Interest rate models

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Abstract

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- ☞ The term structure of interest rates.
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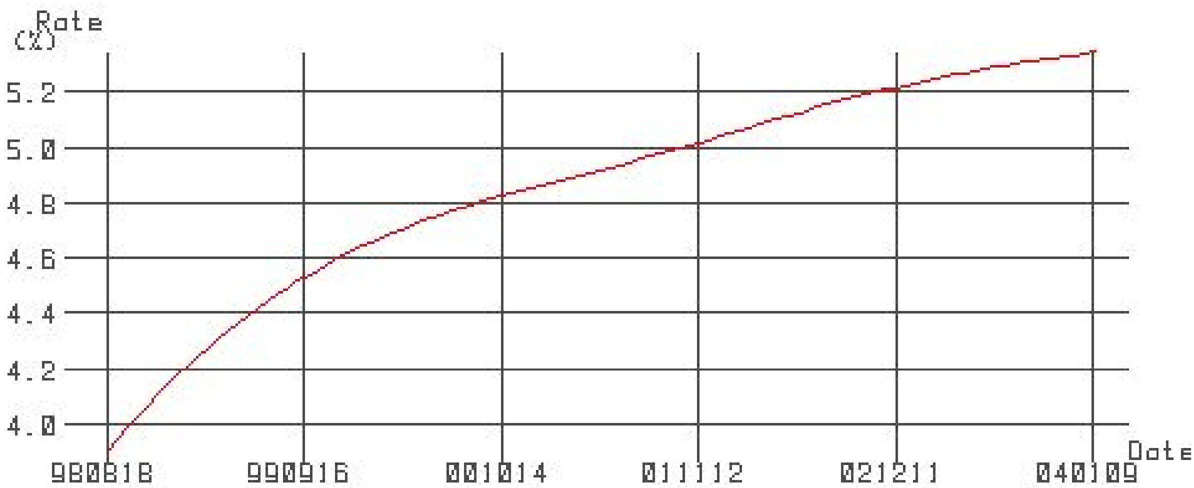
Introduction

Fundamental to the modeling of interest rate patterns is the "term structure" of interest rates. This refers to the relationship between bonds of different terms.

When interest rates of bonds are plotted against their terms, this is called the "yield curve". The shape of the yield curve reflects the market's future expectation for interest rates.



Term structure of interest rates

Term structure models are based on the assumption that the whole term structure of interest rates can be derived from the stochastic behaviour of one or many variables.

The one-parameter models normally use the instantaneous rate as the state variable.

$$dr = \mu(r,t)\,dt + \sigma(r,t)\,dz,$$

where $\mu(r,t)$ is the drift function, and $\sigma(r,t)$ is the volatility function.

Let $\lambda(r,t)$ denotes the market price of risk, and $U(r,t)$ denotes the value of all interest rate contingent claims. Then the next *pricing equation* holds:

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma^2(r,t)\frac{\partial^2 U}{\partial r^2} + (\mu(r,t) - \lambda(r,t)\sigma(r,t))\frac{\partial U}{\partial r} - rU = 0.$$

To find a unique solution to this equation, we must impose one final and two boundary conditions.

The Ho and Lee model

In the Ho and Lee model, the short rate dynamics are represented by:

$$dr = \vartheta(t)\,dt + \sigma\,dz,$$

where σ, the instantaneous standard deviation of the short rate, is constant and $\vartheta(t)$ is

$$\vartheta(t) = \frac{\partial F(0,t)}{\partial t} + \sigma^2 t.$$

Here $F(0,t)$ is the instantaneous for a maturity t as seen at time zero.

In the Ho and Lee model, zero-coupon bonds and European options on zero-coupon bonds can be valued analytically. The price of a zero-coupon bond at time t is

$$P(t,T) = A(t,T)e^{-r(t)(T-t)},$$

where

$$\ln A(t,T) = \ln\frac{P(0,T)}{P(0,T)} - (T-t)\frac{\partial \ln P(0,T)}{\partial t} - \frac{1}{2}\sigma^2 t(T-t)^2.$$

The price at time zero of a call option that matures at time T on a zero-coupon bond maturing at time s is

$$LP(0,s)N(h) - XP(0,T)N(h - \sigma_P),$$

where L is the principal of the bond, X is its strike price,

$$h = \frac{1}{\sigma_P} \ln \frac{LP(0,s)}{P(0,T)X} + \frac{\sigma_P}{2}$$

and

$$\sigma_P = \sigma(s - T)\sqrt{T}.$$

The price of a put option on the bond is

$$p = XP(0,T)N(\sigma_P - h) - LP(0,s)N(-h).$$

Example

The Hull and White model

In the Hull and White model, the short rate dynamics are represented by:

$$dr = (\vartheta(t) - ar)\,dt + \sigma\,dz,$$

where a and σ are constants. It can be characterised as the Ho and Lee model with mean reversion at rate a. The Ho and Lee model is a particular case of the Hull and White model with $a = 0$.

The $\vartheta(t)$ function can be calculated from the initial term structure:

$$\vartheta(t) = \frac{\partial F(0,t)}{\partial t} + aF(0,t) + \frac{\sigma^2}{2a}(1 - e^{-2at}t).$$

Bond prices at time t in the Hull and White model are given by

$$P(t,T) = A(t,T)e^{-B(t,T)r(t)},$$

where

$$B(t,T) = \frac{1 - e^{-a(T-t)}}{a}$$

and

$$\ln A(t,T) = \ln \frac{P(0,T)}{P(0,t)} - B(t,T)\frac{\partial \ln P(0,T)}{\partial t} - \frac{1}{4a^3}\sigma^2(e^{-aT} - e^{-at})^2(e^{2at} - 1).$$

The price at time zero of a call option that matures at time T on a zero-coupon bond maturing at time s is

$$LP(0,s)N(h) - XP(0,T)N(h - \sigma_P),$$

where L is the principal of the bond, X is its strike price,

$$h = \frac{1}{\sigma_P} \ln \frac{LP(0,s)}{P(0,T)X} + \frac{\sigma_P}{2}$$

and

$$\sigma_P = \frac{\sigma}{a}[1 - e^{-a(t-T)}]\sqrt{\frac{1 - e^{-2aT}}{2a}}.$$

The price of a put option on the bond is

$$XP(0,T)N(\sigma_P - h) - LP(0,s)N(-h).$$

Example

A unified approach to interest rate models

Introduce the following notation:

t_0	valuation date
t_s	start date
t_e	end date
X	strike price
σ	volatility
a	reversion rate
$N(x)$	normal distribution function
r_{ir}	instantaneous rate
P	term structure discount function

Then we can write a unified formula for all interest rate models in PRIME:

$$P(r_{\text{ir}}, t_s, t_e) = A(r_{\text{ir}}, t_s, t_e)\exp(-r_{\mathbf{ir}}B(t_s, t_e)),$$

where

$$
\begin{aligned}
\ln A(r_{\mathrm{ir}}, t_s, t_e) &= \ln \frac{d(t_0, t_e)}{d(t_0, t_s)} \\
&- \frac{B(t_s, t_e)[d(t_0, t_s + 1/365) - d(t_0, t_s - 1/365)] \cdot 365}{2d(t_0, t_s)} \\
&- \frac{B^2(t_s, t_e) g(t_e) \sigma^2}{2},
\end{aligned}
$$

and B, g, and σ are model-dependent.

In particular, for the case of the Ho and Lee model we have

$$
\begin{aligned}
B(t_s, t_e) &= t_e - t_s, \\
g(t_e) &= t_e, \\
\sigma(t_s, t_e) &= \sigma \sqrt{t_s - t_0}(t_e - t_s).
\end{aligned}
$$

For the case of the Hull and White model we have

$$B(t_s, t_e) = \frac{1 - \exp(-a(t_e - t_s))}{a},$$

$$g(t_e) = \frac{1 - \exp(-2at_e)}{2a},$$

$$\sigma(t_s, t_e) = \sqrt{\frac{\sigma^2(1 - \exp(-2a(t_s - t_0)))}{2a}} \cdot \frac{1 - \exp(-a(t_e - t_s))}{a}.$$

Unified formulas for interest rate derivatives

The value of the call option on a zero coupon discount bond is

$$c(r_{\mathrm{ir}}, t_s, t_e, X) = \max\{d(t_0, t_e)N(h) - Xd(t_0, t_s)N(h - \sigma(t_s, t_e)), 0\},$$

and the value of the put option is

$$p(r_{\mathrm{ir}}, t_s, t_e, X) = \max\{Xd(t_0, t_s)N(\sigma - h) - d(t_0, t_e)N(-h), 0\}$$

where

$$h = \frac{1}{\sigma(t_s, t_e)} \ln \frac{d(t_0, t_e)}{d(t_0, t_s)X} + \frac{\sigma(t_s, t_e)}{2}.$$

Let X_r denotes the strike price expressed in rate, and let X_d denotes the strike price expressed in discount factor. The value of a caplet is determined as

$$C = \frac{p(r_{\mathrm{ir}}, t_s, t_e, X_d)}{X_r d(t_s, t_e)(t_e - t_s)} \cdot \text{period} \cdot \text{nominal}.$$

Consider the next notation:

c_i	cash flow amount of cash flow i
t_i	cash flow payday of cash flow i
t_{exp}	option expiration date
t_{sett}	strike payday

The value of a call option on a bond is determined as

$$C = \sum_{i=0}^{n} c_i c(r_{\text{ir}}, t_{\text{exp}}, t_i, X_i^*) - Xc(r_{\text{ir}}, t_{\text{exp}}, t_{\text{sett}}, X_{\text{sett}}^*),$$

where

$$X_i^* = P(r_{\text{ir}}^*, t_{\text{exp}}, t_i)$$

and r_{ir}^* is the solution to the equation

$$\sum_{i=0}^{n} c_i P(r_{\text{ir}}^*, t_{\text{exp}}, t_i) - XP(r_{\text{ir}}^*, t_{\text{exp}}, t_{\text{sett}}) = 0.$$

Example

Constant maturity contracts

Constant maturity contracts, that is instruments using a floating rate based on a swap index (i.e. the par rate of a generic swap), are valued in PRIME using the forward measure technology based on term structure models.

Let the value of such a CMS contract equal $g(R_f(T_p, T_1, T_2))$ at payday T_p, where

$$R_f(T_p, T_1, T_2) = \frac{1}{t} \cdot \frac{1 - p(T_1, T_n)}{\sum_{i=2}^{n} p(T_1, T_i)}$$

is the swap rate, having

$p(T_1, T_i)$	zero coupon bond price at T_1 of bond maturing at T_i
t	reset period
T_1	reset day
T_2	payday.

Let the dynamics of the instantaneous rate under the measure Q is described by the Ho and Lee model

$$dr = \vartheta(t)\,dt + \sigma\,dz$$

or by the Hull and White model:

$$dr = (\vartheta(t) - ar)\,dt + \sigma\,dz.$$

Define the *forward measure* Q^T as

$$\frac{d\mathsf{Q}^T}{d\mathsf{Q}} = \frac{\exp\left\{-\int_0^T r(s)\,ds\right\}}{\mathsf{E}^{\mathsf{Q}}\left[\exp\left\{-\int_0^T r(s)\,ds\right\}\right]}.$$

Then the present value of the contract $\mathrm{PV}(t)$ can be expressed as

$$\mathrm{PV}(t) = p(t, T_P)\mathsf{E}^{\mathsf{Q}_{T_p}}[g(R_f(T_p, T_s, T_e))|r(t) = r].$$

Calculating the conditional mathematical expectation, we obtain

$$\mathrm{PV}(t) = p(t, T_P)\frac{1}{\sqrt{2\pi\nu}}\int_{-\infty}^{\infty} g(r)\exp\left(-\frac{(r-m)^2}{2\nu}\right)dr, \tag{1}$$

where

$$m = \mathsf{E}^{\mathsf{Q}_{T_p}}[r(T_r)],$$
$$\nu = \mathsf{Var}^{\mathsf{Q}_{T_p}}[r(T_r)]$$

are model-dependent. In the case of the Ho and Lee model we have

$$m = -\frac{\partial}{\partial s} \ln p(0, s)\Big|_{s=T_r} + \sigma^2 T_r(T_r - T_p),$$
$$\nu = \sigma^2 T_r,$$

while in the case of the Hull and White model

$$m = -\frac{\partial}{\partial s} \ln p(0, s)\Big|_{s=T_r} + \frac{\sigma^2}{2a^2} e^{-aT_r}(e^{-aT_r} - e^{aT_r} + e^{a(2T_t - T_p)} - e^{-aT_p}),$$
$$\nu = \frac{\sigma^2}{2a}(1 - e^{-2aT_r}).$$

Example

Compound options

By compound options, we mean options on option-style instruments. These include:

☞ options on options;

☞ options on caps/floors;

☞ options on free defined cash flows where there is at least one optional cash flow.

The present value of a compound option is calculated by (1), where

$$m = -\frac{\partial}{\partial s} \ln p(0, s)\Big|_{s=T_p}$$

and in the case of the Ho and Lee model

$$\nu = \sigma^2 T_p,$$

while in the case of the Hull and White model

$$\nu = \frac{\sigma^2}{2a}(1 - e^{-2aT_p}).$$

Example

Quanto contracts

Quanto contracts have floating cash flows where the reference rate is a rate index in a currency other than the payout currency. The quanto products that are supported in PRIME include:

- ☞ differential swaps;
- ☞ quanto caps/floors;
- ☞ quanto bond options;
- ☞ swaptions.

Let $Z(t)$, $W^F(t)$, and $X(t)$ be three Wiener processes. Let ρ be the correlation between $Z(t)$ and $W^F(t)$, and let δ be the correlation between $W^F(t)$ and $X(t)$. The domestic rate, $r(t)$, the foreign rate,

$y(t)$, and the exchange rate, $S(t)$ are modeled as

$$dr(t) = \alpha_r \, dt + \sigma_r \, dZ,$$

$$dy(t) = \alpha_y \, dt + \sigma_y \, dW^F,$$

$$\frac{dS(t)}{S(t)} = (r - y) \, dt + \sigma_s \, dX.$$

Differential swaps

Differential swaps are valued using the following formula:

$$\mathrm{PV}(t) = p^{t_1}(t)\left[\frac{1}{q^{t_1}(t_0)} - \frac{1}{p^{t_1}(t_0)}\right] + \sum_{i=2}^{n}(D_{t_{i-1},t_i}(t) - p^{t_{i-1}}(t)),$$

where

$p^T(t)$	the value of a zero coupon bond paying out 1 unit of the domestic currency at time T,
$q^T(t)$	the value of a zero coupon bond paying out 1 unit of the foreign currency at time T,

$$D_{t,\tau}(t) = \frac{p^{\tau}(t)q^T(t)}{q^{\tau}(t)}b(t),$$

and $b(t) = e^{\mathrm{cov}(t,T,\tau)}$ is the “correction factor” that takes the quanto effect into account. This factor is model-dependent. In the Ho and Lee model

$$\mathrm{cov}(t,T,\tau) = \sigma_y(\tau - T)(T - t)\Big[\delta\sigma_s - \sigma_y\tau + \rho\sigma_r T + \frac{\sigma_y - \rho\sigma_r}{2}(T + t)\Big],$$

while in the Hull and White model

$$\mathrm{cov}(t, T, \tau) = C(T, \tau)(I_1 + I_2 + I_3),$$

where

$$\begin{aligned} C(T, \tau) &= \frac{\sigma_y}{a_y}[e^{-a_y(\tau - T)} - 1], \\ I_1 &= \frac{\delta\sigma_s}{a_y}[1 - e^{-a_y(T-t)}], \\ I_2 &= \frac{\sigma_y}{a_y}\left[\frac{1 - e^{-a_y(T-t)}}{a_y} - \frac{1 - e^{-2a_y(T-t)}}{2a_y}\right], \\ I_3 &= \frac{\rho\sigma_r}{a_r}\left[\frac{1 - e^{-a_y(T-t)}}{a_y} - \frac{1 - e^{-(a_y+a_r)(T-t)}}{a_y + a_r}\right]. \end{aligned}$$

The time t_0 does not have to be the starting time of the contract; it can be any reset date.

Example

Quanto caps/floors

The caplet value is:

$$p^T(t)\left[\frac{b(t)q^T(t)}{q^\tau(t)}\Phi(d_+) - (1 + (\tau - T)R)\Phi(d_-)\right],$$

where

$$d_\pm = \frac{\ln\left(\frac{b(t)q^T(t)}{q^\tau(t)(1 + (\tau - T)R)}\right)}{\nu(t)} \pm \frac{\nu(t)}{2},$$

$\nu^2(t)$ is the total variation of the function $f(s)$ on the interval $[t, T]$??, and

$$f = \frac{bq^T}{q^\tau}.$$

Example

Quanto bond options

An option on a portfolio of zero-coupon bonds can be valued as a portfolio of options on zero-coupon bonds. The value of the ith option on a zero coupon bond is then:

$$p^T(t)\left[\frac{q^{T_i}(t)b(t)}{q^T(T)}\Phi(d_+) - D_i\Phi(d_-)\right],$$

where D_i are discount factors,

$$d_\pm = \frac{\ln(f(t)/D_i)}{\nu_i(t)} \pm \frac{\nu_i(t)}{2},$$

$\nu_i^2(t)$ is the total variation of the function $f_i(s)$ on the interval $[t, T]$??, and

$$f_i = \frac{bq^{T_i}}{q^T}.$$

Example

Cash flow spread options

A spread option gives the holder the difference, if positive, between two foreign interest rates, multiplied by the nominal amount in the domestic currency (USD).

Assume the rate index currencies are DEM and FFR. The dollar pay-off of such a contract at maturity T can be derived to:

$$X = \frac{N}{\tau - T} \max\{f(T) - K_q, 0\},$$

where N is the nominal amount, $K_1 = K(\tau - T)$, K is the strike,

$$f(T) = \frac{1}{d_\tau^D} - \frac{1}{d_\tau^F},$$

and the q variables represent the German and French bond prices.

The variance of the combined forward price that is used in the calculations of the spread option prices in the case of the Ho and Lee model is

$$\nu^2(t) = (T - t)(\tau - T)^2(\sigma_D^2 + \sigma_F^2 - 2\rho\sigma_D\sigma_F),$$

while in the case of the Hull and White model

$$\nu^2(t) = \left[\frac{\sigma_D}{a_D}[e^{-a_D(\tau-T)} - 1]\right]^2 \cdot \left[\frac{1 - e^{-2a_D(T-t)}}{2a_D}\right]$$
$$+ \left[\frac{\sigma_F}{a_F}[e^{-a_F(\tau-T)} - 1]\right]^2 \cdot \left[\frac{1 - e^{-2a_F(T-t)}}{2a_F}\right]$$
$$- 2\rho\left[\frac{\sigma_D}{a_D}[e^{-a_D(\tau-T)} - 1]\right]\left[\frac{\sigma_F}{a_F}[e^{-a_F(\tau-T)} - 1]\right]$$
$$\times \left[\frac{1 - e^{-(a_D+a_F)(T-t)}}{a_D + a_F}\right],$$

where ρ denotes the correlation between two Wiener processes that drive the exchange rates of DEM and FFR.

Bermudan, American and Barrier Options

Let $u(r,t)$ be the value of a contract at time t with a given payout function, if the rate at that time is r. The Hull and White pricing equation is

$$\frac{\partial u}{\partial t} = ru - (\vartheta(t) - r)\frac{\partial u}{\partial r} - \frac{1}{2}\sigma^2\frac{\partial^2 u}{\partial r^2}.$$

The payout function $u(r,T)$ depends on the option and is known. It defines the *final condition*. The *boundary conditions* $u(0,t)$ and $u(r_{\max},t)$ are also known. We want to find $u(r,0)$.

In order to calculate this numerically, we will first make time and rate discrete. The step in time is k and the step in rate is h. Let us write time and rate as:

$$t_j = jk, \quad j = 0, 1, 2, \ldots, T$$

$$r_i = ih, \quad i = 0, 1, 2, \ldots, N.$$

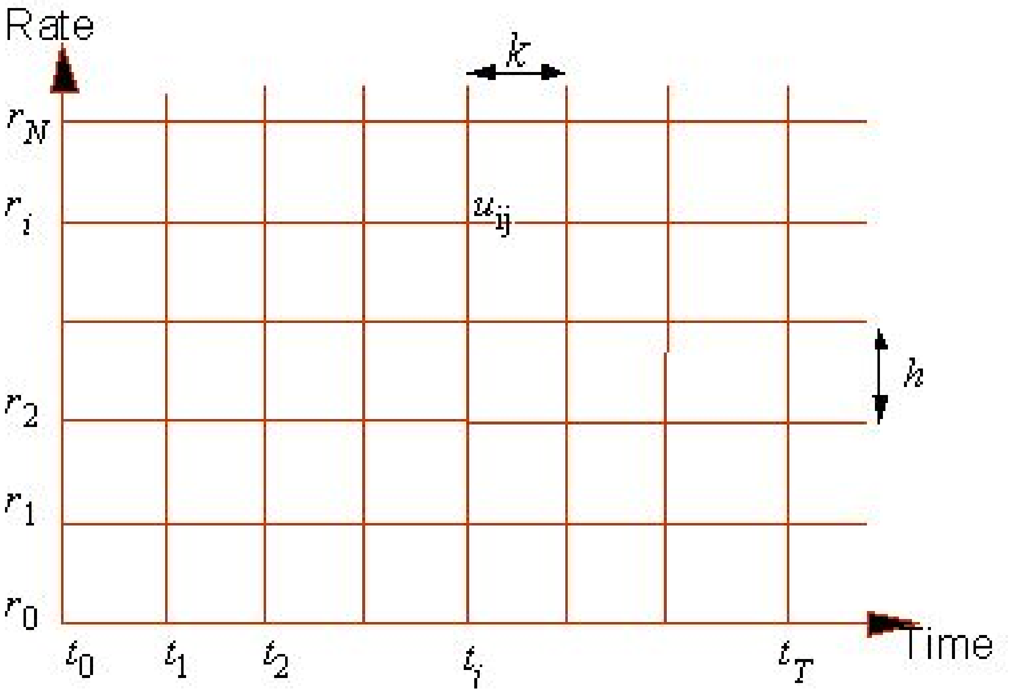


Figure 1: How the time space and rate space are made discrete

The derivatives are approximated with difference quotients:

$$\begin{aligned} u(r_i, t_j) &= u_{i,j}, \\ \frac{\partial u}{\partial t}(r_i, t_j) &= \frac{u_{i,j+1} - u_{i,j}}{k}, \\ \frac{\partial u}{\partial r}(r_i, t_j) &= \frac{u_{i+1,j} - u_{i-1,j}}{2h}, \\ \frac{\partial^2 u}{\partial r^2}(r_i, t_j) &= \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}. \end{aligned}$$

Using these approximations, our equation can be written as

$$\frac{du_{i,j}}{dt} = ru_{i,j} - (\vartheta(t_j) - ar_i)\frac{u_{i+1,j} - u_{i-1,j}}{2h} - \frac{1}{2}\sigma^2\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}.$$

By rearranging this, we obtain

$$\frac{du_{i,j}}{dt} = \left(-\frac{\sigma^2}{2h^2} + \frac{\vartheta(t_j) - ar_i}{2h}\right)u_{i-1,j} + \left(\frac{\sigma^2}{h^2} + r_i\right)u_{i,j} + \left(-\frac{\sigma^2}{2h^2} - \frac{\vartheta(t_j) - ar_i}{2h}\right)u_{i+1,j},$$

where the final values $u(r_i, t_T)$ and boundary values $u(r_0, t_j)$ and $u(r_N, t_j)$ are known.

Let matrix $\mathbf{A}$ contain the time differentiation and the boundary conditions. In matrix form, we get

$$\frac{d\mathbf{u}}{dt} = \mathbf{A}(t)\mathbf{u}(t),$$

where $\mathbf{A}(t)$ is a triagonal $(N+1) \times (N+1)$ matrix

$$\mathbf{A}(t) = \begin{pmatrix} x_{B0}(r_0,t) & y_{B0}(r_0) & z_{B0}(r_0,t) & 0 & \dots & \dots & 0 \\ x(r_1,t) & y(r_1) & z(r_1,t) & 0 & \dots & \dots & 0 \\ 0 & x(r_2,t) & y(r_2) & z(r_2,t) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & x(r_{N-2},t) & y(r_{N-2}) & z(r_{N-2},t) & 0 \\ \dots & \dots & \dots & \dots & x(r_{N-1},t) & y(r_{N-1}) & z(r_{N-1},t) \\ \dots & \dots & \dots & \dots & x_{BN}(r_N,t) & y_{BN}(r_N) & z_{BN}(r_N,t) \end{pmatrix}$$

with

$$x(r_i, t_j) = -\frac{\sigma^2}{2h^2} + \frac{\vartheta(t_j) - ar_i}{2h},$$
$$y(r_i) = \frac{\sigma^2}{h^2} + r_i,$$
$$z(r_i, t_j) = -\frac{\sigma^2}{2h^2} - \frac{\vartheta(t_j) - ar_i}{2h}.$$

If we approximate the time derivative with a difference quotient and apply the trapezoidal rule, we obtain the Crank–Nicholson method:

$$\frac{\mathbf{u}_{j+1} - \mathbf{u}_j}{k} = \frac{\mathbf{A}(t_j)\mathbf{u}_j + \mathbf{A}(t_{j+1})\mathbf{u}_{j+1}}{2}.$$

Solving this equation for $\mathbf{u}_j$ yields

$$\mathbf{u}_j = \left(\mathbf{I} + \frac{k}{2}\mathbf{A}(t_j)\right)^{-1}\left(\mathbf{I} - \frac{k}{2}\mathbf{A}(t_{j+1})\right)\mathbf{u}_{j+1},$$

where $\mathbf{I}$ denotes the identity matrix.

We take one step backward in time. One step further from the day when we had our original payout function and one step closer to today. For each time step we take, we will have to solve a linear system of equations with $N + 1$ unknown variables.

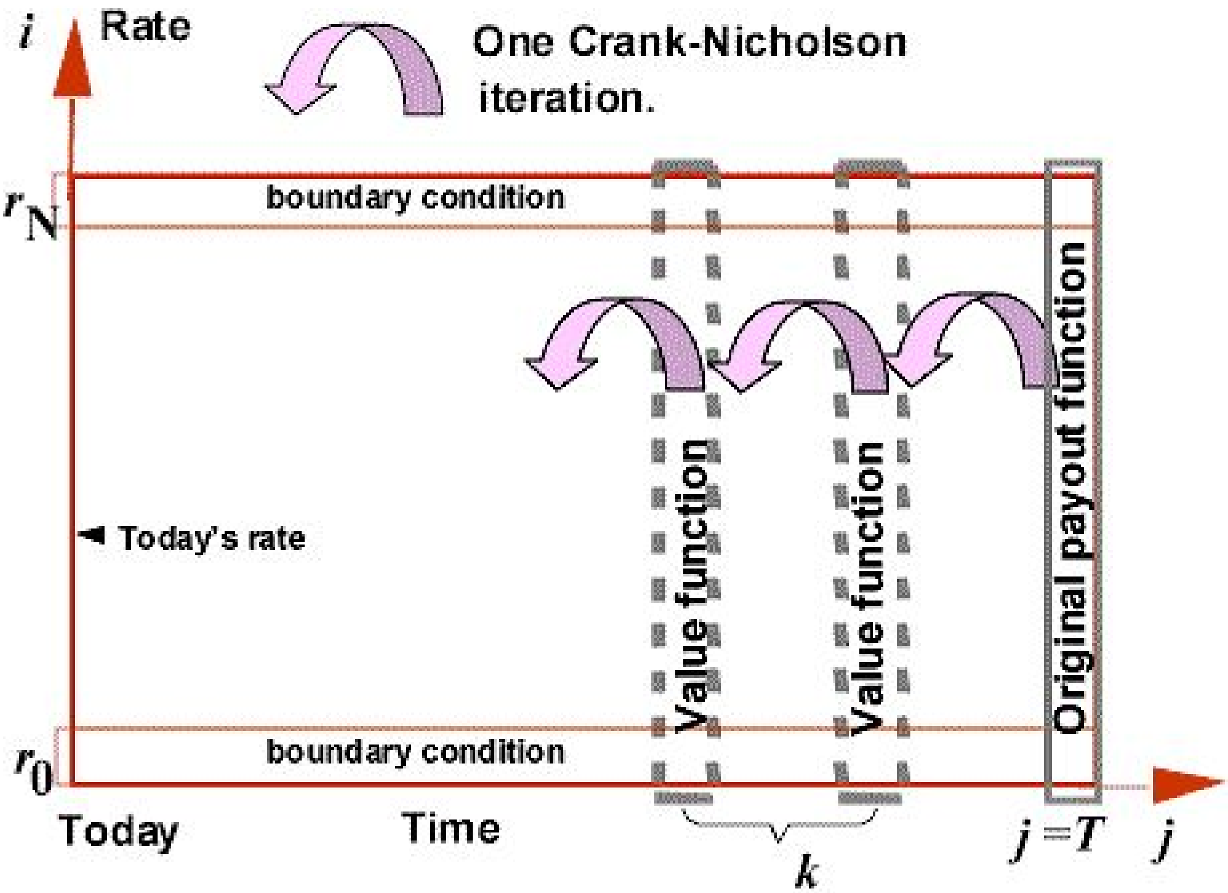


Figure 2: The Crank–Nicholson method to value a given future payout function

These calculations are simplified by the triagonal structure of $\mathbf{A}$. Each iteration generates a new value function valid for a time point one step closer to today. The value function gives the present value of the payout function as a function of the rate.

The iterations continue until we have a value function valid for today. From today's value function we can obtain the value using today's rate. The value is the risk-adjusted present value of the original payout function.

Example

Exercises