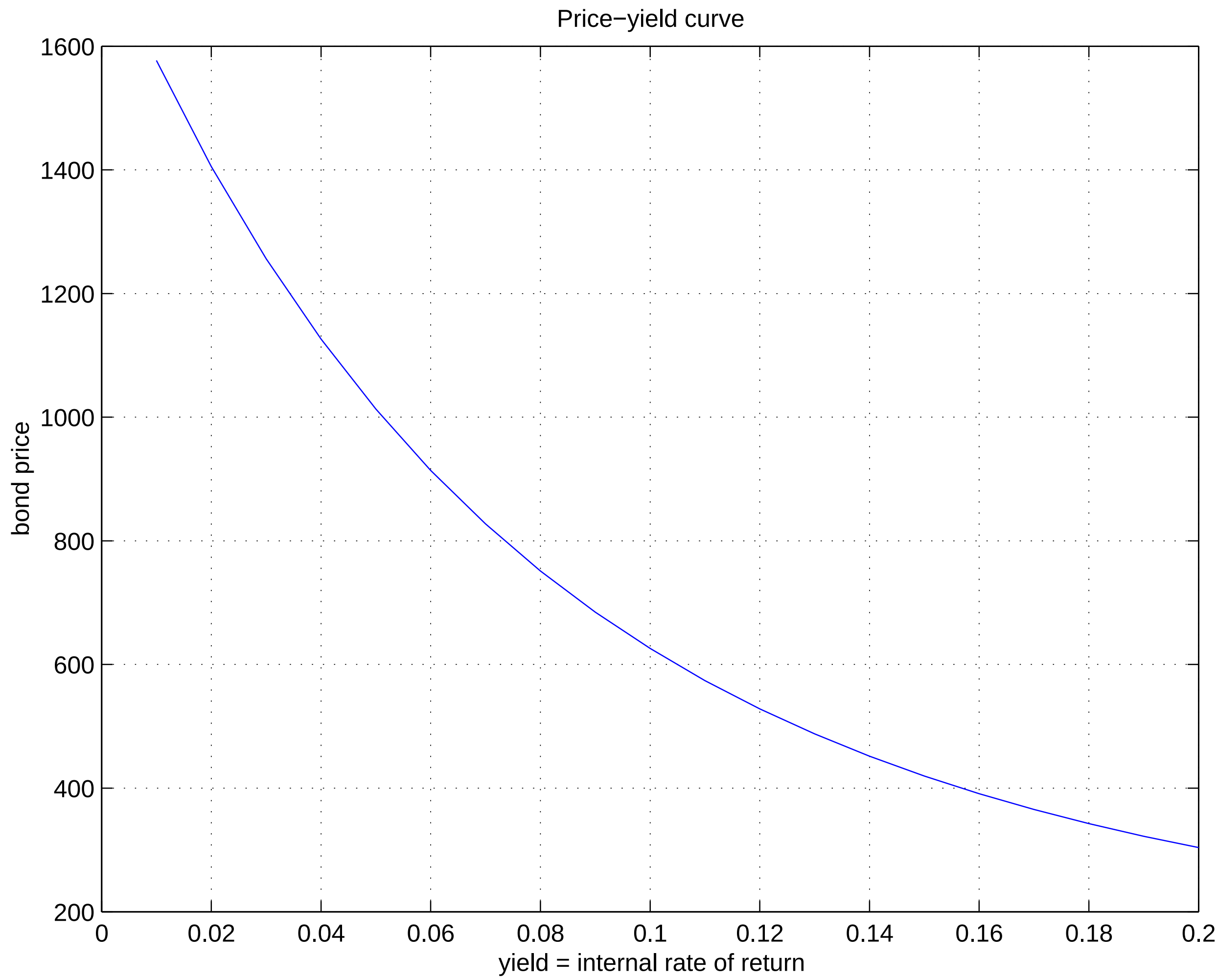
$$-L\frac{F}{P_1} + L(1+r) = L\left(-\frac{F}{P_1} + 1 + r\right) > 0.$$

We value bond by (1). There exist no computational problems.

The yield of a bond

Definition 1. The yield of a bond is the internal rate of return of the bond's cash flow.

The *price-yield curve* is the relationship between the yield and the price of a bond.



The price approximation

The price-yield curve is relatively complicated, because the yield is the solution to a non-linear equation. We want to find a way to approximate the change in price with respect to a change in yield.

Recall Taylor's formula:

$$P(\lambda) \approx P(\lambda_0) + P'(\lambda_0)(\lambda - \lambda_0) + \frac{1}{2}P''(\lambda_0)(\lambda - \lambda_0)^2.$$

Introduce the notation:

$$D = -\frac{P'(\lambda)}{P(\lambda)}, \quad C = \frac{P''(\lambda)}{P(\lambda)}.$$

Now we have

$$P(\lambda) \approx P(\lambda_0) - DP(\lambda_0)(\lambda - \lambda_0) + \frac{1}{2}P(\lambda_0)C(\lambda - \lambda_0)^2.$$

D is the *duration* of a bond, C is the bond's *convexity*.

Managing a portfolio of bonds: example

Example 1. Three different kinds of bonds are available on the market. We want to construct a bond portfolio with $D = 9$ and $C = 110$.

	Bond 1	**Bond 2**	**Bond 3**
Maturity	30.06.07	30.09.12	30.03.18
Coupon rate	0.06	0.05	0.06
Yield	0.062	0.056	0.061

Both the duration and the convexity of the portfolio can be computed as weighted combinations of the corresponding bond characteristics. Strictly speaking, this is not true in general, but can be considered as a simple approximation.

Let C_j and D_j denote the bond durations and convexities, respectively ($1 \le j \le 3$). Then

$$\begin{aligned} D_1 w_1 + \quad D_2 w_2 + \quad D_3 w_3 &= D \\ C_1 w_1 + \quad C_2 w_2 + \quad C_3 w_3 &= C \\ w_1 + \quad w_2 + \quad w_3 &= 1, \end{aligned} \tag{2}$$

where w_j denotes the weight of the jth portfolio.

MATLAB program

```
% File: immunisation.m
% Immunisation of bond portfolio
% Author: Anatoliy Malyarenko
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settle='19-Sep-2002';
maturities = ['30-Jun-2007'
              '30-Sep-2012'
              '30-Mar-2018'];
couponRates = [0.06; 0.05; 0.06];
```

```
yields = [0.062; 0.056; 0.061];
% Compute durations and convexities
durations = bnddury(yields,...
                    couponRates,...
                    settle,...
                    maturities);
convexities = bndconvy(yields,...
                       couponRates,...
                       settle,...
                       maturities);
% Compute portfolio weights
A = [durations'
     convexities'
     1 1 1];
b = [9
     110
     1];
weights = A\b;
% Show the results
disp(' ');
disp('Portfolio 1  Portfolio 2 Portfolio 3');

disp(' ');
disp(sprintf('%6.4f  %6.4f  %6.4f', weights));
```

And the answer is

```
Portfolio 1  Portfolio 2  Portfolio 3

0.0014       0.3228       0.6758
```

Left and right division in MATLAB

Recall that the matrix A^{-1} is called the *inverse* to the matrix A if

$$AA^{-1} = A^{-1}A = I,$$

where I denotes the identity matrix.

MATLAB	**Mathematics**
$A\backslash B$	$A^{-1}B$
A/B	AB^{-1}

Equation (2) can be rewritten as

$$Aw = b.$$

Multiply both hand sides by A^{-1} *from the left*. We obtain

$$w = A^{-1}b.$$

That's why we used *left* division.

Problems

1. Today is September 19, 2002. Three bonds are available on the market (Table 2). Find weights for a bond portfolio with duration 10 years and convexities from 150 to 160 with step 0.1. Show your results graphically.
2. (For pass with distinction). Today is September 19, 2002. Three bonds are available on the market (Table 3). Find weights for a bond portfolio with duration 10 years convexities from 145 to 155 with step 0.1. Show your results graphically. *Hint*. For information on quoted and purchase prices see Financial Toolbox, p. 2-21. Use MATLAB function `bndyield` from Financial Toolbox.

	Bond 1	Bond 2	Bond 3
Maturity	15.06.2015	1.03.2025	1.03.2020
Coupon rate	0.07	0.08	0.06
Yield	0.059	0.075	0.049

Table 2: Available bonds, problem 1

	Bond 1	Bond 2	Bond 3
Maturity	1.03.2025	15.01.2013	1.08.2017
Coupon rate	0.08	0.08	0.075
Face value	250	100	200
Quoted price	264.00	108.36	232.07

Table 3: Available bonds, problem 2