VI

Currency Derivatives

Tomas Björk

Pure Currency Contracts

Consider two markets, domestic (England) and foreign (USA).

$$\begin{aligned} r_d &= \text{domestic short rate} \\ r_f &= \text{foreign short rate} \\ X(t) &= \text{exchange rate} \end{aligned}$$

NB! The exchange rate X is quoted as

$$\frac{\text{units of the domestic currency}}{\text{unit of the foreign currency}}$$

Model

The P-dynamics are given as:

$$
\begin{aligned}
dX &= X\alpha_X dt + X\sigma_X dW, \\
dB_d &= r_d B_d dt, \\
dB_f &= r_f B_f dt,
\end{aligned}
$$

Main Problem:
Find arbitrage free price for currency derivative, Z, of the form

$$Z = \Phi(X(T))$$

Typical example: European Call on X.

$$Z = \max\left[X(T) - K, 0\right]$$

Naive idea:
For the European Call, use the standard Black-Scholes formula, with S replaced by X and r replaced by r_d.

NO!

Why:
When yu buy stock you just keep the paper (until you sell it).
When you buy dollars , these are put into a bank account, giving the interest r_f.

Moral:
Buying a currency is like buying a dividend-paying stock.

Technique:

- Transform all objects into **domestically traded** asset prices.

- Use standard techniques on the transformed model.

Transformed Market:

1. Investing foreign currency in the foreign bank gives value dynamics **in foreign currenc**y according to

$$dB_f = r_f B_f dt.$$

2. B_f units of the foreign currency is worth $X \cdot B_f$ in the domestic currency.

3. Trading in the foreign currency is equivalent to trading in a domestic market with the domestic price process

$$\tilde{B}_f(t) = B_f(t) \cdot X(t)$$

4. Study the domestic market consisting of

$$\tilde{B}_f, \quad B_d$$

Market dynamics:

P-dynamics:

$$\begin{aligned} dX &= X\alpha_X dt + X\sigma_X dW \\ \tilde{B}_f(t) &= B_f(t) \cdot X(t) \end{aligned}$$

Using Itô we have domestic market dynamics

$$\begin{aligned} d\tilde{B}_f &= \tilde{B}_f \left(\alpha_X + r_f\right) dt + \tilde{B}_f \sigma_X dW \\ dB_d &= r_d B_d dt \end{aligned}$$

Standard results gives us Q-dynamics for domestically traded asset prices:
$\mathbf{Q}$:

$$\begin{aligned} d\tilde{B}_f &= \tilde{B}_f r_d dt + \tilde{B}_f \sigma_X dW \\ dB_d &= r_d B_d dt \end{aligned}$$

Itô gives us Q-dynamics for $X = \tilde{B}_f / B_f$:
$\mathbf{Q}$:

$$dX = X(r_d - r_f)dt + X\sigma_X dW$$

Risk neutral Valuation:

Theorem:
The arbitrage free price $\Pi[t;\Phi]$ is given by $\Pi[t;\Phi] = F(t, X(t))$ where

$$F(t,x) = e^{-r_d(T-t)} E^Q_{t,x}[\Phi(X(T)]$$

The Q-dynamics of X are given by

$$dX = X(r_d - r_f)dt + X\sigma_X dW$$

Pricing PDE:

Theorem:
The arbitrage free price $\Pi[t;\Phi]$ is given by $\Pi[t;\Phi] = F(t, X(t))$ where F solves the boundary value problem

$$
\begin{aligned}
\frac{\partial F}{\partial t} + x(r_d - r_f)\frac{\partial F}{\partial x} + \frac{1}{2}x^2\sigma_X^2\frac{\partial^2 F}{\partial x^2} - r_d F &= 0, \\
F(T, x) &= \Phi(x)
\end{aligned}
$$

Currency vs Equity Derivatives

Proposition:

Introduce the notation:

- $F_0(t,x)$ = pricing function $\mathcal{Z} = \Phi(X(T))$, where we interpret X as the price of an ordinary stock without dividends.

- $F(t,x)$ = the pricing function of the same claim when X is interpreted as an exchange rate.

Then the following holds

$$F(t,x) = F_0\left(t, xe^{-r_f(T-t)}\right).$$

Currency Option Formula

The price of a European currency call is given by

$$F(t,x) = xe^{-r_f(T-t)}N\left[d_1\right] - e^{-r_d(T-t)}KN\left[d_2\right],$$

where

$$d_1 = \frac{1}{\sigma_X\sqrt{T-t}}\left\{\ln\left(\frac{x}{K}\right) + \left(r_d - r_f + \frac{1}{2}\sigma_X^2\right)(T-t)\right\}$$

$$d_2 = d_1(t,x) - \sigma_X\sqrt{T-t}$$