IV

Completeness and Hedging

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Problems around Standard Black-Scholes

- We **assumed** that the derivative was traded. How do we price OTC products?

- Why is the option price independent of the expected rate of return α of the underlying stock?

- Suppose that we have sold a call option. Then we face financial risk, so how do we hedge against that risk?

All this has to do with **completeness**.

Definition:
We say that a T-claim X can be **replicated**, alternatively that it is **reachable** or **hedgeable**, if there exists a self financing portfolio h such that

$$V^h(T) = X, \quad P - a.s.$$

In this case we say that h is a **hedge** against X. Alternatively, h is called a **replicating** or **hedging** portfolio. If every contingent claim is reachable we say that the market is **complete**

Basic Idea: If X can be replicated by a portfolio h then the arbitrage free price for X is given by

$$\Pi[t; X] = V^h(t).$$

Trading Strategy

Consider a replicable claim X which we want to sell at $t = 0$..

- Compute the price $\Pi[0; X]$ and sell X at a slightly (well) higher price.

- Buy the hedging portfolio and invest the surplus in the bank.

- Wait until expiration date T.

- The liabilities stemming from X is exactly matched by $V^h(T)$, and we have our surplus in the bank.

Completeness of Black-Scholes

Theorem: The Black-Scholes model is complete.

Proof. Fix a claim $X = \Phi(S(T))$. We want to find processes V, u^0 and $u^\star$ such that

$$dV = V\left\{u^0 \frac{dB}{B} + u^\star \frac{dS}{S}\right\}$$

$$V(T) = \Phi(S(T)).$$

i.e.

$$dV = V\left\{u^0 r + u^\star \alpha\right\} dt + V u^\star \sigma dW,$$

$$V(T) = \Phi(S(T)).$$

Heuristics:
Let us **assume** that X is replicated by $h = (u^0, u^\star)$ with value process V.

Ansatz:

$$V(t) = F(t, S(t))$$

Ito gives us

$$dV = \left\{ F_t + \alpha S F_s + \frac{1}{2}\sigma^2 S^2 F_{ss} \right\} dt + \sigma S F_s dW,$$

Write this as

$$dV = V \left\{ \frac{F_t + \alpha S F_s + \frac{1}{2}\sigma^2 S^2 F_{ss}}{V} \right\} dt + V \frac{S F_s}{V} \sigma dW.$$

Compare with

$$dV = V \left\{ u^0 r + u^\star \alpha \right\} dt + V u^\star \sigma dW$$

Define $u^\star$ by

$$u^\star(t) = \frac{S(t)F_s(t,S(t))}{F(t,S(t))},$$

This gives us the eqn

$$dV = V\left\{\frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF} r + u^\star \alpha\right\} dt + V u^\star \sigma dW.$$

Compare with

$$dV = V\left\{u^0 r + u^\star \alpha\right\} dt + V u^\star \sigma dW$$

Natural choice for u^0 is given by

$$u^0 = \frac{F_t + \frac{1}{2}\sigma^2 S^2 F_{ss}}{rF},$$

The relation $u^0 + u^\star = 1$ gives us the Black-Scholes PDE

$$F_t + rSF_s + \frac{1}{2}\sigma^2 S^2 F_{ss} - rF = 0.$$

The condition

$$V(T) = \Phi\left(S(T)\right)$$

gives us the boundary condition

$$F(T, s) = \Phi(s)$$

Moral: The model is complete and we have explicit formulas for the replicating portfolio.

Main Result

Theorem: Define F as the solution to the boundary value problem

$$\begin{cases} F_t + rsF_s + \frac{1}{2}\sigma^2 s^2 F_{ss} - rF &= 0, \\ F(T,s) &= \Phi(s). \end{cases}$$

Then X can be replicated by the relative portfolio

$$\begin{aligned} u^0(t) &= \frac{F(t,S(t)) - S(t)F_s(t,S(t))}{F(t,S(t))}, \\ u^\star(t) &= \frac{S(t)F_s(t,S(t))}{F(t,S(t))}. \end{aligned}$$

The corresponding absolute portfolio is given by

$$\begin{aligned} h^0(t) &= \frac{F(t,S(t)) - S(t)F_s(t,S(t))}{B(t)}, \\ h^\star(t) &= F_s(t,S(t)), \end{aligned}$$

and the value process V^h is given by

$$V^h(t) = F(t,S(t)).$$

Notes

- Completeness explains unique price - the claim is superfluous!

- Replicating the claim $P-a.s. \Longleftrightarrow$ Replicating the claim $Q-a.s.$ for any $Q \sim P$. Thus the price only depends on the support of P.

- Thus (Girsanov) it will not depend on the drift α of the state equation.

- The completeness theorem is a nice theoretical result, but the replicating portfolio is **continuously rebalanced**. Thus we are facing very high transaction costs.

Completeness vs No Arbitrage

Question:
When is a model arbitrage free and/or complete?

Answer:
Count the number of risky assets, and the number of random sources.

$$
\begin{aligned}
R &= \text{number of random sources} \\
N &= \text{number of risky assets}
\end{aligned}
$$

Intuition:
If N is large, compared to R, you have lots of possibilities of forming clever portfolios. Thus lots of chances of making arbitrage profits. Also many chances of replicating a given claim.

Meta-Theorem

Generically, the following hold.

- The market is arbitrage free if and only if

$$N \leq R$$

- The market is complete if and only if

$$N \geq R$$

Example:
The Black-Scholes model. R=N=1. Arbitrage free and complete.

Parity Relations

Let Φ and Ψ be contract functions for the T-claims $\mathcal{X} = \Phi(S(T))$ and $Y = \Psi(S(T))$. Then for any real numbers α and β we have the following price relation.

$$\Pi\left[t; \alpha\Phi + \beta\Psi\right] = \alpha\Pi\left[t; \Phi\right] + \beta\Pi\left[t; \Psi\right].$$

Proof. Linearity of mathematical expectation.

Consider the following "basic" contract functions.

$$\begin{aligned} \Phi_S(x) &= x, \\ \Phi_B(x) &\equiv 1, \\ \Phi_{C,K}(x) &= \max\left[x - K, 0\right]. \end{aligned}$$

Prices:

$$\begin{aligned} \Pi\left[t; \Phi_S\right] &= S(t), \\ \Pi\left[t; \Phi_B\right] &= e^{-r(T-t)}, \\ \Pi\left[t; \Phi_{C,K}\right] &= c(t, S(t); K, T). \end{aligned}$$

If we have

$$\Phi = \alpha\Phi_S + \beta\Phi_B + \sum_{i=1}^{n} \gamma_i \Phi_{C,K_i},$$

then

$$\Pi\left[t;\Phi\right] = \alpha\Pi\left[t;\Phi_S\right] + \beta\Pi\left[t;\Phi_B\right] + \sum_{i=1}^{n} \gamma_i \Pi\left[t;\Phi_{C,K_i}\right]$$

We may replicate the claim Φ using a portfolio consisting of basic contracts that is **constant** over time, i.e. a **buy-and hold** portfolio:

- α shares of the underlying stock,
- β zero coupon T-bonds with face value \$1,
- γ_i European call options with strike price K_i, all maturing at T.

Put-Call Parity

Consider a European put contract

$$\Phi_{P,K}(s) = \max\,[K - s, 0]$$

It is easy to see (draw a figure) that

$$\begin{aligned} \Phi_{P,K}(x) &= \Phi_{C,K}(x) - s + K \\ &= \Phi_{P,K}(x) - \Phi_S(x) + \Phi_B(x) \end{aligned}$$

We immediately get

Put-call parity:

$$p(t, s; K) = c(t, s; K) - s + Ke^{r(T-t)}$$

Thus you can construct a synthetic put option, using a buy-and-hold portfolio.

Delta Hedging

Consider a fixed claim

$$X = \Phi(S_T)$$

with pricing function

$$F(t, s).$$

Setup:
We are at time t, and have a short (interpret!) position in the contract.

Goal:
Offset the risk in the derivative by buying (or selling) the (highly correlated) underlying.

Definition:
A position in the underlying is a **delta hedge** against the derivative if the portfolio (underlying + derivative) is immune against small changes in the underlying price.

Formal Analysis

$$
\begin{aligned}
-1 &= \text{number of units of the derivative product} \\
x &= \text{number of units of the underlying} \\
s &= \text{today's stock price} \\
t &= \text{today's date}
\end{aligned}
$$

Value of the portfolio:

$$V = -1 \cdot F(t,s) + x \cdot s$$

A delta hedge is characterized by the property that

$$\frac{\partial V}{\partial s} = 0.$$

We obtain

$$-\frac{\partial F}{\partial s} + x = 0$$

Solve for x!

Result:

We should have

$$\widehat{x} = \frac{\partial F}{\partial s}$$

shares of the underlying in the delta hedged portfolio.

Definition:

For any contract, its “delta” is defined by

$$\Delta = \frac{\partial F}{\partial s}.$$

Result:

We should have

$$\widehat{x} = \Delta$$

shares of the underlying in the delta hedged portfolio.

Warning:

The delta hedge must be rebalanced over time. (why?)

Black Scholes

For a European Call in the Black-Scholes model we have

$$\Delta = N[d_1]$$

NB This is **not** a trivial result!

From put call parity it follows (how?) that Δ for a European Put is given by

$$\Delta = N[d_1] - 1$$

Check signs and interpret!

Rebalanced Delta Hedge

- Sell one call option a time $t = 0$ at the B-S price F.

- Compute Δ and by Δ shares. (Use the income from the sale of the option, and borrow money if necessary.)

- Wait one day (week, minute, second..). The stock price has now changed.

- Compute the new value of Δ, and borrow money in order to adjust your stock holdings.

- Repeat this procedure until $t = T$. Then the value of your portfolio (B+S) will match the value of the option almost exactly.

- Lack of perfection comes from discrete, instead of continuous, trading.

- You have created a "synthetic" option. (Replicating portfolio).

Formal result:

The relative weights in the replicating portfolio are

$$u_S = \frac{S \cdot \Delta}{F},$$

$$u_B = \frac{F - S \cdot \Delta}{F}$$

Portfolio Delta

Assume that you have a portfolio consisting of derivatives

$$\Phi_i(S_{T_i}), \quad i = 1, \cdots, n$$

all **written on the same underlying** stock S.

$$\begin{aligned} F_i(t,s) &= \text{pricing function for i:th derivative} \\ \Delta_i &= \frac{\partial F_i}{\partial s} \\ h_i &= \text{units of i:th derivative} \end{aligned}$$

Portfolio value:

$$\Pi = \sum_{i=1}^{n} h_i F_i$$

Portfolio delta:

$$\Delta_\Pi = \sum_{i=1}^{n} h_i \Delta_i$$

Gamma

A problem with discrete delta-hedging is.

- As time goes by S will change.
- This will cause $\Delta = \frac{\partial F}{\partial s}$ to change.
- Thus you are sitting with the wrong value of delta.

Moral:

- If delta is sensitive to changes in S, then you have to rebalance often.
- If delta is insensitive to changes in S you do not need to rebalance so often.

Definition:
Let Π be the value of a derivative (or portfolio). **Gamma** (Γ) is defined as

$$\Gamma = \frac{\partial \Delta}{\partial s}$$

i.e.

$$\Gamma = \frac{\partial^2 \Pi}{\partial s^2}$$

Gamma is a measure of the sensitivity of Δ to changes in S.

Result: For a European Call in a Black-Scholes model, Γ can be calculated as

$$\Gamma = \frac{N'[d_1]}{S\sigma\sqrt{T-t}}$$

Important fact:
For a position in the underlying stock itself we have

$$\Gamma = 0$$

Gamma Neutrality

A portfolio Π is said to be **gamma neutral** if its gamma equals zero, i.e.

$$\Gamma_{\Pi} = 0$$

- Since $\Gamma = 0$ for a stock you can not gamma-hedge using only stocks. item Typically you use some derivative to obtain gamma neutrality.

General procedure

Given a portfolio Π with underlying S. Consider two derivatives with pricing functions F and G.

$$\begin{aligned} x_F &= \text{number of units of } F \\ x_G &= \text{number of units of } G \end{aligned}$$

Problem:
Choose x_F and x_G such that the entire portfolio is delta- and gamma-neutral.

Value of hedged portfolio:

$$V = \Pi + x_F \cdot F + x_G \cdot G$$

Value of hedged portfolio:

$$V = \Pi + x_F \cdot F + x_G \cdot G$$

We get the equations

$$\begin{aligned} \frac{\partial V}{\partial s} &= 0, \\ \frac{\partial^2 V}{\partial s^2} &= 0. \end{aligned}$$

i.e.

$$\begin{aligned} \Delta_\Pi + x_F \Delta_F + x_G \Delta_G &= 0, \\ \Gamma_\Pi + x_F \Gamma_F + x_G \Gamma_G &= 0 \end{aligned}$$

Solve for x_F and x_G!

Particular Case

- In many cases the original portfolio Π is already delta neutral.
- Then it is natural to use a derivative to obtain gamma-neutrality.
- This will destroy the delta-neutrality.
- Therefore we use the underlying stock (with zero gamma!) to delta hedge in the end.

Formally:

$$V = \Pi + x_F \cdot F + x_S \cdot S$$

$$\begin{aligned} \Delta_\Pi + x_F \Delta_F + x_S \Delta_S &= 0, \\ \Gamma_\Pi + x_F \Gamma_F + x_S \Gamma_S &= 0 \end{aligned}$$

We have

$$\begin{aligned} \Delta_\Pi &= 0, \\ \Delta_S &= 1 \\ \Gamma_S &= 0. \end{aligned}$$

i.e.

$$\begin{aligned} \Delta_\Pi + x_F \Delta_F + x_S &= 0, \\ \Gamma_\Pi + x_F \Gamma_F &= 0 \end{aligned}$$

$$\begin{aligned} x_F &= -\frac{\Gamma_\Pi}{\Gamma_F} \\ x_S &= \frac{\Delta_F \Gamma_\Pi}{\Gamma_F} - \Delta_\Pi \end{aligned}$$

Further Greeks

$$\Theta = \frac{\partial \Pi}{\partial t},$$
$$V = \frac{\partial \Pi}{\partial \sigma},$$
$$\rho = \frac{\partial \Pi}{\partial r}$$

V is pronounced "Vega".

NB!

- A delta hedge is a hedge against the movements in the underlying stock, given a **fixed model**.

- A Vega-hedge is not a hedge against movements of the underlying asset. It is a hedge against a **change of the model itself**.