KTH Mathematics

Tentamen i 5B1575 Finansiella Derivat.
Torsdag 25 augusti 2005 kl. 14.00–19.00.

Examinator: Camilla Landén, tel 790 8466.

Tillåtna hjälpmedel: Av institutionen utlånad miniräknare.

Allmänna anvisningar: Lösningarna skall vara lättläsliga och **välmotiverade**. All införd notation skall vara förklarad. Problem rörande integrabilitet behöver ej redas ut.

OBS! Personnummer skall anges på försättsbladet. Endast en uppgift på varje blad. Numrera sidorna och skriv namn på varje blad!

24 poäng inklusive bonuspoäng ger säkert godkänt.

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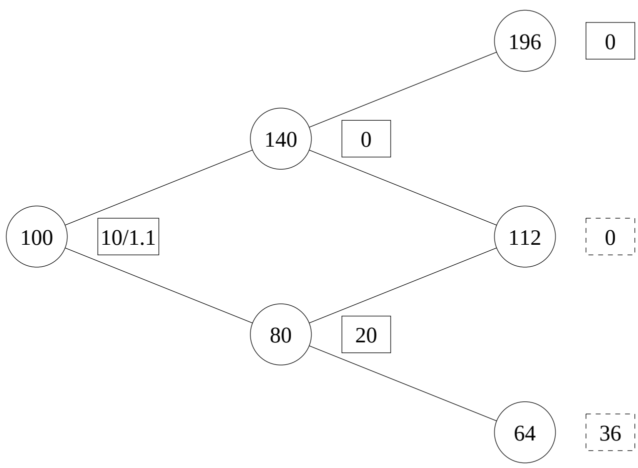
1. (a) In the binomial tree below the price of an *American put option* with strike price $K = 100$ kr and exercise date $T = 2$ years has been computed using the parameters $s_0 = 100$, $u = 1.4$, $d = 0.8$, $r = 10\%$, and $p = 0.75$. (The value of the stock is written in the nodes, and the value of the option is written in the adjacent boxes.)



Note that an early exercise of the option is optimal at the node with stock price 80 (this is why the last option values have been put in dashed boxes)!

Your task is to find the replicating portfolio for this option and to verify that the portfolio is self-financing. .... (3p)

(b) Consider a standard Black-Scholes market, described in detail in Exercise 2. Suppose that you presently own a portfolio with $\Delta = 0$ and $\Gamma = 1$. How many of the underlying stock, and derivatives with payoff

$$X = S_T^2,$$

in $T = 2$ years, and pricing function

$$F(t,s) = s^2 \exp\left\{\left(-\frac{r}{2} - \frac{\sigma^2}{8}\right)(T-t)\right\},$$

should you buy or sell, in order to make your portfolio both delta and gamma neutral? Today's stock price is $S_0 = 10$. Use r=5% and σ=50% on a yearly basis. .... (3p)

(c) i. State the First Fundamental Theorem of arbitrage pricing (a slightly sloppy version will do). .... (2p)

ii. State the Second Fundamental Theorem of arbitrage pricing. .... (2p)

2. Consider a standard Black-Scholes market, i.e., a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB_t &= rB_t dt, \\ B_0 &= 1, \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ S_0 &= s_0. \end{cases}$$

Here W denotes a P-Wiener process and r, α, and σ are assumed to be constants.

A *chooser option* is an agreement which gives the owner the right to choose at some prespecified future date T_0, whether the option is to be a call or put option with exercise price K and remaining time to expiry $T - T_0$. Note that K, T and $T_0 < T$ are all prespecified by the agreement. The only thing the owner can choose is whether the option should be a call or a put option, and this choice has to be made at time T_0.

Compute the arbitrage price of the T_0-claim called a chooser option. .... (10p)

3. Consider the Vasiček model for the short rate, defined under the objective measure P as

$$dr(t) = \kappa\left(\theta^P - r(t)\right)dt + \sigma dW(t),$$

where κ, θ^P, and σ are positive constants, and W denotes a P-Wiener process.

If we assume the market price of risk, λ, to be a constant, this will imply that the dynamics of r under the risk neutral martingale measure Q are given by

$$dr(t) = \kappa\left(\theta^Q - r(t)\right)dt + \sigma dV(t),$$

where $\theta^Q = \theta^P - \sigma\lambda/\kappa$, and V denotes a Q-Wiener process satisfying $dV = \lambda dt + dW$.

Should you feel that you need numbers, then reasonable values are $\theta^P = 0.04$, $\theta^Q = 0.04 \pm 0.02$ (you should choose the sign, see below), $\kappa = 0.5$, and $\sigma = 0.01$.

(a) What sign would you expect the market price of risk λ to have, and why? (Note that $\sigma\lambda/\kappa$ will have the same sign as λ, and this is thus the sign you should use in the expression $\theta^Q = 0.04 \pm 0.02$, should you need the number later on). (2p)

(b) Compute $E^Q[r(T)]$ and $\lim_{T\to\infty} E^Q[r(T)]$. .... (3p)

Remark. It can be shown that the stationary distribution of r has $\lim_{T\to\infty} E^Q[r(T)]$ as expected value.

(c) Recall that the zero-coupon yield, $y(t,T)$, solves

$$p(t,T) = e^{-y(t,T)\cdot(T-t)} \cdot 1.$$

i. Suppose that $r_0 = \theta^Q$. What does the initial yield curve $T \to y(0,T)$ look like then? (You need only give the big picture: where does the yield curve start, what happens as the maturity tends to infinity, and is the slope positive, negative, or none.)
Hint: There are some results at the end of the exam which may be of use when looking at the yield.

ii. Suppose that $r_0 = \theta^P$. What does the initial yield curve look like now (if needed, use the sign of λ you argued for in Exercise (a))?

iii. The first curve can be interpreted as a "Q-typical" shape of the yield curve, just as the second curve can be called a "P-typical" shape of the yield curve. Explain why!
Hint: The remark after Exercise (b) may help.

.... (5p)

4. (a) Forward rate models have to be specified with some care to avoid introducing arbitrage possibilities. It is well known that if the forward rates are modeled as

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t, \quad T \geq 0,$$

where W is an m-dimensional Wiener process, then the the dynamics of the (zero-coupon) bond prices, $p(t,T)$, are given by

$$dp(t,T) = \{r(t) + b(t,T)\}p(t,T)dt + a(t,T)p(t,T)dW(t),$$

where

$$\begin{cases} a(t,T) &= -\int_t^T \sigma(t,u)du, \\ b(t,T) &= -\int_t^T \alpha(t,u)du + \frac{1}{2}\|a(t,T)\|^2. \end{cases}$$

Your task is to derive the Heath-Jarrow-Morton drift condition (under Q). (2p)

(b) Suppose you are to price a foreign bond derivative, for example an option. You then need a term structure model describing the domestic and foreign bond markets, with (zero-coupon) bond prices denoted by $p_d(t,T)$ (in units of domestic currency) and $p_f(t,T)$ (in units of foreign currency), respectively.

i. Start out by assuming that the market consists of a domestic and a foreign risk less asset, with price processes B_d and B_f, respectively. B_d is denominated in units of the domestic currency, and B_f in units of the foreign currency. The dynamics of these price processes are given by

$$dB_d(t) = r_d(t)B_d(t)dt, \text{ and } dB_f(t) = r_f(t)B_f(t)dt,$$

where r_d and r_f are assumed to be adapted stochastic processes. Furthermore, it is assumed that if you buy the foreign currency this is immediately invested in a foreign bank account (i.e. the foreign risk less asset).
The exchange rate process X between the domestic and foreign currency is assumed to be of the form

$$dX(t) = \alpha_X(t)X(t)dt + \sigma_X(t)X(t)dV_t,$$

where α_X is a real-valued adapted process, σ_X is an $\mathbb{R}^m$-valued adapted process, and V is an m-dimensional Wiener process under the domestic martingale measure Q_d.
Derive an expression for α_X under the domestic martingale measure Q_d. If you like you may assume that V is one-dimensional, the result will be the same. .... (3p)

ii. Now, suppose we want the domestic and foreign interest rates to be given by forward rate processes f_d and f_f, respectively, as in the HJM setting. We thus model f_d and f_f under the **domestic** martingale measure in the following way

$$df_d(t,T) = \alpha_d(t,T)dt + \sigma_d(t,T)dV_t,$$

$$df_f(t,T) = \alpha_f(t,T)dt + \sigma_f(t,T)dV_t,$$

where V is an m-dimensional Wiener process under the domestic martingale measure Q_d. Again, in order to bar arbitrage possibilities the forward rate processes have to be modeled with some care. The drift restriction which must be placed on the domestic forward rate process under the **domestic** martingale measure is the well-known HJM drift restriction.
Your task is to derive the drift restriction on the foreign forward rate process f_f under the **domestic** martingale measure Q_d. (The expression for α_X under Q_d obtained in previous exercise is still valid in this new setting).
.... (5p)

5. Consider a given market consisting of a risk free asset B and a risky asset with price process $S(t)$ and cumulative dividend process (utdelningsprocess) $D(t)$. Both $S(t)$ and $D(t)$ are assumed to have stochastic differentials. The short rate r is assumed to be constant and deterministic.

(a) Define the concepts **portfolio**, **value process** and **self-financing portfolio**. .... (2p)

(b) Define what is meant by a **martingale measure** for this model. .... (2p)

(c) Now specialize to the case when the risky asset has a constant dividend yield δ, i.e. $dD(t) = \delta S(t)dt$. Suppose that you own the risky asset S and that you invest all dividends in the risky asset in a self-financing way. How many risky assets will you then own at time T if you started out with exactly one at time $t = 0$, and what will the dynamics of your portfolio look like? .... (6p)
Hint: The concept of a relative portfolio might be useful.

Lycka till!

Hints:
You are free to use the following in any of the above exercises.

- The density function of a normally distributed random variable with expectation m and variance σ^2 is given by

$$\varphi(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/(2\sigma^2)}.$$

- Let Φ denote the cumulative distribution function for the N$(0,1)$ distribution. Then

$$\Phi(-x) = 1 - \Phi(x).$$

- The standard Black-Scholes formula for the price $\Pi(t)$ of a European call option with strike price K and time of maturity T is $\Pi(t) = F(t, S(t))$, where

$$F(t,s) = s\Phi[d_1(t,s)] - e^{-r(T-t)} K \Phi[d_2(t,s)].$$

Here Φ is the cumulative distribution function for the $N(0,1)$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$

- Suppose that there exist processes $X(\cdot, T)$ for every $T \geq 0$ and suppose that Y is a process defined by

$$Y(t) = \int_t^{T_0} X(t,s)ds$$

Then we have the following version of Itô's formula

$$dY_t = -X(t,t)dt + \int_t^{T_0} dX(t,s)ds.$$

- Parameterize the Vasiček model in the following way under the risk neutral martingale measure Q

$$dr(t) = (b - ar)dt + \sigma dV. \tag{1}$$

Proposition 1 (The Vasiček term structure) *In the Vasiček model, parameterized as in (1) under Q, the zero-coupon bond prices are given by*

$$p(t,T) = e^{A(t,T) - B(t,T)r(t)}, \tag{2}$$

where

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}, \tag{3}$$

$$A(t,T) = \frac{[B(t,T) - T + t](ab - \frac{1}{2}\sigma^2)}{a^2} - \frac{\sigma^2 B^2(t,T)}{4a}. \tag{4}$$