III

Black-Scholes

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Typical Setup

Take as given the market price process, $S(t)$, of some underlying asset.

$S(t) =$ price, at t, per unit of underlying asset

Consider a fixed **financial derivative**, e.g. a European call option.

Main Problem: Find the arbitrage free price of the derivative.

European Call Option

The holder of this paper has the right

to buy

1 IBM

on the date

June 30, 2004

at the price

$90

Financial Derivative

- A financial asset which is defined **in terms of** some **underlying** asset.

- Future stochastic claim.

Main problems

- What is a “reasonable” price for a derivative?

- How do you hedge yourself against a derivative.

Philosophy

- The derivative is **defined in terms of** underlying.

- The derivative can be **priced in terms of** underlying price.

- **Consistent** pricing.

- **Relative** pricing.

Portfolios

Consider a market with N assets.

$$S_i(t) = \text{price at } t\text{, of asset No } i.$$

Portfolio: Consider a portfolio strategy

$$h(t) = [h^1(t), \cdots, h^N(t)]$$

$$\begin{aligned} h^i(t) &= \text{number of units of asset } i\text{, in the portfolio a} \\ V(t) &= \text{market value of the portfolio } h \text{ at } t \end{aligned}$$

$$V(t) = \sum_{i=1}^{N} h^i(t) S_i(t)$$

Typically: The portfolio is of the form

$$h(t) = h(t, S_t)$$

i.e. today's portfolio is based on today's prices.

Self financing portfolios

We want to study self financing portfolio straegies, i.e. portfolios where purchase of a “new” asset must be financed through sale of an “old” asset.

How is this formalized?

Definition:
The strategy h is **self financing** if

$$dV(t) = \sum_{i=1}^{N} h^i(t) dS_i(t)$$

Interpret!

Relative weights

$u^i(t) =$ the relative share of the portfolio value, which is invested in asset No i.

$$u^i(t) = \frac{h^i(t)S_i(t)}{V(t)}$$

$$dV(t) = \sum_{i=1}^{N} h^i(t)dS_i(t)$$

Substitute!

$$dV = V \sum_{i=1}^{N} u^i \frac{dS_i}{S_i}$$

Back to Financial Derivatives

Consider e.g. the Black-Scholes model

$$\begin{aligned} dS &= \alpha S dt + \sigma S dW, \\ dB &= rB dt. \end{aligned}$$

We want to price a European call with strike price K and exercise time T. This is a stochastic claim on the future. The future pay-out (at T) is a stochastic variable, X, given by

$$X = \max[S_T - K, 0]$$

More general:

$$X = \Phi(S_T)$$

Main problem: What is a ‘reasonable” price, $\Pi[t; X]$, for X at t?

Main Idea

- We demand **consistent** pricing between derivative and underlying.

- No **mispricing** between derivative and underlying.

- No **arbitrage possibilities** on the market (B, S, Π)

Arbitrage

The portfolio h is an **arbitrage** portfolio if

- The portfolio strategy is self financing.
- $V(0) = 0$.
- $V(T) > 0$ with probability one.

Moral:

- **Arbitrage = Free Lunch**
- **Arbitrage = Serious Mispricing**
- **No arbitrage possibilities in an efficient market.**

Arbitrage test

Suppose that h is self financing portfolio whith portfolio dynamics

$$dV(t) = kV(t)dt$$

- No driving Wiener process
- Detreministic rate of return.
- "Synthetic bank" with rate of return k.

If the market is free of arbitrage we must have:

$$k = r$$

Main Ideas

- Since the derivative is defined in terms of the underlying, the derivative price should be highly correlated with the underlying price .

- We should be able to balance dervative against underlying in our portfolio, so as to cancel the randomness.

- Thus we will obtain a riskless rate of return k on our portfolio.

- Absence of arbitrage must imply

$$k = r$$

Formalized program

- Assume that the derivative price is of the form
$$\Pi\,[t; X] = F(t, S_t).$$

- Form a portfolio based on underlying S an derivative F, with portfolio dynamics
$$dV = V\left\{u^S \cdot \frac{dS}{S} + u^F \cdot \frac{dF}{F}\right\}$$

- Choose u^S an du^F such that the dW-term is wiped out. This gives us
$$dV = V \cdot kdt$$

- Absence of arbitrage implies
$$k = r$$

- This relation will say something about F.

Black-Scholes Analysis

Assumptions:

- The stock price is Geometric Brownian Motion
- Continuous trading.
- Frictionless efficient market.
- Short positions are allowed. Unlimited credit.
- Constant volatility σ.
- Constant short rate of interest r. Flat yield curve.

Back to Black-Scholes:

We assumed

$$\Pi\left[t;X\right]=F(t,S_t)$$

$$dS=\alpha S dt+\sigma S dW$$

Itô's formula gives us the portfolio dynamics

$$\begin{aligned} dF &= \left\{\frac{\partial F}{\partial t}+\alpha S\frac{\partial F}{\partial s}+\frac{1}{2}S^2\sigma^2\frac{\partial^2 F}{\partial s^2}\right\}dt \\ &+ \sigma S\frac{\partial F}{\partial s}dW \end{aligned}$$

Write this as

$$dF=\alpha_F\cdot F dt+\sigma_F\cdot F dW$$

where

$$\begin{aligned} \alpha_F &= \frac{\frac{\partial F}{\partial t}+\alpha S\frac{\partial F}{\partial s}+\frac{1}{2}S^2\sigma^2\frac{\partial^2 F}{\partial s^2}}{F} \\ \sigma_F &= \frac{\sigma S\frac{\partial F}{\partial s}}{F} \end{aligned}$$

Portfolio dynamics:

$$\begin{aligned} dV &= V\left\{u^S \cdot \frac{dS}{S} + u^F \cdot \frac{dF}{F}\right\} \\ &= V\left\{u^S(\alpha dt + \sigma dW) + u^F(\alpha_F dt + \sigma_F dW)\right\} \end{aligned}$$

$$dV = V\left\{u^S\alpha + u^F\alpha_F\right\}dt + V\left\{u^S\sigma + u^F\sigma_F\right\}dW$$

Now we kill the dW-term!

Choose (u^S, u^F) such that

$$\begin{aligned} u^S\sigma + u^F\sigma_F &= 0 \\ u^S + u^F &= 1 \end{aligned}$$

Linear system with solution

$$\begin{aligned} u^S &= \frac{\sigma_F}{\sigma_F - \sigma} \\ u^F &= \frac{-\sigma}{\sigma_F - \sigma} \end{aligned}$$

Plug into dV!

We obtain

$$dV = V\left\{u^S\alpha + u^F\alpha_F\right\}dt$$

"Synthetic bank" with short rate

$$\left\{u^S\alpha + u^F\alpha_F\right\}$$

.

Absence of arbitrage implies

$$\left\{u^S\alpha + u^F\alpha_F\right\} = r$$

Plug in the expressions for u^S, u^F, α_F and simplify!

Black-Schole's PDE

$$\Pi\left[t; X\right] = F\left(t, S_t\right)$$

$$\begin{cases} \frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF & = & 0, \\ F(T, s) & = & \Phi(s). \end{cases}$$

Data needed

- The contract function Φ.
- Today's date t.
- Today's stock price S.
- Short rate r.
- Volatility σ.

Note: The pricing formula does **not** involve the mean rate of return α!

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“Risk neutral valuation”

Appplying Feynman-Kač to the Black-Scholes PDE we obtain

$$\Pi\left[t;X\right]=e^{-r(T-t)}E_{t,s}^{Q}\left[X\right]$$

Q-**dynamics:**

$$\begin{cases} dS = rSdt + \sigma SdW, \\ dB = rBdt. \end{cases}$$

- Price = Expected discounted value of future payments.
- The expectation shall **not** be taken under the “objective” probability measure P, but under the “risk adjusted” measure (“martingale measure”) Q.

Note: $P \sim Q$

Concrete formulas for numerical treatment

$$\Pi\,[0;\Phi] = e^{-rT}\int_{-\infty}^{\infty}\Phi(se^{z})f(z)dz$$

$$f(z) = \frac{1}{\sqrt{2\pi T}}\exp\left\{-\frac{\left[z-(r-\frac{1}{2}\sigma^2)T\right]^2}{2\sigma^2 T}\right\}$$

Black-Schole's formula

European Call

T=date of expiration,
t=today's date, K=strike price,
r=short rate,
s=today's stock price,
σ=volatility.

$$F(t, s) = sN[d_1] - e^{-r(T-t)} KN[d_2].$$

$N[\cdot]$=cdf for $N(0,1)$-distribution.

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left\{ \ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t) \right\},$$

$$d_2 = d_1 - \sigma\sqrt{T-t}.$$

Interpretation of the risk adjusted measure

- **Assume** a risk neutral world.

- Then the following must hold

$$s = S_0 = e^{-rt} E\left[S_t\right]$$

- In our model this means that

$$dS = rSdt + \sigma SdW$$

- The risk adjusted probabilities can be intrepreted as probabilities in a fictuous risk neutral economy.

Moral

- When we compute prices, we can compute **as if** we live in a risk neutral world.

- This does **not** mean that we live (or think that we live) in a risk neutral world.

- The formulas above hold regardless of the investor's attitude to risk, as long as he/she prefers more to less.

- The valuation formulas are therefore called "preference free valuation formulas".

Properties of Q

- $P \sim Q$

- For the price pricess π of any traded asset, derivative or underlying, the process

$$Z_\pi(t) = \frac{\pi(t)}{B(t)}$$

is a Q-martingale.

- Under Q, the price pricess π of any traded asset, derivative or underlying, has r as its local rate of return:

$$d\pi_t = r\pi_t dt + \sigma_\pi \pi_t dW_t$$

- The volatility of π is the same under Q as under P.

Martingale Measures

Consider a market, under an objective probability measure P, with underlying assets

$$B, S_1, \ldots, S_N$$

Definition: A probability measure Q is called a **martingale measure** if

- $P \sim Q$
- For every i, the process
$$Z_i(t) = \frac{S_i(t)}{B(t)}$$
is a Q-martingale.

Theorem: The market is arbitrage free **iff** there exists a martingale measure.

Estimating Volatility

- Historical estimation.
- Implied volatility.