Continuous Time Finance

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II

Stochastic Calculus

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Typical Setup

Take as given the market price process, $S(t)$, of some underlying asset.

$S(t) =$ price, at t, per unit of underlying asset

Consider a fixed **financial derivative**, e.g. a European call option.

Main Problem: Find the arbitrage free price of the derivative.

We Need:

1. Mathematical model for the underlying price process. (The Black-Scholes model)

2. Mathematical techniques to handle the price dynamics. (The Itô calculus.)

Stochastic Processes

- We model the stock price $S(t)$ as a **stochastic process**, i.e. it **evolves randomly over time**.

- We model S as a **Markov process**, i.e. in order to predict the future only the present value is of interest. All past information is already incorporated into today's stock prices. (Market efficiency).

Stochastic variable
Choosing a **number** at random

Stochastic process
choosing a **curve** (trajectory) at random.

Notation

$$\begin{aligned} X(t) &= \text{any random process}, \\ dt &= \text{small time step}, \\ dX(t) &= X(t+dt) - X(t) \end{aligned}$$

- dX is called the **increment** of X over the interval $[t, t+dt]$.

- For any fixed interval $[t, t+dt]$, the increment dX is a stochastic variable.

- If the increments $dX(s)$ and $dX(t)$, over the disjoint intervals $[s, s+ds]$ and $[t, t+dt]$ are independent, then we say that X has **independent increments**.

- If every increment has a normal distribution we say that X is a **normal**, or **Gaussian** process.

The Wiener Process

A stochastic process W is called a **Wiener process** if it has the following properties

- The increments are normally distributed: For $s < t$:

$$W(t) - W(s) \sim N[0, \sqrt{t-s}]$$

$$E[W(t) - W(s)] = 0, \quad Var[W(t) - W(s)] = t - s$$

- W has independent increments.

- $W(0) = 0$

- W has continuous trajectories.

Continuous random walk

Important Fact

Theorem:
A Wiener trajectory is, with probability one, **nowhere differentiable**.

Proof. Hard.

Wiener Process with Drift

A stochastic process X is called a Wiener process with **drift** μ and **diffusion coefficient** σ if it has the following dynamics

$$\begin{aligned} dX &= \mu dt + \sigma dW, \\ X(0) &= x_0 \end{aligned}$$

where x_0, μ and σ are constants.

Summing all increments over the interval $[0, t]$ gives us

$$X(t) - x_0 = \mu t + \sigma W(t)$$

$$X(t) = x_0 + \mu t + \sigma W(t)$$

The distribution of X is thus given by

$$X(t) \sim N[x_0 + \mu t,\ \sigma\sqrt{t}]$$

Stochastic Differential Equations

Take as given two functions $\mu(t,x)$ and $\sigma(t,x)$. We say that the process X is an **diffusion** if it has the local dynamics

$$
\begin{aligned}
dX &= \mu(t, X_t)dt + \sigma(t, X_t)dW, \\
X_0 &= x_0
\end{aligned}
$$

Interpretation:
Over the time interval $[t, t+dt]$, the X-process is driven by two separate terms.

- A locally deterministic velocity $\mu(t, X(t))$.
- An independent Gaussian disturbance term, amplified by the factor $\sigma(t, X(t))$.

How do we make this precise?

Possible Intrepretations

$$dX = \mu(t, X_t)dt + \sigma(t, X_t)dW,$$

(I) "Divide" formally by dt. Then we obtain the stochastic ODE

$$\frac{dX_t}{dt} = \mu\left(t, X_t\right) + \sigma\left(t, X_t\right) v_t$$

where

$$v_t = \frac{dW}{dt}$$

is the formal time derivative of the Wiener process W.

This is impossible, since $\frac{dW}{dt}$ does not exist.

(II) Write the equation on integrated form as

$$X_t = x_0 + \int_0^t \mu\left(s, X_s\right) ds + \int_0^t \sigma\left(s, X_s\right) dW_s$$

How is this interpreted?

$$X_t = x_0 + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s$$

Two terms:

- $$\int_0^t \mu(s, X_s)\, ds$$

  Riemann integral for each X-trajectory.

- $$\int_0^t \sigma(s, X_s)\, dW_s$$

  Stochastic integral. This can **not** be interpreted as a Stieljes integral for each trajectory. We need a new theory for this **Itô integral**.

Information

Let the Wiener process W be given.

Def:

$$\mathcal{F}_t^W = \text{"The information generated by } W \text{ over the interval } [0, t]\text{"}$$

Def: Let Z be a stochastic variable. If the value of Z is completely determined by $\mathcal{F}_t^W$, we write

$$Z \in \mathcal{F}_t^W$$

Ex:
For the stochastic variable Z, defined by

$$Z = \int_0^5 W(s)ds,$$

we have $Z \in \mathcal{F}_5^W$.

We do **not** have $Z \in \mathcal{F}_4^W$.

Adapted Processes

Let W be a Wiener process.

Definition:
A process X is **adapted** to the filtration $\left\{\mathcal{F}_t^W : \ t \geq 0\right\}$ if

$$X_t \in \mathcal{F}_t^W, \quad \forall t \geq 0$$

"**An adapted process does not look into the future**"

Adapted processes are nice integrands for stochastic integrals.

- The process

$$X_t = \int_0^t W_s ds,$$

is adapted.

- The process

$$X_t = \sup_{s \leq t} W_s$$

is adapted.

- The process

$$X_t = \sup_{s \leq t+1} W_s$$

is **not** adapted.

The Itô Integral

We will define the Itô integral

$$\int_a^b g(s)dW(s)$$

for processes $g \in \pounds^2$, i.e.

- The process g is adapted.
- The process g satisfies
$$\int_a^b E\left[g^2(s)\right] ds < \infty$$

This will be done in two steps.

I: Simple Integrands

Definition:
The process g is **simple**, if

- $g \in \pounds^2$

- There exists deterministic points $t_0 \ldots, t_n$ with $a = t_0 < t_1 < \ldots < t_n = b$ such that g is piecewise constant, i.e.

$$g(s) = g(t_k), \quad s \in [t_k, t_{k+1})$$

For simple g we define

$$\int_a^b g(s)dW(s) = \sum_{k=0}^{n-1} g(t_k)\left[W(t_{k+1}) - W(t_k)\right]$$

FORWARD INCREMENTS!

II: General Case

For a general $g \in \mathcal{L}^2$ we do as follows.

1. Approximate g with a sequence of simple g_n such that
$$\int_a^b E\left[\{g_n(s) - g(s)\}^2\right] ds \to 0.$$

2. For each n the integral
$$\int_a^b g_n(s) dW(s)$$
is a well defined stochastic variable Z_n.

3. One can show that the Z_n sequence converges to a limiting stochastic variable.

4. We define $\int_a^b g dW$ by
$$\int_a^b g(s) dW(s) = \lim_{n \to \infty} \int_a^b g_n(s) dW(s).$$

Properties of the Integral

Theorem:
The following relations hold

- $$E\left[\int_a^b g(s)dW(s)\right] = 0$$

- $$E\left[\left(\int_a^b g(s)dW(s)\right)^2\right] = \int_a^b E\left[g^2(s)\right] ds$$

- $$\int_a^b g(s)dW(s) \in \mathcal{F}_b^W$$

Martingales

Definition: An adapted process is a **martingale** if

$$E\left[X_t \middle| \mathcal{F}_s\right] = X_s, \quad \forall s \leq t$$

"A martingale is a process without drift"

Proposition: For $g \in \mathcal{L}^2$, the process

$$X_t = \int_0^t g_s dW_s$$

is a martingale.

Proposition: If X has dynamics

$$dX_t = \mu_t dt + \sigma_t dW_t$$

then X is amartingale **iff** $\mu = 0$.

Stochastic Calculus

General Model:

$$dX_t = \mu_t dt + \sigma_t dW_t$$

Let the function $f(t, x)$ be given, and define the stochastic process Z_t by

$$Z_t = f(t, X_t)$$

Problem: What does $df(t, X_t)$ look like?

The answer is given by the **Itô formula**.

A close up of the Wiener process

Consider an "infinitesimal" Wiener increment

$$dW = W(t + dt) - W(t)$$

We know:

$$dW \sim N[0, \sqrt{dt}]$$

$$E[dW] = 0, \quad Var[dW] = dt$$

From this one can show

$$E[(dW)^2] = dt, \quad Var[(dW)^2] = 3(dt)^2$$

Important observation:
1. Both $E[(dW)^2]$ and $Var[(dW)^2]$ are very small when dt is small .
2. $Var[(dW)^2]$ is negligeable compared to $E[(dW)^2]$.
3. Thus $(dW)^2$ is **deterministic**.

$$(dW)^2 = dt$$

Multiplication table.

- $$(dt)^2 = 0$$

- $$dW \cdot dt = 0$$

- $$(dW)^2 = dt$$

Deriving the Itô formula

$$dX_t = \mu_t dt + \sigma_t dW_t$$

$$Z_t = f(t, X_t)$$

We want to compute $df(t, X_t)$ (i.e. the change in $f(t, X_t)$)

Make a Taylor expansion of $f(t, X_t)$ including second order terms:

$$\begin{aligned} df &= \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX + \frac{1}{2} \frac{\partial^2 f}{\partial t^2} (dt)^2 \\ &+ \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX)^2 + \frac{\partial^2 f}{\partial t \partial x} dt \cdot dX \end{aligned}$$

Plug in the expression for dX, expand, and use the multiplication table!

We get:

$$\begin{aligned} df &= \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}[\mu dt + \sigma dW] + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}[\mu dt + \sigma dW]^2 + \frac{\partial^2 f}{\partial t \partial x}dt \cdot [\mu dt + \sigma dW] \\ &= \frac{\partial f}{\partial t}dt + \mu\frac{\partial f}{\partial x}dt + \sigma\frac{\partial f}{\partial x}dW + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial x^2}[\mu^2(dt)^2 + \sigma^2(dW)^2 + 2\mu\sigma dt \cdot dW] \\ &+ \mu\frac{\partial^2 f}{\partial t \partial x}(dt)^2 + \sigma\frac{\partial^2 f}{\partial t \partial x}dt \cdot dW \end{aligned}$$

Using the multiplikation table this reduces to:

$$\boxed{\begin{aligned} df &= \left\{\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right\}dt \\ &+ \sigma\frac{\partial f}{\partial x}dW \end{aligned}}$$

Itô's formula

$$
\begin{aligned}
dX_t &= \mu_t dt + \sigma_t dW_t \\
Z_t &= f(t, X_t)
\end{aligned}
$$

$$
\boxed{\begin{aligned}
df &= \left\{\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2 f}{\partial x^2}\right\} dt \\
&+ \sigma\frac{\partial f}{\partial x}dW
\end{aligned}}
$$

Alternatively

$$
df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dX)^2,
$$

where we use the multiplication table.

A Useful Trick

Problem: Compute $E\left[Z(T)\right]$.

- Use Itô to get

$$dZ(t) = \mu_Z(t)dt + \sigma_Z(t)dW_t$$

- Integrate.

$$Z(T) = z_0 + \int_0^T \mu_Z(t)dt + \int_0^T \sigma_Z(t)dW_t$$

- Take expectations.

$$E\left[Z(T)\right] = z_0 + \int_0^T E\left[\mu_Z(t)\right]dt + 0$$

- The problem has been reduced to that of computing $E\left[\mu_Z(t)\right]$.

The Black-Scholes model

Price dynamics:(Geometrical Brownian Motion)

$$dS = \alpha S dt + \sigma S dW,$$

Simple analysis:
Assume that $\sigma = 0$. Then

$$dS = \alpha S dt$$

Divide by dt!

$$\frac{dS}{dt} = \alpha S$$

Simple ordinary differential equation with solution

$$S_t = s_0 e^{\alpha t}$$

Conjecture: The solution of the SDE above is a randomly disturbed exponential function.

Economic Interpretation

$$\frac{dS}{S} = \alpha dt + \sigma dW$$

Over a small time interval $[t, t+dt]$ this means:

$$\begin{aligned} \text{Return} &= \text{(mean return)} \\ &+ \sigma \times \text{(Gaussian random disturbance)} \end{aligned}$$

- The asset **return** is a random walk (with drift).

- α = mean rate of return per unit time

- σ = "volatility"

Large σ = large random fluctuations

Small σ = small random fluctuations

We will se that:

$$S(t) = S(0)e^{(\alpha - \frac{1}{2}\sigma^2)t + \sigma W(t)}$$

Stock prices are lognormally distributed.

Returns are normally distributed.

Example GBM

$$dS = \alpha S dt + \sigma S dW$$

We smell something exponential!

Natural Ansatz:

$$\begin{aligned} S(t) &= e^{Z(t)}, \\ Z(t) &= \ln S(t) \end{aligned}$$

Itô on $f(t,s) = \ln(s)$ gives us

$$\frac{\partial f}{\partial s} = \frac{1}{s}, \quad \frac{\partial f}{\partial t} = 0, \quad \frac{\partial^2 f}{\partial s^2} = -\frac{1}{s^2}$$

$$\begin{aligned} dZ &= \frac{1}{S} dS - \frac{1}{2} \frac{1}{S^2} (dS)^2 \\ &= \left(\alpha - \frac{1}{2}\sigma^2 \right) dt + \sigma dW \end{aligned}$$

Integrate!

$$
\begin{aligned}
S(t) - S(0) &= \int_0^t \left(\alpha - \frac{1}{2}\sigma^2\right) d\tau + \sigma \int_0^t dW(s) \\
&= \left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma W(t)
\end{aligned}
$$

Using $S = e^Z$ gives us

$$
S(t) = S(0) e^{\left(\alpha - \frac{1}{2}\sigma^2\right) t + \sigma W(t)}
$$

Since $W(t)$ is $N[0, \sqrt{t}]$, we see that $S(t)$ has a lognormal distribution.

The Connection SDE $\sim$ PDE

Given: $\mu(t,x), \quad \sigma(t,x), \quad \Phi(x), \quad T$

Problem: Find a function F solving the Partial Differential Equation (PDE)

$$
\begin{aligned}
\frac{\partial F}{\partial t}(t,x) + \mathcal{A}F(t,x) &= 0, \\
F(T,x) &= \Phi(x).
\end{aligned}
$$

where $\mathcal{A}$ is defined by

$$
\mathcal{A}F(t,x) = \mu(t,x)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x)
$$

Assume that F solves the PDE.

Fix the point (t, x).

Define the process X by

$$\begin{aligned} dX_s &= \mu(s, X_s)dt + \sigma(s, X_s)dW_s, \\ X_t &= x, \end{aligned}$$

Apply Ito to the process $F(t, X_t)$!

$$\begin{aligned} F(T, X_T) &= F(t, X_t) \\ &+ \int_t^T \left\{ \frac{\partial F}{\partial t}(s, X_s) + \mathcal{A}F(s, X_s) \right\} ds \\ &+ \int_t^T \sigma(s, X_s) \frac{\partial F}{\partial x}(s, X_s) dW_s. \end{aligned}$$

By assumption $\frac{\partial F}{\partial t} + \mathcal{A}F = 0$, and $F(T, x) = \Phi(x)$

Thus:

$$\begin{aligned}\Phi(X_T) &= F(t,x) \\ &+ \int_t^T \sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)dW_s.\end{aligned}$$

Take expectations.

$$F(t,x) = E_{t,x}\left[\Phi\left(X_T\right)\right],$$

Feynman-Kač

The solution $F(t,x)$ to the PDE

$$\begin{aligned}\frac{\partial F}{\partial t}+\mu(t,x)\frac{\partial F}{\partial x}+\frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}-rF &= 0,\\ F(T,x) &= \Phi(x).\end{aligned}$$

is given by

$$F(t,x)=e^{-r(T-t)}E_{t,x}\left[\Phi\left(X_T\right)\right],$$

where X satisfies the SDE

$$\begin{aligned}dX_s &= \mu(s,X_s)dt+\sigma(s,X_s)dW_s,\\ X_t &= x.\end{aligned}$$