V

Extensions of Black-Scholes

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Topics to be covered

1. Options on dividend paying stocks.

2. Futures options.

3. American contracts.

1. Dividends

Black-Scholes model:

$$\begin{aligned} dS &= \alpha S dt + \sigma S dW, \\ dB &= r B dt. \end{aligned}$$

Contingent T-claim:

$$\mathcal{Z} = \Phi\left(S_T\right)$$

New feature:
The underlying stock pays **dividends**.

$D(t)$ = The cumulative dividends over the interval $[0, t]$

Two cases

- Discrete dividends (realistic but messy).
- Continuous dividends (unrealistic but easy to handle).

Continuous Dividend Yield

Dynamics:

$$dS = \alpha S dt + \sigma S dW,$$

The stock pays a **continuous dividend yield** of δ, i.e.

$$dD(t) = \delta S(t) dt$$

Problem:
How does the dividend affect the price of a Euroepean Call? (compared to a non-dividend stock).

Answer:
The price is lower. (why?)

Portfolios and Dividends

Consider a market with N assets.

$$\begin{aligned} S_i(t) &= \text{price at } t\text{, of asset No } i \\ D_i(t) &= \text{cumulative dividends for } S_i \text{ over} \\ &\quad \text{the interval } [0, t] \\ h^i(t) &= \text{number of units of asset } i \\ V(t) &= \text{market value of the portfolio } h \text{ at } t \end{aligned}$$

$$V(t) = \sum_{i=1}^{N} h^i(t) S_i(t)$$

Self financing portfolios

Definition:
The strategy h is **self financing** if

$$dV(t) = \sum_{i=1}^{N} h^i(t) dG_i(t)$$

where the **gains** process G_i is given by

$$dG_i(t) = dS_i(t) + dD_i(t)$$

Interpret!

Relative weights

$u^i(t) =$ the relative share of the portfolio value, which is invested in asset No i.

$$u^i(t) = \frac{h^i(t)S_i(t)}{V(t)}$$

$$dV(t) = \sum_{i=1}^{N} h^i(t)dG_i(t)$$

Substitute!

$$dV = V\sum_{i=1}^{N} u^i \frac{dG_i}{S_i}$$

Black-Scholes with Dividends

$$\begin{aligned} dS &= \alpha S dt + \sigma S dW, \\ dD &= \delta S dt \end{aligned}$$

Gains sprocess:

$$dG = (\alpha + \delta) S dt + \sigma S dW$$

Consider a fixed claim

$$X = \Phi(S_T)$$

and assume that

$$\Pi[t; X] = F(t, S_t)$$

Value dynamics:

$$dV = V \cdot \left\{ u_S \frac{dG_S}{S} + u_F \frac{dF}{F} \right\},$$

$$dG_S = S(\alpha + \delta)dt + \sigma S dW.$$

From Itô

$$dF = \alpha_F F dt + \sigma_F F dW,$$

where

$$\begin{aligned} \alpha_F &= \frac{1}{F} \left\{ \frac{\partial F}{\partial t} + \alpha S \frac{\partial F}{\partial s} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial s^2} \right\}, \\ \sigma_F &= \frac{1}{F} \cdot \sigma S \frac{\partial F}{\partial s}. \end{aligned}$$

Collecting terms gives us

$$\begin{aligned} dV &= V \cdot \{ u_S(\alpha + \delta) + u_F \alpha_F \} dt \\ &+ V \cdot \{ u_S \sigma + u_F \sigma_F \} dW, \end{aligned}$$

Define u_S and u_F by the system

$$\begin{aligned} u_S \sigma + u_F \sigma_F &= 0, \\ u_S + u_F &= 1. \end{aligned}$$

Solution

$$\begin{aligned} u_S &= \frac{\sigma_F}{\sigma_F - \sigma}, \\ u_F &= \frac{-\sigma}{\sigma_F - \sigma}, \end{aligned}$$

Value dynamics

$$dV = V \cdot \{u_S(\alpha + \delta) + u_F \alpha_F\} \, dt.$$

Absence of arbitrage implies

$$u_S(\alpha + \delta) + u_F \alpha_F = r,$$

We get

$$\frac{\partial F}{\partial t} + (r - \delta) S \frac{\partial F}{\partial s} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 F}{\partial s^2} - rF = 0.$$

Proposition:
The pricing function is given by

$$\left\{ \begin{aligned} \frac{\partial F}{\partial t} + (r - \delta) s \frac{\partial F}{\partial s} + \frac{1}{2} \sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF &= 0, \\ F(T, s) &= \Phi(s). \end{aligned} \right.$$

Risk Neutral Valuation

The pricing function has the representation

$$F(t,s) = e^{-r(T-t)} E^{Q}_{t,s}\left[\Phi(S_T)\right],$$

Q-dynamics of S:

$$dS_t = (r-\delta)S_t dt + \sigma(S_t)S_t dW_t.$$

Proposition: Under the martingale measure Q the **normalized gain process**

$$G^Z(t) = e^{-rt}S(t) + \int_0^t e^{-r\tau} dD(\tau)$$

is a Q-martingale.

Note: The result above holds in great generality.

Interpret!

Proposition: Under the martingale measure Q the **normalized gain process**

$$G^Z(t) = e^{-rt}S(t) + \int_0^t e^{-r\tau} dD(\tau)$$

is a Q-martingale.

Interpretation:
In a risk neutral world, today's stock price should be the expected value of all future discounted earnings which arise from holding the stock.

$$S(0) = E^Q\left[\int_0^t e^{-r\tau} dD(\tau) + e^{-rt}S(t)\right],$$

Pricing formula

Pricing formula for claims of the type

$$\mathcal{Z} = \Phi(S_T)$$

We are standing at time t, with dividend yield q. Today's stock price is s.

- Suppose that you have the pricing function

  $$F^0(t, s)$$

  for a non dividend stock.

- Denote the pricing function for the dividend paying stock by

  $$F^q(t, s)$$

- Then you have the relation

  $$F^q(t, s) = F^0\left(t, se^{-q(T-t)}\right)$$

Moral

Use your old formulas, but replace today's stock price s with $se^{-q(T-t)}$.

European Call on Dividend-Paying-Stock

$$F^q(t,s) = se^{-q(T-t)}N\left[d_1\right] - e^{-r(T-t)}XN\left[d_2\right].$$

$$\begin{aligned}
d_1 &= \\
&= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{se^{-q(T-t)}}{X}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\
&= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{X}\right) + \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)\right\} \\
d_2 &= d_1 - \sigma\sqrt{T-t}.
\end{aligned}$$

2. Options on Futures and Forwards

Forward Contract

$$\begin{aligned} T &= \text{Delivery date of the contract.} \\ X &= \text{Contingnent } T\text{-claim} \\ f(t,T) &= \text{Forward price, contracted at } t \\ & \text{for delivery at } T. \end{aligned}$$

Suppose you enter the (long) forward contract at t_0. Then you have the following cash flow.

- At time T you obtain the underlying claim which then is worth X.
- At T you pay $f(t_0, T)$.
- At t_0 your cash flow is ± 0

Thus:

- No payments apart from those at the delivery date T.
- The value of the contract at t_0 is zero.
- The value of the contract at some intermediate time can very well be nonzero.

Underlying dynamics

$$dS = \alpha S dt + \sigma S dW$$

Forward contract:

- At T you receive $X = \Phi(S_T)$
- At T you pay $f(t, T)$. (Forward price at t, for delivery at T).
- $f(t, T)$ is determined at t.

The forward price $f(t,T)$ is determined by the fact that the total contract, valued at t, has value **zero**.

Mathematically this means that

$$\Pi\left[t; X - f(t,T)\right] = 0$$

Thus

$$\Pi\left[t; X\right] = \Pi\left[t; f(t,T)\right]$$

Thus

$$e^{-r(T-t)} E^{Q}_{t,s}\left[X\right] = e^{-r(T-t)} f(t,T)$$

Forward price Formula

Proposition: The forward price is given by

$$f(t,T) = E^Q_{t,s}[X]$$

In particular, if

$$X = S_T$$

we have

$$f(t,T) = e^{r(T-t)} S_t$$

and

$$f(T,T) = S_T$$

Thus a forward (futures) option is equivalent to an option on the underlying. (If date of delivery equals date of maturity).

Forward Contracts

- Not traded standardized at an exchange.
- OTC product.
- Credit risk aspect important.

In order to handle the credit risk associated with forward contracts the market has invented **futures**.

Futures

- Traded at an exchange.

- Standardized product.

- Credit risk almost eliminated.

Basic idea:
Spread the cash flow over the entire contract period $[t_0, T]$, instead of having all payments at the date of delivery T.

Futures - Definition

$$\begin{aligned} T &= \text{Delivery date of the contract.} \\ X &= \text{contingent } T\text{-claim} \\ F(t,T) &= \text{Futures price, contracted at } t \\ &\quad\ \text{for delivery at } T. \end{aligned}$$

Suppose you enter the (long) futureas contract at t_0. Then you have the following cash flow.

- At time T you obtain the underlying claim which then is worth X.
- At T you pay $F(T,T)$.
- Over any intermediate interval $[s,t]$ you receive the net amount $F(t,T)-F(s,T)$ ("marking to market").
- At any intermediary time t the value of the futures contract is zero.

Thus:

The payment stream is spread out over $[t_0,T]$. The futures contract is an asset with zero spot price but with a dividend stream.

$F =$ Futures price process.

Definition:
A European futures call option, with strike price K and exercise date T will, if exercised at T, pay to the holder:

- The amount $F(T,T) - K$ in **cash**.
- A long postition in the underlying futures contract.

NB! The long position above can immediately be closed at no cost.

Institutional fact:
The maturity date of the futures option is typcally very close to the date of delivery of the underlying futures contract.

Why do Futures Options exist?

- On many markets (e.g. commodity markets) the futures market is much more liquid than the underlying market.

- Futures options are typically settled in **cash**. This relieves you from handling the underlying (tons of copper, hundreds of pigs, etc.).

- The market place for futures and futures options is often the same. This facilitates hedging etc.

Pricing Futures Options

Black-76

Recall basic facts:

$$
\begin{aligned}
S &= \text{underlying price} \\
F &= \text{futures price, with fixed delivery date } T
\end{aligned}
$$

Proposition: Assume a **deterministic** and constantshort rate r. Then it holds that

$$\text{futures price} = \text{forward price}$$

i.e.

$$f(t,T) = F(t,T) = E^{Q}_{t,s}[X]$$

Proof:

From risk neutral valuation we know that the normailzed gains process

$$dG^Z(t) = \frac{\Pi(t)}{B(t)} + \int_0^t \frac{1}{B(u)} dF(u,T)$$

is a Q-martingale.

From definition of futures contract:

$$\Pi(t) = 0$$

Thus

$$\frac{1}{B(t)} dF(t,T)$$

is a martingale increment, so F is a Q-martingale.

Thus

$$F(t,T) = E_{t,s}^Q\left[F(T,T)\right] = E_{t,s}^Q\left[X\right] = f(t,T)$$

Black-76

Assumption:
Future on underlying traded asset S. Futures price dynamics are given by

$$dF = \alpha F dt + \sigma D dW$$

Using $S_t = e^{-r(T-t)} F(t,T)$ this is equivalent to underlying dynamics of the form

$$dS = (\alpha - r) S dt + \sigma S dw$$

The contract to be priced is

$$\max\left[F(T,T) - X; 0\right] = \max\left[S_T - X; 0\right]$$

We are back in the Black-Scholes world!

Moral

When dealing with futures options we may proceed as follows.

- We can use the standard Black-Scholes formula in terms of S.
- To express the option price in terms of the futures price, use $S_t = e^{-r(T-t)} F(t,T)$ and simply substitute s by $e^{-r(T-t)} F(t,T)$.

European Call on Forward Contract

Assumption:
The exercise time of option equals the delivery time of the forward contract.

$$c = e^{-r(T-t)} F N\left[d_1\right] - e^{-r(T-t)} X N\left[d_2\right].$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{F}{X}\right) + \frac{1}{2}\sigma^2 (T-t)\right\}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

Put-Call Parity

- Standard Black-Scholes

$$p = c - S + e^{-r(T-t)}X$$

- Continuous dividend yield

$$p = c - Se^{-q(T-t)} + e^{-r(T-t)}X$$

- Futures

$$p = c - Fe^{-r(T-t)} + e^{-r(T-t)}X$$

3. American Options

- A **European** option can be exercised only at the date of maturity.

- An **American** Option can be exercised at **any day prior** to the day of expiration.

NB!
American optionas are traded also in Europe. European options are traded in America. In Sweden, stock options are American, whereas index options are European.

Important fact:
Holding an American option implies that you must find an optimal **exercise strategy**. Thus American option are much harder to analyse than European.

American Call Options

Happy fact:
American calls on an underlying **non dividend** paying stock are easy!

Pricing formula: It is never optimal to exercise an American call (without dividends) before the time of expiration. Thus the

Standard Black-Scholes model.

$$dS = \alpha S dt + \sigma S dW$$

$$\begin{aligned} c(t,s,X,T) &= \text{price of European call} \\ C(t,s,X,T) &= \text{price of American call} \end{aligned}$$

Obviously (why?) we have

$$C(t,s,X,T) \geq c(t,s,X,T).$$

Furthermore we have (why?)

$$c(t,s,X,T) \geq s - e^{-r(T-t)}X$$

If $r > 0$ we also have

$$s - e^{-r(T-t)}X > s - X$$

Thus

$$\begin{aligned} C(t,s,X,T) &> s - X \\ &= \text{value, at } t \text{, of exercising the call} \end{aligned}$$

- The American call is worth more than the value of exercising it.

- Thus it is never optimal to exercise the call before expiration time.

- Thus an American call is always exercised at the time of expiration.

- Thus the American price equals the European price.

NB!
This holds only for underlying stocks **without dividends**!

American Calls with dividends

This is a harder problem.

- Early exercise may be optimal.

- It can however be shown thatan American **call** on an underlying with discrete dividends is **always exercised at a dividend time**. This leads to "compound options".

- An American call on an underlying with continuous dividend yield can not be valued analytically. There are approximative formulas.

American Puts

This is a **very** hard problem.

- Early exercise may be optimal.
- No analytical formulas for pricing.
- No easy put-call parity.

No dividends:
Prices can be computed using binomial trees or "free boundary value problems". Tough work.

Dividends:
Very hard computational problems.