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Completeness and Hedging

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Problems around Standard Black-Scholes

• We assumed that the derivative was traded. How do we price OTC products?

• Why is the option price independent of the expected rate of return $\alpha$ of the underlying stock?

• Suppose that we have sold a call option. Then we face financial risk, so how do we hedge against that risk?

All this has to do with completeness.
Definition:
We say that a $T$-claim $X$ can be \textbf{replicated}, alternatively that it is \textbf{reachable} or \textbf{hedgeable}, if there exists a self financing portfolio $h$ such that

$$V^h(T) = X, \quad P - a.s.$$ 

In this case we say that $h$ is a \textbf{hedge} against $X$. Alternatively, $h$ is called a \textbf{replicating} or \textbf{hedging} portfolio. If every contingent claim is reachable we say that the market is \textbf{complete}.

Basic Idea: If $X$ can be replicated by a portfolio $h$ then the arbitrage free price for $X$ is given by

$$\Pi [t; X] = V^h(t).$$
Trading Strategy

Consider a replicable claim $X$ which we want to sell at $t = 0$..

- Compute the price $\Pi [0; X]$ and sell $X$ at a slightly (well) higher price.

- Buy the hedging portfolio and invest the surplus in the bank.

- Wait until expiration date $T$.

- The liabilities stemming from $X$ is exactly matched by $V^h(T)$, and we have our surplus in the bank.
Completeness of Black-Scholes

**Theorem:** The Black-Scholes model is complete.

**Proof.** Fix a claim \( X = \Phi(S(T')) \). We want to find processes \( V, u^0 \) and \( u^\star \) such that

\[
dV = V \left\{ u^0 \frac{dB}{B} + u^\star \frac{dS}{S} \right\}
\]

\[
V(T) = \Phi(S(T')).
\]

i.e.

\[
dV = V \left\{ u^0 r + u^\star \alpha \right\} dt + Vu^\star \sigma dW,
\]

\[
V(T') = \Phi(S(T')).
\]
Heuristics:
Let us assume that $X$ is replicated by $h = (u^0, u^*)$ with value process $V$.

Ansatz:

$$V(t) = F(t, S(t))$$

Ito gives us

$$dV = \left\{ F_t + \alpha SF_s + \frac{1}{2} \sigma^2 S^2 F_{ss} \right\} dt + \sigma SF_s dW,$$

Write this as

$$dV = V \left\{ \frac{F_t + \alpha SF_s + \frac{1}{2} \sigma^2 S^2 F_{ss}}{V} \right\} dt + V \frac{SF_s}{V} \sigma dW.$$

Compare with

$$dV = V \left\{ u^0 r + u^* \alpha \right\} dt + V u^* \sigma dW.$$
Define $u^*$ by

$$u^*(t) = \frac{S(t)F_s(t, S(t))}{F(t, S(t))},$$

This gives us the eqn

$$dV = V \left\{ \frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF} r + u^* \alpha \right\} dt + Vu^* \sigma dW.$$  

Compare with

$$dV = V \left\{ u^0 r + u^* \alpha \right\} dt + Vu^* \sigma dW$$

Natural choice for $u^0$ is given by

$$u^0 = \frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF},$$
The relation \( u^0 + u^* = 1 \) gives us the Black-Scholes PDE

\[
F_t + rSF_s + \frac{1}{2}\sigma^2 S^2 F_{ss} - rF = 0.
\]

The condition

\[
V(T) = \Phi(S(T))
\]

gives us the boundary condition

\[
F(T,s) = \Phi(s)
\]

**Moral:** The model is complete and we have explicit formulas for the replicating portfolio.
Main Result

**Theorem:** Define $F$ as the solution to the boundary value problem

\[
\begin{aligned}
F_t + rsF_s + \frac{1}{2}\sigma^2 s^2 F_{ss} - rF &= 0, \\
F(T, s) &= \Phi(s).
\end{aligned}
\]

Then $X$ can be replicated by the relative portfolio

\[
\begin{aligned}
&u^0(t) = \frac{F(t, S(t)) - S(t)F_s(t, S(t))}{F(t, S(t))}, \\
&u^*(t) = \frac{S(t)F_s(t, S(t))}{F(t, S(t))}.
\end{aligned}
\]

The corresponding absolute portfolio is given by

\[
\begin{aligned}
h^0(t) &= \frac{F(t, S(t)) - S(t)F_s(t, S(t))}{B(t)}, \\
h^*(t) &= F_s(t, S(t)),
\end{aligned}
\]

and the value process $V^h$ is given by

\[
V^h(t) = F(t, S(t)).
\]
Notes

• Completeness explains unique price - the claim is superfluous!

• Replicating the claim $P - a.s. \iff$ Replicating the claim $Q - a.s.$ for any $Q \sim P$. Thus the price only depends on the support of $P$.

• Thus (Girsanov) it will not depend on the drift $\alpha$ of the state equation.

• The completeness theorem is a nice theoretical result, but the replicating portfolio is continuously rebalanced. Thus we are facing very high transaction costs.
Completeness vs No Arbitrage

Question:
When is a model arbitrage free and/or complete?

Answer:
Count the number of risky assets, and the number of random sources.

\[ R = \text{number of random sources} \]
\[ N = \text{number of risky assets} \]

Intuition:
If \( N \) is large, compared to \( R \), you have lots of possibilities of forming clever portfolios. Thus lots of chances of making arbitrage profits. Also many chances of replicating a given claim.
Meta-Theorem

Generically, the following hold.

- The market is arbitrage free if and only if
  \[ N \leq R \]

- The market is complete if and only if
  \[ N \geq R \]

Example:
The Black-Scholes model. \( R=N=1 \). Arbitrage free and complete.
Parity Relations

Let $\Phi$ and $\Psi$ be contract functions for the $T$-claims $X = \Phi(S(T))$ and $Y = \Psi(S(T))$. Then for any real numbers $\alpha$ and $\beta$ we have the following price relation.

$$\Pi [t; \alpha \Phi + \beta \Psi] = \alpha \Pi [t; \Phi] + \beta \Pi [t; \Psi].$$

Proof. Linearity of mathematical expectation.

Consider the following “basic” contract functions.

$$\Phi_S(x) = x,$$
$$\Phi_B(x) \equiv 1,$$
$$\Phi_{C,K}(x) = \max [x - K, 0].$$

Prices:

$$\Pi [t; \Phi_S] = S(t),$$
$$\Pi [t; \Phi_B] = e^{-r(T-t)},$$
$$\Pi [t; \Phi_{C,K}] = c(t, S(t); K, T).$$
If we have

\[ \Phi = \alpha \Phi_S + \beta \Phi_B + \sum_{i=1}^{n} \gamma_i \Phi_{C,K_i} \]

then

\[ \Pi \left[ t; \Phi \right] = \alpha \Pi \left[ t; \Phi_S \right] + \beta \Pi \left[ t; \Phi_B \right] + \sum_{i=1}^{n} \gamma_i \Pi \left[ t; \Phi_{C,K_i} \right] \]

We may replicate the claim \( \Phi \) using a portfolio consisting of basic contracts that is constant over time, i.e. a **buy-and hold** portfolio:

- \( \alpha \) shares of the underlying stock,
- \( \beta \) zero coupon \( T \)-bonds with face value $1,
- \( \gamma_i \) European call options with strike price \( K_i \), all maturing at \( T \).
Put-Call Parity

Consider a European put contract

$$\Phi_{P,K}(s) = \max [K - s, 0]$$

It is easy to see (draw a figure) that

$$\Phi_{P,K}(x) = \Phi_{C,K}(x) - s + K$$

$$= \Phi_{P,K}(x) - \Phi_{S}(x) + \Phi_{B}(x)$$

We immediately get

**Put-call parity:**

$$p(t, s; K) = c(t, s; K) - s + Ke^{r(T-t)}$$

Thus you can construct a synthetic put option, using a buy-and-hold portfolio.
Delta Hedging

Consider a fixed claim

\[ X = \Phi(S_T) \]

with pricing function

\[ F(t, s). \]

Setup:
We are at time \( t \), and have a short (interpret!) position in the contract.

Goal:
Offset the risk in the derivative by buying (or selling) the (highly correlated) underlying.

Definition:
A position in the underlying is a **delta hedge** against the derivative if the portfolio (underlying + derivative) is immune against small changes in the underlying price.
Formal Analysis

\(-1 = \) number of units of the derivative product
\(x = \) number of units of the underlying
\(s = \) today’s stock price
\(t = \) today’s date

Value of the portfolio:

\[ V = -1 \cdot F(t, s) + x \cdot s \]

A delta hedge is characterized by the property that

\[ \frac{\partial V}{\partial s} = 0. \]

We obtain

\[ -\frac{\partial F}{\partial s} + x = 0 \]

Solve for \(x!\)
Result:
We should have
\[ \hat{x} = \frac{\partial F}{\partial s} \]
shares of the underlying in the delta hedged portfolio.

Definition:
For any contract, its “delta” is defined by
\[ \Delta = \frac{\partial F}{\partial s}. \]

Result:
We should have
\[ \hat{x} = \Delta \]
shares of the underlying in the delta hedged portfolio.

Warning:
The delta hedge must be rebalanced over time. (why?)
Black Scholes

For a European Call in the Black-Scholes model we have

\[ \Delta = N[d_1] \]

NB This is not a trivial result!

From put call parity it follows (how?) that \( \Delta \) for a European Put is given by

\[ \Delta = N[d_1] - 1 \]

Check signs and interpret!
Rebalanced Delta Hedge

• Sell one call option a time $t = 0$ at the B-S price $F$.

• Compute $\Delta$ and by $\Delta$ shares. (Use the income from the sale of the option, and borrow money if necessary.)

• Wait one day (week, minute, second..). The stock price has now changed.

• Compute the new value of $\Delta$, and borrow money in order to adjust your stock holdings.

• Repeat this procedure until $t = T$. Then the value of your portfolio $(B+S)$ will match the value of the option almost exactly.
• Lack of perfection comes from discrete, instead of continuous, trading.

• You have created a “synthetic” option. (Replicating portfolio).

**Formal result:**
The relative weights in the replicating portfolio are

\[ u_S = \frac{S \cdot \Delta}{F}, \]
\[ u_B = \frac{F - S \cdot \Delta}{F} \]
Portfolio Delta

Assume that you have a portfolio consisting of derivatives

\[ \Phi_i(S_{T_i}), \quad i = 1, \ldots, n \]

all written on the same underlying stock \( S \).

\[ F_i(t, s) = \text{pricing function for i:th derivative} \]
\[ \Delta_i = \frac{\partial F_i}{\partial s} \]
\[ h_i = \text{units of i:th derivative} \]

Portfolio value:

\[ \Pi = \sum_{i=1}^{n} h_i F_i \]

Portfolio delta:

\[ \Delta \Pi = \sum_{i=1}^{n} h_i \Delta_i \]
Gamma

A problem with discrete delta-hedging is.

- As time goes by $S$ will change.

- This will cause $\Delta = \frac{\partial F}{\partial s}$ to change.

- Thus you are sitting with the wrong value of delta.

Moral:

- If delta is sensitive to changes in $S$, then you have to rebalance often.

- If delta is insensitive to changes in $S$ you do not need to rebalance so often.
**Definition:**
Let Π be the value of a derivative (or portfolio). **Gamma** (Γ) is defined as

\[ \Gamma = \frac{\partial \Delta}{\partial s} \]

i.e.

\[ \Gamma = \frac{\partial^2 \Pi}{\partial s^2} \]

**Gamma** is a measure of the sensitivity of Δ to changes in S.

**Result:** For a European Call in a Black-Scholes model, Γ can be calculated as

\[ \Gamma = \frac{N'[d_1]}{S\sigma \sqrt{T - t}} \]

**Important fact:**
For a position in the underlying stock itself we have

\[ \Gamma = 0 \]
Gamma Neutrality

A portfolio $\Pi$ is said to be **gamma neutral** if its gamma equals zero, i.e.

$$\Gamma_\Pi = 0$$

- Since $\Gamma = 0$ for a stock you can not gamma-hedge using only stocks. Typically you use some derivative to obtain gamma neutrality.
General procedure

Given a portfolio $\Pi$ with underlying $S$. Consider two derivatives with pricing functions $F$ and $G$.

\[ x_F = \text{number of units of } F \]
\[ x_G = \text{number of units of } G \]

**Problem:**
Choose $x_F$ and $x_G$ such that the entire portfolio is delta- and gamma-neutral.

Value of hedged portfolio:

\[ V = \Pi + x_F \cdot F + x_G \cdot G \]
Value of hedged portfolio:

\[ V = \Pi + x_F \cdot F + x_G \cdot G \]

We get the equations

\[ \frac{\partial V}{\partial s} = 0, \]

\[ \frac{\partial^2 V}{\partial s^2} = 0. \]

i.e.

\[ \Delta_\Pi + x_F \Delta_F + x_G \Delta_G = 0, \]

\[ \Gamma_\Pi + x_F \Gamma_F + x_G \Gamma_G = 0 \]

Solve for \( x_F \) and \( x_G \)! 

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Particular Case

• In many cases the original portfolio Π is already delta neutral.

• Then it is natural to use a derivative to obtain gamma-neutrality.

• This will destroy the delta-neutrality.

• Therefore we use the underlying stock (with zero gamma!) to delta hedge in the end.
Formally:

\[ V = \Pi + x_F \cdot F + x_S \cdot S \]

\[ \Delta \Pi + x_F \Delta F + x_S \Delta S = 0, \]
\[ \Gamma \Pi + x_F \Gamma F + x_S \Gamma S = 0 \]

We have

\[ \Delta \Pi = 0, \]
\[ \Delta S = 1 \]
\[ \Gamma S = 0. \]

i.e.

\[ \Delta \Pi + x_F \Delta F + x_S = 0, \]
\[ \Gamma \Pi + x_F \Gamma F = 0 \]

\[ x_F = -\frac{\Gamma \Pi}{\Gamma F} \]
\[ x_S = \frac{\Delta F \Gamma \Pi}{\Gamma F} - \Delta \Pi \]
Further Greeks

\[ \Theta = \frac{\partial \Pi}{\partial t}, \]
\[ V = \frac{\partial \Pi}{\partial \sigma}, \]
\[ \rho = \frac{\partial \Pi}{\partial r}. \]

\( V \) is pronounced “Vega”.

NB!

- A delta hedge is a hedge against the movements in the underlying stock, given a fixed model.

- A Vega-hedge is not a hedge against movements of the underlying asset. It is a hedge against a change of the model itself.