KTH Mathematics

Tentamen i 5B1575 Finansiella Derivat.
Torsdag 25 augusti 2005 kl. 14.00–19.00.

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Answers and suggestions for solutions.

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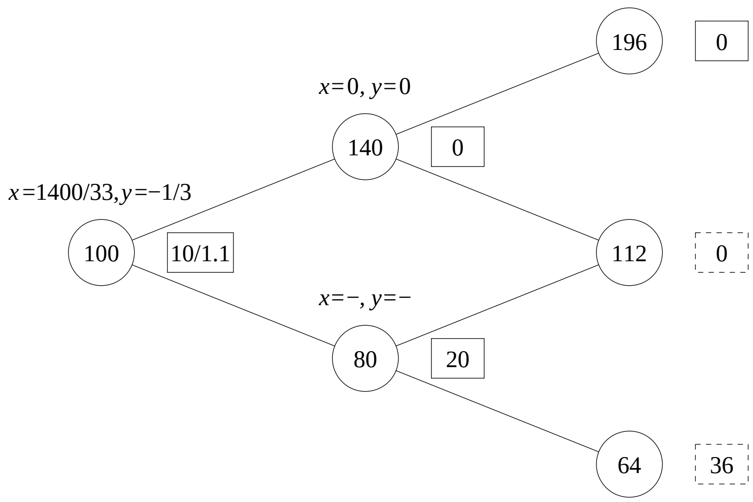
**1.** (a) To obtain the replicating portfolio at $t = 0$ we have to solve the following set of equations

$$\begin{cases} 1.1x + y \cdot 140 &= 0, \\ 1.1x + y \cdot 80 &= 20, \end{cases}$$

since regardless of whether the stock price goes up or down the value of the portfolio should equal the value of the option. This yields

$$x = \frac{1400}{33}, \qquad y = -\frac{1}{3}.$$

Using the same method we find the rest of the replicating portfolio strategy and it is shown in the figure below.




Note that since the option is exercised at the node with stock price 80, you will from that node on no longer hold a portfolio.

That the portfolio strategy is self-financing is seen from the following equation

$$1.1 \cdot \frac{1400}{33} - \frac{1}{3} \cdot 140 = 0 + 0 \cdot 140.$$

(b) Let $x_S$ denote the number of underlying you should add to the portfolio, and let $x_F$ denote the number of derivatives with price function $F$ you should add to the portfolio. For the underlying itself we have that

$$\Delta_S = 1 \qquad \text{and} \qquad \Gamma_S = 0.$$

For the derivatives with price function $F$ we have

$$\begin{aligned}\Delta_F &= \frac{\partial F}{\partial s} = 2s \exp\left\{\left(-\frac{r}{2}-\frac{\sigma^2}{8}\right)(T-t)\right\},\\ \Gamma_F &= \frac{\partial^2 F}{\partial s^2} = 2 \exp\left\{\left(-\frac{r}{2}-\frac{\sigma^2}{8}\right)(T-t)\right\}.\end{aligned}$$

So with $S_0 = 10$, $r = 0.05$, and $\sigma = 0.5$ we have

$$\Delta_F \approx 17.8719 \qquad \text{and} \qquad \Gamma_F \approx 1.7872.$$

In order to make the portfolio both delta and gamma neutral you should solve the following equations

$$\begin{cases}\Delta_P + x_S\Delta_S + x_F\Delta_F = 0\\ \Gamma_P + x_S\Gamma_S + x_F\Gamma_F = 0\end{cases}$$

The solution is

$$\begin{cases}x_F = -\dfrac{\Gamma_P}{\Gamma_F} \approx -0.56,\\ x_S = \dfrac{\Delta_F\Gamma_P}{\Gamma_F} - \Delta_P = 10,\end{cases}$$

that is, you should sell 0.56 derivatives and buy 10 of the underlying stock.

(c) i. **Theorem 1 (First Fundamental Theorem)** *The model is arbitrage free essentially if and only if there exists a (local) martingale measure $Q$.*

ii. **Theorem 2 (Second Fundamental Theorem)** *Assume that the market is arbitrage free. Then the market is complete if and only if the martingale measure is unique.*

**2.** The payoff $X$ of the chooser option at time $T_0$ equals

$$X = \max\left\{C(T_0, S_{T_0}, K, T, r, \sigma), P(T_0, S_{T_0}, K, T, r, \sigma)\right\},$$

where $C(t, s, K, T, r, \sigma)$ denotes the standard Black-Scholes price at time $t$ of a European call option with exercise price $K$ and expiry date $T$, when the current price of the underlying is $s$, the interest rate is $r$, and the volatility of the underlying is $\sigma$. The notation $P(t, s, K, T, r, \sigma)$ is used for the price of the corresponding put option.

Using put-call-parity, $P(t, s, K, T, r, \sigma) = Ke^{-r(T-t)} + C(t, s, K, T, r, \sigma) - s$, this payoff can be written as

$$\begin{aligned}X &= \max\left\{C(T_0, S_{T_0}, K, T, r, \sigma), Ke^{-r(T-T_0)} + C(T_0, S_{T_0}, K, T, r, \sigma) - S_{T_0}\right\}\\ &= C(T_0, S_{T_0}, K, T, r, \sigma) + \max\left\{0, Ke^{-r(T-T_0)} - S_{T_0}\right\}.\end{aligned}$$

The price of the chooser option is therefore given by

$$\begin{aligned}\Pi(t; X) &= e^{-r(T_0-t)}E^Q\left[C(T_0, S_{T_0}, K, T, r, \sigma) + \max\{0, Ke^{-r(T-T_0)} - S_{T_0}\}\Big| \mathcal{F}_t\right]\\ &= e^{-r(T_0-t)}E^Q\left[C(T_0, S_{T_0}, K, T, r, \sigma)|\mathcal{F}_t\right]\\ &\quad + e^{-r(T_0-t)}E^Q\left[\max\left\{0, Ke^{-r(T-T_0)} - S_{T_0}\right\}\Big| \mathcal{F}_t\right].\end{aligned}$$

Now, using that all price processes normalized by the risk free asset $B$ are $Q$-martingales we find that $e^{-r(T_0-t)}E[C(T_0, S_{T_0}, K, T, r, \sigma)|\mathcal{F}_t] = C(t, S_t, K, T, r, \sigma)$. The second term in the price is easily identified as the price at time $t$ of a put option, with exercise date $T_0$, and exercise price $Ke^{-r(T-T_0)}$. The price of the chooser option is thus given by

$$\Pi(t; X) = C(t, S_t, K, T, r, \sigma) + P(t, S_t, Ke^{-r(T-T_0)}, T_0, r, \sigma).$$

Both prices in the above formula can be explicitly computed using Black-Schloes formula, and put-call-parity.

**3.** (a) Recall that if the dynamics of the zero-coupon bond prices, $p(t, T)$, are given by

$$dp(t, T) = \mu(t, T)p(t, T)dt + \nu(t, T)p(t, T)dW(t),$$

under the objective measure $P$, then the market price of risk $\lambda$ is defined as

$$\lambda(t) = \frac{\mu(t, T) - r(t)}{\nu(t, T)}.$$

Now, generally you expect people to be risk averse, and thus requiring risky assets to have higher expected rate of return than the risk less asset, if they are to invest in them. This means that typically $\mu(t, T) > r(t)$, and thus you would expect $\lambda$ to be positive (recall that $\lambda$ is assumed to be a constant here).

(b) We have that

$$r_t = r_0 + \int_0^t \kappa \left(\theta^Q - r_u\right) du + \int_0^t \sigma dV_u.$$

Let $m(t) = E^Q[r_t]$ and take expectations to obtain

$$m(t) = r_0 + \int_0^t \kappa \left[\theta^Q - m(u)\right] + 0.$$

This gives the ODE

$$\begin{cases} \dot{m} &= \kappa \left[\theta^Q - m\right], \\ m(0) &= r_0, \end{cases}$$

with solution $m(t) = r_0 e^{-\kappa t} + \theta^Q\left(1 - e^{-\kappa t}\right)$.
Thus we have that

$$E^Q[r(T)] = r_0 e^{-\kappa T} + \theta^Q\left(1 - e^{-\kappa T}\right),$$

and this tends to $\theta^Q$ as $T$ tends to infinity.

i. The initial zero-coupon yields are given by

$$\begin{aligned} y(0, T) &= -\frac{\ln p(0, T)}{T} = -\frac{\{A(0, T) - B(0, T)r_0\}}{T} \\ &= \frac{B(0, T)r_0}{T} - \frac{A(0, T)}{T}. \end{aligned}$$

From one of the hints we have

$$A(0, T) = \frac{[B(0, T) - T][ab - \frac{1}{2}\sigma^2]}{a^2} - \frac{\sigma^2 B^2(0, T)}{4a},$$

and

$$B(0,T) = \frac{1}{a}\left(1 - e^{-aT}\right),$$

for the parameterization $dr = (b - ar)dt + \sigma dV$. Using these expression we find that

$$\lim_{T\to 0} y(0,T) = r_0.$$

To see this use l'Hospitals rule once, or use the interpretation of the yield. Furthermore we find that

$$\lim_{T\to\infty} y(0,T) = \frac{ab - \frac{1}{2}\sigma^2}{a^2}.$$

If the model is parameterized as $dr = \kappa(\theta^Q - r)dt + \sigma dV$ instead, we have $a = \kappa$, and $b = \kappa \cdot \theta^Q$, and then we have

$$\lim_{T\to\infty} y(0,T) = \theta^Q - \frac{1}{2}\left(\frac{\sigma}{\kappa}\right)^2.$$

Thus if $r_0 = \theta^Q$ the curve starts at $\theta^Q$ and tends to $\theta^Q - \sigma^2/(2\kappa^2)$, with a slight downward slope.

ii. If $r_0 = \theta^P$, and $\lambda > 0$ we have $r_0 = \theta^P > \theta^Q$, and the curve starts at $\theta^P$ and tends to $\theta^Q - \sigma^2/(2\kappa^2)$, with a downward slope.

iii. Assuming that the short rate is stationary its expected value is $\theta^Q$ under $Q$, and $\theta^P$ under $P$. Setting $r_0$ to these values we thus get a picture of what the yield curve will typically look like under $Q$ and $P$, respectively, since typically under $Q$ the short rate will oscillate around $\theta^Q$, just as it typically will oscillate around $\theta^P$ under $P$.

**4.** (a) The drift of any (ideally traded) price process under the risk neutral martingale measure is equal to the short rate (given that the asset pays no dividends). This means that we should have $b(t,T) = 0$. Using the expression for $b(t,T)$ and taking the derivative w.r.t. $T$ we get the HJM drift condition

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,u)'du,$$

where $'$ denotes transpose.

(b) i. Under $Q^d$ the following processes should be martingales

$$\frac{B^d}{B^d}, \quad \text{and} \quad \frac{\widetilde{B}^f}{B^d} = \frac{XB^f}{B^d}.$$

This means that the processes $B^d$, and $\widetilde{B}^f$ should have a local rate of return equal to $r_d$.

Using Itô's formula the $Q^d$-dynamics of $\widetilde{B}^f$ are found to be

$$d\widetilde{B}^f = \widetilde{B}^f(r_f + \alpha_X)dt + \widetilde{B}^f\sigma_X dV.$$

Setting the local rate of return equal to $r_d$ gives the following equation

$$r_f + \alpha_X = r_d,$$

and thus, under $Q^d$ we have

$$\alpha_X = r_d - r_f.$$

ii. Denote by $p_f(t,T)$ the price at time $t$ of a foreign zero-coupon bond. Given the dynamics of $f_f$ the dynamics of $p_f$ are given by

$$dp_f(t,T) = [r_f(t) + b_f(t,T)]p_f(t,T)dt + a_f(t,T)p_f(t,T)dV(t),$$

where

$$\begin{aligned} a_f(t,T) &= -\int_t^T \sigma_f(t,s)ds, \\ b_f(t,T) &= -\int_t^T \alpha_f(t,s)ds + \frac{1}{2}\|a_f(t,T)\|^2. \end{aligned}$$

Furthermore we know from the previous exercise that the $Q_d$-dynamics of $X$ are given by

$$dX = (r_d - r_f)Xdt + \sigma_X X dV.$$

Now, the drift $\alpha_f$ has to be chosen so that the process $\tilde{p}_f(t,T) = X(t)p_f(t,T)$ has a local rate of return equal to $r_d$ under $Q_d$ (or equivalently $\tilde{p}_f/B_d$ is a $Q_d$-martingale). Using Itô's formula the $Q_d$-dynamics of $\tilde{p}_f$ can be found to be

$$\begin{aligned} d\tilde{p}_f(t,T) &= [r_d(t) + b_f(t,T) + a_f(t,T)\sigma_X']\tilde{p}_f(t,T)dt \\ &\quad +[\sigma_X(t) + a_f(t,T)]\tilde{p}_f(t,T)dV, \end{aligned}$$

where $'$ denotes transpose. Setting the local rate of return equal to $r_d$ we obtain the following equation

$$b_f(t,T) + a_f(t,T)\sigma_X(t)' = 0.$$

After inserting the expression for $a_f(t,T)$ and $b_f(t,T)$, and taking the derivative w.r.t. $T$ the equation reads

$$\alpha_f(t,T) = \sigma_f(t,T)\left(\int_t^T \sigma_f(t,s)'ds - \sigma_X(t)'\right).$$

This is the drift restriction on the foreign forward rate process under the domestic martingale measure $Q_d$.

**5.** (a) A portfolio is a vector process $\mathbf{h} = (h_0, h_1)$ which is adapted (really it should be predictable) and sufficiently integrable. Here $h_0$ is the number of risk free assets in the portfolio and $h_1$ is the number of stocks. The value process is given by

$$V(t;\mathbf{h}) = h_0(t)B(t) + h_1(t)S(t).$$

The gain process $G$ of the stock is defined by $G(t) = S(t) + D(t)$ and a portfolio $\mathbf{h}$ is self-financing if it holds that

$$dV(t;\mathbf{h}) = h_0(t)dB(t) + h_1(t)dG(t).$$

(b) A probability measure $Q \sim P$ is a martingale measure if the normalized gain process

$$G^Z(t) = \frac{S(t)}{B(t)} + \int_0^t \frac{1}{B(s)}dD(s),$$

is a $Q$-martingale.

(c) From Exercise (a) we know that the value process of a self-financing portfolio satisfies

$$dV(t;\mathbf{h}) = h_0(t)dB(t) + h_1(t)dG(t).$$

Now, defining the relative portfolio according to

$$u_0(t) = \frac{h_0(t)B(t)}{V(t)}, \qquad u_1 = \frac{h_1(t)S(t)}{V(t)},$$

we have that the value process of a self-financing relative portfolio should satisfy

$$dV(t;\mathbf{u}) = u_0(t)V(t)\frac{dB(t)}{B(t)} + u_1(t)V(t)\frac{dG(t)}{S(t)}.$$

Now, we are interested in the self-financing relative portfolio $\mathbf{u} = (0, 1)$, i.e. the relative portfolio which at all times has all its money invested in the risky asset. Recall that the dynamics under $Q$ of a risky asset paying a constant dividend yield of $\delta$ are given by

$$dS_t = (r - \delta)S_t dt + \sigma S_t dU_t,$$

where $U$ denotes a $Q$-Wiener process. The value process of this portfolio is then seen to solve

$$\begin{aligned} dV_t &= 1 \cdot V_t \cdot \frac{dS_t + dD_t}{S_t} = rV_t dt + \sigma V_t dU_t, && (1) \\ V_0 &= S_0, && (2) \end{aligned}$$

where we have used that we should start out owning exactly one risky asset to begin with. Thus, the value process is given by geometrical Brownian motion, and the explicit solution for $V$ is given by

$$V_t = S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right) t + \sigma U_t\right\}.$$

Finding the number of stocks owned at $t$, $h_1(t)$, is now easy. Simply use that $V(t) = h_1(t)S(t)$ to obtain

$$h_1(t) = \frac{V(t)}{S(t)} = \frac{S_0}{S_t}\exp\left\{\left(r - \frac{1}{2}\sigma^2\right) t + \sigma U_t\right\} = e^{\delta t}. \qquad (3)$$

The number of risky assets owned at time $T$ is thus given by equation (3) evaluated at $t = T$, and the dynamics of the portfolio are given by equation (1).