VII

Several Underlying Assets

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Model

- Under the objective probability measure P, the S-dynamics are given by

$$dS_i(t) = \alpha_i S_i(t)dt + S_i(t) \sum_{j=1}^{n} \sigma_{ij} d\bar{W}_j(t),$$

for $i = 1, \ldots, n$. Here $\bar{W}_1, \ldots, \bar{W}_n$ are independent P-Wiener processes.

- The coefficients α_i and σ_{ij} above are assumed to be known constants.

- The **volatility matrix**

$$\sigma = \left\{\sigma_{ij}\right\}_{i,j=1}^{n}$$

is nonsingular.

Compact Notation

$$dS_i(t) = \alpha_i S_i(t)dt + S_i(t)\sigma_i d\bar{W}(t)$$

$$\bar{W}(t) = \begin{bmatrix} \bar{W}_1(t) \\ \vdots \\ \bar{W}_n(t) \end{bmatrix}$$

$$\sigma_i = [\sigma_{i1}, \ldots, \sigma_{in}]$$

$$dS(t) = D[S(t)]\,\alpha dt + D[S(t)]\,\sigma d\bar{W}(t).$$

$$D[x] = \begin{bmatrix} x_1 & 0 & \ldots & 0 \\ 0 & x_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & x_n \end{bmatrix}$$

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}.$$

Main Problem

Find arbitrage free price for claim of the form

$$\mathcal{X} = \Phi(S_T)$$

Assumption:
The price process is of the form

$$\Pi\left[t; \mathcal{X}\right] = F\left(t, S(t)\right)$$

for some deterministic function

$$F : R_+ \times R^n \to R.$$

Procedure

- Take Φ, F as given.

- Form a self-financing portfolio, based on $S_1, \ldots, S_n, B$ and F. ($n + 1$ degrees of freedom)

- Choose the portfolio weights such that the driving Wiener processes are canceled in the portfolio (uses up n degrees of freedom)

$$dV(t) = V(t)k(t)dt.$$

- Use the remaining degree of freedom in order to force the value dynamics to be of the form

$$dV(t) = (r + \beta)\, V(t)dt,$$

  where β is some fixed nonzero real number. "beat the risk free asset"

- This is possible if and only if a certain matrix is nonsingular.

- Absence of arbitrage possibilities implies that this matrix has to be singular.

- The singularity condition of the matrix leads to a PDE for the pricing function F.

Portfolio dynamics:

$$dV = V\left[\sum_1^n u_i \frac{dS_i}{S_i} + u_F \frac{dF}{F} + u_B \frac{dB}{B}\right],$$

Derivative dynamics:

$$dF = F \cdot \alpha_F dt + F \cdot \sigma_F d\bar{W},$$

where

$$\alpha_F(t) = \frac{1}{F}\left[F_t + \sum_1^n \alpha_i S_i F_i + \frac{1}{2} tr\left\{\sigma^\star D[S] F_{ss} D[S] \sigma\right\}\right]$$

$$\sigma_F(t) = \frac{1}{F}\sum_1^n S_i F_i \sigma_i.$$

Insert dF/F, dS_i/S_i and dB/B in portfolio dynamics and use the relation

$$u_B = 1 - \left(\sum_1^n u_i + u_F\right)$$

Value dynamcis:

$$\begin{aligned} dV &= V \cdot \left[\sum_1^n u_i \left(\alpha_i - r \right) + u_F \left(\alpha_F - r \right) + r \right] dt \\ &+ V \cdot \left[\sum_1^n u_i \sigma_i + u_F \sigma_F \right] d\bar{W}. \end{aligned}$$

Wipe out the noise by choosing weights as

$$\sum_1^n u_i \sigma_i + u_F \sigma_F = 0.$$

Value dynamics:

$$dV = V \cdot \left[\sum_1^n u_i \left(\alpha_i - r \right) + u_F \left(\alpha_F - r \right) + r \right] dt,$$

Now we try to "beat the bank" (by a factor β) by choosing the weights such that

$$\sum_1^n u_i \left(\alpha_i - r \right) + u_F \left(\alpha_F - r \right) + r = r + \beta.$$

On matrix form we thus want to solve:

$$\left[\begin{array}{cccc} \alpha_1 - r & \dots & \alpha_n - r & \alpha_F - r \\ \sigma_1^\star & \dots & \sigma_n^\star & \sigma_F^\star \end{array}\right] \left[\begin{array}{c} u_S \\ u_F \end{array}\right] = \left[\begin{array}{c} \beta \\ 0 \end{array}\right]$$

Absence of arbitrage $\Rightarrow$ this system cannot be solved for $\beta \neq 0$.

Thus the coefficient matrix must be singular.

Proposition: Absence of arbitrage $\Rightarrow$ There exist constants $\lambda_1, \dots, \lambda_n$ such that

$$\begin{aligned} \alpha_i - r &= \sum_{j=1}^{n} \sigma_{ij} \lambda_j, \quad i = 1, \dots, n, \\ \alpha_F - r &= \sum_{j=1}^{n} \sigma_{Fj} \lambda_j, \end{aligned}$$

Interpretation:

$$\lambda_j = \text{“market price of risk” of type } j$$

$\lambda_j =$ risk premium per unit volatility of type j

On matrix form

$$\begin{aligned} \alpha - r\mathbf{1}_n &= \sigma\lambda, \\ \alpha_F - r &= \sigma_F\lambda, \end{aligned}$$

where

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} \qquad \mathbf{1}_n = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

By assumption σ is invertible so

$$\lambda = \sigma^{-1}\left[\alpha - r\mathbf{1}_n\right]$$

Pluging this into

$$\alpha_F - r = \sigma_F\lambda$$

and using

$$\sigma_F = \frac{1}{F} \cdot \left[S_1F_1, \ldots, S_nF_n\right]\sigma.$$

gives us

$$\alpha_F - r = \frac{1}{F} \cdot \left[S_1F_1, \ldots, S_nF_n\right]\left[\alpha - r\mathbf{1}_n\right]$$

Recall

$$\alpha_F - r = \frac{1}{F} \cdot [S_1 F_1, \ldots, S_n F_n] [\alpha - r\mathbf{1}_n]$$

Inserting α_F gives us.

Proposition: Absence of arbitrage implies that F has to satisfy the PDE

$$\begin{aligned} F_t(t,s) &+ \sum_{i=1}^{n} r s_i F_i(t,s) \\ &+ \frac{1}{2} tr \left\{ \sigma^\star D[S] F_{ss} D[S] \sigma \right\} - rF(t,s) = 0, \\ F(T,s) &= \Phi(s) \end{aligned}$$

Risk Neutral Valuation

Proposition: The pricing function F is given by

$$F(t,s) = e^{-r(T-t)} E_{t,s}^{Q}\left[\Phi\left(S(T)\right)\right]$$

Q-dynamics:

$$dS_i = rS_i dt + S_i \sigma_i dW, \quad i = 1, \ldots, n.$$

Alterntive formulation

Dynamics:

$$dS_i = S_i \alpha_i dt + S_i \sigma_i dW_i, \quad i = 1, \cdots, n.$$

Now both σ_i and α_i are scalars.

$$W(t) = \begin{bmatrix} W_1(t) \\ \vdots \\ W_n(t) \end{bmatrix}$$

Here $W_1, \cdots, W_n$ are **correlated** Wiener processes, with unit variance parameter, and covariance structure

$$C_{ij} = Cov\left[dW_i, dW_j\right] = \rho_{ij} dt,$$

i.e.

$$C = E\left[dW \cdot dW^\star\right] = \begin{bmatrix} 1 & \rho_{12} & \cdots & \rho_{1n} \\ \rho_{21} & 1 & \cdots & \rho_{2n} \\ \vdots & & \ddots & \vdots \\ \rho_{n1} & \rho_{n2} & \cdots & 1 \end{bmatrix} dt$$

Pricing PDE

$$
\begin{aligned}
\frac{\partial F}{\partial t} &+ \sum_{i=1}^{n} r s_i \frac{\partial F}{\partial s_i} \\
&+ \frac{1}{2}\sum_{i,j} s_i s_j \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 F}{\partial s_i \partial s_j} - rF = 0, \\
F(T,s) &= \Phi(s)
\end{aligned}
$$

Risk neutral Valuation

$$F(t,s) = e^{-r(T-t)} E^Q_{t,s}\left[\Phi(S(T))\right]$$

Q-dynamics:

$$dS_i = rS_i dt + S_i \sigma_i dW_i,$$

Local rate of return equals r. Same covariance structure under Q as under P.