VIII

Interest Rate Theory 1

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Definitions

Bonds:
T-bond = zero coupon bond, paying 1$ at the date of maturity T.

$$\begin{aligned} p(t,T) &= \text{price, at } t\text{, of a } T\text{-bond.} \\ p(T,T) &= 1. \end{aligned}$$

Main Problem

- Investigate the **term structure**, i.e. how prices of bonds with different dates of maturity are related to each other.

- Compute arbitrage free prices of interest rate derivatives (bond options, swaps, caps, floors etc.)

Risk Free Interest Rates

At time $\mathbf{t}$:

- Sell one S-bond

- Buy exactly $p(t,S)/p(t,T)$ $T-$bonds

- Net investment at t: 0$.

At time $\mathbf{S}$:

- Pay 1$

At time $\mathbf{T}$:

- Collect $p(t,S)/p(t,T) \cdot 1$$

Net Effect

- The contract is made at t.
- An investment of 1 at time S has yielded $p(t,S)/p(t,T)$ at time T.
- The equivalent constant rates, R, are given as the solutions to

Continuous rate:

$$e^{R\cdot(T-S)} \cdot 1 = \frac{p(t,S)}{p(t,T)}$$

Simple rate:

$$[1 + R \cdot (T-S)] \cdot 1 = \frac{p(t,S)}{p(t,T)}$$

Continuous Interest Rates

1. The **forward rate for the period** $[S, T]$**, contracted at** t is defined by

$$R(t; S, T) = -\frac{\log p(t, T) - \log p(t, S)}{T - S}.$$

2. The **spot rate**, $R(S, T)$, for the period $[S, T]$ is defined by

$$R(S, T) = R(S; S, T).$$

3. The **instantaneous forward rate at** T**, conracted at** t is defined by

$$f(t, T) = -\frac{\partial \log p(t, T)}{\partial T} = \lim_{S \to T} R(t; S, T).$$

4. The **instantaneous short rate at** t is defined by

$$r(t) = f(t, t).$$

Simple Rates
(LIBOR)

1. The **simple forward rate L(t;S,T)for the period** $[S,T]$**, contracted at** t is defined by

$$L(t;S,T) = \frac{1}{T-S} \cdot \frac{p(t,S) - p(t,T)}{p(t,T)}$$

2. The **simple spot rate,** $L(S,T)$, for the period $[S,T]$ is defined by

$$L(S,T) = \frac{1}{T-S} \cdot \frac{1 - p(S,T)}{p(S,T)}$$

Practical Formula
(LIBOR)

The **simple spot rate,** $L(T, T+\delta)$, for the period $[T, T+\delta]$ is given by

$$p(T, T+\delta) = \frac{1}{1 + \delta L(T, T+\delta)}$$

i.e.

$$L = \frac{1}{\delta} \cdot \frac{1-p}{p}$$

Bond prices $\sim$ forward rates

$$p(t,T) = p(t,s) \cdot \exp\left\{-\int_s^T f(t,u)du\right\},$$

In particular we have

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}.$$

Caps

Basic idea: Buy an insurance against high interest rates in the future.

1. The contract is written at $t = 0$. At that time also the **principal**, K, and the fixed **cap rate**, R are determined.

2. A cap is a sum of elementary contracts, so called **caplets**.

3. A caplet is active over the period $[T, T+\delta]$,

4. At time $T + \delta$ the holder of the caplet receives

$$X = K\delta \max [L - R, 0] = K\delta (L - R)^+$$

where L is the simple spot rate (LIBOR) for the period $[T, T + \delta]$, i.e.

$$L = L(T, T + \delta)$$

We set $K = 1$ and plug in earlier formulas for LIBOR rates

$$p = \frac{1}{1+\delta L}$$

$$L = \frac{1}{\delta} \cdot \frac{1-p}{p}$$

$$X = \delta\,(L-R)^{+} = \delta \left(\frac{1-p}{p\delta} - R \right)^{+}$$

$$= \left(\frac{1}{p} - (1+\delta R) \right)^{+} = \left(\frac{1}{p} - R^{\star} \right)^{+}.$$

where $R^{\star} = 1 + \delta R$.

At $T+\delta$ the holder thus receives

$$\begin{aligned} X_i &= \left(\frac{1}{p} - R^\star\right)^+ \\ &= R^\star \cdot \frac{1}{p}\left(\frac{1}{R^\star} - p\right)^+ \end{aligned}$$

This is equivalent to receiving

$$Y = R^\star \cdot \left(\frac{1}{R^\star} - p(T, T+\delta)\right)^+$$

at time T.

Result:
A caplet is equivalent to a European Put with excercise date T, on an underlying $(T+\delta)$-bond.

The theoretcial price of the cap will depend upon our choice of interest rate model.

Interest Rate Options

Problem:
We want to price, at t, a European Call, with exercise date S, and strike price K, on an underlying T-bond. ($t < S < T$).

Naive approach: Use Black-Scholes's formula.

$$F(t,p) = pN\left[d_1\right] - e^{-r(S-t)}KN\left[d_2\right].$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{S-t}}\left\{\ln\left(\frac{p}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(S-t)\right\}, \\ d_2 &= d_1 - \sigma\sqrt{S-t}. \end{aligned}$$

where

$$p = p(t,T)$$

Is this allowed?

- p shall be the price of a traded asset. OK!

- The volatility of p must be constant. Here we have a problem because of **pull-to-par**, i.e. the fact that $p(T,T) = 1$. Bond volatilities will tend to zero as the bond approaches the time of maturity.

- The short rate must be **constant** and **deterministic**. Here the approach collapses completely, since the whole point of studying bond prices lies in the fact that interest rates are stochastic.

There is some hope in the case when the remaining time to exercise the option is small in relation to the remaining time to maturity of the underlying bond (why?).

Deeply felt need

A consistent **arbitrage free** model for the bond market

Stochastic interest rates

We assume that the short rate r is a stochastic process.

Money in the bank will then grow according to:

$$\begin{cases} dB(t) &= r(t)B(t)dt, \\ B(0) &= 1. \end{cases}$$

i.e.

$$B(t) = e^{\int_0^t r(s)ds}$$

Models for the short rate

Model: (In reality)

P:

$$
\begin{aligned}
dr &= \mu(t,r)dt + \sigma(t,r)dW, \\
dB &= r(t)Bdt.
\end{aligned}
$$

Question: Are bond prices uniquely determined by the P-dynamics of r, and the requirement of an arbitrage free bond market?

NO!!

WHY?

Stock Models $\sim$ Interest Rates

Black-Scholes:

$$\begin{aligned} dS &= \alpha S dt + \sigma S dw, \\ dB &= r B dt. \end{aligned}$$

Interest Rates:

$$\begin{aligned} dr &= \mu(t, r) dt + \sigma(t, r) dW, \\ dB &= r(t) B dt. \end{aligned}$$

Question: What is the difference?

Answer: The short rate r is **not the price of a traded asset!**

1. **Meta-Theorem:**

   $$\begin{aligned} N &= 0, \text{ (no risky asset)} \\ R &= 1, \text{ (one source of randomness, } W\text{)} \end{aligned}$$

   We have $M < R$. The exongenously given market, consisting only of B, is incomplete.

2. **Replicating portfolios:**
   We can only invest money in the bank, and then sit down passively and wait.

We do **not** have **enough underlying assets** in order to price bonds.

- There is **not** a unique price for a **particular** $T-$bond.

- In order to avoid arbitrage, bonds of **different maturities** have to satisfy internal **consistency** relations.

- If we take **one** "benchmark" T_0-bond as given, then all other bonds can be priced **in terms of** the market price of the benchmark bond.

Assumption:

$$
\begin{aligned}
p(t,T) &= F(t,r(t);T) \\
p(t,T) &= F^T(t,r(t)), \\
F^T(T,r) &= 1.
\end{aligned}
$$

Program:

- Form portfolio based on $T-$ and $S-$bonds. Use Itô on $F^T(t, r(t))$ to get bond- and portfolio dynamics.

$$dV = V\left\{u_T \frac{dF^T}{F^T} + u_S \frac{dF^S}{F^S}\right\}$$

- Choose portfolio weights such that the $dW-$term vanishes. Then we have

$$dV = V \cdot kdt,$$

(“synthetic bank” with k as the short rate)

- Absence of arbitrage $\Rightarrow$ $k = r$.

- Read off the relation $k = r$!

From Itô:

$$dF^T = F^T \alpha_T dt + F^T \sigma_T d\tilde{W},$$

where

$$\begin{cases} \alpha_T &= \frac{F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T}{F^T}, \\ \sigma_T &= \frac{\sigma F_r^T}{F^T}. \end{cases}$$

Portfolio dynamics

$$dV = V\left\{u^T \frac{dF^T}{F^T} + u^S \frac{dF^S}{F^S}\right\}.$$

Reshuffling terms gives us

$$dV = V\cdot\left\{u^T \alpha_T + u^S \alpha_S\right\} dt + V\cdot\left\{u^T \sigma_T + u^S \sigma_S\right\} dW.$$

Let the portfolio weights solve the system

$$\begin{cases} u^T + u^S &= 1, \\ u^T \sigma_T + u^S \sigma_S &= 0. \end{cases}$$

$$\begin{cases} u^T &= -\frac{\sigma_S}{\sigma_T - \sigma_S}, \\ u^S &= \frac{\sigma_T}{\sigma_T - \sigma_S}, \end{cases}$$

Portfolio dynamics

$$dV = V \cdot \left\{ u^T \alpha_T + u^S \alpha_S \right\} dt.$$

i.e.

$$dV = V \cdot \left\{ \frac{\alpha_S \sigma_T - \alpha_T \sigma_S}{\sigma_T - \sigma_S} \right\} dt.$$

Absence of arbitrage requires

$$\frac{\alpha_S \sigma_T - \alpha_T \sigma_S}{\sigma_T - \sigma_S} = r$$

which can be written as

$$\frac{\alpha_S(t) - r(t)}{\sigma_S(t)} = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)}.$$

$$\frac{\alpha_S(t) - r(t)}{\sigma_S(t)} = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)}.$$

Note!
The quotient does **not** depend upon the particular choice of maturity date.

Result:
Assume that the bond market is free of arbitrage. Then there exists a universal process λ, such that

$$\frac{\alpha_T(t) - r(t)}{\sigma_T(t)} = \lambda(t),$$

holds for all t and for every choice of maturity T.

NB: The same λ for all choices of T.

$$\begin{aligned} \lambda &= \text{ Risk premium per unit of volatility} \\ &= \text{ "Market Price of Risk" (cf. CAPM).} \end{aligned}$$

Slogan:
"On an arbitrage free market all bonds have the same market price of risk."

The relation

$$\frac{\alpha_T - r}{\sigma_T} = \lambda$$

is actually a PDE!

The Term Structure Equation

$$
\begin{aligned}
F_t^T + \{\mu - \lambda\sigma\} F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T &= 0, \\
F^T(T, r) &= 1.
\end{aligned}
$$

P-**dynamics:**

$$dr = \mu(t, r)dt + \sigma(t, r)dW.$$

$$\lambda = \frac{\alpha_T - r}{\sigma_T}, \quad \text{for all } T$$

In order to solve the TSE we need to know λ.

General Term Structure Equation

Contingent claim:

$$X = \Phi(r(T))$$

Result:
The price is given by

$$\Pi\left[t; X\right] = F(t, r(t))$$

where F solves

$$\begin{aligned} F_t + \{\mu - \lambda\sigma\} F_r + \frac{1}{2}\sigma^2 F_{rr} - rF &= 0, \\ F(T, r) &= \Phi(r). \end{aligned}$$

In order to solve the TSE we need to know λ.

Question:

Who determines λ?

Answer:

THE MARKET!

Moral

- Since the market is incomplete the requirement of an arbitrage free bond market will not lead to unique bond prices.

- Prices on bonds and other interest rate derivatives are determined by two main factors.

  1. **Partly** by the requirement of an arbitrage free bond market (the pricing functions satisfies the TSE).

  2. **Partly** by supply and demand on the market. These are in turn determined by attitude towards risk, liquidity consideration and other factors. All these are aggregated into the particular λ used (implicitly) by the market.

Risk Neutral Valuation

Using Feynmac–Kač we obtain

$$F(t, r; T) = E^{Q}_{t,r}\left[\mathsf{exp}\left\{-\int_t^T r(s)ds\right\} \times 1\right].$$

Q**-dynamics:**

$$dr = \{\mu - \lambda\sigma\}dt + \sigma dW$$

Risk Neutral Valuation

$$\Pi\left[t; X\right] = E_{t,r}^{Q}\left[e^{-\int_t^T r(s)ds} \times X\right]$$

Q-**dynamics:**

$$dr = \{\mu - \lambda\sigma\}dt + \sigma dW$$

- Price = expected value of future payments
- The expectation should **not** be taken under the "objective" probabilities P, but under the "risk adjusted" probabilities Q.

Interpetation of the risk adjusted probabilities

- The risk adjusted probabilities can be interpreted as probabilities in a (fictuous) risk neutral world.

- When we **compute prices**, we can calculate **as if** we live in a risk neutral world.

- This does **not** mean that we live in, or think that we live in, a risk neutral world.

- The formulas above hold regardless of the attitude towards risk of the investor, as long as he/she prefers more to less.