Finance II

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Answers and solutions

1. **(i)** Standard techniques gives us

$$F(t,x) = E_{t,x}\left[\Phi(X_T)\right] + E_{t,x}\left[\int_t^T k(s,X_s)ds\right]$$

where

$$dX_s = \mu(s,X_s)ds + \sigma(s,X_s)dW_s.$$

(ii) In this case the problem is to compute

$$F(t,x) = E_{t,x}\left[\int_t^T X_s ds\right] = \int_t^T E_{t,x}\left[X_s\right] ds$$

where

$$dX_s = \alpha X_s ds + \beta X_s dW_s.$$

We easily obtain

$$E_{t,x}\left[X_s\right] = xe^{\alpha(s-t)}$$

so

$$F(t,x) = x\int_t^T e^{\alpha(s-t)}ds = \frac{x}{\alpha}\left\{e^{\alpha(T-t)} - 1\right\}.$$

2. We use S_2 as the numeraire and obtain

$$\Pi\left(t;X\right) = S_2(t)E^{S_2}\left[\min\left\{Z(T),1\right\}\middle|\,\mathcal{F}_t\right]$$

where

$$Z(t) = \frac{S_1(t)}{S_2(t)}$$

using a figure you easily see that

$$\min\left\{z,1\right\} = z - \max\left\{z-1,0\right\},$$

which corresponds to a long position in the underlying Z and a short position in a call with strike price $K = 1$.

Under Q^{S_2} the drift of Z equals zero, and the Ito formula easily gives us

$$dZ(t) = Z(t)\left\{\delta d\tilde{W} - \gamma d\tilde{W}\right\}$$

where $\tilde{W}$ and $\tilde{V}$ are Wiener under the relevant measure. We can thus write

$$dZ(t) = Z(t)\sigma d\hat{W}$$

where $\hat{W}$ is Q^{S_2} Wiener and

$$\sigma = \sqrt{\delta^2 + \gamma^2}$$

This corresponds to a world with zero short rate, so we obtain

$$\Pi(t; X) = S_2(t)\{Z(t) - c(t, z(t), 1, \sigma, 0)\} = S_1(t) - S_2(t)c(t, z(t), 1, \sigma, 0)$$

with σ as above.

3.

(a)

$$f(t, T) = -\frac{\partial}{\partial T}\ln p(t, T)$$

(b)

$$\alpha(t, T)dt = \sigma(t, T)\int_t^T \sigma(t, s)ds$$

4. If the T bond has the pricing function $F^T(t, r)$, so

$$p(t, T) = F^T(t, r(t))$$

then by the assumptions we have

$$F^T(t, r) = e^{-r\cdot(T-t)}$$

Thus we have

$$\begin{aligned} F_t^T &= rF^T, \\ F_r^T &= -(T-t)F^T, \\ F_{rr} &= (T-t)^2F^T \end{aligned}$$

Plugging this into the term structure equation

$$F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T = 0.$$

gives us, after simplification,

$$\mu(t,r) = (T-t)\sigma^2(t,r).$$

If this holds for all t and T, then since the lhs does not depend on T we must have $\sigma = 0$ and hence also $\mu = 0$. Thus, the only short rate model which gives us a flat term structure model is the degenerate case when the short rate is constant and deterministic.

5. **(i)** See the textbook, Chapter 26.

$$F(t,T) = E^Q\left[\mathcal{Y}|\,\mathcal{F}_t\right]$$

(ii) Let Z_t be the number of units of the risk free asset B (bank account) and let X_t be the number of units of the T-futures contract. Then, since by definition all money is in the bank, we have

$$V_t = Z_t B_t.$$

Furthermore, the general self financing condition is

$$dV = \sum_i h^i(dS^i + dG^i)$$

where h is the absolute portfolio, S^i is the spot price of the asset, and G^i is the cumulative dividend process. In our case we have two assets. For the risk free asset we have $S = B$ and $G = 0$. For the futures contract we have $S = 0$ and $G = F$. We thus have the dynamics

$$dV_t = Z_t dB_t + X_t dF_t$$

and since $dB = rBdt$ we have

$$dV_t = rZ_tB_tdt + X_tdF_t$$

so

$$dV_t = rV_tdt + X_tdF_t.$$

(iii) Our job is to find a value process V and a process X such that

$$dV_t = rV_tdt + X_tdF_t,$$

and

$$V_T = \Phi(S_T).$$

If V exists, then by general theory we should have $V_t = \Pi(t;\mathcal{Y})$ which would imply

$$V_t = e^{-r(T-t)} E^Q [\mathcal{Y}| \mathcal{F}_t] = e^{-r(T-t)} F_t.$$

Based on this, let us now **define** V by $V_t = e^{-r(T-t)} F_t$. From Ito we then have

$$dV_t = re^{-r(T-t)} F_t dt + e^{-r(T-t)} dF_t = rV_t dt + e^{-r(T-t)} dF_t$$

We have the correct dynamics and the boundary condition is obviously satisfied so we are done. We can also identify X as

$$X_t = e^{-r(T-t)}.$$