# Finance II

## May 28, 2004

## 10.00-15.00

All notation should be clearly defined.
Arguments should be complete and careful.

1. Consider the following boundary value problem in the domain $[0,T]\times R$ for an unknown function $F(t,x)$.

$$\begin{aligned} \frac{\partial F}{\partial t}(t,x) + \mu(t,x)\frac{\partial F}{\partial x}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) + F(t,x)k(t,x) &= 0, \\ F(T,x) &= \Phi(x). \end{aligned}$$

   Here $\mu$, $\sigma$, $k$ and $\Phi$ are assumed to be known functions.

   Derive a Feynman-Kač representation for this problem. In this formula it must be quite clear exactly at which points the various functions should be evaluated. In other words - if you suppress variables you must explain **very** clearly exactly what you mean. ........(20p)

   If you think this problem is hard you may (with loss of 7 points) assume that $k = 0$.

2. Consider the standard Black-Scholes model

$$\begin{aligned} dS_t &= \alpha S_t dt + \sigma S_t dW_t, \\ dB_t &= r B_t dt. \end{aligned}$$

   Derive the Black-Scholes partial differential equation for the price of a contract of the form $\Phi(S_T)$. ................................(20p)

3. Consider two markets: a domestic market with short rate $r_d$, and a foreign market with short rate $r_f$. The exchange rate $X$ is defined as the domestic price of one unit of the foreign currency.

We take as given the following dynamics (under the objective probability measure $P$)

$$\begin{aligned} dX &= X\alpha_X dt + X\sigma_X dW, && (1) \\ dB_d &= r_d B_d dt, && (2) \\ dB_f &= r_f B_f dt, && (3) \end{aligned}$$

where $r_d$, $r_f$,$\alpha_X$, $\sigma_X$ are deterministic constants, and $W$ is a scalar Wiener process.

We want to price (in domestic terms) a European Call Option on one unit of the foreign currency, with strike price $K$ and exercise date $T$. The contract $Y$ is thus given by

$$Y = max[X(T) - K, 0]$$

Derive an explicit pricing formula for the option. The standard Black-Scholes formula for stock options (see below) can be used without further motivation. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (20p)

4. Consider a bond model under the objective probability measure $P$ of the form

$$dp(t,T) = \mu(t,T)p(t,T)dt + p(t,T)\gamma(t,T)dW(t),$$

where $W$ is a Wiener process. Now let $Q$ denote a martingale mesure for the bond market above, and consider a *consol bond*, i.e. a bond which forever gives you an income stream of 1 dollar per unit time. In other words: your income during an interval of the type $[t, t+dt]$ is given by $1 \cdot dt$. Denote the arbitrage free price of a consol by $C_t$, and derive the $Q$-dynamics of $C$. . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . (20p)

5. Consider a model with a stock price process $S_t$, and a corresponding cumulative dividend process $D_t$. The $P$-dynamics are given by

$$\begin{aligned} dS &= \alpha S dt + \sigma S dW, \\ dD &= \beta S dt + \gamma S dW \end{aligned}$$

where $\alpha$, $\sigma$, $\beta$ and $\gamma$ are given constants. Note that the same Wiener process $W$ is driving boh $S$ and $D$.

Now assume that you start at time $t = 0$ with a wealth of $S_0$, so you can buy exactly one unit of the stock. Consider the following portfolio strategy:

- The only asset you ever hold in your portfolio is the stock.
- All gains from the dividends are reinvested in the stock.
- The portfolio is self financing.

Now let $h_t$ as denote the number of units of the stock $t$ in the portfolio above, so in particular we have $h_0 = 1$. Derive an expression (as explicit as possible) for $h_t$. .................................. (20p)

**Black-Scholes formula:**

$$F(t,s) = sN\left[d_1(t,s)\right] - e^{-r(T-t)}KN\left[d_2(t,s)\right].$$

Here $N$ is the cumulative distribution function for the $N[0,1]$ distribution and

$$\begin{aligned} d_1(t,s) &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{s}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2(t,s) &= d_1(t,s) - \sigma\sqrt{T-t}. \end{aligned}$$