Analysing and computing cash flow streams

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Abstract

Contents of the lecture:

☞ Present value.
☞ Rate of return.
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Regular cash flows

Definition 1. The *regular cash flow* is a stream of *periodic* payments C_t at discrete-time instants $t = 0, 1, \dots, n$.

Remark 1. Usually we have $C_0 < 0$ corresponding to an initial cash outlay.

Analysing cash flows is a typical problem of the *investment analysis*.

Present value

Let r denotes the risk-free interest rate.

Definition 2. The *present value* of the regular cash flow is determined as

$$P = \sum_{t=0}^{n} \frac{C_t}{(1+r)^t}. \quad (1)$$

Example 1. Consider the yearly cash flow of an investment:

$$-1000, -1200, 800, 900, 800.$$

The risk-free interest rate is 6% per year. What is the present value of this cash flow?

Solution. This cash flow means that you invest \$1000 today and \$1200 one year later. The investment fund promises to pay \$800 two years later, \$900 three years later and \$900 after four years from now. Is it profitable for you?

The answer is **yes**, if the present value is *positive*. The answer is **no**, if the present value is *negative or equal to zero*.

In order to calculate the present value, we use the function `pvvar` from the MATLAB Financial Toolbox:

```
>> CashFlow=[-1000 -1200 800 900 800];
>> Rate=0.06;
>> PresentVal=pvvar(CashFlow,Rate)

PresentVal =

  -30.7460
```

The present value of the cash flow is *negative*. It means that it is better to invest your money to the bank. □

Irregular cash flows

Definition 3. The *irregular cash flow* consists of two streams:

- ☞ the stream of *times* $t_0, t_1, \ldots, t_n$;
- ☞ the stream of *money* $C_{t_0}, C_{t_1}, \ldots, C_{t_n}$.

The present value of the irregular cash flow is calculated as

$$P = \sum_{k=0}^{n} \frac{C_{t_k}}{(1+r)^{t_k - t_0}}.$$

Example 2. The annual interest rate is 9%. Calculate the present value of this cash flow.

Cash flow	Dates
-$10000	January 12, 1987
$2500	February 14, 1988
$2000	March 3, 1988
$3000	June 14, 1988
$4000	December 1, 1988

Table 1: The irregular cash flow

Solution. We use the function `pvvar` with the additional parameter:

```
>> CashFlow=[-10000 2500 2000 3000 4000];
>> Rate=0.09;
>> Dates=['01/12/1987'
          '02/14/1988'
          '03/03/1988'
          '06/14/1988'
          '12/01/1988'];
>> PresentVal=pvvar(CashFlow,Rate,Dates)

PresentVal = 142.1648
```

It is profitable to invest money to this investment fund.

□

Rate of return

Definition 4. The *internal rate of return* is a value x of the risk-free interest rate such that the present value of the cash flow is zero.

In words, if the risk-free interest rate is equal to x, then both the bank and the investment fund give you the same profit.

Mathematically, the rate of return x of the *regular* cash flow is the solution to the equation

$$f(x) = \sum_{t=0}^{n} \frac{C_t}{(1+x)^t} = 0. \quad (2)$$

How one can solve equation (2)? This is our first example of the *computational problem*: to find the zero of a function $f(x)$.

Newton's method

We study a technique for approximating the real zeros of a function. The technique is called **Newton's method**, and it uses tangent lines to approximate the graph of the function near its x-intercept.

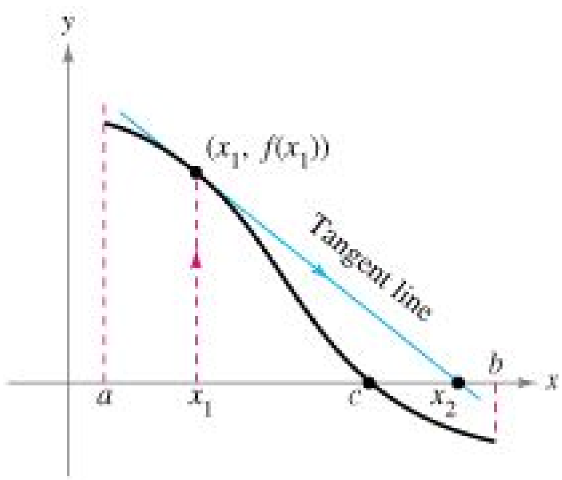

Figure 1: Newton's method: the first two estimates

To see how Newton's method works, consider a function f that is continuous on the interval $[a, b]$ and differentiable on the interval (a, b). If $f(a)$ and $f(b)$ differ in sign, then, by the Intermediate Value Theorem from calculus, f must have at least one zero in the interval (a, b). Suppose you estimate this zero to occur at

$$x = x_1 \qquad \text{First estimate}$$

as shown in Figure 1.

Newton's method is based on the assumption that the graph of f and and the tangent line at $(x_1, f(x_1))$ both cross the x-axis at *about* the same point. Because you can easily calculate the x-intercept for this tangent line, you can use it as a second (and, usually, better) estimate for the zero of f. The tangent line passes through the point $(x_1, f(x_1))$ with a slope of $f'(x_1)$. In point-slope form, the equation of the tangent line is therefore

$$y - f(x_1) = f'(x_1)(x - x_1)$$
$$y = f'(x_1)(x - x_1) + f(x_1).$$

Letting $y = 0$ and solving for x produces

$$x = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

So, from the initial estimate x_1 you obtain a new estimate

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$ Second estimate (see Figure 1)

You can improve on x_2 and calculate yet a third estimate

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$ Third estimate (see Figure 2)

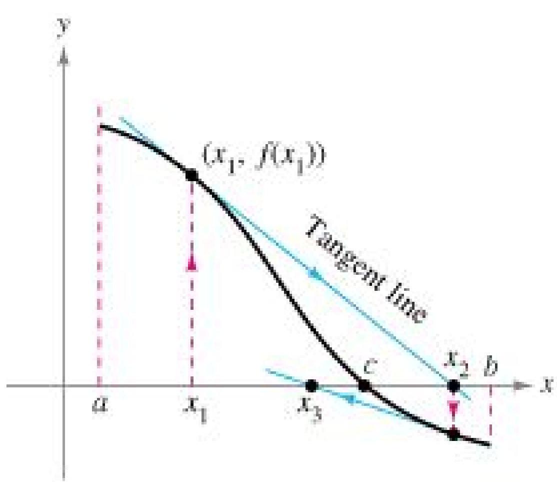

Figure 2: Newton's method: the first three estimates

Repeated application of this process is called Newton's method.

Newton's method: summary

Let $f(c) = 0$, where f is differentiable on an open interval containing c. Then, to approximate c, use the following steps:

① Make an initial estimate x_1 that is "close" to c (a graph is helpful).

② Determine a new approximation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{3}$$

③ If $|x_n - x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedure is called an **iteration**.

Rate of return: example

Example 3. Calculate the rate of return in Example 1.

Solution 1. In order to find the initial guess x_1, we build the graph of the present value, using the next M-file:

```
CashFlow=[-1000 -1200 800 900 800];
Rate=linspace(0,0.1);
PresentValue=ones(size(Rate));
for k=1:length(Rate)
 PresentValue(k)=pvvar(CashFlow,Rate(k));
end
plot(Rate,PresentValue);
grid;

xlabel('Risk-free interest rate');
ylabel('Present value');
title('Present value')
```

The result is shown in Figure 3.

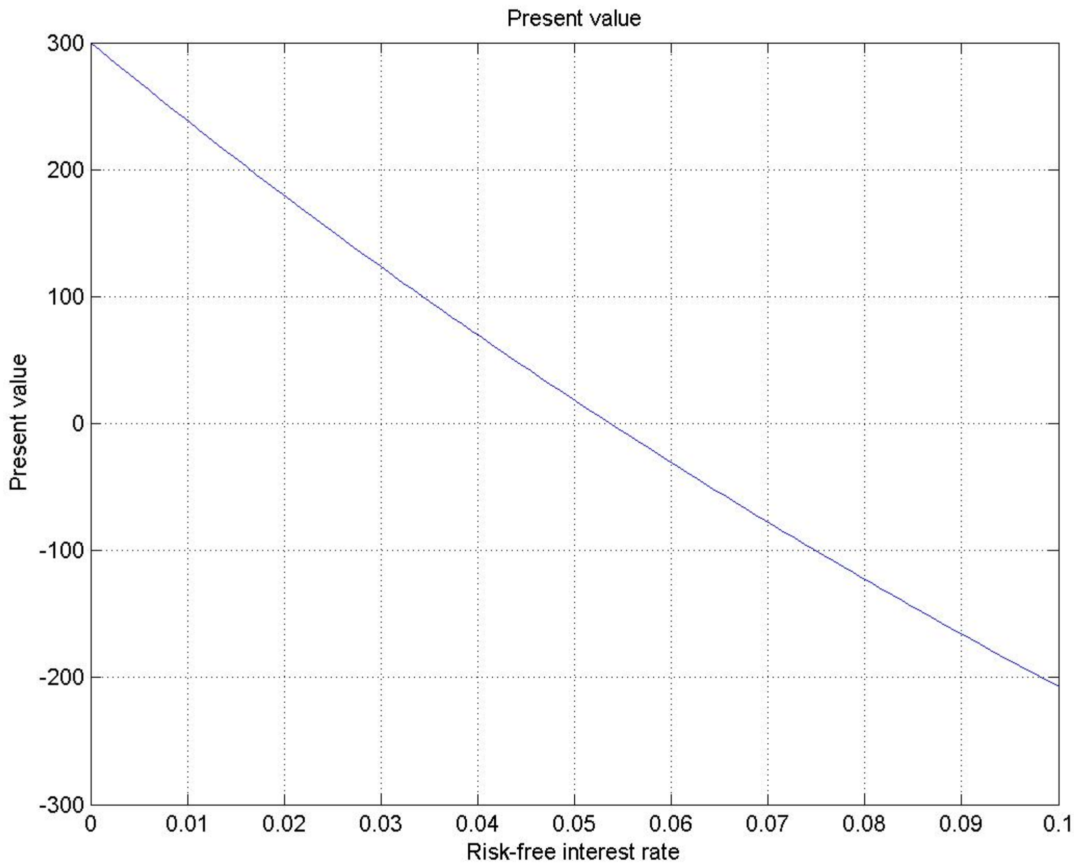

Figure 3: The present value

We can choose $x = 0.05$ as the initial guess.

How to implement Newton's method in MATLAB? We can compute the value $f(x_n)$ in (3), using the `pvvar` function. But how to calculate $f'(x_n)$?

The function $f(x)$ has the form

$$f(x) = -1000 - \frac{1200}{1+x} + \frac{800}{(1+x)^2} + \frac{900}{(1+x)^3} + \frac{800}{(1+x)^4}.$$

It is easy to calculate the derivative $f'(x)$:

$$f'(x) = \frac{1200}{(1+x)^2} - \frac{1600}{(1+x)^3} - \frac{1800}{(1+x)^4} - \frac{3200}{(1+x)^5}.$$

Fortunately, this function has the form (1) and therefore can be calculated, using the `pvvar` function. The script looks like:

```
CashFlow=[-1000 -1200 800 900 800];

CashFlowDer=[0 0 1200 -1600 -1800 -3200];
tol=0.0001;
current=0.05;
next=current-pvvar(CashFlow,current)...
     /pvvar(CashFlowDer,current);
iter=1;
while (abs(next-current)>=tol)
    current=next;
    next=current-pvvar(CashFlow,current)...
         /pvvar(CashFlowDer,current);
    iter=iter+1;
end
next
iter
```

The variable `tol` is the desired accuracy. The variable `current` contains x_n, the variable `next` contains x_{n+1}. Finally, the variable `iter` describes, how many iterations we calculated. The result is:

```
next=
    0.0537
iter=
    4
```

□

Solution 2. We use MATLAB function `irr`.

```
>> CashFlow=[-1000 -1200 800 900 800];
>> Return=irr(CashFlow)

Return =
    0.0537
```

i.,e., the rate of return is 5.37%. Two methods gave the same result up to 4 digits. □

The case of an irregular cash flow

In the case of an irregular cash flow, the rate of return x is the solution of the equation

$$\sum_{k=0}^{n} \frac{C_{t_k}}{(1+x)^{t_k - t_0}} = 0.$$

Example 4. Calculate the rate of return in Example 2.

Solution. We use the MATLAB function `xirr`.

```
>> CashFlow=[-10000 2500 2000 3000 4000];
>> Dates=['01/12/1987'
          '02/14/1988'
          '03/03/1988'
          '06/14/1988'
          '12/01/1988'];
>> Return=xirr(CashFlow,Dates)

Return = 0.1006
```

□

The MATLAB function `xirr`

The description of the function `xirr` has the form

```
Return = xirr(CashFlow,
              CashFlowDates,
              Guess,
              MaxIterations)
```

`Guess` — initial estimate of the expected return;

`MaxIterations` — number of iterations used by Newton's method to solve for `Return`.

Problems

1. Write an M-file that calculates the rates of return for cash flows shown in Tables 2 and 3. Use Newton's method for cash flow I and the MATLAB function `xirr` for cash flow II. What flow is better for you?
2. (For pass with distinction). $150 is paid monthly into a saving account. The payments are made at the end of the month for ten years. What is the present value of these payments when the annual interest rate is changed from 2% to 9% with the step 0.1%? Show your results graphically. *Hint*: use the function `pvfix` from Financial Toolbox.

Dates	**Cash flow**
Initial	-$4,400
Year 1	$800
Year 2	$800
Year 3	$1800
Year 4	$800
Year 5	$1400

Table 2: Cash flow I

Cash flow	**Dates**
-$10,000	January 1, 2002
$800	March 10, 2003
$800	April 14, 2004
$1800	June 26, 2005
$800	August 31, 2006
$1400	November 15, 2007

Table 3: Cash flow II