Finance II

May 27, 2005

10.00-15.00

All notation should be clearly defined.
Arguments should be complete and careful.

1. **(a)** Solve the boundary value problem

$$\begin{aligned}\frac{\partial F}{\partial t}(t,x) + \alpha x \frac{\partial f}{\partial x} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2}(t,x) &= 0,\\ F(T,x) &= \ln(x).\end{aligned}$$

Here σ and α are a known constants. You may use the Feynman-Kač Representation Theorem without proving it. 5p

(b) Let X and Y be any processes possessing stochastic differentials. We let

$$\int_0^t Y(s)dX(s)$$

denote the standard Itô integral. Now we define the *Stratonovich Integral*

$$\int_0^t Y(s) \circ dX(s)$$

by the following formula

$$\int_0^t Y(s) \circ dX(s) = \int_0^t Y(s)dX(s) + \frac{1}{2}\int_0^t \left[dX(s) \cdot dY(s)\right].$$

In the last integral we use the standard multiplication rules for products of differentials, i.e. $dWdt = 0$, $(dW)^2 = dt$, etc. Now assume that X has a differential of the form

$$dX(t) = \mu(t)dt + \sigma(t)dW(t),$$

and that the deterministic function $F(t,x)$ is continuously differentiable, once in the t-variable, and *three* times in the x-variable. The nice thing about the Stratonovich integral is that for this integral concept we have the standard form of the chain rule, as opposed to the Itô formula with the irritating second order term. More precisely the following hold

$$dF(t, X(t)) = \frac{\partial F}{\partial t} dt + \frac{\partial F}{\partial x} \circ dX.$$

Prove this formula! 10p

(c) Let X be an arbitrary strictly prositive Itô process possessing a stochastic differential. Compute, in terms of X, the Stratonovich integral

$$\int_0^t \frac{1}{X(s)} \circ dX(s)$$

.... 5p

2. Consider the standard Black-Scholes model

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

Prove that every contract of the form $\Phi(S_T)$ can be replicted. .... (20p)

3. Consider the following SDE under the objective probability measure P.

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$$

where $\mu(x)$ and $\sigma(x)$ are deterministic given functions. We interpret X as a *non priced* underlying object, and now we want to price derivatives defined in terms of X. One possibility is to interpret X_t as the temerature (at time t) at a particular beach at Tylösand. Now consider the contingent claim

$$Z = \Phi(X_T)$$

where $\Phi(x)$ is some given deterministic function. In our interpetation this could be interpreted as a holidy insurance, which gives the holder an amount of money if the temperature is below some benchmark value.

(a) Is it possible to obtain a unique arbitrage free price process $\Pi(t; Z)$ for the derivative above? You must motivate your answer. 5p

(b) Analyze the situation above as completely as possible. In particular you should come up with a PDE for pricing of derivatives. Indicate clarly which objects in the PDE that are known to you, and how it in priciple would be possible to obtain further necessary information. . . . . (15p)

4. Consider a standard Black-Scholes market, i.e., a market consisting of a risk free asset, B, with P-dynamics given by

$$\begin{cases} dB_t &= rB_t dt, \\ B_0 &= 1, \end{cases}$$

and a stock, S, with P-dynamics given by

$$\begin{cases} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ S_0 &= s_0. \end{cases}$$

Here W denotes a P-Wiener process and r, α, and σ are assumed to be constants.

(a) Recall that a forward contract on S_T contracted at time t, with time of delivery T, and with the forward price $f(t; T, S_T)$ can be seen as a contingent T-claim X with payoff

$$X = S_T - f(t; T, S_T),$$

where the forward price is determined at time t in such a way that the price of X is zero at time t, i.e. $\Pi(t; X) = 0$.

Compute the forward price $f(t; T, S_T)$ in the Black-Scholes model. . . . . (5p)

(b) A *break forward contract* is a T-claim designed to limit the potential loss of a long position in a forward contract by a prespecified amount. More specifically the payoff X from a long position in a break forward is defined by

$$X = \max\{S_T, f(0; T, S_T)\} - K,$$

where $f(0; T, S_T)$ is the forward price of the stock for settlement at time T, and $K > f(0; T, S_T)$ is some constant. The delivery price K is set in such a way that the break forward contract is worthless when it is entered into.

Compute the delivery price K. . . . . (5p)

(c) Compute the arbitrage price of an *exchange option* with exercise date T_0. This is a contract which gives the owner the right, but not the obligation, to exchange a put option written on the stock for a call option written on the stock Both the call and the put option are assumed to have exercise date $T > T_0$, and exercise price K. . . . . (5p)

(d) A *chooser option* is an agreement which gives the owner the right to choose at some prespecified future date T_0 whether the option is to be a call or put option with exercise price K and remaining time to expiry $T - T_0$. Note that K, T and $T_0 < T$ are all prespecified by the agreement. The only thing the owner can choose is whether the option should be a call or a put option, and this choice has to be made at time T_0. Compute the arbitrage price of a chooser option. . . . . (5p)

Note: The Black Scholes formulas for put and call options are assumed to be known and can (if needed) be referred to without proof.

5. Consider an interest model described (under Q) by

$$dr = \mu(t, r)dt + \sigma(t, r)dV$$

(a) Define what is meant by an *Affine Term Structure* (ATS), and derive conditions which are sufficient to guarantee the existence of an ATS for the model above. 10p

(b) Now consider the Ho-Lee model

$$dr = \Phi(t)dt + \sigma dV$$

Show explicitely how this model can be fitted to an initially observed price curve $\{p^\star(0, T);\ T \geq 0\}$ which is assumed to be smooth enough. 10p

Black-Scholes formula:

$$F(t, s) = sN\left[d_1(t, s)\right] - e^{-r(T-t)}KN\left[d_2(t, s)\right].$$

Here N is the cumulative distribution function for the $N[0,1]$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s)-\sigma\sqrt{T-t}. \end{aligned}$$