VIII

Interest Rate Theory 2

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Martingale Modelling

- All prices are determined by Q-dynamics of r.

- Model dr directly under Q!

Problem: Parameter estimation!

Pricing under risk adjusted probabilities

Q**-dynamics:**

$$dr = \mu(t,r)dt + \sigma(t,r)dW$$

$$\begin{aligned} \Pi[0;X] &= E^Q\left[e^{-\int_t^T r(s)ds} \times X\right] \\ p(0,T) &= E^Q\left[e^{-\int_t^T r(s)ds} \times 1\right] \end{aligned}$$

The case $\mathbf{X} = \Phi(r(T))$:

price given by

$$\Pi[t;X] = F(t, r(t))$$

$$\begin{cases} F_t + \mu F_r + \frac{1}{2}\sigma^2 F_{rr} - rF &= 0, \\ F(T,r) &= \Phi(r(T)). \end{cases}$$

1. Vasiček

$$dr = (b - ar)\,dt + \sigma dW,$$

2. Cox-Ingersoll-Ross

$$dr = (b - ar)\,dt + \sigma\sqrt{r}dW,$$

3. Dothan

$$dr = ardt + \sigma r dW,$$

4. Black-Derman-Toy

$$dr = \Phi(t)rdt + \sigma(t)rdW,$$

5. Ho-Lee

$$dr = \Phi(t)dt + \sigma dW,$$

6. Hull-White (extended Vasičec)

$$dr = \{\Phi(t) - ar\}\,dt + \sigma dW,$$

Bond Options

European call on a T-bond with strike price K and delivery date S.

$$
\begin{aligned}
X &= \max\left[p(S,T) - K,\ 0\right] \\
X &= \max\left[F^T(S, r(S)) - K,\ 0\right]
\end{aligned}
$$

$$
\begin{aligned}
F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T &= 0, \\
F^T(T,r) &= 1.
\end{aligned}
$$

$$
\Phi(r) = \max\left[F^T(S,r) - K,\ 0\right]
$$

$$
\begin{aligned}
F_t + \mu F_r + \frac{1}{2}\sigma^2 F_{rr} - rF &= 0, \\
F(S,r) &= \Phi(r(S)).
\end{aligned}
$$

$$
\Pi\left[t; X\right] = F(t, r(t))
$$

Affine Term Structures

Lots of equations!

Need analytic solutions.

We have an **Affine Term Structure** if

$$F(t, r; T) = e^{A(t,T) - B(t,T)r},$$

where A and B are deterministic functions.

Problem: How do we specify μ and σ in order to have an ATS?

Proposition: Assume that μ and σ are of the form

$$\begin{aligned} \mu(t,r) &= \alpha(t)r + \beta(t), \\ \sigma^2(t,r) &= \gamma(t)r + \delta(t). \end{aligned}$$

Then the model admits an affine term structure

$$F(t,r;T) = e^{A(t,T)-B(t,T)r},$$

where A and B satisfy the system

$$\begin{cases} B_t(t,T) = -\alpha(t)B(t,T) + \frac{1}{2}\gamma(t)B^2(t,T) - 1, \\ B(T;T) = 0. \end{cases}$$

$$\begin{cases} A_t(t,T) = \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T), \\ A(T;T) = 0. \end{cases}$$

Parameter Estimation

Suppose that we have chosen a specific model, e.g. H-W . How do we estimate the parameters $a,\ b,\ \sigma$?

Naive answer:
Use standard methods from statistical theory.

WRONG!!

- The parameters are Q-parameters.
- Our observations are **not** under Q, but under P.
- Standard statistical techniques can **not** be used.
- We need to know the market price of risk (λ).
- Who determines λ?
- **The Market!**
- We must get **price information from the market** in order to estimate parameters.

Inversion of the Yield Curve

Q-dynamics with parameter list α:

$$dr = \mu(t, r; \alpha)dt + \sigma(t, r; \alpha)dW$$

$$\Downarrow$$

Theoretical term structure

$$p(0, T; \alpha); \quad T \geq 0$$

Observed term structure

$$p^{\star}(0, T); \quad T \geq 0.$$

Requirement I:
A model such that the **theoretical** prices of today coincide with the **observed** prices of today. We want to choose tha parameter vector α such that

$$p(0, T; \alpha) \approx p^{\star}(0, T);\ \forall T \geq 0$$

Number of equations $= \infty$ (one for each T).
Number of unknowns = number of parameters.

Need:
Infinite parameter list.

The time dependent function Φ in Hull-White is precisely such an infinite parameter list (one parameter for every t).

Result: Hull-White can be calibrated exactly to any initial term strucutre. The calibrated model has the form

$$p(t,T) = \frac{p^\star(0,T)}{p^\star(0,t)} \times e^{C(t,r(t))}$$

where C is given by

$$B(t,T)f^\star(0,t) - \frac{\sigma^2}{2a^2}B^2(t,T)\left(1 - e^{-2aT}\right) - B(t,T)r(t)$$

There are analytical formulas for interest rate options.

Short rate models

Pro:

- Easy to model r.
- Analytical formulas for bond prices and bond options.

Con:

- Inverting the yield curve can be hard work.
- Hard to model a flexible volatility structure for forward rates.
- With a one factor model, all points on the yield curve are perfectly correlated.

Heath-Jarrow-Morton

Ide: Model the dynamics of the **entire yield curve**.

The yield curve itself (rather than the short rate r) is the explanatory variable.

Model forward rates. Use the observed yield curve as initial data.

Q-dynamics:

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW(t), \\ f(0,T) &= f^{\star}(0,T). \end{aligned}$$

One SDE for each maturity date T.

Forward rate dynamics:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

$$f(t,T) = \frac{\partial \log p(t,T)}{\partial T}$$

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}$$

Thus:

Specifying forward rates.

$$\Longleftrightarrow$$

Specifying bond prices.

Thus:

Absence of arbitrage on the bond market

$$\Downarrow$$

restrictions on α and σ.

Which?

Toolbox

Proposition:
If the forward rate dynamics (under any measure) are given by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW$$

Then the bond dynamics are given by

$$\begin{aligned} dp(t,T) &= p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2\right\}dt \\ &+ p(t,T)S(t,T)dW \end{aligned}$$

$$\begin{cases} A(t,T) &= -\int_t^T \alpha(t,s)ds, \\ S(t,T) &= -\int_t^T \sigma(t,s)ds \end{cases}$$

Main Theorem: (HJM:s drift condition)
Under the martingale measure Q, the following must hold

$$\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s) ds.$$

Moral: The volatility can be specified freely. The forward rate drift is then uniquely specified.

Forward rate models

Pro:

- Easy to model a flexible volatility structure for forward rates.

- Easy to include multiple factors.

Con:

- The short rate will generically not be a Markov process.

- Hard computational problems.