X

Change of Numeraire

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Change of Numeraire

(Geman, Jamshidian, El Karoui)

Valuation formula:

$$\Pi\left[t; X\right] = E^{Q}_{t,r}\left[e^{-\int_t^T r(s)ds} \times X\right]$$

Hard to compute. Double integral.

Note: If X and r are **independent** then

$$\begin{aligned}\Pi\left[t; X\right] &= E^{Q}_{t,r}\left[e^{-\int_t^T r(s)ds}\right] \cdot E^{Q}_{t,r}\left[X\right] \\ &= p(t,T) \cdot E^{Q}_{t,r}\left[X\right].\end{aligned}$$

Nice! We do not have to compute $p(t,T)$. It can be observed directly on the market!
Single integral!

Sad Fact: X and r are (almost) never independent!

Idea: Use T-bond (for a fixed T) as numeraire. Define the **T-forward measure** Q^T by the requirement that

$$\frac{\Pi(t)}{p(t,T)}$$

is a Q^T-martingale for every price process $\Pi(t)$.

Then

$$\frac{\Pi[t;X]}{p(t,T)} = E^T\left[\left.\frac{\Pi[T;X]}{p(T,T)}\right|\mathcal{F}_t\right]$$

$$\Pi[T;X] = X, \qquad p(T,T) = 1.$$

$$\Pi[t;X] = p(t,T)E^T[X\,|\mathcal{F}_t]\,s$$

Do such measures exist?.

“The forward measure takes care of the stochastics over the interval $[t, T]$.”

Enormous computational advantages.

Useful for interest rate derivatives, currency derivatives and derivatives defined by several underlying assets.

We carry out this within a **factor model**.

A Factor Model

Given: Non traded underlying factor process X.

P-dynamics:

$$dX(t) = \mu\left(t, X(t)\right) dt + \sigma\left(t, X(t)\right) d\bar{W}(t).$$

Bank:

$$dB(t) = rB(t)dt,$$

Problem: Price claims of the form

$$\mathcal{Y} = \Phi\left(X(T)\right)$$

Incomplete market!

(Compare with short rate models)

- We **cannot** say anything about the price of any **particular** derivative.

- To avoid arbitrage possibilities, prices of **different derivatives** must satisfy certain **internal consistency relations**.

(Compare with short rate models)

Pricing PDE

Theorem: The pricing function $F(t, x)$ of the T-claim $\Phi(X(T))$ solves

$$\begin{aligned} F_t + \{\mu - \lambda\sigma\} F_x + \frac{1}{2}\sigma^2 F_{xx} - rF &= 0, \\ F(T, x) &= \Phi(x), \end{aligned}$$

The market price of risk $\lambda(t, x)$ is universal for all derivatives.

Risk neutral Valuation

Theorem: The pricing function is given by

$$F(t,x) = e^{-r(T-t)} E^{Q}_{t,x}\left[\Phi(X(T))\right],$$

where the Q dynamics of X are

$$dX = \{\mu(t,X) - \lambda(t,X)\sigma(t,X)\}\,dt + \sigma(t,X)\,dW.$$

The Roles of Q and λ

Let S be the price process of any traded asset.

P-dynamics:

$$dS = \alpha_s S dt + S\sigma_s dW$$

Then

- Q-dynamics:

$$dS = rSdt + S\sigma_s dW$$

- The process

$$Z(t) = \frac{S(t)}{B(t)}$$

is a Q-martingale.

- The market price of risk is given by

$$\lambda(t) = \frac{\alpha_s(t) - r}{\sigma_s(t)}$$

Basic Model

Factor process:

$$dX(t) = \mu(t, X(t))\,dt + \delta(t, X(t))\,d\bar{W}(t).$$

Price process vector:

$$S = (S_0, S_1, \ldots, S_n), \quad \text{where} \quad S_n(t) = B(t)$$

Write S-dynamcis under Q as

$$dS_i = rS_i dt + S_i \sigma_i dW$$

Assumption:

- The model is given under a fixed martingale measure Q.

- For the "numeraire process" S_0 we have

$$S_0(t) > 0, \quad S_0(0) = 1$$

Problem: Price a T-claim Y.

The Normalized Economy

Use S_0 as numeraire (accounting unit).

$$Z = \frac{S}{S_0} = [Z_0, Z_1, \ldots, Z_n] = \left[\frac{S_0}{S_0}, \frac{S_1}{S_0}, \ldots, \frac{S_n}{S_0}\right]$$

A portfolio can now interpreted in the S-economy or the Z-economy.

Definition:

$$\begin{aligned} V^S(t;h) &= \sum_{i=0}^{n} h_i(t) S_i(t) \\ V^Z(t;h) &= \sum_{i=0}^{n} h_i(t) Z_i(t) \end{aligned}$$

Invariance Lemma:

- A portfolio h is S-self-financing **iff** it is Z-self-financing.

- The value process are connected by

$$V^Z(t;h) = \frac{1}{S_0(t)} V^S(t;h)$$

- A claim $\mathcal{Y}$ is S-reachable **iff**

$$\frac{\mathcal{Y}}{S_0(T)}$$

  is Z-reachable

- The price process Π is S-arbitrage free **iff**

$$\Pi^Z = \frac{\Pi}{S_0}$$

  is Z-arbitrage free.

Pricing

Pricing Formula: Let Π denote S-price process for derivative. Then:

$$\Pi[t; \mathcal{Y}] = S_0(t) \cdot \Pi^Z\left(t; \frac{\mathcal{Y}}{S_0(T)}\right)$$

Moral: Computation of $\Pi[t; \mathcal{Y}]$ is reduced to computation of $\Pi^Z\left(t; \frac{\mathcal{Y}}{S_0(T)}\right)$

Main Idea: In the Z-economy

$$Z = \frac{S}{S_0} = [Z_0, Z_1, \ldots, Z_n] = \left[\frac{S_0}{S_0}, \frac{S_1}{S_0}, \ldots, \frac{S_n}{S_0}\right]$$

the asset Z_0 is **riskless** with

$$Z_0 = 1$$

Thus:

In the Z-economy

the short rate equals zero!

Q-dynamics of S

$$dS_i = S_i r dt + S_i \sigma_i dW$$

Q-dynamics of Z

$$dZ_i = Z_i \left[\sigma_0^2 - \sigma_i \sigma_0 \right] dt + Z_i \left(\sigma_i - \sigma_0 \right) dW$$

Q^0-dynamics of Z (zero short rate!)

$$dZ_i = Z_i \left[\sigma_i - \sigma_0 \right] dW^0, \quad i = 0, \ldots, n.$$

Thus

$$\lambda = \frac{\left(\sigma_0^2 - \sigma_i \sigma_0 \right) - 0}{\sigma_i - \sigma_0} = -\sigma_0$$

Theorem:

- For every T-claim $\mathcal{Y}$, we have

$$\Pi\left[t;\mathcal{Y}\right] = S_0(t)E^0_{t,X(t)}\left[\frac{\mathcal{Y}}{S_0(T)}\right],$$

- Q^0-dynamics of Z

$$dZ_i = Z_i\left[\sigma_i - \sigma_0\right]dW^0, \quad i = 0,\ldots,n.$$

- Q^0-dynamics of S

$$dS_i = S_i\left(r + \sigma_i\sigma_0\right)dt + S_i\sigma_i dW^0,$$

where W^0 is a Q^0-Wiener process.

- Q^0-dynamics of X

$$dX(t) = \{\mu + \delta\sigma_0\}\,dt + \delta dW^0(t).$$

Forward mesures

Use a T-bond (for a fixed T as numeraire!

$$S_0(t) = p(t,T)$$

Denote the corresponding measure by Q^T: the "T-forward neutral measure".

We obtain

$$\Pi\left[t; \mathcal{Y}\right] = p(t,T) E^T_{t,X(t)}\left[\frac{\mathcal{Y}}{p(T,T)}\right],$$

Pricing Formula:

$$\Pi\left[t; \mathcal{Y}\right] = p(t,T) E^T_{t,X(t)}\left[\mathcal{Y}\right],$$

Connections Between Q and Q^T

We have

$$\begin{aligned} \Pi[0;\mathcal{Y}] &= E^Q\left[\exp\left\{-\int_0^T r(s)ds\right\}\mathcal{Y}\right] \\ \Pi[0;\mathcal{Y}] &= p(0,T)E^T[\mathcal{Y}]. \end{aligned}$$

Thus

$$E^T[\mathcal{Y}] = E^Q\left[\frac{\exp\left\{-\int_0^T r(s)ds\right\}}{p(0,T)}\cdot\mathcal{Y}\right],$$

$$E^T_{t,X_t}[\mathcal{Y}] = \frac{E^Q_{t,X_t}\left[\exp\left\{-\int_0^T r(s)ds\right\}\cdot\mathcal{Y}\right]}{p(t,T)},$$

Corollary: If r is **deterministic**, then

$$Q = Q^T$$

An Expectation Hypothesis

Silly conjecture:

$$f(0,T) = E^P[r(T)]$$

Common conjecture:

$$f(0,T) = E^Q[r(T)]$$

Neither of these conjectures are true. Instead the following holds.

Lemma:

$$f(0,T) = E^T[r(T)]$$

Note: Different measures for different T.

An Example

Exchange option: Fix T_1 and T_2

$$X = \max\left[p(T, T_2) - \beta \cdot p(T, T_1),\ 0\right]$$

$$\Pi\left[t; X\right] = p(t, T_1) E^{T_1}_{t, X_t}\left[\max\left[Z(T) - 1,\ 0\right]\right]$$

where

$$Z(t) = \frac{p(t, T_2)}{p(t, T_1)}$$

European Call on Z with strike price K. Zero interest rate.

If we have a linear model for r, then bond volatilities are deterministic.

Piece of cake!

A new look on option pricing

(Geman, El Karoui, Rochet)

European call on asset S with strike price K and maturity T.

$$X = \max\left[S(T) - K,\ 0\right]$$

Write this as

$$\begin{aligned} X &= [S(T) - K] \cdot I\left\{S(T) \geq K\right\} \\ &= S(T) \cdot I\left\{S(T) \geq K\right\} - K \cdot I\left\{S(T) \geq K\right\} \end{aligned}$$

Use $S(t)$ and $p(t,T)$ as numeraires.

Theorem:

$$\Pi\left[0; X\right] =$$

$$S(0) \cdot Q^S\left[S(T) \geq K\right] - K \cdot p(0,T) \cdot Q^T\left[S(T) \geq K\right]$$

(Compare with Black-Scholes)

Analytical Results

Assumption: Assume that $Z_{S,T}$, defined by

$$Z_{S,T}(t) = \frac{S(t)}{p(t,T)},$$

has dynamics

$$dZ_{S,T}(t) = Z_{S,T}(t) m_T^S(t) dt + Z_{S,T}(t) \sigma_{S,T}(t) dW,$$

where $\sigma_{S,T}(t)$ is **deterministic**.

We have to compute

$$Q^T \left[S(T) \geq K \right]$$

and

$$Q^S \left[S(T) \geq K \right]$$

$$
\begin{aligned}
Q^T\left(S(T) \geq K\right) &= Q^T\left(\frac{S(T)}{p(T,T)} \geq K\right) \\
&= Q^T\left(Z_{S,T}(T) \geq K\right)
\end{aligned}
$$

By definition $Z_{S,T}$ is a Q^T-martingale, so Q^T-dynamics are given by

$$
dZ_{S,T}(t) = Z_{S,T}(t)\sigma_{S,T}(t)dW^T,
$$

with the solution

$$
Z_{S,T}(T) =
$$

$$
\frac{S(0)}{p(0,T)} \times \exp\left\{-\frac{1}{2}\int_0^T \|\sigma_{S,T}(t)\|^2 dt + \int_0^T \sigma_{S,T}(t)dW^T\right\}
$$

Lognormal distribution!

The integral

$$\int_0^T \sigma_{S,T}(t) dW^T$$

is Gaussian, with zero mean and variance

$$\Sigma^2_{S,T}(T) = \int_0^T \|\sigma_{S,T}(t)\|^2 dt$$

Thus

$$Q^T (S(T) \geq K) = N[d_2],$$

$$d_2 = \frac{\ln\left(\frac{S(0)}{Kp(0,T)}\right) - \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}}$$

$$Q^S\left(S(T) \geq K\right) = Q^S\left(\frac{p(T,T)}{S(T)} \leq \frac{1}{K}\right) = Q^S\left(Y_{S,T}(T) \leq \frac{1}{K}\right),$$

$$Y_{S,T}(t) = \frac{p(t,T)}{S(t)} = \frac{1}{Z_{S,T}(t)}.$$

$Y_{S,T}$ is a Q^S-martingale, so Q^S-dynamics are

$$dY_{S,T}(t) = Y_{S,T}(t)\delta_{S,T}(t)dW^S.$$

$$Y_{S,T} = Z_{S,T}^{-1}$$

$$\Downarrow$$

$$\delta_{S,T}(t) = -\sigma_{S,T}(t)$$

$$Y_{S,T}(T) =$$

$$\frac{p(0,T)}{S(0)} \exp\left\{-\frac{1}{2}\int_0^T \sigma_{S,T}^2(t)dt - \int_0^T \sigma_{S,T}(t)dW^S\right\},$$

$$Q^S\left(S(T) \geq K\right) = N[d_1],$$

$$d_1 = d_2 + \sqrt{\Sigma_{S,T}^2(T)}$$

Proposition: Price of call is given by

$$\Pi\left[0; X\right] = S(0)N[d_2] - K \cdot p(0,T)N[d_1]$$

$$d_2 = \frac{\ln\left(\frac{S(0)}{Kp(0,T)}\right) - \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}}$$

$$d_1 = d_2 + \sqrt{\Sigma^2_{S,T}(T)}$$

$$\Sigma^2_{S,T}(T) = \int_0^T \|\sigma_{S,T}(t)\|^2 dt$$

Bond Options in the Hull-White Model

Option on T_2-bond with expiration date T_1

Q-dynamics:

$$dr = \{\Phi(t) - ar\} dt + \sigma dW.$$

Affine term structure:

$$p(t,T) = e^{A(t,T) - B(t,T)r(t)},$$

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}.$$

Check if Z has deterministic volatility

$$Z = \frac{S(t)}{p(t,T_1)}, \qquad S(t) = p(t,T_2)$$

$$Z(t) = \frac{p(t,T_2)}{p(t,T_1)},$$

$$Z(t) = \exp\left\{\Delta A(t) - \Delta B(t) r(t)\right\},$$

$$\begin{aligned}
\Delta A(t) &= A(t,T_2) - A(t,T_1), \\
\Delta B(t) &= B(t,T_2) - B(t,T_1),
\end{aligned}$$

$$dZ(t) = Z(t)\left\{\cdots\right\} dt + Z(t) \cdot \sigma_z(t) dW,$$

$$\sigma_z(t) = -\sigma \Delta B(t) = \frac{\sigma}{a} e^{at} \left[e^{-aT_1} - e^{-aT_2} \right]$$

Deterministic volatility!

Bond Option Pricing Formula

Proposition: In the Hull-White model we have

$$\Pi\left[0;\mathcal{X}\right] = p(0,T_2)N[d_1] - K \cdot p(0,T_1)N[d_2],$$

where

$$\begin{aligned} d_2 &= \frac{\ln\left(\frac{p(0,T_2)}{Kp(0,T_1)}\right) - \frac{1}{2}\Sigma^2}{\sqrt{\Sigma^2}}, \\ d_1 &= d_2 + \sqrt{\Sigma^2}, \\ \Sigma^2 &= \frac{\sigma^2}{2a^3}\left\{1 - e^{-2aT_1}\right\}\left\{1 - e^{-a(T_2 - T_1)}\right\}^2. \end{aligned}$$