VII

Incomplete Markets

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Typical Factor Model Setup

Given:

- An underlying factor process X, which is **not** the price process of a traded asset, with P-dynamics

$$dX_t = \mu(t, X_t)\,dt + \sigma(t, X_t)\,dW_t.$$

- A risk free asset with dynamics

$$dB_t = rB_t dt,$$

Problem:

Find arbitrage free price $\Pi[t; \mathcal{Z}]$ of a derivative of the form

$$\mathcal{Z} = \Phi(X_T)$$

Question: Is the price $\Pi[t; \mathcal{Z}]$ uniquely determined by the P-dynamics of X, and the requirement of an arbitrage free derivatives market?

NO!!

WHY?

Stock Price Model $\sim$ Factor Model

Black-Scholes:

$$\begin{aligned} dS &= \alpha S dt + \sigma S dw, \\ dB &= r B dt. \end{aligned}$$

Factor Model:

$$\begin{aligned} dX &= \mu(t, X) dt + \sigma(t, X) dW, \\ dB &= r B dt. \end{aligned}$$

Question: What is the difference?

Answer:

- X is **not the price of a traded asset!**
- We can not form a portfolio based on X.

1. **Meta-Theorem:**

   $$\begin{aligned} N &= 0, \text{ (no risky asset)} \\ R &= 1, \text{ (one source of randomness, } W) \end{aligned}$$

   We have $M < R$. The exongenously given market, consisting only of B, is incomplete.

2. **Replicating portfolios:**
   We can only invest money in the bank, and then sit down passively and wait.

We do **not** have **enough underlying assets** in order to price X-derivatives.

- There is **not** a unique price for a **particular** derivative.

- In order to avoid arbitrage, **different** derivatives have to satisfy **internal consistency** relations.

- If we take **one** "benchmark" derivative as given, then all other derivatives can be priced **in terms of** the market price of the benchmark.

We consider two given claims $\Phi(X_T)$ and $\Gamma(X_T)$. We assume they are traded with prices

$$\Pi[t;\Phi] = F(t, X_t)$$
$$\Pi[t;\Gamma] = G(t, X_t)$$

Program:

- Form portfolio based on Φ and Γ. Use Itô on F and G to get portfolio dynamics.

$$dV = V \left\{ u_F \frac{dF}{F} + u_G \frac{dG}{G} \right\}$$

- Choose portfolio weights such that the $dW-$ term vanishes. Then we have

$$dV = V \cdot k dt,$$

(“synthetic bank” with k as the short rate)

- Absence of arbitrage $\Rightarrow$ $k = r$.

- Read off the relation $k = r$!

From Itô:

$$dF = F\alpha_F dt + F\sigma_F dW,$$

where

$$\begin{cases} \alpha_F &= \frac{F_t + \mu F_x + \frac{1}{2}\sigma^2 F_{xx}}{F}, \\ \sigma_F &= \frac{\sigma F_x}{F}. \end{cases}$$

Portfolio dynamics

$$dV = V\left\{u_F \frac{dF}{F} + u_G \frac{dG}{G}\right\}.$$

Reshuffling terms gives us

$$dV = V \cdot \{u_F\alpha_F + u_G\alpha_G\}\, dt + V \cdot \{u_F\sigma_F + u_G\sigma_G\}\, dW.$$

Let the portfolio weights solve the system

$$\begin{cases} u_F + u_G &= 1, \\ u_F\sigma_F + u_G\sigma_G &= 0. \end{cases}$$

$$\begin{cases} u_F &= -\frac{\sigma_G}{\sigma_F - \sigma_G}, \\ u_G &= \frac{\sigma_F}{\sigma_F - \sigma_G}, \end{cases}$$

Portfolio dynamics

$$dV = V \cdot \{u_F \alpha_F + u_G \alpha_G\} \, dt.$$

i.e.

$$dV = V \cdot \left\{ \frac{\alpha_G \sigma_F - \alpha_F \sigma_G}{\sigma_F - \sigma_G} \right\} dt.$$

Absence of arbitrage requires

$$\frac{\alpha_G \sigma_F - \alpha_F \sigma_G}{\sigma_F - \sigma_G} = r$$

which can be written as

$$\frac{\alpha_G(t) - r}{\sigma_G(t)} = \frac{\alpha_F(t) - r}{\sigma_F(t)}.$$

$$\frac{\alpha_G - r}{\sigma_G} = \frac{\alpha_F - r}{\sigma_F}.$$

Note!
The quotient does **not** depend upon the particular choice of contract.

Result:
Assume that the market for X-derivatives is free of arbitrage. Then there exists a universal process λ, such that

$$\frac{\alpha_F(t) - r}{\sigma_F(t)} = \lambda(t),$$

holds for all t and for every choice of contract F.

NB: The same λ for all choices of F.

$$\begin{aligned} \lambda &= \text{Risk premium per unit of volatility} \\ &= \text{"Market Price of Risk" (cf. CAPM).} \end{aligned}$$

Slogan:
"On an arbitrage free market all X-derivatives have the same market price of risk."

The relation

$$\frac{\alpha_F - r}{\sigma_F} = \lambda$$

is actually a PDE!

Pricing Equation

$$\begin{aligned} F_t + \{\mu - \lambda\sigma\} F_x + \frac{1}{2}\sigma^2 F_{xx} - rF &= 0, \\ F(T,x) &= \Phi(x), \quad \forall x. \end{aligned}$$

P-**dynamics:**

$$dX = \mu(t,X)dt + \sigma(t,X)dW.$$

$$\lambda = \frac{\alpha_F - r}{\sigma_F}, \quad \text{for all } F$$

In order to solve the TSE we need to know λ.

Question:

Who determines λ?

Answer:

THE MARKET!

Moral

- Since the market is incomplete the requirement of an arbitrage free market will **not** lead to unique prices for X-derivatives.

- Prices on derivatives are determined by two main factors.

  1. **Partly** by the requirement of an arbitrage free derivative market (the pricing functions satisfies the PDE).

  2. **Partly** by supply and demand on the market. These are in turn determined by attitude towards risk, liquidity consideration and other factors. All these are aggregated into the particular λ used (implicitly) by the market.

Risk Neutral Valuation

Using Feynmac–Kač we obtain

$$F(t, x; T) = e^{-r(T-t)} E^Q_{t,x} \left[\Phi(X_T) \right].$$

Q**-dynamics:**

$$dX = \{\mu - \lambda\sigma\} dt + \sigma dW$$

Risk Neutral Valuation

$$\Pi[t; X] = e^{-r(T-t)} E_{t,x}^{Q}[\Phi(X_T)]$$

Q-**dynamics:**

$$dX = \{\mu - \lambda\sigma\}dt + \sigma dW$$

- Price = expected value of future payments
- The expectation should **not** be taken under the "objective" probabilities P, but under the "risk adjusted" probabilities Q.

Interpretation of the risk adjusted probabilities

- The risk adjusted probabilities can be interpreted as probabilities in a (fictuous) risk neutral world.

- When we **compute prices**, we can calculate **as if** we live in a risk neutral world.

- This does **not** mean that we live in, or think that we live in, a risk neutral world.

- The formulas above hold regardless of the attitude towards risk of the investor, as long as he/she prefers more to less.